INTRODUCTION

The purpose of this thesis is to construct a new theory of foliation invariants using Sullivan's theory of minimal models [63], to relate the new invariants obtained this way with the theory of secondary characteristic classes of foliations, and then to use this relationship to answer questions raised by both theories. The dual homotopy invariants which we construct can be thought of as a homotopy theoretic version of the cohomology characteristic classes of a foliation. It is not surprising that a link can be forged between a theory of invariants developed via the theory of minimal models, and the secondary characteristic class theory as developed by Kamber and Tondeur [38], since both theories rely heavily on the techniques of H. Cartan [12] and Koszul [41], and have a common root in the work of Chern and Simons [15].

This marriage of techniques from Sullivan's theory and that of Kamber and Tondeur proves very fruitful; we are able to establish the non-triviality of the secondary classes of foliations in many geometric contexts.

We show for the first time that there exists a foliated manifold with a non-trivial rigid class (Proposition 8.3), and are thus able to answer a question posed by Lawson (Remark 8.4). We also give a considerable extension of the results of Heitsch [32] on the independence and variability of the secondary classes for foliations (Corollary 8.13 and Theorem 8.14).

We prove that all of the dual homotopy invariants of Riemannian foliations are non-trivial, which implies that all of the indecomposable secondary classes of Riemannian foliations are non-trivial (Theorem 9.4).

Since the introduction by Haefliger of the classifying space $B\widetilde{\Gamma}_G$ of G-foliations, a central problem has been to determine the homotopy groups of this space [24], [43]. This problem is addressed repeatedly in this thesis, beginning in Chapter 6 where we show $B\tilde{\Gamma}_G$ and BG have the same homotopy groups in degrees less than q, the codimension of the foliation. That is, the fiber $F\tilde{\Gamma}_G$ of the map $\nu: B\tilde{\Gamma}_G \to BG$ is shown to be (q-1)-connected.

Various theorems regarding the rank of the homotopy groups $\pi_n(B\tilde{\Gamma}_G)$ are given; one of our main results asserts there are epimorphisms of abelian groups $\pi_n(B\tilde{\Gamma}_G) \to \mathbb{R}^{v_n}$, where $\{v_n\}$ is a sequence of integers with a subsequence tending to infinity.

An Outline of the Thesis

We now give a brief survey of the theory developed in this thesis. Let M be a manifold with a G-foliation $\mathfrak{F}$ of codimension q. The Bott Vanishing Theorem [5] implies the Chern-Weil homomorphism h of the normal bundle of $\mathfrak{F}$ defines a differential algebra map $h: I(G)_q \to \Omega(M)$ from the truncated polynomial algebra $I(G)_q$ to the deRham complex of M. It is shown in Theorem 4.4 that this determines a characteristic map

$$h^{\#}: \pi^*(I(G)_q) \to \pi^*(M)$$

of the concordance class of $\mathfrak{F}$. Here, $\pi^*(I(G)_q)$ denotes the algebraic dual homotopy of $I(G)_q$; this is an infinite dimensional space for $q > 1$. If M is simply connected, then there is a natural isomorphism [62]

$$\pi^*(M) \cong \mathrm{Hom}(\pi_*(M), \mathbb{R}) \ .$$

For this reason, the elements in the image of $h^{\#}$ are called the dual homotopy invariants of the foliation.

Depending on the foliation $\mathfrak{F}$, there exists a subgroup H of G, where H is usually taken to be either the trivial group $\{e\}$ or a maximal compact subgroup of G, for which a characteristic map is defined [38],

$$\Delta_*: H^*(A(G,H)_q) \to H^*(M) \ ,$$

whose image consists of the secondary characteristic classes of $\mathfrak{F}$. The dual of the Hurewicz homomorphism gives a map $\mathcal{H}^*: H^*(M) \to \pi^*(M)$, and the composition $\mathcal{H}^* \circ \Delta_*$ defines dual homotopy invariants of $\mathfrak{F}$. These are related to the elements in the image of $h^\#$ by Proposition 4.24, which asserts there is a universal map

$$\zeta: H^*(A(G,H)_q) \to \pi^*(I(G)_q)$$

such that $\mathcal{H}^* \circ \Delta_* = h^\# \circ \zeta$. In terms of the classifying spaces of G-foliations, this is given by the following commutative diagram

$$\begin{array}{ccc} \pi^*(I(G)_q) & \xrightarrow{\tilde{h}^\#} & \pi^*(B\tilde{\Gamma}_G) \\ \zeta \Big\uparrow & & \Big\uparrow \mathcal{H}^* \\ H^*(A(G,\{e\})_q) & \xrightarrow{\tilde{\Delta}_*} & H^*(F\tilde{\Gamma}_G) \end{array} \quad .$$

From this diagram, it is seen that the determination of the map $\tilde{h}^\#$ yields information about the map $\tilde{\Delta}_*$. Conversely, conditions for the injectivity of $\mathcal{H}^*$ are established in Theorem 5.1 so that results on $\tilde{\Delta}_*$ translate into similar results about $\tilde{h}^\#$. This give and take process is used repeatedly to gain information about the non-triviality of both maps, $\tilde{h}^\#$ and $\tilde{\Delta}_*$. This is the idea behind Propositions 5.5 through 5.7 and 5.10 through 5.12.

To facilitate the analysis of the map $\tilde{h}^{\#}$ for the case of Riemannian foliations, where $G = SO(q)$ or $O(q)$, we were led to Theorem 6.1, which states that the classifying space $F\tilde{\Gamma}_G$ is $(q-1)$-connected for any group G.

Using Theorem 6.1 with $G = SO(q)$, we are able to show $\tilde{h}^{\#}$ is injective (Theorem 9.1), hence the kernel of $\tilde{\Delta}_*$ is contained in the kernel of ζ, which consists of the decomposable secondary classes. Further results on the variability of the classes in the image of

$$\tilde{\Delta}_*: H^*(A(SO(q),e)_{[q/2]}) \to H^*(F\tilde{\Gamma}^q_{SO(q)})$$

are also established (Theorem 9.4), using a theorem of Lazarov and Pasternack [46].

For the classifying space of real foliations, $B\Gamma^q$, where $G = G\ell(q,\mathbb{R})$, we show the results of Heitsch [32] on the variability and independence of the classes in the image of $\tilde{\Delta}_*$ imply the image of $\tilde{h}^{\#}$ contains many non-trivial, variable classes (Proposition 8.11 and Theorem 8.12). From this, we obtain a significant extension of the results of Heitsch on the variability and independence of the classes in the image of Δ_* (Corollary 8.13 and Theorem 8.14).

Chapter 10 gives a similar extension of the results of Baum and Bott [3] on the independence of the secondary classes in the image of Δ_* for integrable complex foliations (Theorem 10.7).

For each of the three types of G-foliations considered, namely real, Riemannian or integrable complex, it is shown that the homotopy groups of the corresponding classifying space $B\tilde{\Gamma}_G$ are uncountably generated; there are epimorphisms of abelian groups $\pi_n(B\tilde{\Gamma}_G) \to \mathbb{R}^{v_n}$, where $\{v_n\}$ is a sequence of non-negative integers depending on G, and has a subsequence $\{v_{n_j}\}$ which tends to infinity (Theorems 8.15, 9.6 and 10.8). These theorems can be interpreted as saying that for n arbitrarily large, there exists an open manifold M with the homotopy type of an n-sphere such that there are $\mathbb{R}^{v_n}$ distinct concordance classes of foliations on M. This follows because there are v_n distinct dual homotopy invariants for these foliations which can assume any real value independently. The dual homotopy invariants thus provide a means for showing, given a fixed codimension q, that there are foliated manifolds with arbitrarily high dimension and CW-dimension having non-trivial foliations. This contrasts to the secondary cohomology invariants, which lie in the cohomology groups of M of degree $\leq q^2 + 2q$. For M with the homotopy type of an n-sphere, these invariants would all vanish.

Finally, in Chapter 7, it is shown that if a codimension q foliation $\mathfrak{F}$ on a manifold M is defined by a mapping $f:M \to N$ to a p-dimensional foliated manifold N with $p \leq 2q$, then all the dual homotopy and secondary invariants of $\mathfrak{F}$ are rational valued (Theorems 7.1 and 7.3). For Riemannian or integrable complex foliations, it is required that p = q and f be a submersion. In the interpretation of Theorems 8.15, 9.6 and 10.8 given above, this has the consequence that almost all of the $\mathbb{R}^{v_n}$-distinct foliations on $M \simeq S^n$ are not defined by a submersion, or by a map to a manifold N of dimension $\leq 2q$.

Notation and Conventions

Throughout this thesis the following conventions will be used. We let $\mathbb{k}$ denote either the field of rational numbers $\mathbb{Q}$, of real numbers $\mathbb{R}$ or of complex numbers $\mathbb{C}$. Given a real number x, the symbol [x] denotes the greatest integer $n \leq x$. For a given subset $\mathcal{V}$ of a vector space V, we denote by $\langle\mathcal{V}\rangle$ the subspace of V spanned by the set $\mathcal{V}$.

If V is a graded vector space, the subspace of elements of degree p in V is denoted by V^p, so that $V \cong \oplus V^p$. An element of V^p is said to be homogeneous of degree p.

All algebras have an identity denoted by 1. An algebra A is associative, graded and commutative, unless otherwise noted. For two elements a in A^p and b in A^q, the commutativity of A is given by $a \cdot b = (-1)^{pq} b \cdot a$. For an element a in A^p, we use Ja to denote $(-1)^p a$. A <u>differential algebra</u> is an algebra A with a derivation d_A of degree $+1$ satisfying $d_A^2 = 0$ and $d_A(a \cdot b) = d_A a \cdot b + Ja \cdot d_A b$. A differential algebra will be denoted (A, d_A) when it is necessary to emphasize the differential d_A.

An algebra A is said to be of <u>finite type</u> if each vector space A^p is finite dimensional. An algebra A is said to be <u>connected</u> if $A^0 \cong k$, and <u>n-connected</u> if it is connected and $A^p = \{0\}$ for $1 \leq p \leq n$.

A differential algebra (A, d_A) is <u>h-connected</u> if $H^*(A)$ is a connected algebra.

A map $f: A \to B$ of differential algebras is a <u>weak isomorphism</u> if $f_*: H^*(A) \to H^*(B)$ is an isomorphism of algebras.

We use $A \otimes B$ to denote the tensor product algebra of A and B.

All topological spaces are assumed to be connected, and all maps between spaces are continuous. When necessary, it is assumed that each space has a base point and that maps preserve base points. Given topological spaces X and Y, let $[X,Y]$ denote the set of (pointed) homotopy classes of maps from X to Y.

For a topological space X, we use $H^*(X)$ to denote the singular cohomology of X with coefficients in k. By $H^*(X;R)$ we mean the singular cohomology of X with coefficients in a ring R. A space X is said to be of <u>finite-type</u> if the algebra $H^*(X)$ is of finite-type.

All manifolds are C^∞ and maps between manifolds are C^∞. Unless otherwise noted, a manifold M is assumed to be paracompact and Hausdorff. We use $\Omega^*(M)$ to denote the deRham complex of M.

CHAPTER 1

G-FOLIATIONS

The objects of our study will be the G-foliations on a manifold M, a generalization of the usual notion of a foliation on M.

1.1 DEFINITION. A <u>codimension</u> q <u>foliation</u> on a manifold M^n is an integrable subbundle $\mathfrak{F}$ of rank (n-q) of the tangent bundle TM.

An equivalent way of defining a foliation on M is given by the Frobenius Theorem [40]:

1.2 THEOREM. Let $\mathfrak{F}$ be a codimension q foliation on M. Then there is an open covering $\{U_\alpha | \alpha \in \mathcal{A}\}$ of M and local submersions $\phi_\alpha : U_\alpha \to \mathbb{R}^q$ such that $\mathfrak{F}|_{U_\alpha} = \ker d\phi_\alpha$ for each $\alpha \in \mathcal{A}$.

The collection $\{\phi_\alpha : U_\alpha \to \mathbb{R}^q | \alpha \in \mathcal{A}\}$ is called a <u>defining system of charts</u> for $\mathfrak{F}$. For each $\alpha, \beta \in \mathcal{A}$ with $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \phi$, and each $x \in U_{\alpha\beta}$, by the Implicit Function Theorem there is a local diffeomorphism $\gamma_{\alpha\beta}(x)$ from a neighborhood of $\phi_\alpha(x) \in \mathbb{R}^q$ to a neighborhood of $\phi_\beta(x) \in \mathbb{R}^q$ with $\phi_\beta(y) = \gamma_{\alpha\beta}(x) \circ \phi_\alpha(y)$ for y in a neighborhood of x. The transition functions $\gamma_{\alpha\beta}(x)$ give rise to a well-defined map

$$\gamma_{\alpha\beta} : U_{\alpha\beta} \to \Gamma^q \overset{\text{def}}{=} \{\text{germs of local diffeomorphisms of } \mathbb{R}^q\} .$$

The set of data $\{U_\alpha, \phi_\alpha, \gamma_{\alpha\beta} | \alpha, \beta \in \mathcal{A}\}$ is called a non-singular Γ^q-cocycle defining the foliation $\mathfrak{F}$.

We want to study a generalization of the above construction of a foliation $\mathfrak{F}$ on M, in which the normal bundle $Q \overset{\text{def}}{=\!=} TM/\mathfrak{F}$ is required to have some additional geometric structure. Let G be a closed subgroup of the general linear group $G\ell(q,\mathbb{R})$. Let FM denote the $G\ell(q,\mathbb{R})$-frame bundle of the tangent bundle TM. We need the following concept.

1.3 DEFINITION [40]. A G-<u>structure</u> on a manifold M is a reduction of the $G\ell(q,\mathbb{R})$-bundle $FM \to M$ to a principal G-bundle $\pi: P \to M$. That is, at each point $x \in M$ there is specified a set of frames of TM_x, on which G acts transitively and freely, and the set $P_x = \pi^{-1}(x) \subseteq FM_x$ varies smoothly with x.

A G-structure on M is equivalently determined by giving a section s of the quotient bundle $P/G \underset{s}{\overset{\pi}{\rightleftarrows}} M$, [40].

The manifold $\mathbb{R}^q$ has a canonical G-structure, unique up to the action of $G\ell(q,\mathbb{R})$. Let $\{x_1,\ldots,x_q\}$ be linear coordinates on $\mathbb{R}^q$, and let $\{\partial/\partial x_1,\ldots,\partial/\partial x_q\}$ be the corresponding vector fields. These vector fields define a global framing of $T\mathbb{R}^q$. The right action of G on this frame defines a G-reduction of the frame bundle $F\mathbb{R}^q$. This is called the <u>flat</u> G-structure on $\mathbb{R}^q$.

A G-structure $P \to M^q$ is said to be <u>integrable</u> if for each $x \in M^q$ there is an open neighborhood U_x of x and a coordinate chart $\phi_x: U_x \to \mathbb{R}^q$ such that $P|_{U_x}$ is given by the pull-back via ϕ_x of the flat G-structure on $\mathbb{R}^q$.

We next introduce the notion of a G-foliation. The basic idea is illustrated by the following example. Let M be a manifold with a G-structure $P' \to M$. A submersion $f: N \to M$ defines a foliation $\mathfrak{F}$ of N, whose leaves are the components of the fibers of f. The normal bundle $Q = TN/\mathfrak{F}$ is then isomorphic to the pull-back via f of the tangent bundle TM.

Let FQ denote the $G\ell(q,\mathbb{R})$-frame bundle of Q. If the G-structure on M is defined by a section $s':M \to FM/G$, there is a G-reduction of $FQ \to N$ determined by a map s which satisfies

$$\begin{array}{ccc} FQ/G & \xrightarrow{\bar{f}} & FM/G \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s'} \\ N & \xrightarrow{f} & M \end{array} \quad . \tag{1.4}$$

If $U \subseteq M$ is a coordinate neighborhood, then $FM|_U \cong U \times G\ell(q,\mathbb{R})$. Setting $V = f^{-1}(U)$, this defines a trivialization $FQ|_V \cong V \times G\ell(q,\mathbb{R})$. It follows from (1.4) that the composition

$$V \xrightarrow{s|_V} V \times G\ell(q,\mathbb{R})/G \xrightarrow{\text{proj}} G\ell(q,\mathbb{R})/G$$

is constant along the leaves of $\mathfrak{F}$.

More generally, a foliation $\mathfrak{F}$ on M is a G-foliation if the normal frame bundle FQ admits a G-reduction whose defining section $s:M \to FQ/G$ has the above local constancy along the leaves of $\mathfrak{F}$:

1.5 DEFINITION [24], [25], [17]. A codimension q foliation $\mathfrak{F}$ on M is said to be a <u>G-foliation</u> if:

(a) There is given a q-dimensional manifold B with a G-structure $P' \to B$.

(b) There is an open covering $\{U_\alpha\}$ of M and local submersions $\phi_\alpha:U_\alpha \to B$ such that $\ker d\phi_\alpha = \mathfrak{F}|_{U_\alpha}$.

(c) For each $x \in U_\alpha \cap U_\beta$ the transition function $\gamma_{\alpha\beta}(x)$ is a local G-morphism of B. That is, the differential $d\gamma_{\alpha\beta}(x)$ maps the G-frames in a neighborhood of $\phi_\alpha(x) \in B$ to G-frames in a neighborhood of $\phi_\beta(x)$.

Locally, a G-foliation is given by a submersion into a manifold with a G-structure, and condition (c) implies the local reductions of the frame bundle FQ are compatible.

The manifold B in Definition 1.5 is called the <u>model</u> for $\mathfrak{F}$, and the set of maps $\{\phi_\alpha : U_\alpha \to B\}$ is called a <u>defining system of charts</u> for the G-foliation.

A G-foliation is said to be <u>integrable</u> if the model manifold can be taken to be $\mathbb{R}^q$ with the flat G-structure.

<u>Examples of G-foliations</u>

We describe some contexts in which G-foliations arise. Of course, for $G = G\ell(q,\mathbb{R})$, any foliation of codimension q is an integrable G-foliation. Some more interesting cases are as follows.

1.6 EXAMPLE. Let M be a manifold with a G-structure. The point foliation on M is clearly a G-foliation. More generally, if $f: N \to M$ is a submersion, then the fibers of f define a G-foliation on N.

1.7 EXAMPLE. An O(q)-foliation is called a Riemannian foliation in the literature. Riemannian foliations were first defined and studied by B. Reinhart [56].

(a) A Riemannian metric on a manifold M determines an O(q)-structure, so that examples of type (1.6) always give rise to Riemannian foliations. It will be shown in Chapter 7 that not all Riemannian foliations arise in this fashion.

(b) Let M be a Riemannian manifold with a nowhere-vanishing Killing vector field V, [40]. Then the span $\langle V \rangle \subseteq TM$ defines a Riemannian foliation, as the normal bundle $Q = TM/\langle V \rangle$ has a Riemannian metric which is invariant under the flow of V.

(c) Let K be a compact Lie group which acts effectively on M. Then the action of K defines a Riemannian foliation on M, as any metric on the normal bundle Q can be averaged over the action of K to give an invariant metric [51]. Typical examples of this are obtained by choosing a compact subgroup K of a Lie group H and letting K act on H via the left group action.

1.8 EXAMPLE. (a) An integrable $G\ell(n,\mathbb{C})$-foliation, for $q = 2n$, is called a complex foliation. A complex manifold with the point foliation provides one type of example. Another common family of examples of complex foliations arises from submersions $f:N \to M$ into a complex manifold M.

(b) If M is a complex manifold with a nowhere-vanishing complex vector field V, then $\langle V \rangle$ defines a complex foliation on M, [3].

1.9 EXAMPLE. Let G be a non-trivial, closed subgroup of a Lie group $\bar{G}$. Then the left cosets of G in $\bar{G}$ determine a G-foliation $\mathfrak{F}$ on $\bar{G}$. The G-reduction $P \to \bar{G}$ of the normal frame bundle is given by the pull-back diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \bar{G} \arrow[d] \\ \bar{G} \arrow[r] & \bar{G}/G \end{tikzcd} \quad .$$

Note that $P \to \bar{G}$ is isomorphic, as a G-bundle, to the trivial bundle $\bar{G} \times G \to \bar{G}$. However, the section of $P \to \bar{G}$ defining this trivialization will not be parallel along the leaves of $\mathfrak{F}$, so that this coset foliation of $\overline{G}$ is not an $\{e\}$-foliation. Many examples of foliations of this type can be found in the works of Kamber and Tondeur [37], [38].

A Universal G-foliated Manifold

We describe a construction due to Haefliger [24], [25], [44] of a semi-simplicial, non-Hausdorff, non-paracompact manifold $B\tilde{\Gamma}_G$ with a universal G-foliation: Every foliation on a manifold M is induced by a classifying map $f:M \to B\tilde{\Gamma}_G$.

Recall that $F\mathbb{R}^q \to \mathbb{R}^q$ is the frame bundle of $\mathbb{R}^q$. Let $\mathcal{H}(G)$ be the sheaf of local C^∞-sections of the quotient bundle $F\mathbb{R}^q/G \to \mathbb{R}^q$. The space $\mathcal{H}(G)$ has a C^∞-manifold structure, though it is neither paracompact nor Hausdorff.

1.10 LEMMA. $\mathcal{H}(G)$ has a canonical G-structure.

Proof. For an open set $U \subseteq \mathbb{R}^q$ and a section $s \in \Gamma(U, F\mathbb{R}^q/G)$, let s_x denote the germ of s at $x \in U$. A basis for the topology of $\mathcal{H}(G)$ is given by the collection of sets $U_s = \{s_x | x \in U\}$ for $U \subseteq \mathbb{R}^q$ open and $s \in \Gamma(U, F\mathbb{R}^q/G)$. There is a canonical map $p:\mathcal{H}(G) \to \mathbb{R}^q$ defined by $p(s_x) = x$. Note that $p|_{U_s}:U_s \to U$ is a homeomorphism, and in fact defines the C^∞-structure on $\mathcal{H}(G)$.

For a basic open set U_s, the corresponding open set $U \subseteq \mathbb{R}^q$ has a G-structure determined by the section s. Let U_s have the G-structure induced by $p:U_s \to U$. The definition of the sheaf topology on $\mathcal{H}(G)$ implies that these locally defined G-structures are compatible, giving $\mathcal{H}(G)$ a G-structure. □

The space $\mathcal{N}(G)$ is a universal model manifold for G-foliations. It is clear from Definition 1.4 that every G-foliation on M has a defining system of charts $\{\phi_\alpha : U_\alpha \to \mathcal{N}(G)\}$. The transition function $\gamma_{\alpha\beta}(x)$ for these charts are local G-diffeomorphisms of $\mathcal{N}(G)$. This is the motivation for the next definition.

1.11 DEFINITION. Let $\mathcal{S}_G$ denote the pseudogroup of all local, C^∞, G-diffeomorphisms of $\mathcal{N}(G)$. We define $\tilde{\Gamma}_G$ to be the associated topological groupoid of $\mathcal{S}_G$,[25]. For an element f in $\mathcal{S}_G$ and x in the domain of f, let f_x denote the germ of f at x. Then the underlying set of $\tilde{\Gamma}_G$ is given by

$$\tilde{\Gamma}_G = \{f_x | f \in \mathcal{S}_G \text{ and } x \in \text{domain } f\} .$$

There is a natural map $p_0 : \tilde{\Gamma}_G \to \mathcal{N}(G)$, defined by setting $p_0(f_x) = x$. The space $\tilde{\Gamma}_G$ is given the sheaf topology with respect to the map p_0. The map p_0 is then a local homeomorphism and defines a C^∞-manifold structure on $\tilde{\Gamma}_G$.

For a G-foliation $\mathfrak{F}$ on M with defining system of charts $\{\phi_\alpha : U_\alpha \to \mathcal{N}(G)\}$, the transition functions $\gamma_{\alpha\beta}(x)$ induce maps $\gamma_{\alpha\beta} : U_{\alpha\beta} \to \tilde{\Gamma}_G$. The data $\{U_\alpha, \phi_\alpha, \gamma_{\alpha\beta}\}$ is called a non-singular $\tilde{\Gamma}_G$-cocycle on M. By abuse of language, this cocycle is often denoted simply by $\{\gamma_{\alpha\beta}\}$. The $\tilde{\Gamma}_G$-cocycle associated to a foliation $\mathfrak{F}$ completely determines $\mathfrak{F}$.

1.12 PROPOSITION [25], [17]. Let $\{U_\alpha\}$ be an open covering of a manifold M, and suppose functions $\gamma_{\alpha\beta} : U_{\alpha\beta} \to \tilde{\Gamma}_G$ are given which satisfy:

(a) (cocycle condition) For all $x \in U_\alpha \cap U_\beta \cap U_\lambda$, we have $\gamma_{\beta\lambda}(x) \circ \gamma_{\alpha\beta}(x) = \gamma_{\alpha\lambda}(x)$ when this composition is defined.

(b) (non-singularity) The maps $p_0 \circ \gamma_{\alpha\alpha}: U_\alpha \to \mathcal{N}(G)$ are smooth submersions.

Then setting $\phi_\alpha = p_0 \circ \gamma_{\alpha\alpha}$, the charts $\{\phi_\alpha : U_\alpha \to \mathcal{N}(G)\}$ are a defining system for a G-foliation $\mathfrak{F}$ on M, whose non-singular $\tilde{\Gamma}_G$-cocycle is precisely $\{\gamma_{\alpha\beta}\}$.

This proposition implies the process of associating to a G-foliation on M its non-singular $\tilde{\Gamma}_G$-cocycle gives an exact classification of the G-foliations on M. A weaker but more practical classification is accomplished by introducing topological methods. Let $\{U_\alpha, \phi_\alpha, \gamma_{\alpha\beta}\}$ be a non-singular $\tilde{\Gamma}_G$-cocycle on M. Let $\mathcal{U}$ be the set of all non-empty intersections $U_{\alpha\beta\cdots\gamma} = U_\alpha \cap U_\beta \cap \cdots \cap U_\gamma$. The set $\mathcal{U}$ has a groupoid structure, where the composition is defined to be intersections of the form $U_{\alpha\beta\cdots\gamma} \times U_{\gamma\delta\cdots\lambda} = U_{\alpha\beta\cdots\lambda}$, if the resulting set if non-empty. Giving $\mathcal{U}$ the discrete topology, we obtain a topological groupoid. The cocycle $\{\gamma_{\alpha\beta}\}$ gives rise to a map $\gamma : \mathcal{U} \to \tilde{\Gamma}_G$ of topological groupoids [7]. The Milnor join construction [7], [44] defines a functor, denoted by B, from the category of topological groupoids to the category of topological spaces and homotopy classes of maps. For the above situation, applying B gives a continuous map $B\gamma : B\mathcal{U} \to B\tilde{\Gamma}_G$. It is well-known that there is a natural equivalence $M \simeq B\mathcal{U}$,[7], and we let $f : M \to B\tilde{\Gamma}_G$ denote the map induced by $B\gamma$. The homotopy class of f depends only on the cocycle $\{\gamma_{\alpha\beta}\}$ on M, so this construction defines a set map

$$\{\text{G-foliations on M}\} \to [M, B\tilde{\Gamma}_G] \ .$$

It is natural to ask what is the "kernel" of this set mapping; this is a question which will be answered after some additional constructions are introduced.

An element $f_x \in \tilde{\Gamma}_G$ can be represented as a local G-morphism $f:U_s \to U'_{s'}$, of $\mathcal{N}(G)$. The local diffeomorphisms $p:U_s \to U \subseteq \mathbb{R}^q$ and $p:U'_{s'} \to U' \subseteq \mathbb{R}^q$ define coordinate charts for $\mathcal{N}(G)$. We denote by $J(f)_x$ the Jacobian of f with respect to these charts. Then $J(f)_x \in G$, and the correspondence $f_x \to J(f)_x$ defines a map $\mathrm{Jac}:\tilde{\Gamma}_G \to G$ of topological groupoids. The corresponding map of classifying spaces is denoted $\nu = \mathrm{BJac}:B\tilde{\Gamma}_G \to BG$. The map ν has the following interpretation. If $f:M \to B\tilde{\Gamma}_G$ classifies a G-foliation $\mathfrak{F}$ on M, then $\nu \circ f:M \to BG$ classifies the G-reduction $P \to M$ of the normal frame bundle FQ.

Associated to the map $\nu:B\tilde{\Gamma}_G \to BG$ is a topological space $F\tilde{\Gamma}_G$, the homotopy theoretic fiber of ν, and a diagram

$$\begin{tikzcd} F\tilde{\Gamma}_G \arrow[r] & B\tilde{\Gamma}_G \arrow[d, "\nu"] \\ & BG \end{tikzcd}$$

which is a fibration in the homotopy category [60]. If a map $f:M \to B\tilde{\Gamma}_G$ is given for which $\nu \circ f$ is homotopic to a constant, then there is an induced map $\bar{f}:M \to F\tilde{\Gamma}_G$ and a homotopy commutative diagram

$$\begin{tikzcd} F\tilde{\Gamma}_G \arrow[r] & B\tilde{\Gamma}_G \\ M \arrow[u, "\bar{f}"] \arrow[ur, "f"'] & \end{tikzcd}$$

If a map $f:M \to B\tilde{\Gamma}_G$ classifies a G-foliation $\mathfrak{F}$ on M, then the composition $\nu \circ f$ will be homotopic to a constant precisely when the G-bundle $P \to M$ associated to $\mathfrak{F}$ admits a section. The space $F\tilde{\Gamma}_G$ is therefore seen to classify the G-foliations on M with a trivial associated G-bundle.

We return now to the problem of interpreting the set $[M,B\tilde{\Gamma}_G]$. The following notion will play an essential role.

1.13 DEFINITION. Two codimension q G-foliations $\mathfrak{F}_0$ and $\mathfrak{F}_1$ on M are said to be <u>concordant</u> if there is a codimension q G-foliation $\mathfrak{F}$ on $M \times \mathbb{R}$ such that the inclusions $i_t: M \cong M \times \{t\} \subseteq M \times \mathbb{R}$ induce an equivalence $i_t^*\mathfrak{F} = \mathfrak{F}_t$ for $t = 0,1$.

We say $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are <u>integrably homotopic</u> if, in addition, i_t is transverse to $\mathfrak{F}$ for all $t \in \mathbb{R}$.

Note that integrable homotopy implies concordance.

1.14 LEMMA [25]. If $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are concordant G-foliations on M, then their classifying maps $f_0, f_1: M \to B\tilde{\Gamma}_G$ are homotopic.

There is a converse to this lemma, which gives the answer to our question. First, note that a G-foliation $\mathfrak{F}$ on M defines a splitting of the tangent bundle $TM = Q \oplus \mathfrak{F}$. Therefore, the classifying map of TM factors as

$$(g_1,g_2): M \to BG \times BO(n-q) \subseteq BO(n) .$$

If $f: M \to B\tilde{\Gamma}_G$ classifies $\mathfrak{F}$, then there is a commutative diagram

$$\begin{tikzcd} & B\tilde{\Gamma}_G \times BO(n-q) \arrow[d, "{(\nu,id)}"] \\ M \arrow[ur, "{(f,g_2)}"] \arrow[r, "{(g_1,g_2)}"'] & BG \times BO(n-q) \subseteq BO(n) \end{tikzcd} \tag{1.15}$$

We will denote by $[M,B\tilde{\Gamma}_G \times BO(n-q); BO(n)]$ the set of homotopy classes of maps (f,g_2) which satisfy (1.15), where the composition $M \to BO(n)$ classifies

TM. There is a mapping

$$[M, B\tilde{\Gamma}_G \times BO(n-q);\ BO(n)] \to [M, B\tilde{\Gamma}_G]$$

given by $(f, g_2) \to f$. A very deep result of Gromov, Haefliger and Phillips provides the following converse to Lemma 1.14.

1.16 THEOREM [23], [24], [52]. Let M be an <u>open</u> manifold. There is a bijection between the set of <u>integrable</u> <u>homotopy</u> classes of G-foliations on M and the set $[M, B\tilde{\Gamma}_G \times BO(n-q);\ BO(n)]$.

This is a result which we will use often. Note that it implies that the integrable homotopy class of a G-foliation $\mathfrak{F}$ on an open manifold M is uniquely determined by the homotopy class of the classifying map $f: M \to B\tilde{\Gamma}_G$.

For $G = G\ell(q, \mathbb{R})$, a result corresponding to Theorem 1.16 for compact manifolds has been shown by Thurston.

1.17 THEOREM [66], [67]. Let M be a compact manifold. There is a bijection between the set of <u>concordance</u> classes of $G\ell(q, \mathbb{R})$-foliations on M and the set $[M, B\tilde{\Gamma}_{G\ell(q,\mathbb{R})} \times BO(n-q);\ BO(n)]$.

1.18 REMARK. Suppose a finite CW complex X and a continuous map $g: X \to B\tilde{\Gamma}_G$ are given. By a construction of Haefliger [44], there is an open manifold M with a G-foliation $\mathfrak{F}$ classified by $f: M \to B\tilde{\Gamma}_G$, and a homotopy equivalence $i: X \to M$ such that the diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "i"'] & & B\tilde{\Gamma}_G \\ & M \arrow[ur, "f"'] & \end{tikzcd}$$

homotopy commutes. The map g determines a singular $\tilde{\Gamma}_G$-cocycle on X; the manifold M can be thought of as an open thickening of X on which the $\tilde{\Gamma}_G$-cocycle determined by g has been desingularized.

It follows from Theorem 1.16 that the integrable homotopy class of $\mathfrak{F}$ on M depends only on the homotopy class of g. Therefore, for a finite CW complex X, the set $[X,B\tilde{\Gamma}_G]$ is in 1-1 correspondence with the set of integrable homotopy classes of G-foliations on open thickenings M of X. It is in this sense which we will speak of a G-foliation on a CW complex X determined by a map $g:X \to B\tilde{\Gamma}_G$.

