Math 313 , Introduction to Analysis
uniform continuity - worked example 2

Example: Show that the function $f(x) = \ln(x)$ on the interval $[1, \infty)$ is uniformly continuous.

Solution: Given $\epsilon > 0$ we want to find $\delta > 0$ such that for all $1 \leq c$ and any $|h| < \delta$, then $|f(c+h) - f(c)| < \epsilon$. By the definition of $f(x)$ this means we need

$$|f(c+h) - f(c)| = |\ln(c+h) - \ln(c)| < \epsilon$$

The first problem is now, we need a definition of the function $\ln(x)$! Recall from Calculus I that $\ln(x)$ is the unique differentiable function defined for $x > 0$ satisfying the two conditions

$$\ln(1) = 0 \text{ and } \ln'(x) = \frac{1}{x}$$

Since $\ln(x)$ is differentiable at every point $x > 0$, it is also continuous at every point $x > 0$.

We also need the rules of $\ln(x)$, that $\ln(a \cdot b) = \ln(a) + \ln(b)$.

Then since $\ln(1) = 0$, this implies $0 = \ln(b/b) = \ln(b) - \ln(1/b)$, and so $\ln(1/b) = -\ln(b)$.

Combining these two properties we get $\ln(a/b) = \ln(a) - \ln(b)$.

Now we can rewrite the estimate we need to be

$$|f(c+h) - f(c)| = |\ln(c+h) - \ln(c)| = |\ln\left(\frac{c+h}{c}\right)| = |\ln\left(1 + \frac{h}{c}\right)| < \epsilon$$

When $h \to 0$ the argument $1 + \frac{h}{c} \to 1$, so we need continuity of $\ln(x)$ at $x = 1$, or continuity of the composition $y \mapsto \ln(1+y)$ at $y = 0$. This is true since $\ln(1+y)$ has a derivative at $y = 0$ so it is continuous at $y = 0$. We write out what this means:

For $\epsilon > 0$ given, there is some $\lambda > 0$ so that

$$|y| = |y - 0| < \lambda \implies |\ln(1+y) - \ln(1)| = |\ln(1+y)| < \epsilon$$

This means that if $|\frac{h}{c}| < \lambda$ then $|\ln(1 + \frac{h}{c})| < \epsilon$. Rewrite this as $|h| < c \cdot \lambda$ then $|\ln(1 + \frac{h}{c})| < \epsilon$.

Since $c \geq 1$, if we assume $|h| < \lambda$ then $|h| < c \cdot \lambda$, and then $|\ln(1 + \frac{h}{c})| < \epsilon$.

We need to find $\delta > 0$ so the conclusion $|\ln(1 + \frac{h}{c})| < \epsilon$ is true, so take $\delta = \lambda$, the same λ chosen using continuity of $\ln(1+y)$ at $y = 0$. Then

$$|h| < \delta \Longrightarrow |\ln(1 + \frac{h}{c})| < \epsilon \Longleftrightarrow |\ln(c+h) - \ln(c)| < \epsilon$$