Math 445, Fall 2009 Exercise Set #7 Turn in: November 20

1. Let X be a topological space, and suppose that $A, B \subset X$ are compact subsets. Show that $A \cup B$ is again compact.

2. Let X, Y be topological spaces, $f\colon X \to Y$ a continuous map, and assume that X is *compact* and Y is *Hausdorff*. Show that f is a closed map. That is, if $F \subset X$ is closed, then $f(F) \subset Y$ is closed.

3. [#3, page 103] Let $f\colon X \to Y$ be a continuous one-to-one mapping of a compact metric space X onto a metric space Y. Prove that $f^{-1}\colon Y \to X$ is continuous (and thus f is a homeomorphism.)

4. Show that a topological space X is Hausdorff if and only if the diagonal

$$\Delta(X) = \{(x, x) \mid x \in X\} \subset X \times X$$

is closed for the product topology.

The graph of a function $f\colon X \to Y$ is the set $\mathcal{G}_f = \{(x, f(x)) \mid x \in X\}$.

5. Let X, Y be topological spaces, and suppose that Y is compact Hausdorff.

a) Show that if $f\colon X \to Y$ is continuous, then the graph $\mathcal{G}_f$ of f is closed in $X \times Y$.

b) Show that if the graph $\mathcal{G}_f$ of f is closed, then $f\colon X \to Y$ is continuous.

6. [#5, page 104] Let $A, B \subset X$ be disjoint subsets of a metric space X. Suppose that A is closed, and B is compact. Prove that the distance between A and B is positive. That is, show that

$$\inf \{D(a, b) \mid a \in A,\ b \in B\} > 0$$

7. Let $\{X, d\}$ be a compact metric space. Suppose that $f\colon X \to X$ is a function which satisfies

$$d(f(x), f(y)) \ < \ d(x, y) \text{ for all } x \neq y$$

Prove that f has a unique fixed point.