In the next chapters, we will produce invariants for a G-foliation $\mathfrak{F}$ on M which depend only on the concordance class of $\mathfrak{F}$ and the homotopy type of M. The above remarks allow us to consider these invariants as being defined for pairs (X,f), where X is a finite CW complex and f is an element of $[X,B\tilde{\Gamma}_G]$. For a fixed X, we obtain invariants defined on the set $[X,B\tilde{\Gamma}_G]$. For example, letting X be the n-sphere S^n, this produces invariants defined on the homotopy group $\pi_n(B\tilde{\Gamma}_G)$. By studying these invariants, we will obtain results on the homotopy groups of the space $B\tilde{\Gamma}_G$.

All of the above constructions can be repeated for integrable G-foliations. Using the manifold $\mathbb{R}^q$ with the flat G-structure in place of $\mathcal{R}(G)$, we let Γ_G denote the topological groupoid of germs of local G-diffeomorphisms of $\mathbb{R}^q$. The associated space $B\Gamma_G$ then classifies integrable G-foliations, and the analog of Theorem 1.16 is valid. There is a map $\nu:B\Gamma_G \to BG$ classifying the G-structure on the normal bundle of a G-foliation; we denote by $F\Gamma_G$ the homotopy theoretic fiber of ν.

CHAPTER 2

SECONDARY CHARACTERISTIC CLASSES

In this chapter we give a survey of the theory of secondary characteristic classes for G-foliations as developed by Kamber and Tondeur. A detailed exposition of this theory is given in [38], which will be our general reference. For the groups $G = G\ell(q,\mathbb{R})$ and $G\ell(q,\mathbb{C})$ there are alternative constructions of these secondary classes, due to Bott and Haefliger [8], [6], Bernstein and Rozenfield [4] and Gelfand and Fuks [20]. The construction of Kamber and Tondeur is particularly suited to our needs because it is a principal-bundle theoretic approach, and in a uniform way gives secondary classes for any group G.

For this chapter we work over a field $\Bbbk = \mathbb{R}$ or $\mathbb{C}$. Let $G \subseteq G\ell(q,\mathbb{R})$ be a fixed, closed subgroup with a finite number of connected components. We will consider a fixed codimension q G-foliation $\mathfrak{F}$ on a manifold M. Let $\pi: P \to M$ be the principal G-bundle associated to $\mathfrak{F}$. We denote by $\mathfrak{g}$ the Lie algebra of G.

The analytical construction which is the basis for the theory of secondary classes is the notion of an adapted connection on P.

2.1 DEFINITION [38]. A G-connection ω on P is said to be <u>adapted</u> to $\mathfrak{F}$ if there is a foliation $\tilde{\mathfrak{F}}$ on P such that:

(a) $\tilde{\mathfrak{F}}$ lies in the horizontal distribution of ω; i.e., for each $p \in P$, the space $\tilde{\mathfrak{F}}_p$ lies in the kernel of $\omega: TP_p \to \mathfrak{g}$.

(b) For each $x \in M$ and $p \in P$ with $\pi(p) = x$, the differential $d\pi: TP_p \to TM_x$ is an isomorphism from $\tilde{\mathfrak{F}}_p$ to $\mathfrak{F}_x$.

(c) The distribution $\tilde{\mathfrak{F}}$ is invariant under the right action of G on P.

An adapted connection ω on P is <u>basic</u> if, in addition to (a), (b) and (c) above, we also have:

(d) $\theta(\tilde{u})\omega = 0$ for all vector fields $\tilde{u} \in \Gamma(P,\tilde{\mathfrak{F}})$.

Adapted connections always exist. One construction goes as follows. Let $\{\phi_\alpha : U_\alpha \to B\}$ be a defining system of charts for $\mathfrak{F}$ with $P' \to B$ the associated G-structure on B. Choose a G-connection ω' on P'. There is a natural isomorphism between $P|_{U_\alpha}$ and the pull-back via ϕ_α of the bundle $P'|_{\phi_\alpha(U_\alpha)}$. Let ω'_α be the connection on $P|_{U_\alpha}$ induced from ω' under this isomorphism. Choose a partition of unity $\{\lambda_\alpha\}$ subordinate to the cover $\{U_\alpha\}$ of M. Then

$$\omega \overset{\text{def}}{=} \sum_\alpha (\lambda_\alpha \circ \pi) \cdot \omega'_\alpha$$

defines a G-connection on $P \to M$. A straightforward check, using the fact that $\mathfrak{F}|_{U_\alpha} = \ker d\phi_\alpha$, shows that ω is adapted to $\mathfrak{F}$.

We next fix some notation. For $\mathfrak{g}^* = \mathrm{Hom}(\mathfrak{g}, k)$, denote by:

- $\Lambda\mathfrak{g}^*$, the exterior algebra on $\mathfrak{g}^*$, with $\Lambda^1\mathfrak{g}^* = \mathfrak{g}^*$
- $S\mathfrak{g}^*$, the symmetric algebra on $\mathfrak{g}^*$, with $S^2\mathfrak{g}^* = \mathfrak{g}^*$
- $I(\mathfrak{g}) = (S\mathfrak{g}^*)^{\mathfrak{g}}$, the $\mathrm{ad}\,\mathfrak{g}$-invariant elements of $S\mathfrak{g}^*$
- $I(G) = (S\mathfrak{g}^*)^G$, the Ad G-invariant elements of $S\mathfrak{g}^*$.

Let $W(\mathfrak{g})$ denote the Weil algebra of $\mathfrak{g}$, [12], which as an algebra is isomorphic to $\Lambda\mathfrak{g}^* \otimes S\mathfrak{g}^*$. Define a filtration of $W(\mathfrak{g})$ by letting $F^\ell W(\mathfrak{g})$ be the ideal in $W(\mathfrak{g})$ generated by $\bigoplus_{p \geq \ell} S^{2p}\mathfrak{g}^*$. The <u>truncated</u> <u>Weil</u> <u>algebra</u> is the quotient

$$W(\mathfrak{g})_\ell = W(\mathfrak{g})/F^{\ell+1}W(\mathfrak{g}) .$$

This is again a differential algebra with $i(\mathfrak{g})$ and Ad G-actions [38].

For any closed subgroup $H \subseteq G$, the relative truncated Weil algebra $W(\mathfrak{g},H)_\ell \overset{\text{def}}{=\!=} (W(\mathfrak{g})_\ell)_H$ is defined to be the H-basic elements in $W(\mathfrak{g})_\ell$.

Given a connection ω on $P \to M$, there is defined a differential algebra map $k(\omega):W(\mathfrak{g}) \to \Omega(P)$, called the Weil homomorphism [12]. The map $k(\omega)$ commutes with the $i(\mathfrak{g})$ and Ad G-actions on both algebras. The distinguishing property of adapted connections is given by the important result:

2.2 THEOREM [38]. If ω be a connection adapted to $\mathfrak{F}$ on P, then $F^{q+1}W(\mathfrak{g})$ is contained in the kernel of $k(\omega)$.

2.3 COROLLARY. Given a closed subgroup H of G, an integer $\ell \geq q$ and an adapted connection ω, there is a well-defined differential algebra map

$$k(\omega):W(\mathfrak{g},H)_\ell \to \Omega(P)_H \cong \Omega(P/H) .$$

The existence of secondary characteristic classes of the foliation $\mathfrak{F}$ is a consequence of Theorem 2.2 and Corollary 2.3. The untruncated Weil homomorphism

$$k(\omega)_*:H^*(W(\mathfrak{g},H)) \to H^*(P/H)$$

defines the primary characteristic classes of the H-bundle $P \to P/H$. It is in the cohomology of the truncated Weil algebra that secondary classes arise. The algebra $H^*(W(\mathfrak{g},H)_\ell)$ is described in more detail later in this chapter.

When $H = G$, the algebra $W(\mathfrak{g},G)$ is just the ring $I(G)$ of Ad G-invariant polynomials on $\mathfrak{g}$. In this case, the map $k(\omega)$ is called the Chern-Weil homomorphism and denoted by

$$h(\omega):I(G) \to \Omega(P/G) \cong \Omega(M) .$$

The induced map in cohomology, $h(\omega)_*:I(G) \to H^*(M)$, defines the primary characteristic classes of the bundle $P \to M$. Theorem 2.2 implies that if ω is an adapted connection, then $h(\omega)$ anihilates the ideal $F^{q+1}I(G)$ in $I(G)$. The ideal $F^{q+1}I(G)$ consists of the elements of degree $> 2q$, so we obtain the famous Bott Vanishing Theorem:

2.4 THEOREM [5]. If ω is a connection adapted to $\mathfrak{F}$, then $h(\omega)$ anihilates the elements of degree $> 2q$ in $I(G)$.

If a foliation $\mathfrak{F}$ has a basic connection ω, then the results of Theorem 2.2 can be sharpened.

2.5 THEOREM [38]. Let ω be a basic connection on P. Set $q' = [q/2]$; then $F^{q'+1}W(\mathfrak{g})$ is contained in the kernel of $k(\omega)$.

Given the preceding constructions, we can now state the principal result in the theory of secondary characteristic classes.

2.6 THEOREM [36]. Let $H \subseteq G$ be a closed subgroup, $\ell \geq q$ any integer and ω a G-connection adapted to $\mathfrak{F}$ on $P \to M$. Then the map of cohomology algebras

$$k(\omega)_*:H^*(W(\mathfrak{g},H)_\ell) \to H^*(P/H)$$

is independent of the choice of adapted connection ω, and depends only on the concordance class of $\mathfrak{F}$.

2.7 REMARK. If ω is a basic connection for $\mathfrak{F}$ on P, then for any $\ell \geq q' = [q/2]$ there is a map $k(\omega)$ as above. However, for $\ell < q$ the map $k(\omega)_*$ is not necessarily independent of the choice of basic connection ω

and may depend on the representative $\mathfrak{F}$ chosen within a given concordance class of G-foliations.

For Riemannian foliations, where $G = O(q)$ or $SO(q)$, basic connections always exist. In fact, for a given $O(q)$-foliation $\mathfrak{F}$, there is a unique torsion-free $O(q)$-connection ω on $P \to M$, [51]. Because of this, Theorem 2.6 remains valid when $G = O(q)$, ω is basic and $\ell \geq q'$.

Construction of the Secondary Classes

Theorem 2.6 gives us invariants of the concordance class of $\mathfrak{F}$, but these invariants take their values in the cohomology of the space P/H. To produce foliation invariants in the vector space $H^*(M)$, it is necessary to assume that for some closed subgroup H of G, the G-bundle $P \to M$ admits an H-reduction. Let this reduction be defined by a section $s:M \to P/H$. It is not being assumed that the section s is parallel along the leaves of $\mathfrak{F}$, so that we are not assuming $\mathfrak{F}$ to be an H-foliation. In fact, if $\mathfrak{F}$ is an H-foliation then the invariants to be defined will all vanish [38, Prop. 4.52].

For the given section $s:M \to P/H$, define

$$\Delta(\omega) = ds \circ k(\omega):W(\mathfrak{g},H)_{\ell} \to \Omega(M) \quad .$$

The next proposition then follows from Theorem 2.6.

2.8 PROPOSITION. For any $\ell \geq q$, the map

$$\Delta_* = \Delta(\omega)_*:H^*(W(\mathfrak{g},H)_{\ell}) \to H^*(M)$$

depends only on the concordance class of $\mathfrak{F}$ and the homotopy class of s.

2.9 COROLLARY. If $\ell \geq q$ and K is a maximal compact subgroup of G, then there exists a characteristic map

$$\Delta_*: H^*(W(\mathfrak{g}, K)_\ell) \to H^*(M)$$

which depends only on the concordance class of $\mathfrak{F}$.

Proof. The quotient G/K is contractible, so the bundle $P/K \to M$, with fiber G/K, admits a section s which is unique up to homotopy. □

The image of Δ_* consists of, by definition, the secondary characteristic classes of $\mathfrak{F}$.

2.10 REMARK. Any foliation $\mathfrak{F}$ carries the structure of a $G\ell(q,\mathbb{R})$-foliation. Hence there is always defined a characteristic map

$$\Delta_*: H^*(W(g\ell(q,\mathbb{R}), O(q))_q) \to H^*(M)$$

for $\mathfrak{F}$. The classes in the image of this map are the same as the secondary classes constructed by Bott-Haefliger [8] and Bernstein-Rozenfeld [4], using different techniques.

Note that the map Δ_* is functorial under pullbacks. If $f:N \to M$ is a smooth map which is transverse to the foliation $\mathfrak{F}$ on M, then the diagram below commutes:

$$\begin{tikzcd} H^*(N) & & H^*(M) \arrow[ll, "f^*"'] \\ & H^*(W(\mathfrak{g}, H)_\ell) \arrow[ul, "\Delta_*"] \arrow[ur, "\Delta_*"'] & \end{tikzcd}$$

For K a maximal compact subgroup of G, by Corollary 2.9 and Remark 1.18, it follows that there is induced a universal map

$$\tilde{\Delta}_*: H^*(W(\mathfrak{g}, K)_\ell) \to H^*(B\tilde{\Gamma}_G) .$$

The map $\tilde{\Delta}_*$ has the property that if the G-foliation $\mathfrak{F}$ is classified by a map $f:M \to B\tilde{\Gamma}_G$, then $\Delta_* = f^* \circ \tilde{\Delta}_*$.

An $\{e\}$-reduction of the G-bundle $P \to M$ is a trivialization of P. Therefore, if $P \to M$ admits an $\{e\}$-reduction, then $\mathfrak{F}$ is classified by a map $f: M \to F\tilde{\Gamma}_G$. By Remark 1.18, there is a universal characteristic map

$$\tilde{\Delta}_*: H^*(W(\mathfrak{g},e)_\ell) \to H^*(F\tilde{\Gamma}_G) \ .$$

The determination of the image of $\tilde{\Delta}_*$ in either case is one of the central problems in the theory of secondary invariants for G-foliations. This problem is addressed in Chapters 8, 9 and 10 of this thesis.

For the remainder of this chapter, we consider a fixed subgroup H of G and a fixed positive integer ℓ.

The cohomology of the algebra $W(\mathfrak{g},H)_\ell$ has been studied in detail by Kamber and Tondeur [34], [39]. The most general result is given by the next proposition:

2.11 PROPOSITION. If G satisfies $I(\mathfrak{g}) = I(G)$, then there is an isomorphism

$$H^*(W(\mathfrak{g},H)_\ell) \cong \mathrm{Tor}_{I(G)}(I(H),I(G)_\ell) \ .$$

We will return to this description of $H^*(W(\mathfrak{g},H)_\ell)$ in Proposition 4.33 of this thesis.

In order to describe $H^*(W(\mathfrak{g},H)_\ell)$ more concretely, it is necessary to introduce some additional notation. Let P denote the space of primitives in $\Lambda\mathfrak{g}^*$. Recall that P consists of $\mathrm{ad}_\mathfrak{g}$-invariant elements, and the inclusion $\Lambda P \subseteq (\Lambda\mathfrak{g}^*)^\mathfrak{g}$ is an isomorphism of algebras [21]. Let $\sigma: I(G) \to P$ denote the suspension mapping, and choose a transgression $\tau: P \to I(\mathfrak{g})$.

For the given subgroup H of G, let $i: H \subseteq G$ denote the inclusion mapping. The induced map of algebras $i^*: I(G) \to I(H)$ is just the restriction homomorphism. Define a subspace $\overline{P}_\ell$ of P by setting

$$\overline{P}_\ell \overset{\text{def}}{=\!=} \sigma\{c \in \ker i^*: I(G) \to I(H) \mid \deg c \leq 2\ell\} .$$

The space $\overline{P}_\ell$ consists of the elements in P which transgress to an element in the kernel of i^* of degree $\leq 2\ell$. Note that the elements of $\overline{P}_\ell$ are H-basic, [Proposition 5.103; 38].

We next introduce a differential algebra $\hat{A}(\mathfrak{g},H)_\ell$ which gives a model for $W(\mathfrak{g},H)_\ell$. With the above notation, set:

$$\hat{A}(\mathfrak{g},H)_\ell \overset{\text{def}}{=\!=} \Lambda P \otimes I(\mathfrak{g})^H_\ell \otimes I(H) .$$

The differential in $\hat{A}(\mathfrak{g},H)_\ell$ is determined by setting, for $b \in I(\mathfrak{g})^H_\ell$, $c \in I(H)$ and $y \in P$:

$$d(1 \otimes b \otimes c) = 0$$
$$d(y \otimes 1 \otimes 1) = 1 \otimes \tau(y) \otimes 1 - 1 \otimes 1 \otimes i^*\tau(y) .$$

The following theorem is proven using techniques originating with H. Cartan [12]:

2.12 THEOREM [21], [34]. There exists a differential algebra homomorphism

$$\hat{A}(\mathfrak{g},H)_\ell \to W(\mathfrak{g},H)_\ell$$

which induces an isomorphism of cohomology algebras.

Next we define a differential subalgebra $A(G,H)_\ell$ of $\hat{A}(\mathfrak{g},H)_\ell$. For the subspace $\overline{P}_\ell$ of P as defined above, set:

$$A(G,H)_\ell \overset{\text{def}}{=\!=} \Lambda\overline{P}_\ell \otimes I(G)_\ell .$$

The differential in this algebra is defined by setting, for $c \in I(G)$ and $y \in \overline{P}_\ell$:

$$d(1 \otimes c) = 0$$

$$d(y \otimes 1) = 1 \otimes \tau(y) .$$

If $H = \{e\}$ is the trivial group, then we set $A(G)_\ell \overset{\text{def}}{=\!=} A(G,\{e\})_\ell$.

The natural inclusion $A(G,H)_\ell \subseteq \hat{A}(\mathfrak{g},H)_\ell$ is a map of differential algebras by the choice of the space $\overline{P}_\ell$. The cohomology algebras of these two algebras are then related by the result:

2.13 PROPOSITION [21]. The inclusion $A(G,H)_\ell \subseteq \hat{A}(\mathfrak{g},H)_\ell$ induces an inclusion of algebras

$$H^*(A(G,H)_\ell) \subseteq H^*(\hat{A}(\mathfrak{g},H)_\ell) \cong H^*(W(\mathfrak{g},H)_\ell) .$$

2.14 REMARK. While the universal cohomology invariants of a G-foliation with an H-reduction of the normal bundle are given by the algebra $H^*(W(\mathfrak{g},H)_\ell)$, we are principally interested in those in the subalgebra $H^*(A(G,H)_\ell)$. In Chapter 4, this subalgebra is related to the space of dual homotopy invariants $\pi^*(I(G)_\ell)$ which we define. For an arbitrary pair of Lie groups (G,H), there is, in general, no direct relationship between the algebra $H^*(W(\mathfrak{g},H)_\ell)$ and the space $\pi^*(I(G)_\ell)$. However, in many cases of interest there is an isomorphism

$$H^*(A(G,H)_\ell) \cong H^*(W(\mathfrak{g},H)_\ell) ,$$

so that there is no loss in generality in considering only the algebra $H^*(A(G,H)_\ell)$. This is the situation for the following triples (G,H,ℓ):

$$G = G\ell(q,\mathbb{R}) \;;\; H = O(q) \text{ or } H = \{e\} \;;\; \ell \geq q$$

$$G = G\ell(n,\mathbb{C}) \;;\; H = U(n) \text{ or } H = \{e\} \;;\; \ell \geq n$$

$$G \text{ any connected group} \;;\; H = \{e\} \;;\; \ell \geq q .$$

A Basis for the Algebra $H^*(A(G,H)_\ell)$

A basis of cocycles for the algebras $H^*(A(G)_\ell)$ and $H^*(A(G,H)_\ell)$ is given in this section. To describe this basis, a notation is introduced which will be used repeatedly in later chapters.

We assume that a transgression $\tau: P \to I(\mathfrak{g})$ has been chosen. Pick a basis $\{y_1,\ldots,y_r\}$ of P^G, the Ad G-invariant elements in P, that satisfies the conditions:

(a) There is a subset $\{y_{\alpha_1},\ldots,y_{\alpha_\nu}\}$ which is a basis of $\overline{P}_\ell$.

(b) Define $c_i = \tau y_i \in I(G)$ for $1 \leq i \leq r$; then the inequality $\deg c_i \leq \deg c_j$ is true for all $i \leq j$.

2.15 REMARK. The set $\{c_1,\ldots,c_r\}$ is an algebra basis for the polynomial ring $I(G)$, [21].

We use I to denote an s-tuple $(i_1,\ldots,i_s)$ of integers which satisfy $1 \leq i_1 < \ldots < i_s \leq r$, and J to denote an r-tuple $(j_1,\ldots,j_r)$ of integers which satisfy $j_k \geq 0$ for all k. We then define

$$y_I c_J = y_{i_1} \cdots y_{i_s} \otimes c_1^{j_1} \cdots c_r^{j_r} .$$

If $I = \phi$, we set $y_I c_J = 1 \otimes c_J$.

For each J as above, set $|J| = \frac{1}{2} \cdot \deg c_J$.

2.16 DEFINITION. An element $y_I c_J$ in $A(G)_\ell$ is said to be admissible if I,J satisfy:

(a) $|J| \leq \ell$

(b) $\deg c_{i_1} c_J > 2\ell$

(c) $j_k = 0$ if $\deg c_k < \deg c_{i_1}$.

The set of admissible elements in $A(G)_\ell$ is denoted by $Z(G)_\ell$.

The set $Z(G)_\ell$ is easily seen to consist of cocycles by condition (b). Our next proposition says that $Z(G)_\ell$ is a basis for the vector space $H^*(A(G)_\ell)$.

2.17 PROPOSITION [39]. The vector space $\langle Z(G)_\ell \rangle$ inherits an algebra structure from $A(G)_\ell$, and there is an isomorphism of algebras

$$\langle Z(G)_\ell \rangle \cong H^*(A(G)_\ell) .$$

The basis $Z(G)_\ell$ of $H^*(A(G)_\ell)$ is a generalization of the basis of $H^*(W(g\ell(q,\mathbb{R}),e)_q)$ which was first described by Vey [22], [6]. For this reason, the set $Z(G)_\ell$ is often called a <u>Vey basis</u>.

The induced product in the space $\langle Z(G)_\ell \rangle$ is easy to describe: For any two admissible cocycles $y_I c_J$ and $y_{I'} c_{J'}$ in $Z(G)_\ell$, their product satisfies $y_I c_J \cdot y_{I'} c_{J'} = 0$. From condition 2.16(c), we have that $\deg c_J \geq \deg c_{i_1}$. Therefore, condition 2.16(b) implies that $2 \cdot \deg c_J > 2\ell$, or $\deg c_J > \ell$. Similarly, we conclude that $\deg c_{J'} > \ell$. It follows that $\deg c_J + \deg c_{J'} > 2\ell$ and hence $c_J \cdot c_{J'} = 0$.

There is a corresponding basis of $H^*(A(G,H)_\ell)$ which consists of elements in $A(G,H)_\ell$ satisfying conditions similar to 2.15(a)-(c). Recall that $\{y_{\alpha_1},\ldots,y_{\alpha_\nu}\}$ is a basis of $\overline{P}_\ell$.

2.18 DEFINITION. An element $y_I c_J$ in $A(G,H)_\ell$ is said to be <u>admissible</u> if I,J satisfy

(a) $|J| \leq \ell$

(b) $\deg c_{i_1} c_J > 2\ell$

(c) $j_{\alpha_k} = 0$ if $\deg c_{\alpha_k} < \deg c_{i_1}$

(d) If $I = \phi$, then $j_{\alpha_k} = 0$ for $1 \leq k \leq \nu$.

The set of admissible elements in $A(G,H)_\ell$ is denoted by $Z(G,H)_\ell$.

The set $Z(G,H)_\ell$ again consists of cocycles, by condition (b), and the vector space $\langle Z(G,H)_\ell \rangle$ inherits an algebra structure from $A(G,H)_\ell$.

2.19 PROPOSITION [39]. There is an isomorphism of algebras $\langle Z(G,H)_\ell \rangle \cong H^*(A(G,H)_\ell)$.

Note that Definition 2.18 reduces to Definition 2.16 for $H = \{e\}$, so that Proposition 2.17 is actually a special case of Proposition 2.19.

The algebra $H^*(A(G,H)_\ell)$ will, in general, have many non-trivial products. If H is not a discrete group and ℓ is sufficiently large, then this algebra contains as a non-trivial subalgebra the quotient $I(G)_\ell/\mathrm{Ideal}\{c_{i_1},\ldots,c_{i_\nu}\}$. Further, the cocycles of the form $y_I c_J$ in $Z(G,H)_\ell$ need not have trivial product.

CHAPTER 3

MINIMAL MODELS AND THE HOMOTOPY THEORY OF ALGEBRAS

We discuss the homotopy theory of algebras and the special role played by minimal algebras. This theory was developed by Sullivan as a solution to the commutative cochain problem, and represents an extension of Quillen's work on rational homotopy theory [54]. The end result is that there is a functor $\mathcal{M}$ from the category of topological spaces and homotopy classes of maps to the category of minimal algebras and algebra homotopy classes of maps; the functor $\mathcal{M}$ is a "rational" equivalence of categories. It is this relationship which makes the theory to be described so powerful.

All differential algebras are assumed to be homologically connected. Recall that a map $f:(A,d_A) \to (B,d_B)$ is said to be a weak isomorphism if the map $f_*:H^*(A) \to H^*(B)$ is an algebra isomorphism.

We are interested in studying a category whose objects are algebras, but whose maps are certain equivalence classes of algebra maps, an algebraic counterpart of the homotopy equivalence of topological maps. For the purpose of introducing this relation, we define an algebra $\{t,dt\}$ which will play the role of the unit interval.

Let $\Bbbk[t]$ be the polynomial algebra on a variable t of degree zero. Let $\Lambda(dt)$ be the exterior algebra on a variable dt of degree one. We define $\{t,dt\} = \Lambda(dt) \otimes \Bbbk[t]$, with a differential determined by setting $d(1 \otimes t) = dt \otimes 1$. For any algebra (B,d_B) and $r \in \Bbbk$ there is a differential algebra map $e_r:B \otimes \{t,dt\} \to B$, defined by setting $e_r(b \otimes t) = r \cdot b$ and $e_r(b \otimes dt) = 0$. It follows that e_r is the identity when restricted to $B \otimes 1$ and is the evaluation at $t = r$ when restricted to $1 \otimes \{t,dt\}$. Using these constructions we next define the basic notion of algebra homotopy.

3.1 DEFINITION. Two differential algebra maps $f_0, f_1 : A \to B$ are <u>algebra homotopic</u>, written $f_0 \underset{a}{\simeq} f_1$, if there is a differential algebra map $F : A \to B \otimes \{t, dt\}$ such that $e_1 \circ F = f_1$ and $e_0 \circ F = f_0$.

Algebra homotopy is clearly a reflexive and symmetric relation. It is not, in general, a transitive relation. However, if A belongs to the class of minimal algebras, to be defined shortly, then $\underset{a}{\simeq}$ is an equivalence relation.

For a connected algebra A, let $\overline{A}$ denote the ideal $\bigoplus_{p>0} A^p$ in A of elements of positive degree. We denote by $(\overline{A})^r$ the ideal in A generated by the r-fold products of elements in $\overline{A}$.

Given an algebra (A, d_A), the differential d_A is said to be <u>decomposable</u> if $d_A(\overline{A}) \subseteq (\overline{A})^2$. This is equivalent to saying that for every $x \in \overline{A}$, the differential $d_A(x)$ is a sum of products of elements of positive degree.

For a graded vector space V, the free graded commutative algebra over V is denoted by $\Lambda(V)$. Decomposing V into a sum of odd and even degree elements, $V \cong V^{odd} \oplus V^{even}$, we have that $\Lambda(V)$ is isomorphic to the exterior algebra on V^{odd} tensored with the symmetric algebra on V^{even}. An algebra A is <u>free</u> if it is isomorphic to the free algebra on a graded vector space V.

Let (A, d_A) be a differential algebra. A differential algebra (B, d_B) is said to be an <u>elementary extension</u> of (A, d_A) if there is an isomorphism $B \cong A \otimes \Lambda(V)$, for some graded vector space V, such that $d_B|_A = d_A$ and $d_B(V) \subseteq A$. If d_A is decomposable, then d_B is decomposable exactly if $d_B(V) \subseteq (\overline{A})^2$.

Following the work of Halperin [29], we give a generalization of the idea of an elementary extension. Let V be a graded vector space with an ordered homogeneous basis $\{x_\alpha\}_{\alpha \in \mathcal{A}}$. Then we write $\Lambda_{<\alpha}$ (resp. $\Lambda_{\leq\alpha}$) for the subalgebras of $\Lambda(V)$ generated by the x_β with $\beta < \alpha$ (resp. $\beta \leq \alpha$). Note that $\Lambda_{\leq\alpha} \cong \Lambda_{<\alpha} \otimes \Lambda(x_\alpha)$.

3.2 DEFINITION. An algebra (B,d_B) is said to be a <u>Koszul-Sullivan</u> (KS) <u>extension</u> of an algebra (A,d_A) if there exists a graded vector space V and a homoogeneous basis $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ of V, indexed by a well-ordered set $\mathcal{A}$, such that $B \cong A \otimes \Lambda(V)$ with

$$d_B|_A = d_A$$

$$d_B(x_\alpha) \in A \otimes \Lambda_{<\alpha}\ ;\ \alpha \in \mathcal{A}\ .$$

Each algebra $A \otimes \Lambda_{\leq\alpha}$ is an elementary extension of $A \otimes \Lambda_{<\alpha}$.

A KS-extension (B,d_B) of (A,d_A) is <u>minimal</u> if the well-ordered homogeneous basis $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ of V can be chosen such that $\deg x_\alpha < \deg x_\beta$ implies $\alpha < \beta$. If (B,d_B) is a minimal KS-extension of (A,d_A) and d_A is decomposable, then d_B is also decomposable.

3.3 DEFINITION. A connected algebra $(\mathcal{M},d_{\mathcal{M}})$ is said to be <u>minimal</u> if it is a minimal KS-extension of the algebra $(\mathbb{k},d_{\mathbb{k}} = 0)$.

A minimal algebra is always free and has a decomposable differential. If a 1-connected algebra $(\mathcal{M},d_{\mathcal{M}})$ is free and $d_{\mathcal{M}}$ is decomposable, then it is minimal. For let $\mathcal{M} \cong \Lambda(V)$ and choose a basis $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ of V consisting of homogeneous elements. Order the set $\mathcal{A}$ by setting $\alpha < \beta$ if $\deg x_\alpha < \deg x_\beta$, and by choosing an arbitrary ordering of the elements x_α within a given degree. For this ordering of $\mathcal{A}$, the algebra $(\Lambda(V),d_{\mathcal{M}})$ is a minimal KS-extension of $(\mathbb{k},d_{\mathbb{k}} = 0)$.

The minimal algebras are distinguished in that they are <u>cofibrant</u> [62], [11]:

3.4 PROPOSITION [Thm. 9.19; 29]. Let $\mathcal{M}$ be a minimal algebra, and suppose a diagram of differential algebras is given,

$$
\begin{array}{ccc}
 & & B \\
 & & \downarrow \psi \\
\mathcal{M} & \xrightarrow[\phi]{} & A
\end{array}
$$

where ψ is a weak isomorphism. Then there is a lift $\tilde{\phi}:\mathcal{M} \to B$, unique up to algebra homotopy, such that $\psi \circ \tilde{\phi} \underset{a}{\simeq} \phi$. If ψ is surjective, then $\tilde{\phi}$ may be chosen such that $\psi \circ \tilde{\phi} = \phi$.

It follows from Proposition 3.4 that $\underset{a}{\simeq}$ is a transitive relation when the domain is a minimal algebra [16]:

3.5 COROLLARY [Prop. 5.14; 29]. Let $(\mathcal{M},d_{\mathcal{M}})$ be a minimal algebra and let (A,d_A) be any algebra. Then algebra homotopy is an equivalence relation on the set of maps $\{\phi:(\mathcal{M},d_{\mathcal{M}}) \to (A,d_A)\}$.

We use $[\mathcal{M},A]$ to denote the set of algebra homotopy classes of differential maps from $\mathcal{M}$ to A. We define a category <u>Min Alg</u>, whose objects are minimal algebras and whose morphisms are <u>algebra</u> <u>homotopy</u> <u>classes</u> <u>of</u> <u>maps</u>. It follows from Propositions 5.14 and 5.15 of [29] that this is a well-defined category. Let <u>Alg</u> denote the category of h-connected differential algebras and differential algebra maps. We next head toward the construction of a functor from <u>Alg</u> to <u>Min Alg</u>.

3.6 DEFINITION. A <u>minimal</u> <u>model</u> of an h-connected algebra (A,d_A) consists of a minimal algebra $(\mathcal{M}_A,d_{\mathcal{M}})$ and a weak isomorphism $\phi:\mathcal{M}_A \to A$. A minimal model is denoted by $(\mathcal{M}_A,d_{\mathcal{M}},\phi_A)$, or by abuse of notation just by $\mathcal{M}_A$.

A central result is the following theorem of Sullivan [62]:

3.7 THEOREM [Thm. 6.1; 29]. If (A,d_A) is an h-connected algebra, then it has a minimal model $(\mathcal{M}_A, d_{\mathcal{M}}, \phi_A)$. Given another minimal model $(\mathcal{M}'_A, d'_{\mathcal{M}}, \phi'_A)$ of (A,d_A), there is an isomorphism $\psi: (\mathcal{M}_A, d_{\mathcal{M}}) \to (\mathcal{M}'_A, d'_{\mathcal{M}})$ such that $\phi'_A \circ \psi \underset{a}{\simeq} \phi_A$.

An immediate corollary of this theorem is that if $\phi: (\mathcal{M}, d_{\mathcal{M}}) \to (\mathcal{M}', d'_{\mathcal{M}})$ is a weak isomorphism of minimal algebras, then ϕ is an isomorphism.

Let A be any connected algebra. Define a differential by setting $d_A = 0$; then $H^*(A) = A$. It follows from Theorem 3.7 that (A,d_A) has a minimal model $(\mathcal{M}_A, d_{\mathcal{M}}, \phi_A)$ with $H^*(\mathcal{M}_A) \cong A$. A minimal algebra $(\mathcal{M}, d_{\mathcal{M}})$ of this type is said to be <u>formal</u>, and for every connected algebra A there is a corresponding formal minimal algebra. However, not all minimal algebras are formal [30], [61].

