Improper Integrals [1]

The integrals considered so far $\int_a^b f(x)\,dx$ assume implicitly that a and b are finite numbers and that the function $f(x)$ is nicely behaved on the interval. *Improper* integrals arise when

- The function $f(x)$ blows up (goes to $\pm\infty$) at one of the endpoints, or
- One of the end points a and/or b is infinite,
- A combination of both of the above

Examples.

- Find the total area under the curve $y = xe^{-x}$, $0 \le x < \infty$.

The integral to calculate is

$$\int_0^\infty xe^{-x}\,dx.$$

Since $\int xe^{-x}\,dx = -xe^{-x} - e^{-x}$, the *area out to* b is $\int_0^b te^{-t}\,dt = (-te^{-t} - e^{-t})|_{t=0}^{t=b} = -be^{-b} - e^{-b} + 1$, which tends to 1 as b tends to ∞. Thus the area is finite and should be said to be 1. We say the improper integral $\int_0^\infty xe^{-x}\,dx$ *converges* to the value 1.

We are really calculating $\int_0^\infty xe^{-x}\,dx = (-xe^{-x} - e^{-x})|_0^\infty$ and interpreting the expression $(-xe^{-x} - e^{-x})$ at $x = \infty$ as 0 *in the sense of limits*.

- Find the total area under the curve $y = \frac{1}{x\ln(x)}$, $e \le x < \infty$.

The integral to calculate is

$$\int_e^\infty \frac{1}{x\ln(x)}\,dx.$$

Since $\int \frac{1}{x\ln(x)}\,dx = \ln(\ln(x)) + C$, $\int_e^\infty \frac{1}{x\ln(x)}\,dx = \ln(\ln(x))|_{x=e}^{x=\infty}$, and $\ln(\ln(\infty))$ is to be interpreted as ∞ *in the sense of limits*.

We thus say that the improper integral *diverges* [to ∞] and the total area is infinite. Note that

$$\int_1^\infty \frac{1}{x\ln(x)}\,dx$$

is improper for an additional reason - at the initial end point $x = 1$, $\ln(1) = 0$.

- The improper integral $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$

converges because $\arcsin(x)|_{x=0}^{x=1}$ can be evaluated because $\arcsin(1) = \frac{\pi}{2}$ *in the sense of limits*. The integral is improper since the integrand *blows up* near the right hand end point.

The Method

- In each of the examples, we took the lim as $x \to$ singular point of the anti-derivative function created *after* the integration.

[1] This note was written by J. Lewis, and slightly revised by S. Hurder 1/23/2001

Comparison Tests

For non-negative functions $f(x)$, the improper integrals $\int_{\cdot}^{\infty} f(x)\,dx$ converges if and only if the approximating integrals $\int_{\cdot}^{b} f(t)\,dt$ are bounded as $b \to \infty$. This is because $\int_{\cdot}^{b} f(t)\,dt = F(b) - F(\cdot)$ and $F(b)$ is increasing and $F(b)$ has a limit if and only if $F(b)$ is bounded.

A similar statement can be made for integrals of the form $\int_a^{\cdot} f(x)\,dx$ ($f(x)$ blows up at a) or $\int_{\cdot}^{b} f(x)\,dx$ (f(x) blows up at b).

- If $0 \le f(x) \le g(x)$, then $0 \le \int_{\cdot}^{\cdot} f(x)\,dx \le \int_{\cdot}^{\cdot} g(x)\,dx$
- If $0 \le f(x) \le g(x)$, and $\int_{\cdot}^{\cdot} g(x)\,dx$ converges, then $\int_{\cdot}^{\cdot} f(x)\,dx$ converges also.
- If $0 \le f(x) \le g(x)$, and $\int_{\cdot}^{\cdot} f(x)\,dx$ diverges, then $\int_{\cdot}^{\cdot} g(x)\,dx$ diverges also.

p-tests for improper integrals:

- $\int_a^{\infty} \frac{1}{x^p}\,dx \begin{cases} \text{converges if } & \text{p} > 1, \\ \text{diverges if } & \text{p} \le 1. \end{cases}$
- $\int_0^{b} \frac{1}{x^p}\,dx \begin{cases} \text{converges if } & \text{p} < 1, \\ \text{diverges if } & \text{p} \ge 1. \end{cases}$
- $\int_0^{\infty} \frac{1}{x^p}\,dx$ diverges for all p.

Examples

- $\int_1^{\infty} \frac{\cos^2(\phi)}{\phi^2}\,d\phi$ converges by comparison with $\int_1^{\infty} \frac{1}{\phi^2}\,d\phi$ or simply by the $p-$ test with $p = 2$.
- $\int_1^{\infty} \frac{\cos^2(\phi)}{\phi}\,d\phi$ diverges, but cannot be handled directly by the comparison with $\int_1^{\infty} \frac{1}{\phi}\,d\phi$.
- $\int_0^{\infty} e^{\frac{-x^2}{2}}\,dx$ The integral converges since $e^{\frac{-x^2}{2}} \ll e^{-x}$ for large x, and $\int_{\cdot}^{\infty} e^{-x}\,dx$ converges.
- $\int_{-4}^{3} \frac{1}{x^2}\,dx$ DIVERGES.