The next proposition is a basic result which follows directly from Proposition 3.4.

3.8 PROPOSITION. Let a map $\psi: (A,d_A) \to (B,d_B)$ be given. Let $(\mathcal{M}_A, d_{\mathcal{M}}, \phi_A)$ and $(\mathcal{M}_B, d_{\mathcal{M}}, \phi_B)$ be minimal models of (A,d_A) and (B,d_B), respectively. Then there is a map $\tilde{\psi}$, unique up to homotopy, such that the diagram below commutes up to algebra homotopy:

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle \phi_A}\uparrow & & \uparrow{\scriptstyle \phi_B} \\ \mathcal{M}_A & \xrightarrow{\tilde{\psi}} & \mathcal{M}_B \end{array} \quad .$$

Further, if $\psi_0 \underset{a}{\simeq} \psi_1 : (A,d_A) \to (B,d_B)$ are two algebra homotopic maps, then the induced maps $\tilde{\psi}_0$ and $\tilde{\psi}_1$ are algebra homotopic.

Equivalently, the map ϕ_B induces a bijection of sets $(\phi_B)_{\#}:[\mathcal{M}_A,\mathcal{M}_B] \leftrightarrow [\mathcal{M}_A,B]$.

With this preparation, we define a functor $\mathcal{M}$ from the category Alg of h-connected algebras to the category Min Alg. For each object (A,d_A) in Alg, choose a minimal model $(\mathcal{M}_A,d_{\mathcal{M}},\phi_A)$. Then $\mathcal{M}:(A,d_A) \to (\mathcal{M}_A,d_{\mathcal{M}})$ is the map of objects. For each differential algebra map $\psi:(A,d_A) \to (B,d_B)$ of objects in Alg, let $\tilde{\psi}:\mathcal{M}_A \to \mathcal{M}_B$ be a map chosen as in Proposition 3.8. Then $\mathcal{M}(\psi)$ is defined to be the algebra homotopy class of $\tilde{\psi}$. By Proposition 3.8 we have that $\mathcal{M}:$Alg $\to$ Min Alg is a well-defined functor. Note that $\mathcal{M}$ has the additional property that if $\psi_0 \underset{a}{\simeq} \psi_1$, then $\mathcal{M}(\psi_0) = \mathcal{M}(\psi_1)$.

The Minimal Model of a Topological Space

Define a category Top whose objects are the pointed, connected topological spaces with the homotopy type of a CW complex. The morphisms of Top are the pointed homotopy classes of maps. We are going to construct a functor, again denoted by $\mathcal{M}$, from Top to Min Alg which defines a "rational" equivalence of categories. The rest of this chapter is essentially devoted to the study of this functor $\mathcal{M}$.

For a topological space X, let $\mathcal{S}(X)$ denote the semi-simplicial complex of singular simplices in X. Let $\mathcal{E}^*(X)$ denote the differential algebra of k-valued, compatible polynomial forms on $\mathcal{S}(X)$ [18], [63]. The algebra $\mathcal{E}^*(X)$ can be thought of as a polynomial deRham complex of $|\mathcal{S}(x)|$, the lean realization of $\mathcal{S}(X)$ [7].

For a manifold M, there are several alternative complexes available. Let $\mathcal{S}_\infty(M)$ denote the semi-simplicial complex of smooth simplices in M, and let $\mathcal{E}^*_\infty(M)$ denote the differential algebra of k-valued, compatible polynomial forms on $\mathscr{A}_\infty(M)$. If k is $\mathbb{R}$ or $\mathbb{C}$, then we also have the algebra $\tilde{\mathcal{E}}^*_\infty(M)$ of compatible, k-valued smooth forms on $\mathscr{A}_\infty(M)$ as well as the algebra $\Omega^*(M)$, the deRham complex of M. These differential algebras are related by inclusions

$$\Omega^*(M) \subseteq \tilde{\mathcal{E}}^*_\infty(M) \supseteq \mathcal{E}^*_\infty(M) \supseteq \mathcal{E}^*(M) \ . \tag{3.9}$$

The following theorem of Dupont-Sullivan-Swan asserts that these inclusions are weak isomorphisms:

3.10 THEOREM (Generalized deRham [18], [63]).

(a) For any field k and X in Top, there is an isomorphism $H^*(X;k) \cong H^*(\mathcal{E}^*(X))$.

(b) For $k = \mathbb{R}$ or $\mathbb{C}$ and M a manifold, the inclusions in (3.9) induce isomorphisms of cohomology algebras.

Let $f:X \to Y$ be a map of topological spaces. This induces a map $\mathscr{A}(f):\mathscr{A}(X) \to \mathscr{A}(Y)$ of complexes, and hence a map $\mathcal{E}(f):\mathcal{E}^*(Y) \to \mathcal{E}^*(X)$ of differential algebras. Similarly, if $f:M \to N$ is a smooth map of manifolds, then there are induced differential algebra maps

$$\mathcal{E}_\infty(f):\mathcal{E}^*_\infty(N) \to \mathcal{E}^*_\infty(M)$$

$$\tilde{\mathcal{E}}_\infty(f):\tilde{\mathcal{E}}^*_\infty(N) \to \tilde{\mathcal{E}}^*_\infty(M)$$

$$df:\Omega^*(N) \to \Omega^*(M) \ .$$

We now come to the point of these constructions. Suppose X is an object in Top; then $\mathcal{E}^*(X)$ is an object in Alg, an h-connected differential algebra. By Theorem 3.7, we can choose a minimal model of $\mathcal{E}^*(X)$, denoted $(\mathcal{M}_X, d_{\mathcal{M}}, \phi_X)$. This defines a map from objects in Top to objects in

Min Alg, $X \to (\mathcal{M}_X, d_{\mathcal{M}})$. If $f:X \to Y$ represents a morphism in Top, then there is induced a map $\mathcal{E}(f):\mathcal{E}^*(Y) \to \mathcal{E}^*(X)$ and by Proposition 3.8 this determines an algebra homotopy class of maps $\mathcal{M}\mathcal{E}(f):\mathcal{M}_Y \to \mathcal{M}_X$. One of the fundamental results is that the algebra homotopy class of $\mathcal{M}\mathcal{E}(f)$ depends only on the homotopy class of $f:X \to Y$. Defining $\mathcal{M}f$ to be the algebra homotopy class of $\mathcal{M}\mathcal{E}(f)$, we see that $f \to \mathcal{M}f$ defines a map from the morphisms in Top to the morphisms in Min Alg. Thus we have:

3.11 THEOREM [63], [29]. $\mathcal{M}$ is a contravariant functor from Top to Min Alg.

If M is a manifold and $k = \mathbb{R}$ or $\mathbb{C}$, then there are alternative constructions of $\mathcal{M}_M$, obtained by choosing a minimal model of one of the algebras $\mathcal{E}^*_\infty(M)$, $\tilde{\mathcal{E}}^*_\infty(M)$ or $\Omega^*(M)$. Correspondingly, a smooth map $f:M \to N$ induces a morphism $\mathcal{M}f$ from the algebra maps $\mathcal{E}_\infty(f)$, $\tilde{\mathcal{E}}_\infty(f)$ or df. This again gives rise to functors $\mathcal{M}'$, $\mathcal{M}''$ and $\mathcal{M}'''$ respectively. However, it follows from Theorem 3.10(b) that each of these functors is naturally equivalent to $\mathcal{M}$. We therefore identify these four functors, and speak only of the model $(\mathcal{M}_M, d_{\mathcal{M}})$ of a space or manifold.

A Rational Adjoint to $\mathcal{M}$

For this section all algebras are over the field $k = \mathbb{Q}$. A topological space X is said to be rational if the homotopy groups of X have a $\mathbb{Q}$-vector space structure: for all $n > 0$, $\pi_n(X) \cong \pi_n(X) \otimes_Z \mathbb{Q}$.

Define full subcategories of Top whose objects X are given by

Top-sc : X is simply connected

$\text{Top}_{\mathbb{Q}}$: X is rational

$\text{Top}_{\mathbb{Q}}$-sc: X is rational and simply connected.

For any category $\mathcal{C}$, we denote by $f\mathcal{C}$ the full subcategory of $\mathcal{C}$ whose objects are of finite type. Thus, we have full subcategories f Top, f $\underline{\text{Top}}_{\mathbb{Q}}$, f Alg and f Min Alg of their respective categories.

Bousfield and Gugenheim have constructed in [11] a contravariant functor $\mathcal{R}$ from f Min Alg to f $\underline{\text{Top}}_{\mathbb{Q}}$, called the realization functor, which is a "rational" adjoint to $\mathcal{M}$. For any space X in f Top, set $X_{\mathbb{Q}} = \mathcal{R}\mathcal{M}_X$. Then there exist a map $i_X : X \to X_{\mathbb{Q}}$ such that $i_X^* : H^*(X_{\mathbb{Q}}) \to H^*(X)$ is an isomorphism of algebras. If X is simply connected, then for each $n > 0$ the groups $\pi_n(X) \otimes_Z \mathbb{Q}$ and $\pi_n(X_{\mathbb{Q}})$ are isomorphic. In this case, $X_{\mathbb{Q}}$ is called the rationalization of X. If X is in f $\underline{\text{Top}}_{\mathbb{Q}}$-sc, then $i_X : X \to X_{\mathbb{Q}}$ is a homotopy equivalence, and the functors $\mathcal{M}$ and $\mathcal{R}$ set up an equivalence of categories [11]:

$$\mathcal{M}\text{:f } \underline{\text{Top}}_{\mathbb{Q}}\text{-sc} \rightleftarrows \text{f } \underline{\text{Min Alg}}\text{-sc}:\mathcal{R} .$$

For a general X in f Top, the space $X_{\mathbb{Q}}$ is called the $\mathbb{Q}$-localization of X [11], and has the properties:

(a) $i_{X_{\mathbb{Q}}} : X_{\mathbb{Q}} \to (X_{\mathbb{Q}})_{\mathbb{Q}}$ is a homotopy equivalence,

(b) $\pi_1(X_{\mathbb{Q}})$ is the rational nilpotent completion of $\pi_1(X)$.

The rational homotopy theory of a space X is the study of the spaces $\pi_n(X) \otimes \mathbb{Q}$ and the homotopy operations defined on them. For a simply connected space X of finite type, this is equivalent to studying the homotopy properties of $X_{\mathbb{Q}}$, which are determined by the minimal model $(\mathcal{M}_X, d_{\mathcal{M}})$. We will discuss in the next section how the homotopy properties of $X_{\mathbb{Q}}$ can be read off directly from $(\mathcal{M}_X, d_{\mathcal{M}})$.

Similarly, for $k = \mathbb{R}$ or $\mathbb{C}$ one defines the k-homotopy theory of a simply connected space X to be the invariants of the homotopy type of X which are determined by the minimal model $(\mathcal{M}_X, d_{\mathcal{M}})$ defined over k. There is not a corresponding concept of $\mathfrak{k}$-localizing a space, so there can be no realization functor $\mathcal{R}$; the analysis of $(\mathcal{M}_X, d_{\mathcal{M}})$ we make in the next section will proceed along purely algebraic paths. Note that by Theorem 3.10, the model $(\mathcal{M}_X, d_{\mathcal{M}})$ defined over $\mathfrak{k}$ is obtained, up to isomorphism, by tensoring by $\mathfrak{k}$ the minimal model model of X defined over $\mathbb{Q}$. Therefore, the rational homotopy theory of X determines the $\mathfrak{k}$-homotopy theory.

Whitehead Products and Graded Lie Algebras

We work over a field $\mathfrak{k} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

For a topological space X and any positive integers p,q there is a Z-bilinear pairing [68]: $[\ ,\ ]: \pi_p(X) \otimes \pi_q(X) \to \pi_{p+q-1}(X)$. The bracket $[\alpha,\beta]$ of the homotopy classes α in $\pi_p(X)$ and β in $\pi_q(X)$ is called the Whitehead product of α and β. Up to a sign, $[\alpha,\beta]$ is the obstruction to extending the map $f \vee g: S^p \vee S^q \to X$ to a map $S^p \times S^q \to X$ where $f: S^p \to X$ and $g: S^q \to X$ are representatives of α and β, respectively. The Whitehead product satisfies two general relations:

3.12 PROPOSITION [68]. If X is a topological space, if p,q,r are integers and if $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$ and $\gamma \in \pi_{r+1}(X)$, then

(a) $[\alpha,\beta] = -(-1)^{pq}[\beta,\alpha]$,

(b) $$(-1)^{pr}[[\alpha,\beta],\gamma] + (-1)^{pq}[[\beta,\gamma],\alpha] + (-1)^{qr}[[\gamma,\alpha],\beta] = 0 \; . \tag{3.13}$$

A graded vector space $L = \bigoplus_{p \geq 0} L^p$ is a graded Lie algebra if there is a bilinear pairing, for each p,q positive integers, $[\ ,\ ]: L^p \otimes L^q \to L^{p+q}$, and satisfying the relations, for all $x \in L^p$, $y \in L^q$, $z \in L^r$:

$$[x,y] = -(-1)^{pq}[y,x]$$
$$(-1)^{pr}[[x,y],z] + (-1)^{pq}[[y,z],x] + (-1)^{rq}[[z,x],y] = 0 \;. \tag{3.14}$$

Denote by Lie Alg the category with objects the graded Lie algebras and morphisms the grade preserving Lie algebra maps. Recall that Vect is the category of graded vector spaces over k and grade preserving linear maps. Define a functor s^{-1}:Vect $\to$ Vect, which maps an object V to the graded vector space $s^{-1}V$ with $(s^{-1}V)^p = V^{p+1}$, and which is the "identity" on maps. Proposition 3.12 then asserts that the functor $s^{-1}\pi_* \otimes k$:Top $\to$ Vect actually takes values in the category Lie Alg.

We next construct a functor Π from the category Min Alg to the category of graded vector spaces with a pairing [,] satisfying conditions (3.13). It then follows that $s^{-1}\Pi$:Min Alg $\to$ Lie Alg and the main theorem of this section is that the functors $s^{-1}\pi_* \otimes k$ and $s^{-1}\Pi\mathcal{M}$ from f Top-sc to Lie Alg are naturally equivalent.

For a connected algebra A, let QA denote the graded vector space of indecomposable elements in A: $QA = \overline{A}/(\overline{A})^2$. For an element a in $\overline{A}$, let $\overline{a}$ denote its image in QA. If A has a differential d_A, then QA has an induced differential Qd_A. If d_A is decomposable, then $Qd_A = 0$ by definition. A map of connected algebras ϕ:A $\to$ B induces a map of graded vector spaces $Q\phi$. The following result is the basis for our next definition:

3.15 PROPOSITION [Lemma 8.3; 29]. If $\mathcal{M}$ and B are connected differential algebras, if $\mathcal{M}$ is minimal and if $f_0 \underset{a}{\simeq} f_1 : \mathcal{M} \to B$ are algebra homotopic maps, then the induced maps on indecomposables are equal:

$$Qf_0 = Qf_1 : Q\mathcal{M} \to QB \;.$$

This proposition implies Q is a functor from Min Alg to Vect, and so $Q\mathcal{M}$ defines a functor π^* from either Top or Alg to Vect. The dual homotopy of an algebra (A,d_A) in Alg is defined to be the graded vector space

$$\pi^*(A,d_A) \overset{\text{def}}{=\!=} Q\mathcal{M}_A .$$

The dual homotopy of a space X in Top is the graded vector space

$$\pi^*(X) \overset{\text{def}}{=\!=} Q\mathcal{M}_X .$$

Let Hom:Vect $\to$ Vect denote the contravariant functor Hom $(\ ,\Bbbk)$. We define a covariant functor $\Pi_* = \text{Hom} \circ \pi^*$ from either Top or Alg to Vect. Given a space X (resp. algebra (A,d_A)), we will construct a bracket operation $[\ ,\ ]$ on $\Pi_*(X)$ (resp. $\Pi_*(A)$) which satisfies conditions (3.13). It then follows that $s^{-1}\Pi_*$ is a functor from Top or Alg to Lie Alg.

Let X in Top be given and let $(\mathcal{M}_X,d_{\mathcal{M}})$ be its minimal model. Choose an algebra basis $\{x_1,x_2,\ldots\}$ of $\overline{\mathcal{M}}_X$; their residues $\text{mod}(\overline{\mathcal{M}}_X)^2$ then give a basis $\{\bar{x}_1,\bar{x}_2,\ldots\}$ of $Q\mathcal{M}_X$. The differential $d_{\mathcal{M}}$ is decomposable, so there are numbers a_{ij}^k in $\Bbbk$ such that

$$d_{\mathcal{M}}(x_k) = \frac{1}{2}\sum_{i,j} a_{ij}^k\, Jx_i \wedge x_j \qquad \text{mod}(\overline{\mathcal{M}}_X)^3 ,$$

where for an element x in $\mathcal{M}_X$, Jx denotes $(-1)^{\deg x}\cdot x$. By requiring that

$$a_{ij}^k = -(-1)^{(\deg x_i - 1)(\deg x_j - 1)}\, a_{ji}^k , \tag{3.16}$$

we find that the a_{ij}^k are uniquely determined. Define a linear map $d^*: Q\mathcal{M}_X \to Q\mathcal{M}_X \otimes Q\mathcal{M}_X$ of degree 1 by setting

$$d^*(\bar{x}_k) = \sum_{i,j} a^k_{ij}\, \bar{x}_i \otimes \bar{x}_j \ .$$

The map d^* is well-defined, independent of the choice of basis $\{x_1, x_2, \ldots\}$ and depends only on the map $Q(d): Q\bar{\mathcal{M}}_X \to (\bar{\mathcal{M}}_X)^2/(\bar{\mathcal{M}}_X)^3$.

The map d^* dualizes to an operation

$$[\ ,\ ]_d : \mathrm{Hom}(Q\mathcal{M}_X, k) \otimes \mathrm{Hom}(Q\mathcal{M}_X, k) \to \mathrm{Hom}(Q\mathcal{M}_X, k) \ ,$$

which satisfies 3.13(a) by the symmetry condition (3.16) and satisfies 3.13(b) because $d^2_{\mathcal{M}} = 0$. We give $\Pi_*(X)$ the bracket operation defined by $[\ ,\ ]_d$. With this preparation, we can state the fundamental theorem of minimal model theory, due to Sullivan [62], [63]:

3.17 THEOREM [16], [1]. Given a simply connected space X of finite type, there is a natural isomorphism

$$Q\mathcal{M}_X = \pi^*(X) \xrightarrow{\psi} \mathrm{Hom}(\pi_*(X), k) \ ,$$

such that the diagram commutes:

$$\begin{tikzcd} Q\mathcal{M}_X \arrow[r, "\psi"] \arrow[d, "d^*"'] & \mathrm{Hom}(\pi_*(X), k) \arrow[d, "{[\ ,\ ]^*}"] \\ Q\mathcal{M}_X \otimes Q\mathcal{M}_X \arrow[r, "\psi \otimes \psi"] & \mathrm{Hom}(\pi_*(X), k) \otimes \mathrm{Hom}(\pi_*(X), k) \end{tikzcd} \quad . \qquad (3.18)$$

Here $[\ ,\ ]^*$ denotes the dual of the Whitehead product in $\pi_*(X)$.

There is an equivalent formulation of this theorem in terms of the functors constructed above.

3.19 THEOREM. There is a natural equivalence between the functors $s^{-1}\pi_* \otimes \Bbbk$ and $s^{-1}\Pi_*\mathcal{M}$ from the category f Top-sc to the category Lie Alg.

It follows as a corollary to Theorem 3.17 that the Whitehead product in $\pi_*(X)$ determines the map d^* via Diagram 3.18, and therefore the coefficients a^k_{ij} in the expansion of $d_{\mathcal{M}}(x_k)$. In this sense, it is said that the Whitehead product in $\pi_*(X)$ determines the quadratic terms of the differential in $\mathcal{M}_X$, and vice-versa.

The Hurewicz Homomorphism

Let X be a space in Top with minimal model $\mathcal{M}_X$. The quotient map $\overline{\mathcal{M}}_X \to Q\overline{\mathcal{M}}_X = \pi^*(X)$ is a map of differential, graded vector spaces, hence induces a map $H^*(\overline{\mathcal{M}}_X) \to \pi^*(X)$. By Theorem 3.10, the cohomology $H^*(\overline{\mathcal{M}}_X)$ is naturally isomorphic to the reduced cohomology $\tilde{H}^*(X)$. The composition

$$\overline{\mathcal{H}}^*: \tilde{H}^*(X) \to \pi^*(X)$$

is called the (algebraic) dual Hurewicz homomorphism. If X is in f Top-sc, then $\overline{\mathcal{H}}^*$ is related to the usual Hurewicz homomorphism $\mathcal{H}: \pi_*(X) \to H_*(X;Z)$ by a natural commutative diagram:

$$\begin{tikzcd} H^*(X) \arrow[r, "\overline{\mathcal{H}}^*"] \arrow[d, "\cong"'] & \pi^*(X) \arrow[d, "\psi"] \\ \mathrm{Hom}(H_*(X;Z), \Bbbk) \arrow[r, "\mathcal{H}^*"] & \mathrm{Hom}(\pi_*(X), \Bbbk) \end{tikzcd} \quad .$$

3.20 REMARK. The functors π^*: Top → Vect and $s^{-1}\Pi_*$: Top → Lie Alg are related to $s^{-1}\pi_* \otimes \Bbbk$: Top → Vect when restricted to f Top-sc by Theorems 3.17 and 3.19. This relation can be extended to the category

Top-sc. Any object X in Top-sc is the direct limit of spaces in f Top-sc, say $X = \varinjlim_{\alpha \in \mathcal{A}} (X_\alpha, f_\alpha)$ where $f_\alpha : X_\alpha \to X$. The category Vect has inverse limits, and we have $\pi^*(X) \cong \varprojlim_{\alpha \in \mathcal{A}} \pi^*(X_\alpha)$. For each $\alpha \in \mathcal{A}$, Theorem 3.17 gives a natural isomorphism $\psi_\alpha : \pi^*(X_\alpha) \to \mathrm{Hom}(\pi_*(X_\alpha), k)$. Passing to limits we obtain

$$\begin{aligned} \pi^*(X) &\cong \varprojlim_{\alpha \in \mathcal{A}} \pi^*(X_\alpha) \\ &\overset{\psi_\alpha}{\cong} \varprojlim_{\alpha \in \mathcal{A}} \mathrm{Hom}(\pi_*(X_\alpha), k) \\ &= \mathrm{Hom}(\varinjlim_{\alpha \in \mathcal{A}} \pi_*(X_\alpha), k) \\ &\cong \mathrm{Hom}(\pi_*(X), k) \ . \end{aligned}$$

Similarly, Diagram 3.18 is preserved under limits and so Theorem 3.17 remains valid for any X in Top-sc.

Note that a similar extension is not possible for Theorem 3.19 because $\Pi_*(X)$ and $\varinjlim_{\alpha \in \mathcal{A}} \Pi_*(X_\alpha)$ are not isomorphic if X is not of finite type.

If Y is an arbitrary simply connected space, then there is a space X in Top-sc with a weak homotopy equivalence $f: X \to Y$. The space X has a unique homotopy type and f is unique up to homotopy [Thm. 7.8.1; 60]. For such a pair (X,f), we define $\mathcal{M}_Y$ and $\pi^*(Y)$ to be $\mathcal{M}_X$ and $\pi^*(X)$, respectively. Thus, we can extend the functors $\mathcal{M}$ and π^* to even more general spaces and preserve the relation of Theorem 3.17 with the functor π_*. This extension is necessary in order to work with the various classifying spaces of G-foliations that will arise.

Higher Order Whitehead Products

Given a topological space X, the Whitehead product in $\pi_*(X)$ is an example of a second order operation which is defined for all pairs in $\pi_*(X) \otimes \pi_*(X)$. There are also s-th order Whitehead products in $\pi_*(X)$, for any $s \geq 2$, but which are defined only for subsets of $\pi_*(X) \otimes \cdots \otimes \pi_*(X)$, s factors. If this product is defined for a set $(\alpha_1,\ldots,\alpha_s)$ of elements in $\pi_*(X)$ and if the product $[\alpha_1,\ldots,\alpha_s] \in \pi_*(X)$ is non-zero when tensored by k, then it can be "detected" by an element of $\pi^*(X)$. We will describe how this element can be found in terms of the differential in $\mathcal{M}_X$. We first give the definition of an s-th order Whitehead product.

3.21 DEFINITION [1], [53]. An element α in $\pi_n(X)$ is an s-th order Whitehead product of type $(\alpha_1,\ldots,\alpha_s)$, and we write $\alpha = [\alpha_1,\ldots,\alpha_s]$, if the data in (a), (b) and (c) below are given, for which (d) holds:

(a) There is a partition $n_1 + \ldots + n_s = n + 1$.

(b) There are maps $f_j : S^{n_j} \to X$ representing $\alpha_j \in \pi_{n_j}(X)$ for $1 \leq j \leq s$.

Define $P = S^{n_1} \times \ldots \times S^{n_s}$ and let $W = S^{n_1} \vee \ldots \vee S^{n_s} \subseteq P$ be the wedge of the factors. Choose a point $p_1 \in P$ not in W and set $T = P - \{p_1\}$. The space T is homotopy equivalent to the fat wedge $T(S^{n_1},\ldots,S^{n_s})$, [1]; in particular, for $s = 2$ we have $T \simeq S^{n_1} \vee S^{n_2}$.

(c) The map $f_1 \vee \ldots \vee f_s : W \to X$ extends to a map $f : T \to X$.

Let $D \subseteq P$ be a neighborhood of p_1 diffeomorphic to the closed unit disc in $\mathbb{R}^{n+1}$.

(d) α is represented by the composition

$$\overline{f} : S^n = \partial D \subseteq T \xrightarrow{f} X .$$

The obstruction to extending f to a map $\tilde{f}:P \to X$ is exactly $\alpha \in \pi_n(X)$. For $s = 2$, it follows that $\alpha = [\alpha_1,\alpha_2]$ is, up to a sign, the Whitehead product defined earlier.

3.22 REMARK. For a given set of elements $(\alpha_1,\ldots,\alpha_s)$ in $\pi_*(X)$, Porter has shown [53] there exists an extension $f:T \to X$ of $f_1 \vee \ldots \vee f_s:W \to X$ if and only if for each subset $(\alpha_{j_1},\ldots,\alpha_{j_r})$, for $2 \leq r < s$, the Whitehead product $[\alpha_{j_1},\ldots,\alpha_{j_r}]$ exists and is equal to zero.

An element x in a connected algebra A is said to have <u>order</u> p if $dx \in (\bar{A})^p$ but $dx \notin (\bar{A})^{p+1}$.

Let X be a simply connected space, and suppose a Whitehead product $\alpha = [\alpha_1,\ldots,\alpha_s]$ in $\pi_n(X)$ is given. For an indecomposable element x in $\mathcal{M}_X$ of order s we give a formula, due to Andrews and Arkowitz [1], for determining the value of $\psi(\bar{x}) \in \mathrm{Hom}(\pi_n(X),\mathbb{k})$ on the element α. This is a very useful result, as it is inductive in nature. Let $\{x_1,x_2,\ldots\}$ be an algebra basis of $\bar{\mathcal{M}}_X$. For $x \in \mathcal{M}_X$ of order s, we can write

$$dx = \sum_{I \in \mathcal{K}} \lambda_I x_I + \beta$$

where $\mathcal{K}$ is a set of indices $I = (i_1,\ldots,i_s)$ with $x_I = x_{i_1} \wedge \ldots \wedge x_{i_s}$ and $i_1 \leq \ldots \leq i_s$, each $\lambda_I \in \mathbb{k}$ for $I \in \mathcal{K}$ and $\beta \in (\bar{\mathcal{M}}_X)^{s+1}$.

The elements $\{x_1,x_2,\ldots\}$ determine functions $\psi(\bar{x}_j)$ in $\mathrm{Hom}(\pi_*(X),\mathbb{k})$, so for each $I \in \mathcal{K}$ and α_k in $(\alpha_1,\ldots,\alpha_s)$ set

$$A^I_{jk} = \psi(\bar{x}_j)(\alpha_k) \ , \quad 1 \leq j,k \leq s \ .$$

Note that A^I_{jk} will be zero unless the degree of x_{i_j} equals n_k, that of α_k. The matrix A^I is a sort of lower order period formed from the basis of

$\pi^*(X)$ and the factors of $\alpha = [\alpha_1,\ldots,\alpha_s]$. For each permutation σ of the set $\{1,\ldots,s\}$, let $\varepsilon(\sigma) \in \{0,1\}$ be determined by the rule

$$x_{i_1} \wedge \cdots \wedge x_{i_s} = (-1)^{\varepsilon(\sigma)} x_{i_{\sigma(1)}} \wedge \cdots \wedge x_{i_{\sigma(s)}} .$$

For each $I \in \mathcal{K}$, define a number in $\mathbb{k}$ by

$$\kappa(A^I) = \sum_{\sigma} (-1)^{\varepsilon(\sigma)} A^I_{1\sigma(1)} \cdots A^I_{s\sigma(s)} .$$

Setting $N = \sum_{1 \le i < j \le s} n_i n_j$, we then have the result:

3.23 THEOREM [1]. Let X be a simply connected space. Let $x \in \mathcal{M}_X$ have order $s \geq 2$. Then $\psi(\bar{x}) \in \mathrm{Hom}(\pi_n(X),\mathbb{k})$ vanishes on all Whitehead products of order less than s in $\pi_n(X)$. For an s-th order product $\alpha = [\alpha_1,\ldots,\alpha_s]$ in $\pi_n(X)$ we have, with notation as above,

$$\psi(\bar{x})(\alpha) = (-1)^N \sum_{I \in \mathcal{K}} \lambda_I \kappa(A^I) . \tag{3.24}$$

CHAPTER 4

CONSTRUCTION OF THE DUAL HOMOTOPY INVARIANTS

We define in this chapter the dual homotopy invariants of a foliated manifold and study their basic properties. We consider a fixed, connected manifold M with a G-foliation $\mathfrak{F}$ of codimension q. It is assumed that G has a finite number of components and that k is either $\mathbb{R}$ or $\mathbb{C}$. Also, it is assumed that ℓ is a positive integer for which one of the following three conditions holds:

4.1(a) G is any group and $\ell \geq q$.

4.1(b) $G = O(q)$ or $SO(q)$ and $\ell \geq [q/2]$.

4.1(c) $G = G\ell(n,\mathbb{C})$ or $U(n)$, the G-foliation $\mathfrak{F}$ is integrable and $\ell \geq n$.

The group G being fixed throughout this chapter, we will often suppress it from the notation. For example, I_ℓ will denote the truncated algebra $I(G)_\ell$ and A_ℓ will denote the algebra $A(G)_\ell$. We will denote by $\phi_I:\mathcal{M}(I_\ell) \to I_\ell$ a minimal model of the algebra (I_ℓ, d_I) where $d_I = 0$.

If ω is a G-connection on $P \to M$, adapted to $\mathfrak{F}$, then the truncated Chern-Weil homomorphism $h(\omega):I_\ell \to \Omega(M)$ is well-defined. The following result is the basis for our construction of invariants of the G-foliation $\mathfrak{F}$.

4.2 PROPOSITION. The algebra homotopy class of the composition $h(\omega) \circ \phi_I:\mathcal{M}(I_\ell) \to \Omega(M)$ is independent of the choice of adapted connection ω and depends only on the G-concordance class of $\mathfrak{F}$.

Proof. First note that the algebra homotopy class of $h(\omega) \circ \phi_I$ is well-defined by Corollary 3.5.

Given two adapted connections ω_0 and ω_1, we show that $h(\omega_0) \underset{a}{\simeq} h(\omega_1)$ and hence $h(\omega_0) \circ \phi_I \underset{a}{\simeq} h(\omega_1) \circ \phi_I$. Let $p:M \times \mathbb{R} \to M$ be the projection of the product manifold onto the first factor. This induces a foliation $p^*\mathfrak{F}$ on $M \times \mathbb{R}$; an adapted G-connection for this foliation is given by $\omega_t = t\omega_1 + (1-t)\omega_0$, where t is the coordinate of $\mathbb{R}$. The curvature of ω_t is given by $\Omega_t = d(\omega_t) + \omega_t \wedge \omega_t$, and this expression involves only dt, powers of t and elements of $\Omega(M)$. Therefore, the Chern-Weil homomorphism is given by a composition

$$h(\omega_t): I_\ell \to \Omega(M) \otimes \{t, dt\} \subseteq \Omega(M \times \mathbb{R}) \; .$$

It is then clear that $e_r \circ h(\omega_t) = h(\omega_r)$ for $r = 0$ or 1, so that $h(\omega_0) \underset{a}{\simeq} h(\omega_1)$ as claimed.

Next, assume that $\mathfrak{F}_0$ is a G-foliation on M which is G-concordant to $\mathfrak{F}_1 = \mathfrak{F}$: There is a G-foliation $\tilde{\mathfrak{F}}$ on $M \times \mathbb{R}$ such that $i_r^*\tilde{\mathfrak{F}} = \mathfrak{F}_r$ for $r = 0$ or 1. Let ω be an adapted connection for $\tilde{\mathfrak{F}}$. For any $r \in \mathbb{R}$ there is an algebra homotopy commutative diagram

$$\begin{tikzcd} I_\ell \arrow[r, "h(\omega)"] & \Omega(M \times \mathbb{R}) \arrow[r, shift left, "di_r"] & \Omega(M) \arrow[l, shift left, "dp"] \\ \mathcal{M}(I_\ell) \arrow[u, "\phi_I"] \arrow[rr, "\psi"] & & \mathcal{M}_M \arrow[u, "\phi_M"'] \end{tikzcd}$$

where ψ is chosen, using Proposition 3.8, so that $h(\omega) \circ \phi_I \underset{a}{\simeq} dp \circ \phi_M \circ \psi$. Then we have

$$di_r \circ h(\omega) \circ \phi_I \underset{a}{\simeq} di_r \circ dp \circ \phi_M \circ \psi = \phi_M \circ \psi \; ,$$

and setting r equal to 0 or 1 gives

$$di_0 \circ h(\omega) \circ \phi_I \underset{a}{\simeq} di_1 \circ h(\omega) \circ \phi_I \; .$$

The composition $di_r \circ h(\omega)$ is clearly the Chern-Weil homomorphism of the foliation $\mathfrak{F}_r$, completing the proof of the proposition. □

4.3 COROLLARY. The algebra homotopy class of the induced map of minimal algebras, $\mathcal{M}h(\omega):\mathcal{M}(I_\ell) \to \mathcal{M}_M$, depends only on the G-concordance class of $\mathfrak{F}$.

We denote the morphism in Min Alg that $h(\omega)$ determines by $\mathcal{M}h$. This morphism is the universal invariant of the concordance class of the foliation $\mathfrak{F}$; from it is derived all the other invariants to be constructed. We next investigate the functoriality of $\mathcal{M}h$. Let $f:N \to M$ be a smooth map of manifolds and suppose that f is transverse to $\mathfrak{F}$. An adapted connection for the G-foliation $\overline{\mathfrak{F}}$ on N that f induces is given by $\overline{\omega} = f^*(\omega)$. This gives a commutative diagram

$$\begin{tikzcd} \Omega(M) \arrow[rr, "df"] & & \Omega(N) \\ & I_\ell \arrow[ul, "h(\omega)"] \arrow[ur, "h(\overline{\omega})"'] & \end{tikzcd}$$

from which we conclude $\mathcal{M}h(\overline{\omega}) \underset{a}{\simeq} \mathcal{M}f \circ \mathcal{M}h(\omega)$. It follows that $\mathcal{M}h$ is functorial with respect to transversal maps $f:N \to M$.

The next theorem is the result which defines the dual homotopy invariants of $\mathfrak{F}$:

4.4 THEOREM. Let M be a connected manifold with a G-foliation $\mathfrak{F}$, and let ℓ satisfy condition (4.1). Then the Chern-Weil homomorphism induces a well-defined characteristic map of graded vector spaces

$$h^{\#}:\pi^*(I_\ell) \to \pi^*(M)$$

which depends only on the G-concordance class of $\mathfrak{F}$. Further, $h^{\#}$ is functorial with respect to maps $f: N \to M$ which are transversal to $\mathfrak{F}$.

Proof. We define $h^{\#}$ by applying the functor π^*:Min Alg $\to$ Vect. The theorem then follows from Corollary 4.3. □

The elements in the image of $h^{\#}$ are by definition the dual homotopy invariants of $\mathfrak{F}$. There is an alternative invariant of $\mathfrak{F}$ which is obtained by applying the functor $s^{-1}\Pi_*$ to $\mathcal{M}h$:

4.5 THEOREM. Let M be a connected manifold with a G-foliation $\mathfrak{F}$, and let ℓ satisfy condition (4.1). Then the Chern-Weil homomorphism induces a well-defined characteristic map of graded Lie algebras

$$s^{-1}h_{\#}: s^{-1}\Pi_*(M) \to s^{-1}\Pi_*(I_\ell)$$

which depends only on the G-concordance class of $\mathfrak{F}$. Further, $s^{-1}h_{\#}$ is functorial with respect to maps $f: N \to M$ which are transversal to $\mathfrak{F}$.

Remark 1.18 and the functoriality of the maps $h^{\#}$ and $s^{-1}h_{\#}$ with respect to pull-backs together imply there exist universal maps, for G and ℓ satisfying 4.1(a) or 4.1(b),

$$\tilde{h}^{\#}: \pi^*(I_\ell) \to \pi^*(B\tilde{\Gamma}_G)$$

$$s^{-1}\tilde{h}_{\#}: s^{-1}\Pi_*(B\tilde{\Gamma}_G) \to s^{-1}\Pi_*(I_\ell) .$$

If G and ℓ satisfy condition 4.1(a) or 4.1(c), then there are also universal maps $\tilde{h}^{\#}$ and $s^{-1}\tilde{h}_{\#}$ for the classifying space $B\Gamma_G$. The maps $\tilde{h}^{\#}$ and $s^{-1}\tilde{h}_{\#}$ have the property that if $f: M \to B\tilde{\Gamma}_G$ classifies the foliation $\mathfrak{F}$ on M, then the diagrams

$$\begin{tikzcd}
\pi^*(M) & & \pi^*(B\widetilde{\Gamma}_G) \arrow[ll, "\pi^*(f)"'] \\
& \pi^*(I_\ell) \arrow[ul, "h^{\#}"] \arrow[ur, "\widetilde{h}^{\#}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
s^{-1}\Pi_*(M) \arrow[rr, "s^{-1}\Pi_*(f)"] \arrow[dr, "s^{-1}h_{\#}"'] & & s^{-1}\Pi_*(B\widetilde{\Gamma}_G) \arrow[dl, "s^{-1}\widetilde{h}_{\#}"] \\
& s^{-1}\Pi_*(I_\ell) &
\end{tikzcd}$$

commute. This is the meaning of the universality of $\widetilde{h}^{\#}$ and $s^{-1}\widetilde{h}_{\#}$.

In Chapter 6 we will show that the fiber $F\widetilde{\Gamma}_G$ of the map $\nu: B\widetilde{\Gamma}_G \to BG$ is $(q-1)$-connected, for any group G. For $G = G\ell(q,\mathbb{R})$ or $S\ell(q,\mathbb{R})$, Haefliger has shown that $F\Gamma_G$ is at least $(q-1)$-connected [24]; for $G = G\ell(n,\mathbb{C})$, Landweber has shown $\Gamma\Gamma_G$ to be $(n-1)$-connected [42]. Consequently, for $q \geq 2$ in these cases we have $\pi_1(B\widetilde{\Gamma}_G)$ or $\pi_1(B\Gamma_G)$ is isomorphic to $\pi_1(BG)$, which is isomorphic to $\pi_0(G)$ as a set. If G is connected, then BG is simply connected. By Remark 3.20 we conclude that

$$\pi^*(B\widetilde{\Gamma}_G) \cong \mathrm{Hom}_Z(\pi_*(B\widetilde{\Gamma}_G), k) \ ,$$

so that the universal invariants defined by $\widetilde{h}^{\#}$ are exactly dual homotopy classes.

For a non-connected group G, let G_0 be the connected component of the identity. Then $B\widetilde{\Gamma}_{G_0} \to B\widetilde{\Gamma}_G$ is the universal covering map, hence $\pi_n(B\widetilde{\Gamma}_{G_0}) \cong \pi_n(B\widetilde{\Gamma}_G)$ for $n > 1$. We can then apply the above remarks to the space $B\widetilde{\Gamma}_{G_0}$.

The Spaces of Universal Invariants, $\pi^*(I(G)_\ell)$ and $s^{-1}\Pi_*(I(G)_\ell)$

Recall that ℓ is a fixed integer satisfying (4.1). Let $\{c_1,\ldots,c_r\}$ be an algebra basis of $I_\ell = I(G)_\ell$ with $\deg c_j \le \deg c_k \le 2\ell$ for $j \le k \le r$. We assume that a closed, reductive subgroup H of G is given. Then $\overline{P}_\ell = \langle y_{\alpha_1},\ldots,y_{\alpha_\nu}\rangle$ denotes the vector space of H-basic primitives of $\mathfrak{g}$ having degree less than 2ℓ. With the notation of Chapter 2, let $\tau: P \to I_\ell$ be a transgression with $\tau(y_j) = c_j$. Then $A(G,H)_\ell$ denotes the algebra $\Lambda\overline{P}_\ell \otimes I_\ell$, equipped with the differential $d(y_{\alpha_j} \otimes 1) = 1 \otimes c_{\alpha_j}$. For $H = \{e\}$ we let $A(G)_\ell$ denote $A(G,\{e\})_\ell$.

Recall that $\phi_I: \mathcal{M}(I_\ell) \to I_\ell$ is a minimal model of I_ℓ, and x_j denotes the homogeneous element in $\mathcal{M}(I_\ell)$ with $\phi_I(x_j) = c_j$. Define a differential algebra $\Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell)$ whose differential d_Λ extends that of $\mathcal{M}(I_\ell)$ by setting $d_\Lambda(y_{\alpha_j} \otimes 1) = 1 \otimes x_{\alpha_j}$ for $1 \le j \le \nu$. This algebra is a KS-extension of the minimal algebra $\mathcal{M}(I_\ell)$. The map $\phi_I: \mathcal{M}(I_\ell) \to I_\ell$ extends to a differential algebra map

$$\hat{\phi}_I \overset{\text{def}}{=\!=} \mathrm{id} \otimes \phi_I : \Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell) \to \Lambda\overline{P}_\ell \otimes I_\ell = A(G,H)_\ell \,. \tag{4.6}$$

An immediate spectral sequence argument shows that $\hat{\phi}_I$ is a weak isomorphism [28]. If we endow $\Lambda\overline{P}_\ell$ with the trivial differential, then there is a commutative diagram of differential agebras

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_\ell & \xrightarrow{\ \ i\ \ } & A(G,H)_\ell & & \\ & & \uparrow{\scriptstyle \phi_I} & & \uparrow{\scriptstyle \hat{\phi}_I} & & \\ 0 & \longrightarrow & \mathcal{M}(I_\ell) & \xrightarrow{\mathcal{M}i} & \Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell) & \longrightarrow & \Lambda\overline{P}_\ell \end{array} \tag{4.7}$$

From the bottom row of (4.7) we obtain an exact sequence of differential, graded vector spaces by passing to indecomposables:

$$0 \to Q\mathcal{M}(I_\ell) \to Q(\Lambda\bar{P}_\ell \otimes \mathcal{M}(I_\ell)) \to Q(\Lambda\bar{P}_\ell) \to 0 . \tag{4.8}$$

The differential Qd_Λ in the middle term of (4.8) is non-trivial. Using that $\hat{\phi}_I$ is a weak isomorphism and the freeness of the algebra $\Lambda\bar{P}_\ell \otimes \mathcal{M}(I_\ell)$, we can identify the cohomology $H^*(Q(\Lambda\bar{P}_\ell \otimes \mathcal{M}(I_\ell))$ with $Q\mathcal{M}(A(G,H)_\ell) = \pi^*(A(G,H)_\ell)$, [29]. Therefore, after passing to cohomology, the exact sequence (4.8) gives a long exact sequence

$$\to \pi^{*-1}(\Lambda\bar{P}_\ell) \xrightarrow{\delta} \pi^*(I_\ell) \xrightarrow{i^*} \pi^*(A(G,H)_\ell) \to \quad .$$

The algebra $\Lambda\bar{P}_\ell$ is minimal, hence $\pi^*(\Lambda\bar{P}_\ell) \cong \bar{P}_\ell$ and the map δ is given by $\delta(y_{\alpha_j}) = \bar{x}_{\alpha_j}$. From these remarks, we conclude:

4.9 PROPOSITION. There is an exact sequence of graded vector spaces

$$0 \to \langle \bar{x}_{\alpha_1}, \ldots, \bar{x}_{\alpha_\nu} \rangle \to \pi^*(I(G)_\ell) \xrightarrow{i^{\#}} \pi^*(A(G,H)_\ell) \to 0 \tag{4.10}$$

where the first map is inclusion and the second is $\pi^*(i)$, for the inclusion $I(G)_\ell \xrightarrow{i} A(G,H)_\ell$.

We draw several important corollaries from this proposition.

4.11 COROLLARY. There is an exact sequence of graded vector spaces

$$0 \to \langle \bar{x}_1, \ldots, \bar{x}_r \rangle \to \pi^*(I(G)_\ell) \underset{b}{\overset{i^{\#}}{\rightleftarrows}} \pi^*(A(G)_\ell) \to 0 , \tag{4.12}$$

with a unique splitting b.

Proof. The exact sequence (4.10) reduces to the exact sequence (4.12) when $H = \{e\}$. A splitting b of (4.12) must be unique, since $\pi^n(A(G)_\ell) = 0$ for $n \leq 2\ell$ and $\deg x_j \leq 2\ell$ for each $1 \leq j \leq r$. $\square$

4.13 COROLLARY. The map $\mathcal{M}i : \mathcal{M}(I(G)_\ell) \to \mathcal{M}(A(G,H)_\ell)$ is surjective.

Proof. The map $Q\mathcal{M}i : Q\mathcal{M}(I_\ell) \to Q\mathcal{M}(A(G,H)_\ell)$ is surjective by Proposition 4.9, hence the image of $\mathcal{M}i$ contains a set of algebra generators of $\mathcal{M}(A(G,H)_\ell)$. $\square$

4.14 PROPOSITION. There is an exact sequence of graded Lie algebras

$$0 \to s^{-1}\Pi_*(A(G,H)_\ell) \xrightarrow{i_\#} s^{-1}\Pi_*(I(G)_\ell) \to s^{-1}\langle \bar{x}^*_{\alpha_1}, \ldots, \bar{x}^*_{\alpha_\nu} \rangle \to 0 . \quad (4.15)$$

Proof. Consider the sequence of minimal algebras

$$k[x_{\alpha_1}, \ldots, x_{\alpha_\nu}] \longrightarrow \mathcal{M}(I(G)_\ell) \xrightarrow{\mathcal{M}i} \mathcal{M}(A(G,H)_\ell) \to 0 . \qquad (4.16)$$

Passing to indecomposables gives the exact sequence (4.10), and applying the functor $s^{-1}\mathrm{Hom}(\ ,k)$ gives the sequence (4.15), which is therefore exact. The maps in (4.16) preserve differentials, so the induced maps in (4.15) preserve the bracket operations. $\square$

Corollary 4.11 reduces the problem of determining $\pi^*(I_\ell)$ to the task of determining $\pi^*(A(G)_\ell)$, which we next undertake. Recall from Chapter 2 that Z_ℓ is the set of admissible cocycles in $A_\ell = A(G)_\ell$, and we have $Z_\ell = \{y_I c_J \mid (I,J) \text{ admissible}\}$. It was observed that $y_I c_J \cdot y_{I'} c_{J'} = 0$ in A_ℓ, so the vector space $\langle Z_\ell \rangle$ inherits a trivial algebra structure from A_ℓ.

Since the inclusion $\langle Z_\ell \rangle \hookrightarrow A_\ell$ of algebras is a weak isomorphism, the minimal model of A_ℓ can be chosen to factor $\mathcal{M}(A_\ell) \to \langle Z_\ell \rangle \subseteq A_\ell$.

The further analysis of the algebra $\mathcal{M}(A_\ell)$ and the vector space $\pi^*(A_\ell)$ depends on the fact that the algebra $\langle Z_\ell \rangle$, and hence A_ℓ, is both formal and coformal [2], [30]. The formality of A_ℓ can be interpreted as follows. For each index (I,J), choose a homogeneous cocycle $u_{I,J}$ in $\mathcal{M}(A_\ell)$ which maps onto $y_I c_J$. Denote by $\bar{u}_{I,J}$ the image of $u_{I,J}$ in $\pi^*(A_\ell)$. For each index (I,J), let $s^{-1}\bar{u}^*_{I,J}$ denote a new variable of degree equal $\deg \bar{u}_{I,J} - 1$. We denote by $\mathcal{L}$ the free, graded Lie algebra on the set of generators $\{s^{-1}\bar{u}^*_{I,J} \mid (I,J) \text{ admissible}\}$. Then there is an isomorphism of graded Lie algebras

$$s^{-1}\mathrm{Hom}(\pi^*(A_\ell),\Bbbk) = s^{-1}\Pi_*(A_\ell) \cong \mathcal{L}$$

which satisfies $\bar{u}^*_{I,J}(\bar{u}_{I',J'}) = \delta_I^{I'}\delta_J^{J'}$, where $\bar{u}^*_{I,J}$ denotes the element in $\mathrm{Hom}(\pi^*(A_\ell),\Bbbk)$ corresponding to $s^{-1}\bar{u}^*_{I,J}$.

The coformality of A_ℓ has the following interpretation. The set of algebra generators $\{s^{-1}\bar{u}^*_{I,J} \mid (I,J) \text{ admissible}\}$ of $\mathcal{L}$ gives rise to a Hall vector space basis of $\mathcal{L}$, which we denote $\{s^{-1}w_1^*, s^{-1}w_2^*, \ldots\}$. We will assume that for some N, $\{s^{-1}w_1^*, \ldots, s^{-1}w_N^*\} = \{s^{-1}\bar{u}^*_{I,J} \mid (I,J) \text{ admissible}\}$. Let w_j^* denote the element of $\mathrm{Hom}(\pi^*(A_\ell),\Bbbk)$ corresponding to $s^{-1}w_j^*$; the set $\{w_1^*, w_2^*, \ldots\}$ is a basis of this vector space, and we denote by $\{w_1, w_2, \ldots\}$ the corresponding dual basis of $\pi^*(A_\ell)$. As an algebra, the minimal model $\mathcal{M}(A_\ell)$ of A_ℓ is isomorphic to the free algebra $\Lambda(\langle w_1, w_2, \ldots \rangle)$. We will use the Lie bracket [,] in $\mathcal{L}$ to define a differential $d_{\mathcal{L}}$ in this free algebra. For i,j,k positive integers, let a_{ij}^k be constants defined by the rule

$$[s^{-1}w_i^*, s^{-1}w_j^*] = \sum_k a_{ij}^k \; s^{-1}w_k^* \; . \tag{4.17}$$

The differential $d_{\mathcal{L}}$ is then defined by setting

$$d_{\mathcal{L}}w_k = \frac{1}{2} \sum_{i,j} a_{ij}^k \; Jw_i \wedge w_j \; . \tag{4.18}$$

The coformality of A_ℓ implies that $\mathcal{M}(A_\ell)$ is isomorphic, as a differential graded algebra, to the algebra $(\Lambda(\langle w_1,w_2,\ldots\rangle),d_{\mathcal{L}})$,[50]. Because of this, we will henceforth identify these two algebras.

The above remarks are summarized in the next proposition.

4.19 PROPOSITION. Let $\mathcal{L}$ be the free, graded Lie algebra on the set $s^{-1}Z_\ell^* = \{s^{-1}\bar{u}_{I,J}^* \mid (I,J) \text{ admissible}\}$. Then there is an isomorphism of differential algebras

$$\mathcal{M}(A_\ell) \cong (\Lambda(s\mathrm{Hom}(\mathcal{L},\Bbbk)),d_{\mathcal{L}})$$

where $d_{\mathcal{L}}$ is given by formula (4.18). There is an isomorphism of graded vector spaces

$$\pi^*(A_\ell) \cong s\mathrm{Hom}(\mathcal{L},\Bbbk) \; .$$

4.20 REMARK. Since $\{s^{-1}w_1^*,\ldots,s^{-1}w_N^*\}$ is the generating set of the Hall basis of $\mathcal{L}$, it follows that $d_{\mathcal{L}}w_j = 0$ for $1 \leq j \leq N$. The cocycles $\{w_1,\ldots,w_N\}$ in $\mathcal{M}(A_\ell)$ map onto the cocyles $\{y_I c_J \mid I,J \text{ admissible}\}$ in A_ℓ, so they form a basis of $H^*(\mathcal{M}(A_\ell))$. Moreover, they form a basis of the cocycles in the span of $\{w_1,w_2,\ldots\}$. Let $w = \sum \lambda_j w_j$ satisfy $d_{\mathcal{L}}w = 0$. Then for some element $w' = \sum_{j=1}^{N} a_j w_j$, the difference $w-w'$ is exact. The differential $d_{\mathcal{L}}$ in $\mathcal{M}(A_\ell)$ being decomposable, it follows that $w-w'$ is in $(\overline{\mathcal{M}(A_\ell)})^2$. The elements $\{\bar{w}_1, \bar{w}_2, \ldots\}$ are a basis of $Q\mathcal{M}(A_\ell)$, so we must have $\bar{w}-\bar{w}' = 0$, implying that w is in the span of $\{w_1, \ldots, w_N\}$.

For $W = \langle w_{N+1}, w_{N+2}, \ldots \rangle$, we conclude from the above discussion that $d_{\mathcal{L}}: W \to (\overline{\mathcal{M}(A_\ell)})^2$ is injective. We rephrase this in terms of the mapping $d_{\mathcal{L}}^*$ of Chapter 3:

4.21 LEMMA. Let W be a complement to the space of closed indecomposables in $Q\mathcal{M}(A_\ell)$. Then $d_{\mathcal{L}}^*: W \to Q\mathcal{M}(A_\ell) \otimes Q\mathcal{M}(A_\ell)$ is injective.

4.22 REMARK. There is a version of Lemma 4.21 for the algebra $\mathcal{M}(I_\ell)$ which will be needed in the next chapter. It was shown in Corollary 4.13 that $\mathcal{M}i: \mathcal{M}(I_\ell) \to \mathcal{M}(A_\ell)$ is surjective, and the kernel of $\mathcal{M}i$ is the ideal $Id\{x_1, \ldots, x_r\}$ generated by $\{x_1, \ldots, x_r\}$. For each $j \geq 1$, choose a homogeneous element $\tilde{w}_j$ in $\mathcal{M}(I_\ell)$ which maps onto w_j in $\mathcal{M}(A_\ell)$. Formula (4.18) then lifts to

$$d\tilde{w}_k = \frac{1}{2} \sum_{i,j} a_{ij}^k \, J\tilde{w}_i \wedge \tilde{w}_j \mod Id\{x_1, \ldots, x_r\} . \qquad (4.23)$$

The set $\{x_1, \ldots, x_r, \tilde{w}_1, \tilde{w}_2, \ldots\}$ is an algebra basis of $\mathcal{M}(I_\ell)$, as its image in $Q\mathcal{M}(I_\ell)$ gives a vector space basis by Corollary 4.12. Let d denote the differential in $\mathcal{M}(I_\ell)$, and suppose that $u \in \mathcal{M}(I_\ell)$ satisfies

$$du \in (\overline{\mathcal{M}(I_\ell)})^3 + Id\{x_1, \ldots, x_r\} .$$

For some constants b_j and c_k, we can write u in the form

$$u = \sum_{j=1}^{r} b_j x_j + \sum_{k \geq 1} c_k \tilde{w}_k \mod (\overline{\mathcal{M}(I_\ell)})^2 .$$

Then

$$du = \sum_{K > N} c_k d\tilde{w}_k \mod (\overline{\mathcal{M}(I_\ell)})^3 + Id\{x_1, \ldots, x_r\} .$$

So by (4.23), Lemma 4.20 and the assumption above, we must have $c_k = 0$ for $K > N$. We summarize these remarks as follows:

4.24 LEMMA. Let u in $\mathfrak{M}(I_\ell)$ satisfy

$$du \in (\overline{\mathfrak{M}(I_\ell)})^3 + \mathrm{Id}\{x_1,\ldots,x_r\} .$$

Then, for some constants b_j and c_k,

$$u = \sum_{j=1}^{r} b_j x_j + \sum_{k=1}^{N} c_k \tilde{w}_k \quad \mathrm{mod}\,(\overline{\mathfrak{M}(I_\ell)})^2 .$$

The Relation Between the Dual Homotopy Invariants and the Secondary Classes

For this section we assume that the G-foliation $\mathfrak{F}$ on M admits an H-reduction of the associated principal G-bundle $P \to M$, defined by a section $s: M \to P/H$. Secondary invariants of $\mathfrak{F}$ are then defined by $\Delta_*: H^*(A(G,H)_\ell) \to H^*(M)$. In Chapter 3, the algebraic dual Hurewicz map $\overline{\mathcal{H}}^*: H^*(M) \to \pi^*(M)$ was introduced. The composition $\overline{\mathcal{H}}^* \circ \Delta_*: H^*(A(G,H)_\ell) \to \pi^*(M)$ defines a set of invariants of $\mathfrak{F}$, and a natural question is to ask what relationship they have with the invariants defined by Theorem 4.4. This is answered in the next proposition.

4.25 PROPOSITION. There exist a map of graded vector spaces $\zeta: H^*(A(G,H)_\ell) \to \pi^*(I(G)_\ell)$ such that, for any G-foliation $\mathfrak{F}$ admitting an H-reduction, the diagram below commutes:

$$\begin{array}{ccc} \pi^*(I(G)_\ell) & \xrightarrow{h^\#} & \pi^*(M) \\ \zeta \uparrow & & \uparrow \overline{\mathcal{H}}^* \\ H^*(A(G,H)_\ell) & \xrightarrow{\Delta_*} & H^*(M) \end{array} \qquad (4.26)$$

4.27 REMARK. Notice that $h^{\#} \circ \zeta$ determines the image of Δ_* up to the kernel of $\overline{\mathcal{K}}^*$. In particular, the dependence of the map Δ_* on the choice of the H-reduction s is restricted to varying the image of Δ_* by elements in the kernel of $\overline{\mathcal{K}}^*$. This is in agreement with various formulas that exist for this variation [55].

<u>Proof</u>. Choose a splitting b of the exact sequence (4.10). The commutative diagram

$$\begin{tikzcd} I(G)_\ell \arrow[d, "i"'] \arrow[dr, "h(\omega)"] & \\ A(G,H)_\ell \arrow[r, "\Delta(\omega)"] & \Omega(M) \end{tikzcd}$$

gives rise to a commutative diagram

$$\begin{tikzcd} \pi^*(I_\ell) \arrow[d, shift left, "i^{\#}"] \arrow[dr, "h^{\#}"] & \\ \pi^*(A(G,H)_\ell) \arrow[u, shift left, "b"] \arrow[r, "\Delta^{\#}"] & \pi^*(M) \end{tikzcd} \quad . \tag{4.28}$$

By the naturality of $\overline{\mathcal{K}}^*$, the following square commutes:

$$\begin{tikzcd} \pi^*(A(G,H)_\ell) \arrow[r, "\Delta^{\#}"] & \pi^*(M) \\ H^*(A(G,H)_\ell) \arrow[u, "\overline{\mathcal{K}}^*"] \arrow[r, "\Delta_*"] & H^*(M) \arrow[u, "\overline{\mathcal{K}}^*"'] \end{tikzcd} \quad . \tag{4.29}$$

We set $\zeta \overset{\text{def}}{=\!=} b \circ \overline{\mathcal{K}}^*: H^*(A(G,H)_\ell) \rightarrow \pi^*(I(G)_\ell)$. Then diagrams (4.28) and (4.29) combine to show that the conditions of the proposition are satisfied. $\square$

We next remark on a special case of this proposition, where H is the trivial group.

4.30 PROPOSITION. The map $\zeta : H^*(A(G)_\ell) \to \pi^*(I(G)_\ell)$ is injective.

Proof. We have $\zeta = b \circ \overline{\mathcal{H}}^*$. The map b is clearly injective, so it will suffice to show $\overline{\mathcal{H}}^*$ is injective. By Remark 4.20, we can factor $\overline{\mathcal{H}}^*$ as

$$H^*(A_\ell) \cong H^*(\mathcal{M}(A_\ell)) \cong \langle w_1,\ldots,w_N\rangle \subseteq \langle w_1,w_2,\ldots\rangle \cong \pi^*(A_\ell) ,$$

from which it is clear that $\overline{\mathcal{H}}^*$ is injective. □

If G is a connected group or if $G = G\ell(q,\mathbb{R})$, and if $\ell \geq q$, then by Remark 2.14 there is an isomorphism $H^*(W(\mathfrak{g},\{e\})_\ell) \cong H^*(A(G)_\ell)$. Combining this remark with Proposition 4.25 and 4.30 we obtain:

4.31 PROPOSITION. Let G be a connected group or $G\ell(q,\mathbb{R})$ and suppose $\ell \geq q$. Then there exists an injective map ζ such that for any G-foliation $\mathfrak{F}$ with trivial normal bundle, the following diagram commutes:

$$\begin{tikzcd} \pi^*(I(G)_\ell) \arrow[r, "h^{\#}"] & \pi^*(M) \\ H^*(W(\mathfrak{g},\{e\})_\ell) \arrow[u, "\zeta"] \arrow[r, "\Delta_*"] & H^*(M) \arrow[u, "\overline{\mathcal{H}}^*"'] \end{tikzcd} .$$

There is an alternative construction of the map ζ in Proposition 4.31 which is of interest, as it relates the vector spaces $H^*(W(\mathfrak{g},\{e\})_\ell)$ and $\pi^*(I_\ell)$ in a natural way. We assume the hypothesis of Proposition 4.31 is in force. Recall from Proposition 2.11 that there is an isomorphism

$$\mathrm{Tor}_{I(G)}(k, I_\ell) \cong H^*(W(\mathfrak{g},\{e\})_\ell)$$

since $I(G) = I(\mathfrak{g})$. This isomorphism is constructed using a projective resolution of the field $\Bbbk$ over the polynomial ring $I(G) \cong \Bbbk[c_1,\ldots,c_r]$. By calculating the space $\mathrm{Tor}_{I(G)}(\Bbbk, I_\ell)$ using a projective resolution of I_ℓ over $I(G)$, we will obtain a relation with $Q\mathcal{M}(I_\ell) = \pi^*(I_\ell)$.

It was pointed out by Stasheff in [61] that the minimal model $\phi_I:\mathcal{M}(I_\ell) \to I_\ell$ is a projective resolution of I_ℓ over $I(G)$. Define an algebra inclusion $I(G) \to \mathcal{M}(I_\ell)$ by mapping the generators $\{c_1,\ldots,c_r\}$ of $I(G)$ to the elements $\{x_1,\ldots,x_r\}$ in $\mathcal{M}(I_\ell)$. As an algebra we have $\mathcal{M}(I_\ell) \cong I(G) \otimes \mathcal{M}(A_\ell)$, so this makes $\mathcal{M}(I_\ell)$ into a free $I(G)$-module. With the appropriate filtering, $\mathcal{M}(I_\ell) \xrightarrow{\phi_I} I_\ell \to 0$ is thus a resolution.

Let $I_{(0)}$ denote the local ring obtained by localizing $I(G)$ at the ideal generated by $\{c_1,\ldots,c_r\}$. The algebra I_ℓ is also a module over $I_{(0)}$, as we have $I_\ell \otimes I_{(0)} \cong I_\ell$. For the localization $\mathcal{M}(I_\ell)_{(0)} = \mathcal{M}(I_\ell) \otimes I_{(0)}$, we therefore obtain a resolution $\mathcal{M}(I_\ell)_{(0)} \xrightarrow{\phi_I} I_\ell \to 0$ of I_ℓ over the local ring $I_{(0)}$. A projective resolution which has an algebra structure over a local ring, as $\mathcal{M}(I_\ell)_{(0)}$ does, is called a graded Tate resolution, and has been studied by Jozefiak [33], [64].

The property of resolutions over a local ring that we are interested in is that there always exists minimal ones: There exists a resolution $\mathcal{K}^\cdot \xrightarrow{\lambda} I_\ell \to 0$ of I_ℓ over $I_{(0)}$ with the property that $\mathcal{K}^\cdot \otimes_{I_{(0)}} \Bbbk$ has trivial differential [57]. Further, there is a universal property of such resolutions which asserts, in our case, that there exist a map Φ of complexes over $I_{(0)}$

$$\begin{array}{ccc} \mathcal{M}(I_\ell)_{(0)} & \xrightarrow{\phi_I} & I_\ell \to 0 \\ \Phi \updownarrow & & \| \\ \mathcal{K}^\bullet & \xrightarrow{\lambda} & I_\ell \to 0 \end{array} \quad . \qquad (4.32)$$

The map Φ is unique up to chain homotopy of $I_{(0)}$-modules [57]. This leads us to the next proposition, which is an alternative version of Proposition 4.31.

4.33 PROPOSITION. There exist a canonical inclusion

$$\bar{\zeta}:\mathrm{Tor}_{I(G)}(k, I_\ell) \to \pi^*(I_\ell) \ .$$

Proof. Let $\mathcal{K}^\bullet \to I_\ell \to 0$ be a minimal resolution of I_ℓ over $I_{(0)}$. Since $\mathcal{K}^\bullet \otimes_{I_{(0)}} k$ has trivial differential, there is a canonical isomorphism $\mathrm{Tor}_{I_{(0)}}(k, I_\ell) \cong \mathcal{K}^\bullet \otimes_{I_{(0)}} k$. Noting that $\mathrm{Tor}_{I(G)}(k, I_\ell)$ and $\mathrm{Tor}_{I_{(0)}}(k, I_\ell)$ are naturally isomorphic, define $\bar{\zeta}$ to be the composition

$$\mathcal{K}^\bullet \otimes_{I_{(0)}} k \xrightarrow{\Phi \otimes 1} \overline{\mathcal{M}(I_\ell)}_{(0)} \otimes_{I_{(0)}} k \longrightarrow Q\mathcal{M}(I_\ell) = \pi^*(I_\ell)$$

where $\Phi:\mathcal{K}^\bullet \to \mathcal{M}(I_\ell)$ is a map of complexes over $I_{(0)}$. A different choice Φ' will produce a map $\bar{\zeta}':\mathcal{K}^\bullet \otimes_{I_{(0)}} k \to \pi^*(I_\ell)$ which is chain homotopic to $\bar{\zeta}$. Since both vector spaces have trivial differential, we see that $\bar{\zeta} = \bar{\zeta}'$, hence the map $\bar{\zeta}$ defined above is unique. □

A Description of ζ in Terms of the Structure of $\mathcal{M}(I_\ell)$

The map $\zeta: H^*(A(G,H)_\ell) \to \pi^*(I_\ell)$ was defined abstractly in the last section. In this section, we give a concrete representation of ζ in terms of the structure of $\mathcal{M}(I_\ell)$. This will prove to be very useful.

Recall that $\phi_I: \mathcal{M}(I_\ell) \to I_\ell$ maps x_j to c_j for $1 \le j \le r$. For a sequence of non-negative integers $J = (j_1,\dots,j_r)$ we have $x_J = x_1^{j_1}\dots x_r^{j_r}$; let $|J|$ denote the integer $\frac{1}{2}\cdot\deg x_J$. We next choose certain elements in $\mathcal{M}(I_\ell)$ by specifying their differentials. The justification for the following process will be given presently.

For each (i,J) with $1 \le i \le r$, $|J| \le \ell$ and $\deg x_i x_J > 2\ell$, choose a homogeneous element $u_{i,J}$ in $\mathcal{M}(I_\ell)$ satisfying $du_{i,J} = -x_i x_J$.

Assume that the elements $u_{I,J}$ have been chosen for $I = (i_1,\dots,i_n)$ with $n < s$. For each $I = (i_1,\dots,i_s)$ with $1 \le i_1,\dots,i_s \le r$, each J with $|J| \le \ell$ and $\deg x_{i_m} x_J > 2\ell$ for $1 \le m \le s$, choose a homogeneous element $u_{I,J}$ in $\mathcal{M}(I_\ell)$ satisfying

$$du_{I,J} = \sum_{m=1}^{s} (-1)^m x_{i_m} u_{I_m,J} \tag{4.34}$$

where $I_m = (i_1,\dots,\hat{i}_m,\dots,i_s)$. For $I = \phi$, we define $u_{I,J} = x_J$; then formula (4.34) is valid for $u_{i,J}$.

The sum on the right-hand side of (4.34) is a cocycle, and $\phi_I: \mathcal{M}(I_\ell) \to I_\ell$ is a weak isomorphism, so it is possible to choose some $u_{I,J}$ satisfying (4.34). Notice that if $u_{I,J}$ and $u'_{I,J}$ are two different choices, then $u_{I,J} - u'_{I,J}$ is a cocycle, must be exact, and is therefore decomposable. Thus, the choice of $u_{I,J}$ is unique modulo $(\overline{\mathcal{M}(I_\ell)})^2$. We denote by $\bar{u}_{I,J}$ the element in $\pi^*(I_\ell)$ determined by $u_{I,J}$.

Recall that the set of admissible cocycles $Z(G,H)_\ell$ in $A(G,H)_\ell$ consists of elements of the form $y_I c_J$ for (I,J) admissible, and gives a basis of $H^*(A(G,H)_\ell)$.

4.35 PROPOSITION. The map $\zeta: H^*(A(G,H)_\ell) \to \pi^*(I(G)_\ell)$ is given by $\zeta: y_I c_J \to \bar{u}_{I,J}$, for $I \neq \phi$.

It follows from this that under the inclusion $\pi^*(A_\ell) \xrightarrow{b} \pi^*(I_\ell)$, the elements $\bar{u}_{I,J}$ in $\pi^*(I_\ell)$ defined above and the elements $\bar{u}_{I,J}$ defined in the last section will coincide.

Proof. The map ζ is given by the composition

$$H^n(A(G,H)_\ell) \xrightarrow{\bar{\mathcal{K}}^*} \pi^n(A(G,H)_\ell) \underset{b}{\overset{i^\#}{\rightleftarrows}} \pi^*(I_\ell) \ ,$$

where we may assume $n > 2\ell$ as $I \neq \phi$. We first construct the map $\bar{\mathcal{K}}^*$ on the element level. Denote by $\phi_A: \mathcal{M}(A(G,H)_\ell) \to A(G,H)_\ell$ a minimal model of $A(G,H)_\ell$. Let $y_I c_J$ in $A(G,H)_\ell$ be an admissible cocycle. Then because ϕ_A is a weak isomorphism, we can choose a cocycle $z_{I,J}$ in $\mathcal{M}(A(G,H)_\ell)$ such that $\phi_A(z_{I,J})$ and $y_I c_J$ differ by a boundary in $A(G,H)_\ell$. The map $\bar{\mathcal{K}}^*$ then sends the cohomology class of $y_I c_J$ to $\bar{z}_{I,J}$, the coset of $z_{I,J}$ in $\pi^*(A(G,H)_\ell)$.

We next show that $i^\#$ maps the coset $\bar{u}_{I,J}$ to the coset $\bar{z}_{I,J}$, implying that $b(\bar{z}_{I,J}) = \bar{u}_{I,J}$ and completing the proof of the proposition. The map $i^\#$ is induced from a map $\mathcal{M}i$, defined to be any map making the square

$$\begin{tikzcd} \mathcal{M}(I_\ell) \arrow[r, "\phi_I"] \arrow[d, "\mathcal{M}i"'] & I_\ell \arrow[d, "i"] \\ \mathcal{M}(A(G,H)_\ell) \arrow[r, "\phi_A"] & A(G,H)_\ell \end{tikzcd} \tag{4.36}$$

algebra homotopy commute. We make a particular choice of $\mathcal{M}i$ to suit our purposes.

Recall that $\Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell)$ is the KS-extension of $\mathcal{M}(I_\ell)$ defined in (4.7); the associated map $\hat{\phi}_I : \Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell) \to A(G,H)_\ell$ is a surjection and a weak isomorphism. By Proposition 3.4, it is possible to choose a lifting $\hat{\phi}_A$ making the diagram commute:

$$\begin{tikzcd} & \Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell) \arrow[d, "\hat{\phi}_I"] \\ \mathcal{M}(A(G,H)_\ell) \arrow[ur, "\hat{\phi}_A"] \arrow[r, "\phi_A"] & A(G,H)_\ell \end{tikzcd} .$$

Note that $\hat{\phi}_A$ is a weak isomorphism. Again using Proposition 3.4, we can choose a map Φ such that the diagram

$$\begin{tikzcd} \mathcal{M}(I_\ell) \arrow[r, "j"] \arrow[d, "\Phi"'] & \Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell) \arrow[d, "\hat{\phi}_I"] \\ \mathcal{M}(A(G,H)_\ell) \arrow[ur, "\hat{\phi}_A"] \arrow[r, "\phi_A"] & A(G,H)_\ell \end{tikzcd} \tag{4.37}$$

algebra homotopy commutes. Since $\hat{\phi}_I \circ j = i \circ \phi_I$, we can take Φ to be the representative of $\mathcal{M}i$ in diagram (4.36).

It was noted that $\hat{\phi}_A$ is a weak isomorphism; because the algebra $\Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell)$ is free, it follows that

$$Q\hat{\phi}_A : \pi^*(A(G,H)_\ell) \to Q(\Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell))$$

is injective [47]. Also, note that

$$Qj : \pi^*(I_\ell) \to Q(\Lambda\overline{P}_\ell \otimes \mathcal{M}(I_\ell)) = \pi^*(I_\ell) \oplus \langle\overline{P}_\ell\rangle$$

is just the inclusion. Therefore, using (4.37), it will suffice to show that $Qj(\overline{u}_{I,J}) = Q\hat{\phi}_A(\overline{z}_{I,J})$ to complete the proof of the proposition.

The final step in the proof consists of exhibiting a cocycle $v_{I,J}$ in $\Lambda\overline{P} \otimes \mathcal{M}(I_\ell)$ which satisfies $Qj(\overline{u}_{I,J}) = \overline{v}_{I,J}$ and $\hat{\phi}_I(v_{I,J}) = y_I c_J$. From this last condition we conclude $\hat{\phi}_A(z_{I,J}) = v_{I,J} + d\alpha$, for some α in $\Lambda\overline{P} \otimes \mathcal{M}(I_\ell)$. The assumption that deg $z_{I,J} = n$ is greater than 2ℓ implies deg $\alpha \geq 2\ell$. Therefore, $d\alpha$ is decomposable and

$$Q\hat{\phi}_A(\overline{z}_{I,J}) = \overline{v}_{I,J} = Qj(\overline{u}_{I,J}) .$$

To define the element $v_{I,J}$, we introduce some notation. Given the index $I = (i_1,\ldots,i_s)$ and integers $1 \leq k_1 < \ldots < k_m \leq s$, set

$$I_{k_1\ldots k_m} = (i_1,\ldots,\hat{i}_{k_1},\ldots,\hat{i}_{k_m},\ldots,i_s) .$$

For any $m \geq 0$, set $\varepsilon(m) = \frac{m(m+1)}{2}$. The element $v_{I,J}$ is then defined by the formula

$$v_{I,J} = \sum_{m=0}^{s} (-1)^{\varepsilon(m)} \sum_{1 \leq k_1 < \ldots < k_m \leq s} (-1)^{k_1+\ldots+k_m}$$

$$\times\, u_{(i_{k_1},\ldots,i_{k_m}),J} \cdot y_{I_{k_1\ldots k_m}} .$$

A check shows that $v_{I,J}$ is a cocycle, and clearly satisfies $\hat{\phi}_I(v_{I,J}) = y_I c_J$. Further, we have

$$\bar{v}_{I,J} = (-1)^{\epsilon(s)}(-1)^{1+2+\ldots+s}\bar{u}_{I,J} = \bar{u}_{I,J} . \quad \square$$

CHAPTER 5

TECHNIQUES

This chapter contains several technical theorems which relate the cohomology of a topological space X with its dual homotopy. These results are then applied to the classifying spaces $F\tilde{\Gamma}_G$ and $B\tilde{\Gamma}_G$ to derive properties of their dual homotopy vector spaces. Note that all of the results of this chapter are also true for the classifying spaces of integrable G-foliations, $B\Gamma_G$ and $F\Gamma_G$. We assume throughout that the ground field $\Bbbk$ is either $\mathbb{R}$ or $\mathbb{C}$.

Generalizations of the Hurewicz Theorem

The first theorem has been in the literature for several years. A proof can be found in the paper [2].

5.1 THEOREM. Let X be an n-connected space.

(a) The rational Hurewicz homomorphism $\mathcal{H}: \pi_m(X) \otimes \mathbb{Q} \to H_m(X;\mathbb{Q})$ is an isomorphism for $m \leq 2n$ and an epimorphism for $m = 2n + 1$.

(b) If $H^m(X;\mathbb{Q})$ is finite-dimensional for $m \leq 2n + 1$, then the dual map $\mathcal{H}^*: H^m(X;\mathbb{Q}) \to \mathrm{Hom}(\pi_m(X),\mathbb{Q})$ is an isomorphism for $m \leq 2n$, and is injective for $m = 2n + 1$.

Let $e: H^m(X) \to \mathrm{Hom}(H_m(X;\mathbb{Q}),\Bbbk)$ be the evaluation map. If X is not of finite type, then 5.1(b) must be weakened:

5.2 COROLLARY. Let X be an n-connected space. Then the kernel of $\mathcal{H}^*: H^m(X) \to \pi^m(X)$ is equal to the kernel of the evaluation map e, for $m \leq 2n + 1$.

Proof. This follows immediately from Theorem 5.1(a), using the Universal Coefficient Theorem [60]. □

The homology groups $H_*(B\tilde{\Gamma}_G;Z)$ of the classifying space $B\tilde{\Gamma}_G$ often turn out to be uncountably generated. We give two definitions which, in a sense, define a measure of how large these groups are.

Let a connected topological space X be given. For a subset $\{z_1,\ldots,z_d\}$ of $H^m(X)$, the evaluation map e induces a homomorphism $z:H_m(X;Z) \to \mathbb{R}^d$ defined as follows: for $c \in H_m(X;Z)$, set

$$z(c) = (e(z_1)(c),\ldots,e(z_d)(c)) \in \mathbb{R}^d .$$

5.3 DEFINITION. A set $\{z_1,\ldots,z_d\} \subseteq H^m(X)$ is said to be independently continuously variable (I.C.V.) if the map $z:H_m(X;Z) \to \mathbb{R}^d$ is surjective.

For a simply connected space X, there is a natural isomorphism $\psi:\pi^m(X) \to \mathrm{Hom}(\pi_m(X),\mathbb{R})$. A set of elements $\{u_1,\ldots,u_d\}$ in $\pi^m(X)$ gives rise to a homomorphism $u:\pi_m(X) \to \mathbb{R}^d$ defined as follows: for $\alpha \in \pi_m(X)$, set

$$u(\alpha) = (\psi(u_1)(\alpha),\ldots,\psi(u_d)(\alpha)) \in \mathbb{R}^d .$$

5.4 DEFINITION. Let X be a simply connected space. A set $\{u_1,\ldots,u_d\} \subseteq \pi^m(X)$ is said to be I.C.V. if the map $u:\pi_m(X) \to \mathbb{R}^d$ is surjective.

The next proposition is an application of the rational Hurewicz Theorem:

5.5 PROPOSITION. Suppose X is an n-connected space and let $\{z_1,\ldots,z_d\} \subseteq H^m(X)$ be an I.C.V. set. If $m \leq 2n + 1$, then

$\{\mathcal{H}^* z_1,\ldots,\mathcal{H}^* z_d\} \subseteq \pi^m(X)$ is an I.C.V. set. That is, the composition $z \circ \mathcal{H}: \pi_n(X) \to \mathbb{R}^d$ is surjective.

Proof. By Theorem 5.1, the composition

$$\pi_m(X) \otimes \mathbb{Q} \xrightarrow{\mathcal{H}} H_m(X;\mathbb{Q}) \xrightarrow{z} \mathbb{R}^d$$

is surjective. The rationals can be written as a limit $\mathbb{Q} = \varinjlim_{p} Z[1/p]$; therefore, for some p, $\pi_m(X) \otimes Z[1/p]$ maps onto an open set in $\mathbb{R}^d$, and the proposition follows. □

Relations Between $\pi^*(B\tilde{\Gamma}_G)$ and $H^*(F\tilde{\Gamma}_G)$

Let G be a fixed group and suppose ℓ satisfies condition (4.1). The following is a very useful result relating $\pi^n(B\tilde{\Gamma}_G)$ with $H^n(F\tilde{\Gamma}_G)$.

5.6 PROPOSITION. Let z be an admissible cocycle in $A(G)_\ell$ of degree n; set $u = \zeta(z) \in \pi^n(I(G)_\ell)$.

(a) Suppose there exists an $\alpha \in \pi_n(B\tilde{\Gamma}_G)$ such that $\nu_\#(\alpha) \in \pi_n(BG)$ is torsion and $\tilde{h}^\#(u)(\alpha) \neq 0$. Then $\tilde{\Delta}_*(z) \in H^n(F\tilde{\Gamma}_G)$ is non-zero.

(b) If $\tilde{h}^\#(u)$ is variable, then $\tilde{\Delta}_*(z)$ is variable.

Proof of (a). Let $N > 0$ be an integer for which $N \cdot \nu_\#(\alpha) = 0$. Choose a map $f: S^n \to B\tilde{\Gamma}_G$ representing $N \cdot \alpha$; the composition $\nu \circ f: S^n \to BG$ is then homotopic to a constant. By Proposition 4.24, we have

$$\mathcal{H}^* \circ \Delta_*(z) = \tilde{h}^\#(u)([f]) = N \cdot \tilde{h}^\#(u)(\alpha)$$

which is non-zero by assumption. Therefore, $\Delta_*(z)$ is non-zero in $H^n(S^n)$ and the claim of part (a) follows.

Proof of (b). Assume that $\tilde{h}^{\#}(u) \in \pi^n(B\tilde{\Gamma}_G)$ is variable. For any $\lambda \in \mathfrak{k}$, this implies there exists a map $f_\lambda : S^n \to B\tilde{\Gamma}_G$ such that $\tilde{h}^{\#}(u)([f_\lambda]) = \lambda$. Since $\pi_n(BG)$ is countable, the homotopy classes of the compositions $\nu \circ f_\lambda : S^n \to BG$, for $\lambda \in \mathfrak{k}$, partition $\mathfrak{k}$ into a countable union of disjoint sets. Therefore, for some λ' the set $U \subseteq \mathfrak{k}$ of λ satisfying $\nu \circ f_{\lambda'} \simeq \nu \circ f_\lambda$ contains an open subset of $\mathfrak{k}$. Without loss of generality, we may assume $\nu \circ f_{\lambda'}$ is homotopic to a constant. Applying Proposition 4.24, we conclude that $e(\tilde{\Delta}_*(z)) : H_n(F\tilde{\Gamma}_G) \to \mathfrak{k}$ maps onto the set U. Since U contains an open subset, this map must be surjective. $\square$

The next set of propositions establishes a bootstrapping process by which, given that some elements in $\pi^*(B\tilde{\Gamma}_G)$ are linearly independent, we can construct a family of others which are linearly independent. These techniques and Proposition 5.6 are at the heart of our methods for showing the non-triviality of the dual homotopy invariants and the secondary classes.

An element $y_I c_J$ in the set $Z_\ell \overset{\text{def}}{=\!=} Z(G)_\ell$ of admissible cocycles is said to have <u>length</u> s if $I = (i_1, \ldots, i_s)$. This gives a grading of Z_ℓ by setting

$$Z_\ell(s) = \{y_I c_J \in Z_\ell \mid y_I c_J \text{ has length } s\},$$

and induces a filtration of Z_ℓ by setting $F^s Z_\ell = \bigcup_{r=1}^{s} Z_\ell(r)$.

Suppose that subsets $\mathcal{J} \subseteq \{1, 2, \ldots, r\}$ and $Z \subseteq Z_\ell(1)$ are given. We then define the <u>extension of</u> Z <u>by</u> $\mathcal{J}$ to be the set

$$Z' \overset{\text{def}}{=\!=} \{y_I c_J \in Z_\ell \mid y_{i_1} c_J \in Z \text{ and } (i_2, \ldots, i_s) \subseteq \mathcal{J}\}.$$

Note that the filtration on Z_ℓ induces one on Z' by setting $F^sZ' = Z' \cap F^sZ_\ell$. With the above notation, we have the following result:

5.7 PROPOSITION. Assume that $B\tilde{\Gamma}_G$ is simply connected, and suppose that $\tilde{h}^{\#} \circ \zeta: \langle Z\rangle \oplus \langle\{c_i \mid i \in S\}\rangle \to \pi^*(B\tilde{\Gamma}_G)$ is injective; then $\tilde{h}^{\#} \circ \zeta: \langle Z'\rangle \to \pi^*(B\tilde{\Gamma}_G)$ is injective.

<u>Proof</u>. For $r = 1$, we are given that $\tilde{h}^{\#} \circ \zeta: \langle F^rZ'\rangle \to \pi^*(B\tilde{\Gamma}_G)$ is injective. Let $s \geq 1$ be given, and assume that the map $\tilde{h}^{\#} \circ \zeta$ is injective for $r = s$; we show it is injective for $r = s + 1$. Suppose that

$$\sum_{\alpha=1}^{\mu} \lambda_\alpha y_{I^\alpha} c_{J^\alpha} \in \langle F^{s+1}Z'\rangle ,$$

of degree n, is in the kernel of $\tilde{h}^{\#} \circ \zeta$, with each $\lambda_\alpha \neq 0$ and $y_{I^\alpha}c_{J^\alpha} \neq y_{I^\beta}c_{J^\beta}$ for $\alpha \neq \beta$. By the inductive hypothesis, some I^α must have length $s + 1 \geq 2$. Let i be the largest integer occurring in the sets $\{I^\alpha \mid 1 \leq \alpha \leq \mu$ and length $I^\alpha \geq 2\}$. Then $i \in \mathscr{I}$, and we can choose an element γ_i in $\pi_d(B\tilde{\Gamma}_G)$ such that

$$\tilde{h}^{\#}(\bar{x}_j)(\gamma_i) = \begin{cases} 0 \text{ for } j \in \mathscr{I} \text{ and } j \neq i \\ \text{a non-zero number for } j = i \end{cases} .$$

Define $\mathscr{A}$ to be the set $\{\alpha \mid i \in I^\alpha\}$. Set $p = n - d + 1$; for any element $\gamma \in \pi_p(B\tilde{\Gamma}_G)$, there is a corresponding Whitehead product $[\gamma_i, \gamma] \in \pi_n(B\tilde{\Gamma}_G)$. By our assumption and (3.24), we have:

$$\tilde{h}^{\#}\Big(\sum_{\alpha=1}^{\mu} \lambda_\alpha u_{I^\alpha, J^\alpha}\Big)([\gamma_i, \gamma])$$

$$= \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \tilde{h}^{\#}(\bar{x}_i)(\gamma_i) \cdot \tilde{h}^{\#}(u_{I^\alpha - i, J^\alpha})(\gamma)$$

$$= \tilde{h}^{\#}(\bar{x}_i)(\gamma_i) \cdot \tilde{h}^{\#}\Big(\sum_{\alpha \in \mathcal{A}} \lambda_\alpha u_{I^\alpha - i, J^\alpha}\Big)(\gamma) \,. \tag{5.8}$$

If $I^\alpha = (i)$, then we have that $\deg u_{I^\alpha - i, J^\alpha} = \deg x_{J^\alpha} \leq 2\ell$. By the choice of i and $\mathcal{A}$, there exists an $\alpha \in \mathcal{A}$ with $I^\alpha \neq (i)$. Therefore, $\deg u_{I^\alpha - i, J^\alpha} > 2\ell$, which implies that $I^\alpha \neq (i)$ for all $\alpha \in \mathcal{A}$. This implies that $\{y_{I^\alpha - i}\, c_{J^\alpha} \mid \alpha \in \mathcal{A}\} \subseteq F^s Z'$, so by the inductive hypothesis $\tilde{h}^{\#}\big(\sum_{\alpha \in \mathcal{A}} \lambda_\alpha u_{I^\alpha - i, J^\alpha}\big)$ is non-zero. Thus, there exists an element γ in $\pi_p(B\tilde{\Gamma}_G)$ making the right-hand side of (5.8) non-zero, contrary to assumption. □

5.9 REMARK. In the proof of the above proposition, it is actually shown that for any $z \in \langle Z' \rangle$ with $z \notin \langle Z \rangle$, there is a Whitehead product $[\gamma_i, \gamma]$ in $\pi_n(B\tilde{\Gamma}_G)$ for which $\tilde{h}^{\#} \circ \zeta(z)([\gamma_i, \gamma]) \neq 0$.

This remark and Proposition 5.7 give us the following extension of Proposition 5.6:

5.10 COROLLARY. Assume that $B\tilde{\Gamma}_G$ is simply connected and let $Z \subseteq Z_\ell(1)$ and $\mathcal{J} \subseteq \{1,2,\ldots,r\}$ be given such that

$$\tilde{h}^{\#} \circ \zeta\colon \langle Z \rangle \oplus \langle \{c_i \mid i \in \mathcal{J}\} \rangle \to \pi^*(B\tilde{\Gamma}_G)$$

is injective. Then $\tilde{\Delta}_*\colon \langle Z' \rangle \to H^*(F\tilde{\Gamma}_G)$ is injective.

Proof. It is well known that the minimal model of BG is a polynomial algebra $[c_1,\ldots,c_t]$ with the trivial differential. Therefore, the homotopy groups of BG in odd degrees are all torsion, and all Whitehead products in $\pi_*(BG)$ are torsion. By Proposition 5.7, we know that $\tilde{h}^\# \circ \zeta\colon \langle Z' \rangle \to \pi^*(B\tilde{\Gamma}_G)$ is injective. Applying these last remarks to the homotopy sequence of the fibration $F\tilde{\Gamma}_G \to B\tilde{\Gamma}_G \to BG$, we conclude that $\tilde{h}^\# \circ \zeta\colon \langle Z' \rangle \to \pi^*(F\tilde{\Gamma}_G)$ is injective. The corollary then follows from Proposition 5.6. □

Let a set $\mathcal{V} \subseteq Z_\ell(1)$ be given for which $\tilde{h}^\# \circ \zeta(\mathcal{V}) \subseteq \pi^*(B\tilde{\Gamma}_G)$ is I.C.V. We define the extension $\mathcal{V}'$ of $\mathcal{V}$ by a set $\mathcal{J}$ as above; then the techniques used in proving Proposition 5.7 can be generalized to show that the set $\tilde{h}^\# \circ \zeta(\mathcal{V}')$ is also I.C.V. This is very useful, as there are many examples of I.C.V. sets in the literature which can thus be extended to larger I.C.V. sets.

5.11 PROPOSITION. Assume that $B\tilde{\Gamma}_G$ is simply connected. Let subsets $\mathcal{J} \subseteq \{1,2,\ldots,r\}$ and $\mathcal{V} \subseteq Z_\ell(1)$ be given. If

$$\tilde{h}^\# \circ \zeta\colon \langle \{c_i \mid i \in \mathcal{J}\} \rangle \to \pi^*(B\tilde{\Gamma}_G)$$

is injective and $\tilde{h}^\# \circ \zeta(\mathcal{V}) \subseteq \pi^*(B\tilde{\Gamma}_G)$ is I.C.V., then $\tilde{h}^\# \circ \zeta(\mathcal{V}')$ is I.C.V.

Proof. It is given that $\tilde{h}^\# \circ \zeta(F^1\mathcal{V}')$ is I.C.V. Assuming that $\tilde{h}^\# \circ \zeta(F^s\mathcal{V}')$ is I.C.V., we will show that the set $\tilde{h}^\# \circ \zeta(F^{s+1}\mathcal{V}')$ is I.C.V. Let V^p be the elements in $F^{s+1}\mathcal{V}'$ of degree p, and let n be an integer for which $V^n \cap \mathcal{V}'(s+1)$ is non-empty. We must show that $\tilde{h}^\# \circ \zeta(V^n)$ is I.C.V. Let i be the largest integer occurring in the index sets $\{I \mid y_I c_J \in V^n \text{ and length } I \geq 2\}$; it follows that $i \in \mathcal{J}$. Enumerate the elements in V^n as $\{y_{I^\alpha}{}^c{}_{J^\alpha} | 1 \leq \alpha \leq \mu\}$, with $i \in I^\alpha$ for $\alpha \leq \alpha_0$ and $i \notin I^\alpha$ for $\alpha > \alpha_0$. As in the proof of Proposition 5.7, we have $I^\alpha \neq (i)$ for $\alpha \leq \alpha_0$ and $\{y_{I^\alpha - i}{}^c{}_{J^\alpha} | \alpha \leq \alpha_0\} \subseteq F^s\mathcal{V}'$.

Let $u_\alpha = \tilde{h}^{\#} \circ \zeta(y_{I^\alpha}{}^c{}_{J^\alpha})$; for the elements $\{u_1, \ldots, u_\mu\}$, form the corresponding evaluation map $u: \pi_n(B\tilde{\Gamma}_G) \to \mathcal{k}^\mu$. Let d be the degree of x_i and set $p = n - d + 1$. By assumption, there exists an element γ_i in $\pi_d(B\tilde{\Gamma}_G)$ for which

$$\tilde{h}^{\#}(\bar{x}_j)(\gamma_i) = \begin{cases} 0 \text{ for } j \in \mathcal{J} \text{ and } j \neq i \\ \text{a non-zero number for } j = i \end{cases} .$$

For $\Pi \overset{\text{def}}{=\!=} \pi_p(B\tilde{\Gamma}_G)$, let X be the space defined by $X = \bigvee_{\gamma \in \Pi} S^n_\gamma$. A natural map $g: X \to B\tilde{\Gamma}_G$ is constructed by letting g restricted to the γ-th factor be the Whitehead product $[\gamma_i, \gamma]: S^n_\gamma \to B\tilde{\Gamma}_G$.

Consider the composition $\pi_n(X) \xrightarrow{g_{\#}} \pi_n(B\tilde{\Gamma}_G) \xrightarrow{u} \mathcal{k}^\mu$. Using (3.24), we see that $\psi(u_\alpha) \circ g_{\#} = 0$ for $\alpha > \alpha_0$. Further, using (3.24), the fact that $\{y_{I^\alpha - i}{}^c{}_{J^\alpha} | \alpha \leq \alpha_0\} \subseteq F^s\mathcal{V}'$ and the inductive hypothesis, we have that $(\psi(u_1), \ldots, \psi(u_{\alpha_0})) \circ g_{\#}: \pi_n(X) \to \mathcal{k}^{\alpha_0}$ is surjective. Therefore, the set $\tilde{h}^{\#} \circ \zeta\{y_{I^\alpha}{}^c{}_{J^\alpha} | \alpha \leq \alpha_0\}$ is I.C.V. and independent of the image of the set $\{y_{I^\alpha}{}^c{}_{J^\alpha} | \alpha > \alpha_0\}$. If this latter set is contained in $F^s\mathcal{V}'$, then we are done. Otherwise, choose a new maximal index $i \in I^\alpha$ for $\alpha > \alpha_0$ and proceed as before. $\square$

The last result of this section is a direct consequence of Propositions 5.6 and 5.11:

5.12 COROLLARY. If $\mathcal{V}$ and $\mathcal{J}$ are sets satisfying the conditions of Proposition 5.11, then the set $\tilde{\Delta}_*(\mathcal{V}') \subseteq H^*(F\tilde{\Gamma}_G)$ is I.C.V.

The Lie Algebra Structure of $s^{-1}\Pi_*(I(G)_\ell)$ and Variability

In this section, we use the Lie algebra structure of $s^{-1}\Pi_*(I(G)_\ell)$ and $s^{-1}\pi_*(B\tilde{\Gamma}_G)$ to construct infinite I.C.V. sets in the image of $\tilde{h}^{\#}$. For a fixed group G and integer ℓ satisfying (4.1), we adopt the abbreviations $I_\ell = I(G)_\ell$, $A_\ell = A(G)_\ell$ and $Z_\ell = Z(G)_\ell$. With the notation of Chapter 4, recall that $\mathcal{W} \overset{\text{def}}{=\!=} \{s^{-1}w_1^*, s^{-1}w_2^*, \ldots\}$ is a Hall basis of $\mathcal{L}$. Via the isomorphism $\mathcal{M}(A_\ell) \cong \mathrm{Hom}(s\mathcal{L}, k)$ of Proposition 4.19, we have that $\mathcal{M}(A_\ell) = \Lambda(\langle w_1, w_2, \ldots\rangle)$ and $\phi_A : \mathcal{M}(A_\ell) \to A_\ell$ maps $\{w_1, \ldots, w_N\}$ to the set $\{z_1, \ldots, z_N\} = Z_\ell$. Finally, we view $\pi^*(A_\ell)$ as a subspace of $\pi^*(I_\ell)$ via the inclusion b of Corollary 4.11.

Let a subset $\mathcal{V}' \subseteq Z_\ell$ be given, which we may assume to be the set $\{z_1, \ldots, z_m\}$ for some $m \leq N$. There is a corresponding subset $\mathcal{W}' = \{s^{-1}w_1^*, \ldots, s^{-1}w_m^*\}$ of the Lie algebra basis $\mathcal{W} = \{s^{-1}w_1^*, \ldots, s^{-1}w_N^*\}$ of $\mathcal{L}$; let $\mathcal{L}'$ denote the free Lie subalgebra of $\mathcal{L}$ which $\mathcal{W}'$ generates. There is an inclusion of Lie algebras $\sigma : \mathcal{L}' \to \mathcal{L}$ corresponding to the inclusion of bases $\mathcal{W}' \subseteq \mathcal{W}$. Define a projection $\rho : \mathcal{L} \to \mathcal{L}'$ by mapping the generators $s^{-1}w_j^*$ to themselves for $1 \leq j \leq m$, and to zero for $j > m$. The composition $\rho \circ \sigma : \mathcal{L}' \to \mathcal{L}'$ fixes the generators of $\mathcal{L}'$, so must be the identity. The map ρ gives rise to an inclusion of minimal algebras $\mathcal{M}(\langle \mathcal{V}' \rangle) \xrightarrow{\rho^*} \mathcal{M}(A_\ell)$ by suspending and dualizing the Lie algebra map. By Proposition 4.19, we can identify $\mathcal{M}(\langle \mathcal{V}' \rangle)$ with the algebra $\Lambda(\mathrm{Hom}(s\mathcal{L}', k))$ whose differential is determined by formula (4.18). For an element w in $\mathcal{L}'$, we see that $d(\rho^*(sw^*))$ is a sum of products of elements in $\rho^*(s\mathcal{L}'^*)$. This observation will be used in the proof of the next proposition.

We use $\mathcal{H}'$ to denote the Hall basis of $\mathcal{L}'$ generated by the set $\mathcal{W}'$; the set $\mathcal{H}'$ will be a subset of the Hall basis $\mathcal{H} = \{s^{-1}w_1^*, s^{-1}w_2^*, \ldots\}$ of $\mathcal{L}$ generated by $\mathcal{W}$. Let $s\mathcal{H}'^* = \{w_{i_1}, w_{i_2}, \ldots\} \subseteq \pi^*(A_\ell)$ be the corresponding subset of the algebra generators $\{w_1, w_2, \ldots\}$ of $\mathcal{M}(A_\ell)$. With this notation, we have the following result:

5.13 PROPOSITION. Let $\mathcal{V}' \subseteq Z_\ell$ be given such that $\tilde{h}^\# \circ \zeta(\mathcal{V}') \subseteq \pi^*(B\tilde{\Gamma}_G)$ is I.C.V. Then the set $\tilde{h}^\#(s\mathcal{H}'^*) \subseteq \pi^*(B\tilde{\Gamma}_G)$ is I.C.V.

If $\mathcal{V}'$ contains at least two elements, then $s\mathcal{H}'^*$ is an infinite set. In fact, if v_n denotes the number of elements in $s\mathcal{H}'^*$ of degree n, then the sequence $\{v_n\}$ has a subsequence tending to infinity [10].

From the definition of an I.C.V. set, we also have the following corollary:

5.14 COROLLARY. With the hypothesis of Proposition 5.13, for each $n > 0$ there is an epimorphism of abelian groups $\pi_n(B\tilde{\Gamma}_G) \to k^{v_n}$.

<u>Proof of Proposition 5.13</u>. Let $\mathcal{V}' = \{z_1, \ldots, z_m\} \subseteq Z_\ell$. For each $1 \leq j \leq m$, let n_j denote the degree of z_j, and form a topological space $Y_j \overset{\text{def}}{=\!=} \bigvee_{\lambda \in k} S_\lambda^{n_j}$. For $Y \overset{\text{def}}{=\!=} \bigvee_{j=1}^{m} Y_j$, the hypothesis of the proposition implies we can choose a map $f: Y \to B\tilde{\Gamma}_G$ such that the composition

$$\pi_{n_j}(Y_j) \subseteq \pi_{n_j}(Y) \xrightarrow{f_\#} \pi_{n_j}(B\tilde{\Gamma}_G) \xrightarrow{\psi\tilde{h}^\#(w_k)} k$$

is surjective for $j = k$, and trivial otherwise. We will show that $h^\#(s\mathcal{H}'^*) \subseteq \pi^*(Y)$ is I.C.V., from which the claim of the proposition follows.

Give the set $s\mathcal{X}'^*$ a grading by setting $s\mathcal{X}'^*_t$ equal to the subset of $s\mathcal{X}'^*$ whose elements are dual to the elements in $\mathcal{X}'$ of bracket length t. For example, $s\mathcal{X}'^*_1$ is precisely the set $s\mathcal{V}'^* = \{w_1,\ldots,w_m\}$ corresponding to $\mathcal{V}'$. Recall that $s\mathcal{X}'^* = \{w_{i_1}, w_{i_2}, \ldots\}$; for each $j \geq 1$, let t_j be the unique integer for which $w_{i_j} \in s\mathcal{X}'^*_{t_j}$.

Give the homotopy groups $\pi_*(Y)$ the lower central filtration of a graded Lie algebra, which is defined by:

$$F_1\pi_*(Y) = \pi_*(Y)$$

$$F_t\pi_*(Y) = s[s^{-1}\pi_*(Y), s^{-1}F_{t-1}\pi_*(Y)] .$$

For each element w_j in the algebra basis $\{w_1,w_2,\ldots\}$ of $\mathcal{M}(A_\ell)$, choose an element $\tilde{w}_j$ in $\mathcal{M}(I_\ell)$ mapping to w_j. From formula (4.23) for the differential d in $\mathcal{M}(I_\ell)$, we have that

$$d\tilde{w}_k = \sum_{i,j} a^k_{ij} J\tilde{w}_i \wedge \tilde{w}_j \quad \text{mod } \mathrm{Id}\{x_1,\ldots,x_r\}$$

for some constants $a^k_{ij} \in k$ determined by the Lie bracket in $\mathcal{L}$. There are two particular cases of this formula of interest: If $k \leq N$, then it was noted in Remark 4.20 that $a^k_{ij} = 0$ for all i,j. That is, $d\tilde{w}_k$ lies in the ideal $\mathrm{Id}\{x_1,\ldots,x_r\}$. If $w_k \in s\mathcal{X}'^*$, then a^k_{ij} is zero if either w_i or w_j is not in $s\mathcal{X}'^*$. This follows from the remarks preceding the statement of Proposition 5.13.

With these constructions, we are in a position to prove the proposition. It is given that $h^\#(s\mathcal{X}'^*_1) \subseteq \pi^*(Y)$ is I.C.V. Further, for each $1 \leq j \leq m$ the element $\tilde{w}_j$ satisfies $d\tilde{w}_j \in \mathrm{Id}\{x_1,\ldots,x_r\}$. Using

Theorem 3.24, we see that the element $\psi h^{\#}(\overline{w}_j)$ in $\mathrm{Hom}(\pi_*(Y),k)$ annihilates the ideal $F_2\pi_*(Y)$.

Next, assume that for all $t \leq \mu$ the following two conditions are true:

$$\psi h^{\#}(s\mathcal{X}_t'^{*}):F_t\pi_*(Y) \rightarrow k^{v_t} \text{ is onto, where } v_t \text{ denotes the number of elements in } s\mathcal{X}_t'^{*}, \tag{5.15}$$

$$\psi h^{\#}(s\mathcal{X}_t'^{*}) \text{ annihilates } F_{t+1}\pi_*(Y) . \tag{5.16}$$

We will show that both (5.15) and (5.16) are then true for $t = \mu + 1$.

We can assume the set $\bigcup_{t=1}^{\mu} s\mathcal{X}_t'^{*}$ is given by $\{w_{i_1},\ldots,w_{i_\eta}\}$, with each w_{i_j} in $s\mathcal{X}_{t_j}'^{*}$, and the set $s\mathcal{X}_{\mu+1}'^{*}$ is given by $\{w_{i_{\eta+1}},\ldots,w_{i_{\eta+v_{\mu+1}}}\}$. Then for $1 \leq k \leq v_{\mu+1}$, there are constants b^k_{cd} defined by formula (4.17) such that

$$d\tilde{w}_{i_{\eta+k}} = \sum_{1 \leq c,d \leq \eta} b^k_{c\,d} J\tilde{w}_{i_c} \wedge \tilde{w}_{i_d} \quad \mathrm{mod}\ \mathrm{Id}\{x_1,\ldots,x_r\} , \tag{5.17}$$

where $b^k_{cd} \neq 0$ implies that $t_c + t_d = \mu + 1$.

By the inductive hypothesis and Theorem 3.23, it follows from (5.17) that $\psi h^{\#}(\overline{w}_{i_{\eta+k}})$ annihilates the ideal $F_{\mu+2}\pi_*(Y)$. This establishes (5.16) for $t = \mu + 1$. It remains to show that (5.15) holds for $t = \mu + 1$.

By definition, the ideal $F_{\mu+1}\pi_*(Y)$ is the image under the Whitehead product of the summand:

$$\bigoplus_{t=1}^{\mu} F_t\pi_*(Y) \otimes F_{\mu-t+1}\pi_*(Y) \xrightarrow{[\ ,\ ]} F_{\mu+1}\pi_*(Y) . \tag{5.18}$$

Using formulas (3.24) and (5.17), the composition of the mapping

$$\psi h^{\#}(s\mathcal{X}'^{*}_{\mu+1}) = \bigoplus_{k=1}^{v_{\mu+1}} \psi h^{\#}(\bar{w}_{i_{\mu+k}})$$

with the map [,] in (5.18) can be written as

$$\bigoplus_{k=1}^{v_{\mu+1}} \Big(\sum_{1 \leq c,d \leq \eta} b^{k}_{cd} \psi h^{\#}(\bar{w}_{i_c}) \otimes \psi h^{\#}(\bar{w}_{i_d}) \Big):$$

$$\bigoplus_{t=1}^{\mu} F_t \pi_*(Y) \otimes F_{\mu-t+1} \pi_*(Y) \rightarrow \mathbb{k}^{v_{\mu+1}} \quad . \tag{5.19}$$

For any term $\psi h^{\#}(\bar{w}_{i_c}) \otimes \psi h^{\#}(\bar{w}_{i_d})$ in the sum on the left-hand side of (5.19) with $b^{k}_{cd} \neq 0$, the inductive hypothesis (5.16) implies that this functional vanishes when restricted to $F_t \pi_*(Y) \otimes F_{\mu-t+1} \pi_*(Y)$, if $t \neq t_c$. On the other hand, by (5.15), there is a subspace of $F_{t_c} \pi_*(Y) \otimes F_{t_d} \pi_*(Y)$ on which the term $\psi h^{\#}(\bar{w}_{i_\xi}) \otimes \psi h^{\#}(\bar{w}_{i_d})$ vanishes for $\xi \neq c$, but for $\xi = c$ takes on all values in $\mathbb{k}$. A similar remark holds with regard to the index d. We conclude from this that the image of the map in (5.19) contains the image of the quadratic mapping:

$$\bigoplus_{k=1}^{v_{\mu+1}} \sum_{1 \leq c,d \leq \eta} b^{k}_{cd} X_c \otimes X_d : \sum_{1 \leq c,d \leq \eta} \mathbb{k}_c \otimes \mathbb{k}_d \rightarrow \mathbb{k}^{v_{\mu+1}} \quad . \tag{5.20}$$

Note that $s\mathcal{X}'^{*}_{\mu+1} \cap s\mathcal{Y}^{*}$ is empty since $\mu + 1 \geq 2$. Therefore, by Lemma 4.21, the quadratic forms

$$\Big\{ \sum_{1 \leq c,d \leq \eta} b^{k}_{cd} X_c \otimes X_d \,\Big|\, 1 \leq k \leq v_{\mu+1} \Big\}$$

are linearly independent. It follows that the image in $\mathcal{k}^{v_{\mu+1}}$ of the mapping in (5.20) is an algebraic variety which is not contained in any hyperplane. By the proposition in the appendix of [3], this algebraic variety must generate $\mathcal{k}^{v_{\mu+1}}$ additively. Since the image of the map in (5.19) is an additive group, the image must be all of $\mathcal{k}^{v_{\mu+1}}$. □

Let a set $\mathcal{V}' \subseteq Z_\ell$ be given such that $\tilde{h}^{\#} \circ \zeta(\mathcal{V}')$ is I.C.V. If $\mathcal{V}'$ contains more than one element, then it was seen in the proof of Proposition 5.13 above that the subgroup of Whitehead products in $\pi_n(B\tilde{\Gamma}_G)$ maps onto $\mathcal{k}^{v_n}$, where v_n is positive for an infinite number of n. A natural question is to ask what this means for the Lie algebra structure of $s^{-1}\pi_*(B\tilde{\Gamma}_G)$: Is there a continuum of Lie subalgebras over the integers in $s^{-1}\pi_*(B\tilde{\Gamma}_G)$ such that their elements of degree n map onto $\mathcal{k}^{v_n}$? This question is addressed in the next proposition.

Let the set $\mathcal{V}'$ be given by $\{z_1,\ldots,z_m\}$, where $\deg z_j = n_j$ for $1 \leq j \leq m$. Define Y to be the space $Y \overset{\text{def}}{=\!=} \bigvee_{j=1}^{m} S^{n_j}$.

Recall that $\mathcal{k}$ is the field $\mathbb{R}$ or $\mathbb{C}$. Let $\hat{\mathbb{Q}}$ be the algebraic closure in $\mathcal{k}$ of the field $\mathbb{Q}$ of rational numbers. Let $\mathcal{K} \overset{\text{def}}{=\!=} \{k_\alpha | \alpha \in \mathcal{A}\} \subseteq \mathcal{k}$ be a transcendence basis for the field $\mathcal{k}$ over the algebraicly closed field $\hat{\mathbb{Q}}$. The set $\mathcal{A}$ is then uncountable, and the set $\mathcal{K}$ is also transcendental over $\mathbb{Q}$.

By the assumption that $\tilde{h}^{\#} \circ \zeta(\mathcal{V}')$ is I.C.V., for each $\alpha \in \mathcal{A}$ and $1 \leq j \leq m$ we can choose a map $f^j_\alpha : S^{n_j} \to B\tilde{\Gamma}_G$ such that

$$\tilde{h}^{\#} \circ \zeta(z_i)([f^j_\alpha]) = k_\alpha \cdot \delta_{ij} , \quad 1 \leq i,j \leq m .$$

Define a map $f_\alpha : Y_\alpha = Y \to B\tilde{\Gamma}_G$ by setting the restriction of f_α to the factor S^{n_j} of Y equal to f_α^j. With this notation, we state our last result in this chapter:

5.21 PROPOSITION. Suppose that $\mathcal{V}' \subseteq Z_\ell$ satisfies $\tilde{h}^{\#} \circ \zeta(\mathcal{V}') \subseteq \pi^*(B\tilde{\Gamma}_G)$ is I.C.V. Then the direct sum of maps

$$\bigoplus_{\alpha \in \mathcal{a}} (f_\alpha)_{\#} : \bigoplus_{\alpha \in \mathcal{a}} \pi_*(Y_\alpha) \otimes \mathbb{Q} \to \pi_*(B\tilde{\Gamma}_G) \otimes \mathbb{Q}$$

is injective.

In other words, the graded Lie algebra $s^{-1}\pi_*(B\tilde{\Gamma}_G)$ contains uncountably many linearly independent, free Lie algebras.

Proof. Suppose that there exists an element

$$\Phi = \sum_{i=1}^{\mu} \lambda_i (f_{\alpha_i})_{\#}(\gamma_i) = 0$$

with $\gamma_i \in \pi_n(Y_{\alpha_i}) \otimes \mathbb{Q}$, each $\lambda_i \in \mathbb{Q}$, and such that $\sum_{i=1}^{\mu} \lambda_i \gamma_i \neq 0$. We will reach a contradition from this assumption.

With the notation of Proposition 5.13, let $s\mathcal{W}'^* = \{w_1, \ldots, w_m\}$ be the set corresponding to $\mathcal{V}'$, and let $\mathcal{L}'_{\mathbb{Q}}$ be the free, graded Lie algebra on the set $\mathcal{W}'$ over the field $\mathbb{Q}$. Then $\pi_*(Y) \otimes \mathbb{Q}$ is isomorphic to $s\mathcal{L}'_{\mathbb{Q}}$. If $\mathcal{X}'$ denotes the Hall basis of $\mathcal{L}'_{\mathbb{Q}}$ constructed from the set $\mathcal{W}'$, then without loss of generality we can assume that each

$$\gamma_i \in \pi_n(Y_\alpha) \otimes \mathbb{Q} = \pi_n(Y) \otimes \mathbb{Q} \cong s\mathcal{L}'_{\mathbb{Q}}$$

is an element of the vector space basis $s\mathcal{X}'$.

Fix an integer i_0 with $1 \leq i_0 \leq \mu$, and let $w \in s\mathcal{V}'^{*}$ be the dual to γ_{i_0}. Define an index set $I = \{i \mid \gamma_i = \gamma_{i_0}\}$. For each $i \in I$, set $\Gamma_i = (f_{\alpha_i})_{\#}(\gamma_{i_0}) \in \pi_n(B\tilde{\Gamma}_G) \otimes \mathbb{Q}$. We can further assume that $\Gamma_i \neq \Gamma_j$ for $i \neq j$. Then the functional $\tilde{h}^{\#}(\bar{w}) \in \pi^n(B\tilde{\Gamma}_G)$ evaluated on Φ gives

$$\tilde{h}^{\#}(\bar{w})(\Phi) = \sum_{i \in I} \lambda_i \tilde{h}^{\#}(\bar{w})(\Gamma_i) . \tag{5.22}$$

We claim that each $\tilde{h}^{\#}(\bar{w})(\Gamma_i)$ is a non-zero polynomial in the ring $\mathbb{Q}[k_{\alpha_i}]$. This follows because the element $\Gamma_i = (f_{\alpha_i})_{\#}(\gamma_i)$ is an iterated Whitehead product of the maps $\{f^j_{\alpha_i} \mid 1 \leq j \leq m\}$, so by Theorem 3.23 and formula (5.17) the evaluation of $\tilde{h}^{\#}(\bar{w})$ on Γ_i yields a polynomial in k_{α_i}. This polynomial is non-zero since $s^{-1}w$ was chosen to be dual to γ_i.

By the assumption that $\sum_{i=1}^{\mu} \lambda_i \gamma_i \neq 0$, we can assume that $\lambda_i \neq 0$ for some $i \in I$. Therefore, the right-hand side of (5.22) is a non-trivial polynomial over $\mathbb{Q}$, and our assumption that $\Phi = 0$ makes this polynomial into a relation between the algebraicly independent elements of the set $\{k_{\alpha_i} \mid i \in I\}$. $\square$

CHAPTER 6

THE HOMOTOPY GROUPS OF $F\tilde{\Gamma}_G$

The classifying spaces $B\tilde{\Gamma}_G$, $F\tilde{\Gamma}_G$, $B\Gamma_G$ and $F\Gamma_G$ defined in Chapter 1 were first introduced by Haefliger in the paper [24]. Therein, the problem was posed of determining the degree of connectivity of the spaces $F\tilde{\Gamma}_G$ and $F\Gamma_G$. In this chapter, we will show that the space $F\tilde{\Gamma}_G$ is always (q-1)-connected. For $G = SO(q)$ or $G = S\ell(q,\mathbb{R})$, this is the best possible result.

The classifying space of foliations with trivial normal bundle, which we denote by $F\Gamma^q \overset{\text{def}}{=\!=} F\Gamma_{G\ell(q,\mathbb{R})}$, was shown by Haefliger to be q-connected [24]. Using entirely different techniques, Thurston and Mather extended Haefliger's result, proving that $F\Gamma^q$ is (q+1)-connected [65]. It has been conjectured that $F\Gamma^q$ is 2q-connected, which is the greatest connectivity this space can possibly have. To see this, suppose more generally that the space $F\Gamma_G$ is m-connected. This implies there is a section of $\nu: B\Gamma_G \to BG$ over the (m+1)-skeleton of the CW complex BG. Therefore, the map $\nu^*: H^n(BG) \to H^n(B\Gamma_G)$ is injective for $n \leq m + 1$. On the other hand, the Bott Vanishing Theorem implies that ν^* is the trivial map for $n > 2q$. While the space $H^{2q+1}(BG)$ is always trivial, the space $H^{2q+2}(BG)$ may not be. Therefore, the connectivity of $F\Gamma_G$ can in general be no greater than 2q. A similar comment applies to the space $F\tilde{\Gamma}_G$.

In some geometric contexts, the Bott Vanishing Theorem can be sharpened. For example, the vanishing theorem of J. Pasternack [51] implies that the map $\nu^*: H^n(BSO(q)) \to H^n(B\tilde{\Gamma}_{SO(q)})$ is trivial for $n > q$. If q is of the form $4k + 3$ for some integer k, then $H^{q+1}(BSO(q)) \neq \{0\}$ and it follows that $F\tilde{\Gamma}_{SO(q)}$ can be at most (q-1)-connected. By the next theorem, this is a sharp upper bound on the connectivity of $F\tilde{\Gamma}_{SO(q)}$:

6.1 THEOREM. The classifying space $F\tilde{\Gamma}_G$ is (q-1)-connected.

For the group $G = SO(q)$, there are many other consequences of this theorem which will be developed in Chapter 9 of this thesis.

An interesting special case of Theorem 6.1 is given by the group $G = S\ell(q,\mathbb{R})$. An $S\ell(q,\mathbb{R})$-foliation $\mathfrak{F}$ on a manifold M consists of a volume form μ, on the normal bundle of the foliation $\mathfrak{F}$, which is parallel along the leaves. It can be shown that this volume form μ is always a local pull-back of the standard volume form $dx_1 \wedge \ldots \wedge dx_q$ on $\mathbb{R}^q$ [Remark 4.2; 17]. Hence, every $S\ell(q,\mathbb{R})$-foliation is integrable and there is a homotopy equivalence between the classifying spaces $F\tilde{\Gamma}_{S\ell(q,\mathbb{R})}$ and $F\Gamma_{S\ell(q,\mathbb{R})}$. We then obtain as a corollary to Theorem 6.1 the following result of Haefliger [25]:

6.2 COROLLARY. The classifying space $F\Gamma_{S\ell(q,\mathbb{R})}$ is (q-1)-connected.

It was noted by Haefliger that there is an epimorphism $\mathrm{vol}:\pi_q(B\Gamma_{S\ell(q,\mathbb{R})}) \to \mathbb{R}$, defined by integrating the volume form μ of an $S\ell(q,\mathbb{R})$-foliation over the manifold S^q. Therefore, Theorem 6.1 is the best possible result when $G = S\ell(q,\mathbb{R})$.

We state one further consequence of Theorem 6.1, which follows directly from Theorem 5.1:

6.3 COROLLARY. The rational Hurewicz homomorphism $\mathcal{H}: \pi_m(F\tilde{\Gamma}_G) \otimes \mathbb{Q} \to H_m(F\tilde{\Gamma}_G;\mathbb{Q})$ is an isomorphism for $m \leq 2q - 2$ and an epimorphism for $m = 2q - 1$.

<u>Proof of Theorem 6.1</u>. Let $M \subseteq \mathbb{R}^q$ be an open subset homotopic to the sphere S^n. Then $\pi_n(F\tilde{\Gamma}_G)$ is isomorphic to $[M,F\tilde{\Gamma}_G]$, the set of homotopy classes of

maps $f:M \to F\tilde{\Gamma}_G$. By Theorem 1.16, there is a bijection between $[M,F\tilde{\Gamma}_G]$ and the set of integrable homotopy classes of G-foliations on M with trivial G-structures. We will show that any two such foliations on M are integrably homotopic.

Fix an integer n with $0 \leq n < q$. Let $(\theta,r) \in \mathbb{R}^{n+1}$ be polar coordinates, with $\theta \in S^n$ and $r \in \mathbb{R}$. For any $a,b \in \mathbb{R}$ with $a < b$, define

$$B(a,b) = \{(\theta,r) \in \mathbb{R}^{n+1} | a < r < b\} \times \mathbb{R}^{q-n-1} .$$

Set $M = B(0,1)$; then $M \subseteq \mathbb{R}^q$ is open and homotopic to S^n.

A codimension q G-foliation on M must be the point foliation with a G-structure on the tangent bundle TM. The tangent bundle is trivial, so the G-structure is characterized by a smooth map $\alpha:M \to Y$, where Y is the coset space $G\ell(q,\mathbb{R})/G$. We dnote by (M,α) the G-foliation on M with characteristic map α. The G-structure on (M,α) is trivial if α is homotopic to the constant map with image the identity coset of Y. For two G-foliations (M,α_0) and (M,α_1) with trivial G-structures, it is apparent that α_0 and α_1 are homotopic.

To prove the theorem, it will suffice to show that if α_0 and α_1 are homotopic to the constant map, then there is an integrable homotopy through G-foliations with trivial G-structure from (M,α_0) to (M,α_1). To do this, we will construct three integrable homotopies on $M \times [0,1]$, $M \times [1,2]$ and $M \times [2,3]$ which combine to give the desired integrable homotopy.

<u>Step 1</u>. Choose a monotone, C^∞-function

$$\phi:[0,1] \to [1/2,1] \text{ with } \phi(t) = \begin{cases} 1 & \text{for } t \leq 1/4 \\ 1/2 & \text{for } t \geq 3/4 \end{cases} .$$

Define $H:M \times [0,1] \to M$ by

$$H_t(\theta,r,v) = (\theta,\phi(t) \cdot (r - 1/2) + 1/2,\ v) \ .$$

For each t, the map H_t is a submersion; H_0 is the identity and H_1 maps M to a subannulus of M. Also, H_t is constant with respect to t for t near 0 or 1 (see Figure 6.4).

Define a G-structure on M by setting $\alpha_0' = \alpha_0 \circ H_1 : M \times \{1\} \to Y$. Then the submersion $H:M \times [0,1] \to (M,\alpha_0)$ defines a G-foliation on $M \times [0,1]$ which is an integrable homotopy from (M,α_0) to (M,α_0').

Step 2. Define $H'':M \times [2,3] \to M$ by setting $H_t'' = H_{3-t}$. Define a G-structure on M by setting $\alpha_1' = \alpha_1 \circ H_2''$. Then the submersion $H'':M \times [2,3] \to (M,\alpha_1)$ defines a G-foliation which is an integrable homotopy from (M,α_1') to (M,α_1).

Step 3. We next produce an integrable homotopy from (M,α_0') to (M,α_1') by constructing a G-foliation (M,α) and a submersion $H':M \times [1,2] \to M$ so that $\alpha_0' = \alpha \circ H_1'$ and $\alpha_1' = \alpha \circ H_2'$.

Define functions f_0 and f_1 as follows:

$$f_0 : B(5/8,1) \to B(0,3/4) \text{ by } f_0(\theta,r,v) = (\theta,2r - 5/4,v)$$

$$f_1 : B(0,3/8) \to B(1/4,1) \text{ by } f_1(\theta,r,v) = (\theta,2r + 1/4,v) \ .$$

Note that f_0 maps $B(3/4,1)$ to the image of H_1 and f_1 maps $B(0,1/4)$ to the image of H_2''. The relationship between all of the maps defined so far is given in Figure 6.4, for the case $q = 2$ and $n = 1$.

There are inclusions

$$i_0 : S^n \times \{3/4\} \times \mathbb{R}^{q-n-1} \subseteq B(5/8,1)$$

$$i_1 : S^n \times \{1/4\} \times \mathbb{R}^{q-n-1} \subseteq B(0,3/8)$$

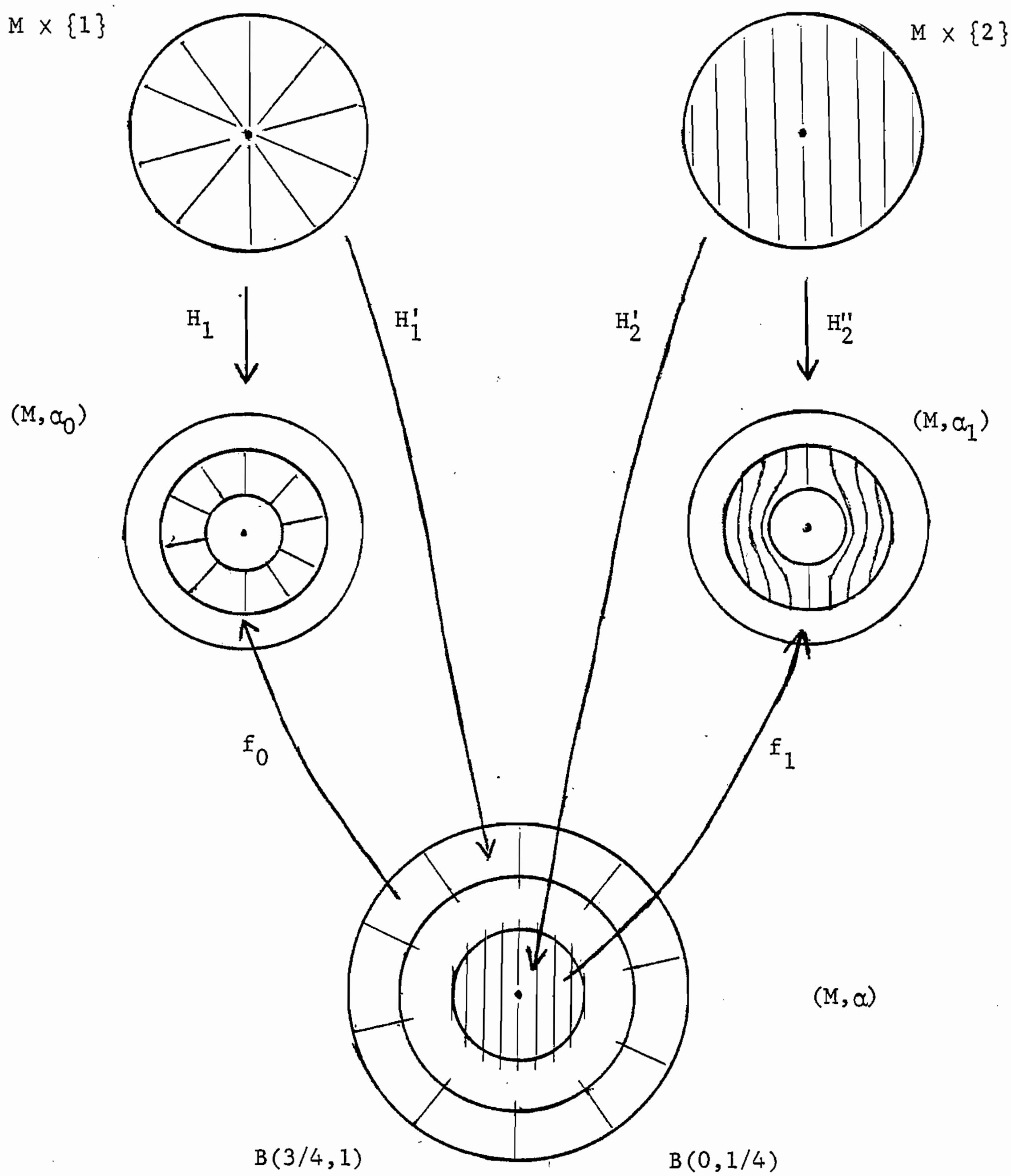

Figure 6.4

and the compositions $\alpha_0 \circ f_0 \circ i_0$ and $\alpha_1 \circ f_1 \circ i_1$ are homotopic by assumption. Therefore, there exists a smooth extension

$$\tilde{\alpha}: S^n \times [1/4,3/4] \times \mathbb{R}^{q-n-1} = \overline{B(1/4,3/4)} \to Y$$

of $\alpha_0 \circ f_0 \circ i_0 \cup \alpha_1 \circ f_1 \circ i_1$. We define a smooth map $\alpha: M \to Y$ by

$$\alpha = \begin{cases} \alpha_0 \circ f_0 & \text{on } B(3/4,1) \\ \tilde{\alpha} & \text{on } \overline{B(1/4,3/4)} \\ \alpha_1 \circ f_1 & \text{on } B(0,1/4) \end{cases} .$$

Finally, we construct the submersion $H': M \times [1,2] \to M$. Choose a monotone, C^∞-function $\varphi: [1,2] \to [0,3]$ with

$$\varphi(t) = \begin{cases} 3 & \text{for } t \leq 5/4 \\ 0 & \text{for } t \geq 7/4 \end{cases} .$$

Then H' at time t is given by

$$H_t'(\theta,r,v) = (\theta, 1/4(r + \varphi(t)), v) .$$

The map H' has the effect of sliding the image of $M \times \{t\}$ from image $f_0^{-1} \circ H_1$ to image $f_1^{-1} \circ H_2''$ as t varies from 1 to 2.

Let $M \times [1,2]$ have the G-foliation defined by the submersion $H': M \times [1,2] \to (M,\alpha)$. This gives an integrable homotopy from $(M, \alpha \circ H_1')$ to $(M, \alpha \circ H_2')$. A straightforward check shows that $f_0 \circ H_1' = H_1$ and $f_1 \circ H_2' = H_2''$. This implies that $\alpha_0' = \alpha \circ H_1'$ and $\alpha_1' = \alpha \circ H_2'$, which finishes Step 3 and completes the proof of Theorem 6.1. $\square$

CHAPTER 7

RATIONALITY AND INTEGRALITY CRITERIA

In this chapter we give criteria for when the secondary classes and dual homotopy invariants of a foliation are rational valued. For example, if a Riemannian foliation is defined by a submersion, the classes $\Delta_*(y_I c_J)$ must be rational. Since there are Riemannian foliations whose characteristic classes are variable, this shows that not all Riemannian foliations can be defined by a submersion. Similar applications in other cases can be given.

Let $\mathfrak{F}$ be a given G-foliation of codimension q on a manifold M of dimension m. Let ℓ be a positive integer for which the characteristic map $h^{\#}:\pi^*(I(G)_\ell) \to \pi^*(M)$ of $\mathfrak{F}$ is well-defined.

The space $\pi^*(I(G)_\ell)$ is defined over the field $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$. For our purposes, we must identify a subspace which represents the rational invariants. It is first necessary to make a normalization. For a topological space X, an element z in $H^*(X)$ is <u>integral</u> if z is in the image of the map $\otimes\mathbb{k}:H^*(X;Z) \to H^*(X)$. We similarly define the <u>rational</u> elements in $H^*(X)$ to be those in the image of the map $\otimes\mathbb{k}:H^*(X;\mathbb{Q}) \to H^*(X)$.

The classifying space BG has a universal Chern-Weil homomorphism $\tilde{h}_*:I(G) \to H^*(BG)$; let $\{c_1,\ldots,c_r\}$ be homogeneous elements of I(G) which form an algebra basis of $I(G)_\ell$ and such that each $\tilde{h}_*(c_j) \in H^*(BG)$ is integral. With respect to this basis we have $I(G)_\ell \cong \mathbb{k}[c_1,\ldots,c_r]_\ell$. Let $I_{\mathbb{Q}}(G)_\ell$ denote the algebra $\mathbb{Q}[c_1,\ldots,c_r]_\ell$. There is a natural inclusion $I_{\mathbb{Q}}(G)_\ell \subseteq I(G)_\ell$, which induces an inclusion $\pi^*(I_{\mathbb{Q}}(G)_\ell) \subseteq \pi^*(I(G)_\ell)$ of vector spaces, where $\pi^*(I_{\mathbb{Q}}(G)_\ell)$ is the rational dual homotopy of the algebra $I_{\mathbb{Q}}(G)_\ell$.

For a topological space X, let $\pi^*_{\mathbb{Q}}(X)$ denote the rational dual homotopy of X. We now state our first result.

7.1 THEOREM. Let $\mathfrak{F}$ be a G-foliation on a manifold M of dimension m. If ℓ satisfies $2\ell \geq m$, then the characteristic map $h^{\#}$ of $\mathfrak{F}$ has a factorization:

$$\begin{tikzcd}
\pi^*(I_{\mathbb{Q}}(G)_{\ell}) \arrow[rr, "h^{\#}"] \arrow[dr, "h^{\#}_{\mathbb{Q}}"'] & & \pi^*(M) \\
& \pi^*_{\mathbb{Q}}(M) \arrow[ur, hook] &
\end{tikzcd} .$$

Further, the map $h^{\#}$ is completely determined by the Chern-Weil homomorphism $h_*: I(G)_{\ell} \to H^*(M)$.

7.2 COROLLARY. Let M, $\mathfrak{F}$ and ℓ be as in Theorem 7.1. If a G-foliation $\mathfrak{F}'$ on a manifold N is defined by a map $f: N \to M$, with f transverse to $\mathfrak{F}$, then the characteristic map $h^{\#}$ of $\mathfrak{F}'$ has a factorization:

$$\begin{tikzcd}
\pi^*(I_{\mathbb{Q}}(G)_{\ell}) \arrow[rr, "h^{\#}"] \arrow[dr, "h^{\#}_{\mathbb{Q}}"'] & & \pi^*(N) \\
& \pi^*_{\mathbb{Q}}(N) \arrow[ur, hook] &
\end{tikzcd} .$$

Proof of Corollary 7.2. The characteristic map $h^{\#}$ is functorial with respect to transversal maps $f: N \to M$, so the corollary follows directly from Theorem 7.1. □

A manifold M of dimension q with a G-structure on TM has a characteristic map $h^{\#}:\pi^*(I(G)_{q'}) \to \pi^*(M)$, where $q' = [q/2]$, defined by giving M the point foliation. For q even, Theorem 7.1 then implies that the dual homotopy invariants of the point foliation on M are completely determined by the primary characteristic classes of the G-structure. Thus, no new information is gained from $h^{\#}$. However, for q odd, Theorem 7.1 only implies that the dual homotopy invariants in the image of $h^{\#}:\pi^*(I(G)_{\ell}) \to \pi^*(M)$ for $\ell > q'$ are determined by the primary classes $h_*(c_i) \in H^*(M)$. The classes in the image of $h^{\#}:\pi^q(I(G)_{q'}) \to \pi^q(M)$ can possibly give finer invariants of the G-structure on M. For example, when $G = O(q)$, the classes of degree q in the image of $h^{\#}$ correspond to the Chern-Simons invariants of the metric on M. These invariants have been extensively studied by Chern and Simons [15], [59], and have been shown to carry geometric information about the manifold M. In addition, it is known they can vary continuously with variations in the metric on M, and therefore need not be rational.

In constructing their invariants, Chern and Simons showed that certain of the secondary classes are integral for a foliation $\mathfrak{F}'$ which satisfies the conditions of Corollary 7.2. We next give a generalization of this result. Let $f:N \to M$ be a smooth map of manifolds with f transverse to the G-foliation $\mathfrak{F}$ on M. The map f induces a G-foliation $\mathfrak{F}'$ on N. Associated to the G-foliation $\mathfrak{F}$ is a principal G-bundle, $P \to M$; let $P' \to N$ denote the pull-back of P along f. The principal bundle $P' \to N$ defines the G-structure on $\mathfrak{F}'$. We assume that for some closed subgroup H of G, the bundle $P' \to N$ admits an H-reduction determined by a section $s':N \to P/H$. Let ℓ be a positive integer for which the characteristic map $\Delta_*:H^*(A(G,H)_{\ell}) \to H^*(N)$ of the foliation $\mathfrak{F}'$ is well-defined.

The space $H^*(A(G,H)_\ell)$ of invariants is defined over $\Bbbk = \mathbb{R}$ or $\Bbbk = \mathbb{C}$. We construct a lattice in this space which represents the integral secondary classes. Let $\{c_1,\ldots,c_r\}$ be the "integral" basis of $I(G)_\ell$ chosen as above; let $\{y_1,\ldots,y_r\} \subseteq (\Lambda\mathfrak{g}^*)^G$ denote the suspensions of these elements. With respect to the subgroup H of G, let $\bar{P}_\ell$ be the subspace of $\langle y_1,\ldots,y_r\rangle$ which was defined in Chapter 2, where $\bar{P}_\ell$ has a basis given by the subset $\{y_{\alpha_1},\ldots,y_{\alpha_\nu}\}$. Denote by $\Lambda_Z(y_{\alpha_1},\ldots,y_{\alpha_\nu})$ the free exterior algebra over Z generated by the set $\{y_{\alpha_1},\ldots,y_{\alpha_\nu}\}$, and define an algebra $A_Z(G,H)_\ell)$ by:

$$A_Z(G,H)_\ell = \Lambda_Z(y_{\alpha_1},\ldots,y_{\alpha_\nu}) \otimes Z[c_1,\ldots,c_r]_\ell \ .$$

There is a natural inclusion $A_Z(G,H)_\ell \subseteq A(G,H)_\ell$ which defines a differential in $A_Z(G,H)_\ell$ such that

$$H^*(A_Z(G,H)_\ell) \otimes \Bbbk \cong H^*(A(G,H)_\ell) \ .$$

In fact, the set $Z(G,H)_\ell$ of admissible cocycles is an algebra basis of $H^*(A_Z(G,H)_\ell)$ over the integers.

We make the following technical assumption, which is satisfied in the situations of interest. The inclusion $H \subseteq G$ induces a map of classifying spaces $BH \to BG$. If $h_*(c_j) \in H^*(BG)$ is mapped to zero under $H^*(BG) \to H^*(BH)$, then we assume that $h_*(c_j) \in H^*(BG;Z)$ is also mapped to zero under $H^*(BG;Z) \to H^*(BH;Z)$.

7.3 THEOREM. Let N have a G-foliation $\mathfrak{F}'$ defined by a map $f:N \to M^m$ as above. If ℓ satisfies $2\ell \geq m$, then:

(a) The characteristic map Δ_* of $\mathfrak{F}'$ has a factorization

$$\begin{tikzcd} H^*(A_Z(G,H)_\ell) \arrow[rr, "\Delta_*"] \arrow[dr, "(\Delta_Z)_*"'] & & H^*(N) \\ & H^*(N;\mathbb{Q}) \arrow[ur] & \end{tikzcd} .$$

In particular, each secondary class $\Delta_*(y_I c_J) \in H^*(N)$ is rational.

(b) For $I = (i)$, each secondary class $\Delta_*(y_i c_J)$ is integral.

(c) For J such that $\deg c_J \geq m$, the secondary class $\Delta_*(y_I c_J)$ is integral.

We state as corollaries the three special cases of this theorem which are most common.

7.4 COROLLARY. Let $\mathfrak{F}'$ be a codimension q foliation with trivial normal bundle on a manifold N. If $\mathfrak{F}'$ is defined by a mpa $f:N \to M^m$ transverse to a foliation $\mathfrak{F}$ on M, where $m \leq 2q$, then for each $y_I c_J \in Z(G\ell(q,\mathbb{R}))_q$, the secondary class $\Delta_*(y_I c_J) \in H^*(N)$ is rational. If $\deg c_J \geq m$, then $\Delta_*(y_I c_J)$ is integral.

7.5 COROLLARY. Let $\mathfrak{F}'$ be an $SO(q)$-foliation with trivial normal bundle on a manifold N. If $\mathfrak{F}'$ is defined by a submersion $f:N \to M^q$, then each secondary class $\Delta_*(y_I c_J) \in H^*(N)$ is rational for $y_I c_J \in Z(SO(q))_{[(q+1)/2]}$, and integral if $\deg c_J = 2 \cdot [(q+1)/2]$.

Note that for q even, Corollary 7.5 applies to all of the secondary classes defined for a Riemannian foliation. For q odd, this corollary only applies to the rigid secondary classes.

7.6 COROLLARY. Let $\mathfrak{F}'$ be an integrable $G\ell(n,\mathbb{C})$-foliation with trivial normal bundle on a manifold N. If $\mathfrak{F}'$ is defined by a submersion $f:N \to M^{2n}$ onto a complex manifold M, then each secondary class $\Delta_*(y_I c_J) \in H^*(N)$ is rational for $y_I c_J \in Z(G\ell(n,\mathbb{C}))_n$, and integral if $\deg c_J = 2n$.

Statement (b) of Theorem 7.3 is a consequence of Corollary 3.17 of [15]. Parts (a) and (c) of Theorem 7.3 are generalizations of this corollary of Chern and Simons. It seems reasonable to conjecture that all of the classes $\Delta_*(y_I c_J)$ are integral, though with our techniques this can only be shown to hold in the E_∞ term of a spectral sequence, where $E_\infty^{*,*} \cong H^*(N)$. Our interest in this theorem began with the example of J. H. C. Whitehead, which calculated the Hopf invariant of $f:S^3 \to S^2$ in terms of the integral of the form $\Delta_*(y_1 c_1)$ over S^3, [14], [69]. This example is in fact the original source for the idea behind this thesis.

<u>Proof of Theorem 7.1</u>. We must show that the map $h^{\#}:\pi^*(I_{\mathbb{Q}}(G)_\ell) \to \pi^*(M)$ factors through $\pi^*_{\mathbb{Q}}(M) \subseteq \pi^*(M)$.

Recall that $\tilde{\mathcal{E}}^*_\infty(M)$ denotes the differential algebra of compatible, $\Bbbk$-valued smooth forms on $\mathcal{S}_\infty(M)$, the semi-simplicial complex of C^∞-simplices in M. Let $\mathcal{E}^*_{\mathbb{Q}}(M)$ denote the differential algebra of compatible, $\mathbb{Q}$-valued polynomial forms on $\mathcal{S}_\infty(M)$. In a natural way, we can view $\mathcal{E}^*_{\mathbb{Q}}(M)$ and the deRham complex $\Omega(M)$ as subalgebras of the complex $\tilde{\mathcal{E}}^*_\infty(M)$. Note that the inclusion $\mathcal{E}^*_{\mathbb{Q}}(M) \subseteq \tilde{\mathcal{E}}^*_\infty(M)$ induces the map $\otimes\Bbbk:H^*(M;\mathbb{Q}) \to H^*(M;\Bbbk)$ in cohomology.

Let ω be an adapted G-connection on the G-bundle $P \to M$ associated to the foliation $\mathcal{F}$ on M. The characteristic map $h^{\#}$ is induced from the Chern-Weil homomorphism

$$h(\omega): I_{\mathbb{Q}}(G)_{\ell} = \mathbb{Q}[c_1,\ldots,c_r]_{\ell} \to \Omega(M) \subseteq \tilde{\mathcal{E}}^*_{\infty}(M) \ .$$

We will construct a homomorphism

$$h_{\mathbb{Q}}: \mathbb{Q}[c_1,\ldots,c_r]_{\ell} \to \mathcal{E}^*_{\mathbb{Q}}(M) \subseteq \tilde{\mathcal{E}}^*_{\infty}(M)$$

and an algebra homotopy

$$H: \mathbb{Q}[c_1,\ldots,c_r]_{\ell} \to \tilde{\mathcal{E}}^*_{\infty}(M) \otimes \{t,dt\}$$

such that $e_0 \circ H = h(\omega)$ and $e_1 \circ H = h_{\mathbb{Q}}$. Because $\pi^*_{\mathbb{Q}}(M)$ is defined using a minimal model, over $\mathbb{Q}$, of the algebra $\mathcal{E}^*_{\mathbb{Q}}(M)$, the claim of the theorem will follow.

For each generator c_j, let $b_j \overset{\text{def}}{=\!=} h(\omega)(c_j)$ in $\Omega(M)$; then b_j is a closed form. By choice, the coset $[b_j] \in H^*(M)$ is the pull-back of an integral class in $H^*(BG)$, hence is integral. In particular, $[b_j]$ is in the image of $H^*(M;\mathbb{Q}) \to H^*(M)$, so there exist a rational homogeneous element ξ_j in $\mathcal{E}^*_{\mathbb{Q}}(M)$ such that $b_j = \xi_j + d\alpha_j$ in $\tilde{\mathcal{E}}^*_{\infty}(M)$ for some α_j.

Define an algebra homomorphism

$$H: \mathbb{Q}[c_1,\ldots,c_r] \to \tilde{\mathcal{E}}^*_{\infty}(M) \otimes \{t,dt\}$$

by setting:

$$H(c_j) = (1-t)b_j + t \cdot \xi_j + \alpha_j \wedge dt \ , \quad 1 \leq j \leq r \ .$$

Note that each element $H(c_j)$ is closed, so H is a map of differential algebras. It only remains to show that $H(c_J) = 0$ for c_J of degree $p > 2\ell$, for then there is induced a map $H: I_{\mathbb{Q}}(G)_{\ell} \to \tilde{\mathcal{E}}^*_{\infty}(M) \otimes \{t, dt\}$ with $e_0 \circ H = h(\omega)$ and $e_1 \circ H \subseteq \mathcal{E}^*_{\mathbb{Q}}(M)$ as desired.

Let $c_J = c_1^{j_1} \cdots c_r^{j_r} \in \mathbb{Q}[c_1, \ldots, c_r]$ have degree p greater than 2ℓ. Since p is even, this implies $p - 1$ is also greater than 2ℓ. We now have:

$$\begin{aligned} H(c_J) &= \prod_{i=1}^{r} H(c_i)^{j_i} \\ &= \prod_{i=1}^{r} \{(1-t)b_i + t\xi_i + \alpha_i dt\}^{j_i} \\ &\in \tilde{\mathcal{E}}^{p-1}_{\infty}(M) \otimes k[t] \wedge dt \oplus \tilde{\mathcal{E}}^{p}_{\infty}(M) \otimes k[t] . \end{aligned}$$

The forms in $\tilde{\mathcal{E}}^*_{\infty}(M)$ are compatible with respect to the degeneracy maps in $\mathcal{S}_{\infty}(M)$; since M has dimension m, the complex $\tilde{\mathcal{E}}^*_{\infty}(M)$ is trivial for $* > m$. In particular, $p - 1 > 2\ell \geq m$ implies that $H(c_J) = 0$.

Finally, we note that $h^{\#}_{\mathbb{Q}}$ depends only on the cohomology class $[\xi_j] \in H^*(N;\mathbb{Q})$. This follows using an argument similar to that above. Since $[\xi_j]$ is equal to $h_*(c_j)$, the j-th Chern class of the G-bundle $P \to M$, we conclude that $h^{\#}$ is determined by the primary classes of this bundle. □

<u>Proof of Theorem 7.3</u>. Recall that $P' \to N$ is the G-bundle associated to the foliation $\mathfrak{F}'$ on N, and there is given a section $s': N \to P'/H$. This data is summarized in the commutative diagram:

$$\begin{tikzcd} P'/H \arrow[r, "\bar{f}"] & P/H \arrow[d, "\pi"] \\ N \arrow[u, "s'"] \arrow[ur, "s"] \arrow[r] & M \end{tikzcd} .$$

Let P/H have the G-foliation determined by the pull-back via π of the G-foliation $\mathfrak{F}$ on M. Then the characteristic homomorphism of $\mathfrak{F}'$ has a factorization:

$$\begin{tikzcd} H^*(A_Z(G,H)_\ell) \arrow[rr, "\Delta_*"] \arrow[dr, "\Delta_*"'] & & H^*(N) \\ & H^*(P/H) \arrow[ur, "s^*"'] & \end{tikzcd} .$$

It therefore suffices to consider just the case N = P/H.

Choose an adapted G-connection ω on $P \to M$. By the work of Narasimhan and Ramanan on universal connections [49], there exists a G-connection θ on the universal G-bundle $EG \to BG$ and a smooth map of G-bundles

$$\begin{tikzcd} P \arrow[r, "\bar{g}"] \arrow[d] & EG \arrow[d] \\ M \arrow[r, "g"] & BG \end{tikzcd}$$

such that $\bar{g}^*(\theta) = \omega$. In this case, we consider BG and EG as a finite dimensional Grassmannian and a Stiefel manifold, respectively, for which EG is 2ℓ-connected. This implies that $H^p(EG;Z) = 0$ for $p \leq 2\ell$.

Let $h(\theta):\mathbb{Q}[c_1,\ldots,c_r] \to \Omega(BG)$ be the Chern-Weil homomorphism with respect to the universal connection θ. For each $1 \leq j \leq r$, set:

$$\beta_j \overset{\text{def}}{=\!=} h(\theta)(c_j) \in \Omega(BG) .$$

Note that by our normalization, the cohomology class $[\beta_j] \in H^*(BG)$ is integral. For each $1 \leq j \leq r$, set:

$$b_j \overset{\text{def}}{=\!=} h(\omega)(c_j) \in \Omega(M) .$$

Then we have that $g^*(\beta_j) = b_j$ for $1 \leq j \leq r$.

Let $\mathcal{E}^*_{\mathbb{Q}}(BG) \subseteq \tilde{\mathcal{E}}^*_{\infty}(BG) \supseteq \Omega(BG)$ be differential algebras as in the proof of Theorem 7.1. Let $C^*(\mathcal{J}_{\infty}(BG);\mathbb{Q})$ denote the cochain complex over $\mathbb{Q}$ on $\mathcal{J}_{\infty}(BG)$, and let

$$\int : \mathcal{E}^*_{\mathbb{Q}}(BG) \to C^*(\mathcal{J}_{\infty}(BG);\mathbb{Q})$$

denote the map of differential cochain complexes given by integration on chains. Since $\beta_j \in \tilde{\mathcal{E}}^*_{\infty}(BG)$ is an integral class, it follows from Example 4 in the paper [13] by H. Cartan that there exists a closed from Φ_j in $\mathcal{E}^*_{\mathbb{Q}}(BG)$ such that

$$\beta_j = \Phi_j + d\alpha_j \text{ , for } \alpha_j \in \tilde{\mathcal{E}}^*_{\infty}(BG)$$

$$\int \Phi_j \in C^*(\mathcal{J}_{\infty}(BG);Z) .$$

That is, Φ_j is in the same cohomology class as β_j and the integration of the form Φ_j over any integral chain in $\mathcal{J}_{\infty}(BG)$ always yields an integer. Thus, Φ_j exhibits $[\beta_j] \in H^*(BG)$ as an integral class in the strongest possible fashion.

Next consider the cohomology of the quotient space EG/H. There is an associated fibration $H \to EG \to EG/H$. The cohomology of the Lie group H is a free algebra with generators of odd degree; since $H^p(EG) = 0$ for $p \leq 2\ell$, the spectral sequence of this fibration shows that $H^{odd}(EG/H) = 0$ in degrees less than 2ℓ.

We next lift the forms defined on BG to forms defined on EG/H. For each $1 \leq j \leq r$, set

$$\tilde{\beta}_j \overset{\text{def}}{=\!=} \pi^*(\beta_j) \in \Omega(EG/H)$$

$$\tilde{\alpha}_j \overset{\text{def}}{=\!=} \pi^*(\alpha_j) \in \Omega(EG/H)$$

$$\tilde{\Phi}_j \overset{\text{def}}{=\!=} \mathcal{S}(\pi)(\Phi_j) \in \mathcal{S}^*_{\mathbb{Q}}(EG/H) \ .$$

The Weil homomorphism with respect to the connection θ gives a differential algebra map

$$\Delta(\theta):A(G,H) \to \Omega(EG/H) \ .$$

In particular, note that for $1 \leq i \leq \nu$

$$d\Delta(\theta)(y_{\alpha_i}) = \Delta(\theta)(c_{\alpha_i}) = \tilde{\beta}_{\alpha_i} \ ,$$

so that the cohomology class $[\tilde{\beta}_{\alpha_i}] \in H^*(EG/H)$ is zero. By assumption, the corresponding integral class $[\tilde{\Phi}_{\alpha_i}] \in H^*(EG/H;Z)$ is also zero. Thus, Example 4 of [13] implies that for each $1 \leq i \leq \nu$, there exists a form $\psi_{\alpha_i} \in \mathcal{S}^*_{\mathbb{Q}}(EG/H)$ such that:

$$d\psi_{\alpha_i} = \tilde{\Phi}_{\alpha_i}$$

$$\int \psi_{\alpha_i} \in C^*(\mathcal{S}_\infty(EG/H);Z) \ .$$

For each $1 \leq i \leq \nu$, set:

$$\tau_{\alpha_i} = \Delta(\theta)(y_{\alpha_i}) \in \Omega(EG/H) \subseteq \tilde{\mathcal{S}}^*_\infty(EG/H) \ .$$

Then in $\tilde{\mathcal{S}}^*_\infty(EG/H)$ we have that

$$d\tau_j = \tilde{\beta}_j = \tilde{\Phi}_j + d\tilde{\alpha}_j = d\psi_j + \tilde{\alpha}_j = d(\psi_j + \tilde{\alpha}_j)$$

for $j = \alpha_i$. Therefore $(\tau_j - \psi_j - \tilde{\alpha}_j) \in \tilde{\mathcal{S}}^*_\infty(EG/H)$ is a closed form of odd degree less than 2ℓ. It is therefore exact; let $\eta_j \in \tilde{\mathcal{S}}^*_\infty(EG/H)$ satisfy $d\eta_j = \tau_j - \psi_j - \tilde{\alpha}_j$. With these constructions, we now have the key lemma.

7.7 LEMMA. For an admissible cocycle $y_I c_J \in Z(G,H)_\ell$, the element $\Delta(\theta)(y_I c_J)$ in $\tilde{\mathcal{S}}^*_\infty(EG/H)$ satisfies

$$\Delta(\theta)(y_I c_J) = \psi_I \tilde{\Phi}_J + \text{exact} + \mathcal{S}(\pi)(\alpha) ,$$

where $\alpha \in \tilde{\mathcal{S}}^*_\infty(BG)$ has degree greater than 2ℓ.

<u>Proof</u>. Expand $\Delta(\theta)(y_I c_J) = \tau_I \tilde{\beta}_J$ in terms of $\tau_j = \psi_j + \tilde{\alpha}_j + d\eta_j$ and $\tilde{\beta}_j = \tilde{\Phi}_j + d\tilde{\alpha}_j$. □

The proof of part (a) of the theorem now follows immediately:

$$\begin{aligned}\Delta(\omega)(y_I c_J) &= \mathcal{E}(\bar{g})(\Delta(\theta)(y_I c_J)) \\ &= \mathcal{E}(\bar{g})(\psi_I \tilde{\Phi}_J) + \mathcal{E}(\bar{g}) \circ \mathcal{E}(\pi)(\alpha) + \text{exact} \\ &= \mathcal{E}(\bar{g})(\psi_I \tilde{\Phi}_J) + \mathcal{E}(\pi) \circ \mathcal{E}(g)(\alpha) + \text{exact} .\end{aligned}$$

Since $\tilde{\mathcal{E}}^p_\infty(M) = 0$ for $p > m$, the condition $\deg \alpha > 2\ell \geq m$ implies that $\mathcal{E}(g)(\alpha) = 0$. Therefore,

$$\Delta_*(y_I c_J) = [\mathcal{E}(\bar{g})(\psi_I \tilde{\Phi}_J)] \in H^*(P/H;\mathbb{Q}) .$$

As remarked earlier, part (b) of the theorem follows directly from Corollary 3.17 in [15].

Part (c) of Theorem 7.3 follows from a more careful analysis of the class $\mathcal{E}(\bar{g})(\psi_I\tilde{\Phi}_J)$ in $\mathcal{E}^*_{\mathbb{Q}}(P/H)$. By construction, each of the forms

$$\bar{\psi}_j \overset{\text{def}}{=\!=} \mathcal{E}(\bar{g})(\psi_j)$$

$$\bar{\Phi}_j \overset{\text{def}}{=\!=} \mathcal{E}(\bar{g})(\tilde{\Phi}_j)$$

in $\mathcal{E}^*_{\mathbb{Q}}(P/H)$ corresponds to an integral cochain under the mapping $\int$. We would like to be able to assert that the product of forms $\bar{\psi}_I\bar{\Phi}_J$ also corresponds to an integral cochain. This would prove that all of the classes $\Delta_*(y_Ic_J)$ are integral. However, we can only assert that

$$\int \bar{\psi}_I\bar{\Phi}_J \in C^*(\mathcal{S}_\infty(P/H);Z[1/\sigma!]) ,$$

where σ is an integer depending only on the index (I,J). So in this sense, the rational classes $\Delta_*(y_Ic_J)$ have a uniform bound on their denominators.

We now return to the proof of (c). The cohomology of the space P/H can be computed from the Serre spectral sequence [60] of the fibration $G/H \to P/H \to M$. This induces a grading on the cohomology of P/H which we will denote by $\mathcal{J}^sH^*(P/H;R)$, where R is the coefficient ring. The naturality of the spectral sequence with respect to change of coefficients gives a commutative diagram:

$$\begin{array}{ccc} \mathcal{J}^sH^{s+t}(P/H;Z) & \cong & E_\infty^{s,t}(P/H;Z) \\ \downarrow & & \downarrow \\ \mathcal{J}^sH^{s+t}(P/H;\mathbb{Q}) & \cong & E_\infty^{s,t}(P/H;\mathbb{Q}) \end{array} \tag{7.8}$$

For some s,t we have that $\Delta_*(y_I c_J) \in \mathscr{F}^s H^{s+t}(P/H;\mathbb{Q})$, and this class is represented by the cocycle $\overline{\psi}_I\overline{\Phi}_J$. Viewing this class as being in $E_\infty^{s,t}(P/H;\mathbb{Q})$, we see that $[\overline{\psi}_I\overline{\Phi}_J]$ originates in the spectral sequence as the class

$$[\overline{\psi}_I] \otimes [\overline{\Phi}_J] \in H^t(G/H;\mathbb{Q}) \otimes H^s(M;\mathbb{Q})$$

$$\cong E_2^{s,t}(P/H;\mathbb{Q}) .$$

Noting that both $[\overline{\psi}_I]$ and $[\overline{\Phi}_J]$ represent integral cohomology classes, it follows from (7.8) that $\Delta_*(y_I c_J) = [\overline{\psi}_I\overline{\Phi}_J]$ is in the image of the map

$$\mathscr{F}^s H^{s+t}(P/H;Z) \to \mathscr{F}^s H^{s+t}(P/H;\mathbb{Q}) .$$

For $s \geq m = \dim M$, it is standard that both of these graded spaces are subspaces of the ungraded cohomology $H^*(P/H;R)$, and (c) follows. □

CHAPTER 8

THE HOMOTOPY THEORY OF $B\Gamma^q$

In this chapter we apply the machinery developed in the preceding chapters to study the cohomology and homotopy algebras of the classifying space of $G\ell(q,\mathbb{R})$-foliations. We adopt the standard notations:

$$B\Gamma^q \overset{\text{def}}{=\!=} B\Gamma_{G\ell(q,\mathbb{R})}$$

$$F\Gamma^q \overset{\text{def}}{=\!=} F\Gamma_{G\ell(q,\mathbb{R})} \ .$$

Recall that $\nu: B\Gamma^q \to BO(q)$ is the classifying map of the normal bundle of the Γ^q-structure on $B\Gamma^q$, and $F\Gamma^q$ is the homotopy theoretic fiber of ν.

Let $B\Gamma^q_+$ denote the classifying space of foliations with an orientable normal bundle. The classifying map of the normal bundle is given by $\nu: B\Gamma^q_+ \to BSO(q)$, and this map again has homotopy theoretic fiber $F\Gamma^q$. It follows from Theorem 8.1 below that the space $B\Gamma^q_+$ is simply connected. Because of this, we actually study the space $B\Gamma^q_+$ in this chapter. The topology of $B\Gamma^q_+$ can be easily related to that of $B\Gamma^q$: there is a natural "covering" map $B\Gamma^q_+ \to B\Gamma^q$ whose homotopy theoretic fiber is the group $\{\pm 1\}$. So for n greater than one, there is an isomorphism $\pi_n(B\Gamma^q_+) \cong \pi_n(B\Gamma^q)$. The simple connectivity of $B\Gamma^q_+$ and Theorem 3.17 imply there is a chain of isomorphisms

$$\pi^*(B\Gamma^q_+) \cong \mathrm{Hom}(\pi_*(B\Gamma^q_+),\mathbb{R}) \cong \mathrm{Hom}(\pi_*(B\Gamma^q),\mathbb{R}) \ .$$

The results we obtain about $\pi^*(B\Gamma^q_+)$ thus can be related to the topology of $B\Gamma^q$.

For this chapter, we adopt the standard notation

$$W_q \overset{\text{def}}{=\!=} A(G\ell(q,\mathbb{R}))_q \,.$$

The foundation upon which the results in this chapter are based is the following theorem:

8.1 THEOREM. The classifying space $F\Gamma^q$ is $(q+1)$-connected.

It was originally shown by Haefliger in [24] that $F\Gamma^q$ is q-connected, and this was extended by Thurston and Mather to the above result [65].

Theorem 8.1 is equivalent to the assertion that $\nu_{\#}: \pi_n(B\Gamma^q_+) \to \pi_n(BSO(q))$ is an isomorphism for $n \leq q+1$, and an epimorphism for $n = q+2$. In particular, since $BSO(q)$ is simply connected, it follows that $B\Gamma^q_+$ is simply connected.

The rational homotopy groups of $BSO(q)$ can be computed using the minimal model of this space. Recall that there are algebra isomorphisms [48]:

$$H^*(BSO(q)) \cong \begin{cases} \mathbb{R}[p_1,\ldots,p_r] & q = 2r+1 \\ \mathbb{R}[p_1,\ldots,p_{r-1},e_r] & q = 2r \end{cases}$$

where $\deg p_j = 4j$ and $\deg e_r = 2r$. The minimal model of $BSO(q)$ is isomorphic to the cohomology algebra $H^*(BSO(q))$, [47]. Therefore, $\pi^*(BSO(q))$ has a vector space basis with elements in degrees:

$$0,4,\ldots,4(r-1) \text{ and } 4r \;, \quad q = 2r+1$$

$$0,4,\ldots,4(r-1) \text{ and } 2r \;, \quad q = 2r \;.$$

We set $k \overset{\text{def}}{=\!=} [\![(q+2)/4]\!]$. For each $1 \leq i \leq k$, choose a map representing a generator of $\pi_{4i}(BSO(q)) \otimes \mathbb{R}$, given by $g_i: S^{4i} \to BSO(q)$ with $g_i^*(p_i) \in H^{4i}(S^{4i})$ non-zero. Theorem 8.1 implies that there exists a lift $\tilde{g}_i: S^{4i} \to B\Gamma_+^q$ of g_i, and the composition $\nu \circ \tilde{g}_i$ satisfies $(\nu \circ \tilde{g}_i)^*(p_i) \neq 0$. Equivalently, we see that the map $\tilde{h}^{\#}: \langle \bar{x}_2, \ldots, \bar{x}_{2k} \rangle \to \pi^*(B\Gamma_+^q)$ is injective.

The calculation of the spaces $\pi^*(BSO(q))$ above also determines the spaces $\pi_*(BSO(q)) \otimes \mathbb{R}$: for n odd or for $n > 2q$, the space $\pi_n(BSO(q)) \otimes \mathbb{R}$ is zero, and the corresponding group $\pi_n(BSO(q))$ must be finite.

We draw one more conclusion from Theorem 8.1, using Theorem 5.1:

8.2 THEOREM. The rational Hurewicz homomorphism $\mathcal{H}: \pi_n(F\Gamma^q) \otimes \mathbb{Q} \to H_n(F\Gamma^q; \mathbb{Q})$ is an isomorphism for $n \leq 2q + 2$ and an epimorphism for $n = 2q + 3$.

A Non-trivial Rigid Class and a Problem of Lawson

The first example we give of a non-trivial secondary invariant relates to a problem derived from a result of Heitsch [31]. A cocycle $y_I c_J$ in $H^*(W_q)$ is said to be <u>variable</u> if $\deg y_{i_1} c_J = 2q + 1$, and <u>rigid</u> if $\deg y_{i_1} c_J > 2q + 1$. For a foliation $\mathcal{F}$ of codimension q on M, the Heitsch Rigidity Theorem states that the characteristic class $\Delta_*(y_I c_J) \in H^*(M)$ is invariant under deformations of $\mathcal{F}$ if $y_I c_J$ is rigid. If $y_I c_J$ is variable, then $\Delta_*(y_I c_J)$ can vary as the foliation $\mathcal{F}$ is deformed. The examples of foliated manifolds in the literature with non-trivial secondary classes establish that some of the variable classes are non-trivial and do vary. However, there are no examples for which a rigid class is non-trivial. It has been announced by Fuks in [19] that every secondary class $y_I c_J$ can be realized non-trivially for some foliated manifold. The proof of this theorem has not appeared. It has been conjectured, in fact, that the rigid secondary classes must vanish for all foliations. The next proposition shows this is not true.

8.3 PROPOSITION. For $q = 4k - 2$ or $4k - 1$, the element $\tilde{h}^{\#} \circ \zeta(y_{2k}c_{2k})$ in $\pi^*(B\Gamma_+^q)$ is non-zero. Consequently, $\tilde{\Delta}_*(y_{2k}c_{2k}) \in H^*(F\Gamma^q)$ is non-zero.

Note that when q is even, the class $y_{2k}c_{2k}$ is rigid.

Proof. The second statement of the proposition follows from the first, using Proposition 5.6.

Let $\zeta(y_{2k}c_{2k}) = \bar{u}_{2k,J}$ in $\pi^*(I(G\ell(q,\mathbb{R}))_q)$; the algebra representative of $\bar{u}_{2k,J}$ satisfies $du_{2k,J} = -x_{2k}^2$. Recall that the map $\tilde{g}_k : S^{4k} \to B\Gamma_+^q$ defined above satisfies $(\nu \circ \tilde{g}_k)^*(p_k) \neq 0$. We evaluate the element $\tilde{h}^{\#}(\bar{u}_{2k,J})$ on the Whitehead product $[\tilde{g}_k,\tilde{g}_k] \in (B\Gamma_+^q)$, using (3.24):

$$\begin{aligned}\tilde{h}^{\#}(\bar{u}_{2k,J})([\tilde{g}_k,\tilde{g}_k]) &= -2\{\tilde{h}^{\#}(\bar{x}_{2k})(\tilde{g}_k)\}^2 \\ &= -2\{\int_{S^{4k}} (\nu \circ \tilde{g}_k)^*(p_k)\}^2 \\ &\neq 0 . \qquad \square\end{aligned}$$

The proposition shows that the Whitehead product $[\tilde{g}_k,\tilde{g}_k]$ in $\pi_*(B\Gamma_+^q) \otimes \mathbb{R}$ is non-trivial. This fact was originally proved by Schweitzer and Whitman [58], who used a theory of residues for singular foliations. These residues correspond to a representation of the functional $\tilde{h}^{\#}(\bar{u}_{2k,J})$

in terms of forms derived from the curvature of an adapted connection. In fact, all of the various residue theories for foliations with a singularity at a point [3], [46], [58] can be viewed as special cases of the theory of dual homotopy invariants constructed in this thesis.

8.4 REMARK. The existence of a foliation with a non-trivial rigid class can be used to answer a question posed by Lawson in the survey article [43]. Problem 3 in this paper asks whether there exists, on some manifold M, two non-homotopic foliations with homotopic plane fields.

The answer is "yes", for $q = 4k - 2$. To see this, let $p = 8k - 1$ and consider the open manifold $M = S^p \times \mathbb{R}^q$. Choose a non-zero integer λ for which $\lambda \cdot \nu_{\#}([\tilde{g}_k, \tilde{g}_k]) \in \pi_p(BSO(q))$ is zero. Let $f_0 : S^p \to B\Gamma_+^q$ represent $\lambda \cdot [\tilde{g}_k, \tilde{g}_k]$ and let $f_1 : S^p \to B\Gamma_+^q$ represent $2\lambda \cdot [\tilde{g}_k, \tilde{g}_k]$. Let $g_0 : S^p \to BO(p)$ classify the tangent bundle of S^p. Then the maps

$$(f_0, g_0), (f_1, g_0) : M \to B\Gamma_+^q \times BO(p)$$

satisfy Diagram 1.15. Therefore, by Theorem 1.16, they determine foliations $\mathfrak{F}_0$ and $\mathfrak{F}_1$ on M whose normal bundles are homotopic to the bundle $S^p \times \mathbb{R}^q$. The plane fields of $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are easily seen to be homotopic. However, the foliations $\mathfrak{F}_0$ and $\mathfrak{F}_1$ have distinct rigid classes $\Delta_*(y_{2k} c_{2k}) \in H^p(M)$, hence cannot be deformed into one another.

8.5 REMARK. A slightly stronger statement than given in Proposition 8.3 can be shown. Consider the foliation on S^{4k} induced by the map $\tilde{g}_k$. Because $\pi_*(S^{4k}) \otimes \mathbb{R}$ is spanned by the elements $[id] \in \pi_{4k}(S^{4k})$ and the Whitehead product $[id, id] \in \pi_p(S^{4k})$, where $p = 8k - 1$, we can use

Theorem 3.23 to determine the image of $h^{\#} \circ \zeta$ in $\pi^*(S^{4k})$ completely. In fact, if $y_I c_J \in H^p(W_q)$ is an admissible cocycle other than $y_{2k}c_{2k}$, then $h^{\#} \circ \zeta(y_I c_J)$ is zero. Evaluating this functional on $[id,id]$, we see by (3.24) that the result is zero unless $du_{I,J}$ contains a non-trivial summand of x_{2k}^2. The only admissible cocycle satisfying this condition is $y_{2k}c_{2k}$.

For the foliation on S^p induced by the map $[\tilde{g}_k, \tilde{g}_k]$, we conclude from the above discussion that $h^{\#} \circ \zeta(y_I c_J) \in \pi^*(S^p)$ is zero unless $y_I c_J = y_{2k}c_{2k}$. Since the map $\mathcal{K}^*: H^*(S^p) \to \pi^*(S^p)$ is an isomorphism, the class $\Delta_*(y_I c_J) \in H^*(S^p)$ is zero unless $y_I c_J = y_{2k}c_{2k}$. Therefore, the universal class $\tilde{\Delta}_*(y_{2k}c_{2k})$ in $H^*(F\Gamma^q)$ is non-zero <u>and</u> independent of all the other secondary classes.

Using techniques similar to those in Proposition 8.3 and Remark 8.5, and by making a careful analysis of the homotopy groups $\pi_n(B\Gamma_+^q)$ for $n \leq 2q$, one can show:

8.6 PROPOSITION. The following classes are independent in the images of the maps $\tilde{h}^{\#} \circ \zeta: H^*(W_q) \to \pi^*(B\Gamma_+^q)$ and $\Delta_*: H^*(W_q) \to H^*(F\Gamma^q)$:

(a) for $q = 5$, $y_2 c_2^2$,

(b) for $q = 7$, $y_2 c_2^3$, $y_2 c_2 c_4$, $y_2 y_4 c_2^3$, $y_2 y_4 c_2 c_4$,

(c) for $q = 9$, $y_2 c_2^4$, $y_2 c_2^2 c_4$, $y_2 c_4^2$

$y_2 y_4 c_2^4$, $y_2 y_4 c_2^2 c_4$, $y_2 y_4 c_4^2$.

If the space $F\Gamma^q$ could be shown to be $(q+2)$-connected, then the above list could be greatly extended.

Independent Variation of the Secondary Classes

A more systematic study of the vector spaces $\pi^*(B\Gamma_+^q)$ and $H^*(F\Gamma^q)$ requires the use of the machinery developed in Chapter 5. Our approach to the problem is as follows. It has been shown that there exists a set $\mathcal{V}$ in $H^{2q+1}(W_q)$ with $\tilde{\Delta}_*(\mathcal{V}) \subseteq H^*(F\Gamma^q)$ an I.C.V. set. The space $F\Gamma^q$ is (q+1)-connected, so there is a corresponding I.C.V. set in $\pi^*(F\Gamma^q)$. Using the isomorphism $\pi^n(B\Gamma_+^q) \cong \pi^n(F\Gamma^q)$ for $n > 2q$, we obtain an I.C.V. set $\tilde{h}^{\#} \circ \zeta(\mathcal{V})$ in $\pi^{2q+1}(B\Gamma_+^q)$. The maps $\{\tilde{g}_1,\ldots,\tilde{g}_k\}$ constructed earlier allow us to extend the set $\mathcal{V}$ to a larger I.C.V. set, $\tilde{h}^{\#} \circ \zeta(\mathcal{V}') \subseteq \pi^*(B\Gamma_+^q)$. This, in turn, gives an I.C.V. set $\tilde{\Delta}_*(\mathcal{V}')$ in $H^*(F\Gamma^q)$. By this means, we show that many of the secondary classes of foliations are independently variable.

Our results have some overlap with those of Heitsch [32]. However, we note that the independent variation results we obtain are complementary, in part, to those obtained by Heitsch; these two different approaches combine to give us Theorem 8.14 below. Also, in the process of establishing the independent variation of the secondary classes in $\mathcal{V}' \subseteq H^*(W_q)$, it is shown that these classes are non-trivial for foliations on an open manifold with the homotopy type of a sphere, a result unique to our approach.

Finally, we use the I.C.V. set $\tilde{h}^{\#} \circ \zeta(\mathcal{V}')$ in $\pi^*(B\Gamma_+^q)$ to construct surjections of $\pi_n(B\Gamma_+^q)$ onto $\mathbb{R}^{v_n}$, where v_n is positive for infinitely many n, and to prove the existence of uncountably many linearly independent, free Lie algebras in $s^{-1}\pi_*(B\Gamma_+^q)$.

We recall the following theorem of Heitsch [32].

8.7 THEOREM. For each odd integer $q \geq 3$, there exists a set $\mathcal{V} \subseteq Z_q$ of admissible cocycles of degree $2q + 1$ such that $\tilde{\Delta}_*(\mathcal{V}) \subseteq H^{2q+1}(F\Gamma^q)$ is I.C.V., and $\mathcal{V}$ contains at least two elements.

The exact number of elements in $\mathcal{V}$ varies with q. A precise description of this set is given in [32].

A more general result than Theorem 8.7 has been announced by Fuks in [19], [20]:

8.8 THEOREM. The composition

$$H^*(W_q) \xrightarrow{\tilde{\Delta}_*} H^*(F\Gamma^q) \xrightarrow{e} \mathrm{Hom}(H_*(F\Gamma^q;Z),\mathbb{R})$$

is injective, and the image of a basis of the variable classes in $H^*(W_q)$ is mapped to an I.C.V. set in $H^*(F\Gamma^q)$.

Recall that Z_q denotes the set of admissible cocycles in W_q. The set Z_q is called the Vey basis of $H^*(W_q)$, as it was first derived by the late J. Vey [22]. If we let Z denote the set of elements in Z_q having degree $\leq 2q + 3$, then using Theorem 8.2, we have as a consequence of Theorem 8.8:

8.9 COROLLARY. The map $\tilde{h}^{\#} \circ \zeta : \langle Z \rangle \to \pi^*(B\Gamma_+^q)$ is injective.

For $k = [(q+2)/4]$, let $\mathcal{J}$ denote the set $\{2,4,\ldots,2k\}$. Form the extension Z' of the set Z by $\mathcal{J}$, as in Chapter 5. It was shown earlier that the map $\tilde{h}^{\#} : \langle \bar{x}_2,\ldots,\bar{x}_{2k} \rangle \to \pi^*(B\Gamma_+^q)$ is injective; by Proposition 5.7, we can draw the following conclusion from Theorem 8.8:

8.10 COROLLARY. The map $\tilde{h}^{\#} \circ \zeta\langle Z' \rangle \to \pi^*(B\Gamma_+^q)$ is injective.

These two corollaries assert that using Fuk's Theorem, it follows that the secondary classes in Z' can be realized independently on open manifolds with the homotopy type of a sphere.

We next consider the independent variability of the secondary classes in $H^*(W_q)$. Let $\mathscr{V} \subseteq Z_q$ be the largest set of admissible cocycles of degree $2q + 1$ such that $\Delta_*(\mathscr{V}) \subseteq H^*(F\Gamma^q)$ is an I.C.V. set.

8.11 PROPOSITION. The set $\tilde{h}^{\#} \circ \zeta(\mathscr{V}) \subseteq \pi^*(B\Gamma_+^q)$ is I.C.V.

<u>Proof</u>. Proposition 5.5 and Theorem 8.1 imply that $\tilde{h}^{\#} \circ \zeta(\mathscr{V}) \subseteq \pi^*(F\Gamma^q)$ is I.C.V. From the homotopy sequence of the fibration $F\Gamma^q \to B\Gamma_+^q \to BSO(q)$, the map $\pi_n(F\Gamma^q) \to \pi_n(B\Gamma_+^q)$ is seen to be an isomorphism modulo torsion for n greater than 2q. Therefore, the set $\tilde{h}^{\#} \circ \zeta(\mathscr{V}) \subseteq \pi^*(B\Gamma_+^q)$ is also I.C.V. □

Let $\mathscr{V}'$ denote the extension of $\mathscr{V}$ by the set $\mathscr{A}$, where we recall the definition:

$$\mathscr{V}' \overset{\text{def}}{=\!=} \{y_I c_J \in Z_q \mid y_{i_1} c_J \in \mathscr{V} \text{ and } (i_2,\ldots,i_s) \subseteq (2,4,\ldots,2k)\} .$$

The conditions of Proposition 5.11 are then satisfied for the sets $\mathscr{V}$ and $\mathscr{A}$, so we conclude:

8.12 THEOREM. The set $\tilde{h}^{\#} \circ \zeta(\mathscr{V}') \subseteq \pi^*(B\Gamma_+^q)$ is I.C.V.

This theorem yields, by Corollary 5.12, the following result on the independent variability of the secondary classes of foliations:

8.13 COROLLARY. The set $\tilde{\Delta}_*(\mathcal{V}') \subseteq H^*(F\Gamma^q)$ is I.C.V. In particular, the map $\tilde{\Delta}_*: \langle \mathcal{V}' \rangle \to H^*(F\Gamma^q)$ is injective.

Using Theorem 8.7 above on the existence of a set $\mathcal{V}$ with $\tilde{\Delta}_*(\mathcal{V})$ an I.C.V. set, and forming the extension $\mathcal{V}'$ as above, Corollary 8.13 then implies that $\tilde{\Delta}_*(\mathcal{V}')$ consists of independently variable classes in $H^*(F\Gamma^q)$. We note that Heitsch has shown [Theorem 6.12; 32] that there exists a larger set $\overline{\mathcal{V}}$ than $\mathcal{V}$, with $\mathcal{V} \subseteq \overline{\mathcal{V}} \subseteq H^*(W_q)$, for which $\tilde{\Delta}_*(\overline{\mathcal{V}})$ is I.C.V. The set $\overline{\mathcal{V}}$ of Heitsch and our set $\mathcal{V}'$ have many elements in common, but do not coincide. For example, the set $\mathcal{V}'$ contains the cocycles of the type $y_I c_J = y_{i_1} \cdots y_{i_{s-1}} y_{2k} c_J$ for $y_{i_1} c_J \in \mathcal{V}$ and $(i_2,\ldots,i_{s-1}) \subseteq (2,4,\ldots,2k-2)$, and many of these are not in the set $\overline{\mathcal{V}}$. Therefore, Corollary 8.13 represents an extension of the results in [32].

In fact, a much stronger statement can be shown. For an index $I = (i_1,\ldots,i_s)$, set $I_1 \overset{\text{def}}{=\!=} (i_2,\ldots,i_s)$. Then for any $y_I c_J \in \overline{\mathcal{V}}$ with $I_1 \not\subseteq (2,4,\ldots,2k)$, the class $\tilde{\Delta}_*(y_I c_J)$ varies independently of the set $\tilde{\Delta}_*(\mathcal{V}')$ in $H^*(F\Gamma^q)$. The proof of this uses the same techniques as in Remark 8.5. Note that the proof of Proposition 5.11 actually shows that the set $\tilde{h}^{\#} \circ \zeta(\mathcal{V}')$ is independently variable on the iterated Whitehead products in $\pi_*(B\Gamma_+^q)$ formed from the elements of $\pi_{2q+1}(B\Gamma_+^q)$ and the maps $\{\tilde{g}_1,\ldots,\tilde{g}_k\}$. If $f: S^n \to B\Gamma^q$ represents one of these iterated Whitehead products, then it follows from Proposition 4.35 and Theorem 3.23 that $\tilde{h}^{\#} \circ \zeta(y_I c_J)([f]) = 0$ if $I_1 \not\subseteq (2,4,\ldots,2k)$. Our claim then follows as in Remark 8.5. Thus, setting

$$\mathcal{V}'' \overset{\text{def}}{=\!=} \mathcal{V}' \cup \{y_I c_J \in \overline{\mathcal{V}} \mid I_1 \not\subseteq (2,4,\ldots,2k)\} ,$$

and combining the above remarks with Theorem 6.12 of Heitsch [32], we obtain the following theorem on the variability of the secondary classes:

8.14 THEOREM. For $\mathcal{V}'$ as defined above, the set $\tilde{\Delta}_*(\mathcal{V}') \subseteq H^*(F\Gamma^q)$ is I.C.V. In particular, the map $\tilde{\Delta}_*: \langle \mathcal{V}' \rangle \to H^*(F\Gamma^q)$ is injective.

We finish this chapter with some results on the ranks of the higher homotopy groups of $B\Gamma^q_+$. With the notation of Chapter 5, there is a subset $\mathcal{W}' = s^{-1}\mathcal{V}'^*$ of the set $\mathcal{W} = s^{-1}Z^*_q$. The set $\mathcal{W}'$ corresponding to $\mathcal{V}'$ generates a free, graded Lie subalgebra $\mathcal{L}'$ of the Lie algebra $\mathcal{L}$, and there is a canonical inclusion of the dual space to $\mathcal{L}'$ into the space of dual homotopy invariants, $s\mathcal{L}'^* \subseteq \pi^*(I(G\ell(q,\mathbb{R}))_q)$. We let $\mathcal{X}'$ denote the Hall basis of $\mathcal{L}'$ which is generated by the set $\mathcal{W}'$. Then by Proposition 5.13, we have the result:

8.15 THEOREM. The set $h^{\#}(s\mathcal{X}'^*) \subseteq \pi^*(B\Gamma^q_+)$ is I.C.V.

The set $\mathcal{X}'$ will be infinite if $\mathcal{V}'$ contains at least two elements. If we let v_n denote the number of elements in $s\mathcal{X}'^*$ of degree n, then the sequence $\{v_n\}$ has a subsequence tending to infinity. Theorem 8.15 asserts that for each positive integer n, there is an epimorphism of abelian groups $\pi_n(B\Gamma^q_+) \to \mathbb{R}^{v_n}$. It follows that the homotopy groups $\pi_n(B\Gamma^q_+)$ become extremely large as n tends to infinity, not only in the sense of being uncountable, but in that they have a vector space structure whose dimension tends to infinity.

Finally, let the set $\mathcal{V}'$ be enumerated as $\{z_1,\ldots,z_d\}$, where z_j has degree n_j. Let Y be a wedge of spheres, defined by $Y \overset{\text{def}}{=\!=} \bigvee_{j=1}^{d} S^{n_j}$. With the notation of Proposition 5.22, the following result is a consequence of Theorem 8.12 and Proposition 5.22:

8.16 THEOREM. There exists an uncountable set $\mathcal{A}$ and maps $f_\alpha: Y \to B\Gamma_+^q$ for $\alpha \in \mathcal{A}$ such that the direct sum of maps

$$\bigoplus_{\alpha \in \mathcal{A}} (f_\alpha)_\#: \bigoplus_{\alpha \in \mathcal{A}} \pi_*(Y) \otimes \mathbb{Q} \to \pi_*(B\Gamma_+^q) \otimes \mathbb{Q}$$

is injective.

In other words, the rational, graded Lie algebra $s^{-1}\pi_*(B\Gamma_+^q) \otimes \mathbb{Q}$ contains uncountably many, linearly independent, free graded Lie subalgebras. We remark that Haefliger established a method in [27] for showing that each of the maps $(f_\alpha)_\#$ is injective. Theorem 8.16 represents a continuation of the ideas in [27], where we use the more extensive machinery of Chapter 5 to achieve a sharper result.

CHAPTER 9

THE HOMOTOPY THEORY OF $B\tilde{\Gamma}_{SO(q)}$

The classifying space of SO(q)-foliations is denoted by $B\tilde{\Gamma}_{SO(q)}$. In the literature, an SO(q)-foliation is also called an orientable Riemannian foliation, and this classifying space is sometimes denoted by $BR\Gamma^q_+$, [45]. The map $\nu : B\tilde{\Gamma}_{SO(q)} \to BSO(q)$ classifying the normal bundle has homotopy theoretic fiber $F\tilde{\Gamma}_{SO(q)}$. We will show that all of the dual homotopy invariants of SO(q)-foliations are non-trivial in $\pi^*(B\tilde{\Gamma}_{SO(q)})$, and variable when possible; all of the indecomposable secondary classes are non-trivial in $H^*(F\tilde{\Gamma}_{SO(q)})$, and variable when possible.

For this chapter we assume $q > 1$ and work over the ground field $\mathcal{k} = \mathbb{R}$. Since the space BSO(q) is simply connected, Theorem 6.1 implies that $B\tilde{\Gamma}_{SO(q)}$ is simply connected. Certain indices occur frequently, so we adopt them as part of our notation. Set

$$q' = [q/2] \; ; \; \lambda = [(q-1)/2] \; ; \; k = [q/4] + 1 \; .$$

We use the notation $A_{q'} \overset{\text{def}}{=\!=} A(SO(q))_{q'}$.

The vector space of universal secondary invariants defined for an SO(q)-foliation with trivial normal bundle is given by the algebra $H^*(W(so(q),\{e\})_{q'})$. We next describe a factorization of this algebra. Recall that there is an algebra isomorphism

$$I(SO(q)) \cong \begin{cases} \mathbb{R}[p_1,\ldots,p_\lambda] & q = 2q'+1 \\ \mathbb{R}[p_1,\ldots,p_\lambda,e_q] & q = 2q' \end{cases} ,$$

where p_j has degree $4j$ and e_q has degree q. Let y_j denote the suspension of p_j. Then there is an isomorphism of graded algebras

$$H^*(W(so(q),\{e\})_{q'}) \cong H^*(A_{q'}) \otimes \Lambda\langle y_k,\ldots,y_\lambda\rangle .$$

We let $Z_{q'}$ denote the set of admissible cocycles in $A_{q'}$. An element $y_I c_J$ in $Z_{q'}$ is said to be <u>variable</u> if the degree of $y_{i_1} c_J$ equals $2q' + 1$. Let $\mathcal{V} \subseteq Z_{q'}$ denote the subset of variable elements; note that $\mathcal{V}$ is empty unless $q = 4k - 2$ or $q = 4k - 1$.

The first theorem of this chapter asserts that all of the dual homotopy invariants of SO(q)-foliations are non-trivial:

9.1 THEOREM. The characteristic homomorphism

$$\tilde{h}^{\#}: \pi^*(I(SO(q))_{q'}) \to \pi^*(B\tilde{\Gamma}_{SO(q)})$$

is injective.

Let $V^{2q'+1}$ denote the subset of $\mathcal{V}$ consisting of elements of degree $2q' + 1$. The following result was proven by Lazarov and Pasternack [46]:

9.2 THEOREM. For $q = 4k - 2$, the set

$$\tilde{h}^{\#} \circ \zeta(V^{2q'+1}) \subseteq \pi^{2q'+1}(B\tilde{\Gamma}_{SO(q)})$$

is I.C.V.

Using Theorem 9.2, we establish the complete variability of the variable classes in the image of $\tilde{h}^{\#} \circ \zeta$:

9.3 THEOREM. For all q, the set

$$\tilde{h}^{\#} \circ \zeta(\mathcal{V}) \subseteq \pi^*(B\tilde{\Gamma}_{SO(q)})$$

is I.C.V.

As a consequence of Theorems 9.1 and 9.3, we show that all of the <u>indecomposable</u> secondary classes of SO(q)-foliations are non-trivial, and variable when possible:

9.4 THEOREM

(a) The restriction of the characteristic homomorphism,

$$\tilde{\Delta}_*: H^*(A_{q'}) \oplus \Lambda\langle y_k, \ldots, y_\lambda\rangle \to H^*(F\tilde{\Gamma}_{SO(q)}) \ ,$$

is injective.

(b) The set $\tilde{\Delta}_*(\mathcal{V}) \subseteq H^*(F\tilde{\Gamma}_{SO(q)})$ is I.C.V.

The degree of each class y_j is $4j - 1$, so for $k \leq j \leq \lambda$ we have that $\deg y_j \leq 2q - 1$. We conclude from Corollary 6.3 that each class $\tilde{\Delta}_*(y_j) \in H^{4j-1}(F\tilde{\Gamma}_{SO(q)})$ is non-zero on a spherical class. This observation, together with Theorem 9.1, yields the following result:

9.5 PROPOSITION. The homomorphism

$$\tilde{\Delta}^{\#}: \pi^*(W(so(q),\{e\})_{q'}) \to \pi^*(F\tilde{\Gamma}_{SO(q)})$$

is injective.

Finally, we apply Propositions 5.13 and 5.22 to case G = SO(q). Let $s\mathcal{H}'^* \subseteq \pi^*(I(SO(q))_{q'})$ be the set corresponding to the Hall basis of the free, graded Lie algebra $\mathfrak{L}'$ generated by the set $\mathcal{H}' = s^{-1}\mathcal{V}^*$. Let v_n denote the number of elements in $s\mathscr{X}'^*$ of degree n. If $q = 4k - 2$ with $k \geq 2$ or if $q = 4k - 1$ with $k \geq 3$, then the sequence of integers $\{v_n\}$ has a subsequence which tends to infinity. The next theorem follows directly from Proposition 5.13.

9.6 THEOREM. The set $\tilde{h}^{\#}(s\mathscr{X}'^*) \subseteq \pi^*(B\tilde{\Gamma}_{SO(q)})$ is I.C.V. In particular, for each $n > 0$ there is an epimorphism of abelian groups

$$\pi_n(B\tilde{\Gamma}_{SO(q)}) \to \mathbb{R}^{v_n} .$$

Let $\mathscr{V}$ be given by $\{z_1,\ldots,z_d\} \subseteq Z_{q}'$, where z_j has degree n_j. Let Y be the space defined by $Y = \bigvee_{j=1}^{d} S^{n_j}$. With the notation of Proposition 5.22, we then have the last result of this chapter.

9.7 THEOREM. There exists an uncountable set $\mathscr{A}$ and maps $f_\alpha : Y \to B\tilde{\Gamma}_{SO(q)}$ for $\alpha \in \mathscr{A}$ such that the direct sum of maps

$$\bigoplus_{\alpha \in \mathscr{A}} (f_\alpha)_{\#} : \bigoplus_{\alpha \in \mathscr{A}} \pi_*(Y) \otimes \mathbb{Q} \to \pi^*(B\tilde{\Gamma}_{SO(q)}) \otimes \mathbb{Q}$$

is injective.

<u>Proof of Theorem 9.1</u>. Let X be a q-dimensional, simply connected CW complex with a map $f: X \to BSO(q)$ inducing an algebra isomorphism $f^*: H^n(BSO(q)) \to H^n(X)$ for $n \leq q$. By Theorem 6.1, there is a lifting $\tilde{f}$ of f:

$$\begin{tikzcd} & B\tilde{\Gamma}_{SO(q)} \arrow[d, "\nu"] \\ X \arrow[ur, "\tilde{f}"] \arrow[r, "f"'] & BSO(q) \end{tikzcd} .$$

The Chern-Weil homomorphism $h:I(SO(q))_{q'} \to \mathcal{E}^*(X)$ is a weak isomorphism; therefore, the composition

$$\mathcal{M}(I(SO(q))_{q'}) \xrightarrow{\phi_I} I(SO(q))_{q'} \xrightarrow{h} \mathcal{E}^*(X)$$

is a minimal model for X. Any two minimal models for X are isomorphic, so the induced map $h^{\#}:\pi^*(I(SO(q))_{q'}) \to \pi^*(X)$ is an isomorphism. □

Proof of Theorem 9.3. First note that for $q = 4k - 4$ or $q = 4k - 3$, the set $\mathcal{V}$ is empty so there is nothing to prove.

For $q = 2$ (resp. $q = 3$), the set $\mathcal{V}$ consists of a single element χe_2 (resp. y_1) of degree 3, and these classes were shown by Lazarov and Pasternack [45] to take on a continuum of values for a family of foliations on S^3.

For $q = 4k - 2$ with $k \geq 2$, Theorem 9.2 states that $\tilde{h}^{\#} \circ \zeta(V^{2q'+1}) \subseteq \pi^*(B\tilde{\Gamma}_{SO(q)})$ is I.C.V. It follows from Theorem 6.1 that the map $\tilde{h}^{\#} \circ \zeta:\langle p_1,\ldots,p_{k-1}\rangle \to \pi^*(B\tilde{\Gamma}_{SO(q)})$ is injective. The set $\mathcal{V}$ is the extension of $V^{2q'+1}$ by the set $\mathcal{J} = \{1,2,\ldots,k-1\}$, so by Proposition 5.11 we are done.

The last case, for $q = 4k - 1$ with $k \geq 2$, follows from the next lemma, whose proof is obvious.

9.8 LEMMA. For each $q > 1$, there is a commutative diagram

$$\begin{array}{ccc} \pi^*(I(SO(q-1))_{q'}) & \xrightarrow{\tilde{h}^{\#}} & \pi^*(B\tilde{\Gamma}_{SO(q-1)}) \\ \uparrow & & \uparrow \\ \pi^*(I(SO(q))_{q'}) & \xrightarrow{\tilde{h}^{\#}} & \pi^*(B\tilde{\Gamma}_{SO(q)}) \end{array} .$$

To complete the proof of Theorem 9.3, it only remains to note that for $q = 4k - 1$, the set $\mathcal{V} \subset Z_{q'}$ is mapped bijectively to the corresponding set $\mathcal{V} \subseteq Z_{(q-1)'}$ in the above diagram. □

<u>Proof of Theorem 9.4</u>. (a) The injectivity of $\tilde{\Delta}_*: H^*(A_{q'}) \to H^*(F\tilde{\Gamma}_{SO(q)})$ follows directly from Theorem 9.1, Proposition 5.6 and the injectivity of $\zeta: H^*(A_{q'}) \to \pi^*(I(SO(q))_{q'})$. It is thus only necessary to show that the vector space $\tilde{\Delta}_*: (\Lambda(y_k, \ldots, y_\lambda)) \subseteq H^*(F\tilde{\Gamma}_{SO(q)})$ is linearly independent with respect to the vector space $\tilde{\Delta}_* H^*(A_{q'})$.

We first consider the case $q = 2q'$. It was shown by Kamber and Tondeur [Theorem 6.52; 37] that $\tilde{\Delta}_*$ is injective when restricted to the ideal generated by $\Lambda(\chi e_q) \otimes \Lambda(y_k, \ldots, y_\lambda)$ in $H^*(W(so(q), \{e\})_{q'})$. Suppose that a non-zero element $y \in \Lambda^n(y_k, \ldots, y_\lambda)$ exists such that $\tilde{\Delta}_*(y)$ is in the space $\tilde{\Delta}_* H^n(A_{q'}) \subseteq H^n(F\tilde{\Gamma}_{SO(q)})$. Observing that $e_q \cdot A_{q'} = 0$ in $W(so(q), \{e\})_{q'}$, we conclude that $\tilde{\Delta}_*(y \cdot \chi e_q) = \tilde{\Delta}_*(y) \cdot \tilde{\Delta}_*(\chi e_q) = 0$. This contradicts the injectivity of $\tilde{\Delta}_*$ on the ideal generated by χe_q.

The case $q = 2q' + 1$ is a consequence of the injectivity of the natural mapping

$$H^*(W(so(q), \{e\})_{q'}) \to H^*(W(so(q-1), \{e\})_{q'})$$

for $q = 2q' + 1$, and the following lemma which corresponds to Lemma 9.8:

9.9 LEMMA. For all $q > 1$, there is a commutative diagram:

$$\begin{array}{ccc} H^*(W(so(q-1), \{e\})_{q'}) & \xrightarrow{\tilde{\Delta}_*} & H^*(F\tilde{\Gamma}_{SO(q-1)}) \\ \uparrow & & \uparrow \\ H^*(W(so(q), \{e\})_{q'}) & \xrightarrow{\tilde{\Delta}_*} & H^*(F\tilde{\Gamma}_{SO(q)}) \end{array} .$$

Part (b) of Theorem 9.4 follows from Theorem 5.9 and Proposition 5.6. $\square$

CHAPTER 10

THE HOMOTOPY THEORY OF $B\Gamma^n_{\mathbb{C}}$

The classifying space of integrable $G\ell(n,\mathbb{C})$-foliations will be denoted by $B\Gamma^n_{\mathbb{C}} \overset{\text{def}}{=\!=} B\Gamma_{G\ell(n,\mathbb{C})}$. The classifying map of the $G\ell(n,\mathbb{C})$-structure on the normal bundle is given by the map $\nu: B\Gamma^n_{\mathbb{C}} \to BU(n)$; we let $F\Gamma^n_{\mathbb{C}}$ denote the homotopy theoretic fiber of ν. The ground field will be $\mathcal{k} = \mathbb{C}$ throughout. We adopt the standard notation $W_n = A(G\ell(n,\mathbb{C}))_n$, and let $Z_n = Z(G\ell(n,\mathbb{C}))_n$ be the set of admissible cocycles in W_n.

The results we obtain on the spaces $B\Gamma^n_{\mathbb{C}}$ and $F\Gamma^n_{\mathbb{C}}$ will be based on the work of Baum and Bott on residues of holomorphic foliations [3], and the following theorem of Landweber [42]:

10.1 THEOREM. The classifying space $F\Gamma^n_{\mathbb{C}}$ is (n-1)-connected.

The complex Bott Vanishing Theorem [5] states that the map $\nu^*: H^m(BU(n)) \to H^m(B\Gamma^n_{\mathbb{C}})$ vanishes for $m > 2n$. Therefore, the space $F\Gamma^n_{\mathbb{C}}$ can be at most 2n-connected. If it can be shown that $F\Gamma^n_{\mathbb{C}}$ is 2n-connected, then we point out that the techniques used in Chapter 9 to analyze $B\tilde{\Gamma}_{SO(q)}$ can be applied to show that all of the dual homotopy invariants are non-trivial in $\pi^*(B\Gamma^n_{\mathbb{C}})$, and all of the secondary classes are non-trivial in $H^*(F\Gamma^n_{\mathbb{C}})$.

We next give three corollaries of Theorem 10.1.

10.2 COROLLARY. For $n \geq 1$, the space $B\Gamma^n_{\mathbb{C}}$ is simply connected.

Proof. Observe that BU(n) is simply connected for all $n \geq 1$, as U(n) is always a connected group. For $n > 1$, the corollary then follows directly from Theorem 10.1. For $n = 1$, it was shown by Haefliger that $F\Gamma^1_{\mathbb{C}}$ is simply connected [25], so that $B\Gamma^1_{\mathbb{C}}$ is also simply connected. □

10.3 COROLLARY. The rational Hurewicz homomorphism

$$\mathcal{H}: \pi_m(F\Gamma^n_{\mathbb{C}}) \otimes \mathbb{Q} \to H_m(F\Gamma^n_{\mathbb{C}};\mathbb{Q})$$

is an isomorphism for $m \leq 2n - 2$, and an epimorphism for $m = 2n - 1$.

Proof. Apply Theorem 5.1. □

Recall that the cohomology of the space $BU(n)$ is isomorphic to a polynomial algebra

$$H^*(BU(n)) \cong \mathbb{C}[c_1,\ldots,c_n] \cong I(G\ell(n,\mathbb{C})) ,$$

where c_j is the j-th universal Chern class with $\deg c_j = 2j$, [48]. The minimal algebra $\mathcal{M}(I(G\ell(n,\mathbb{C}))_n)$ then contains homogeneous elements $\{x_1,\ldots,x_n\}$, where x_j maps to the Chern class c_j.

10.4 COROLLARY. For $1 \leq j \leq [n/2]$, the element $\tilde{h}^{\#}(\bar{x}_j)$ in $\pi^{2j}(B\Gamma^n_{\mathbb{C}})$ is non-zero.

Proof. For each $1 \leq j \leq n$, there is a map $g_j: S^{2j} \to BU(n)$ such that $(g_j)^*(c_j) \in H^{2j}(S^{2j})$ is non-zero. For $1 \leq j \leq [n/2]$, Theorem 10.1 implies that there exists a lifting $\tilde{g}_j: S^{2j} \to B\Gamma^n_{\mathbb{C}}$ of g_j. The map $\tilde{g}_j$ then satisfies $(\nu \circ \tilde{g}_j)^*(c_j) \neq 0$. □

There is a second result which we need for our analysis of the space $B\Gamma^n_{\mathbb{C}}$. Recall that an admissible cocycle $y_I c_J$ in W_n is of the form $y_I c_J = y_{i_1}\cdots y_{i_s} \otimes c_1^{j_1}\cdots c_n^{j_n}$, where:

$$1 \le i_1 < \ldots < i_s \le n$$

$$|J| \le n , \quad |J| + i_1 \ge n + 1$$

$$\ell < i_1 \Rightarrow j_\ell = 0 .$$

An admissible cocycle $y_I c_J$ is said to be <u>variable</u> if $|J| + i_1 = n + 1$, and is said to be <u>rigid</u> otherwise. Note that for a variable cocycle $y_I c_J$, the cocycle $y_{i_1} c_J$ has degree $2n + 1$. Let $\mathcal{V}$ denote the subset of Z_n consisting of elements of degree $2n + 1$; the set $\mathcal{V}$ is equivalently defined to be the variable cocycles $y_i c_J$ in Z_n. It was shown by Baum and Bott that the elements of $\mathcal{V}$ map to independently variable elements in $\pi^*(B\Gamma^n_{\mathbb{C}})$:

10.5 THEOREM [3]

(a) The set $\tilde{h}^\# \circ \zeta(\mathcal{V}) \subseteq \pi^{2n+1}(B\Gamma^n_{\mathbb{C}})$ is I.C.V.

(b) The set $\tilde{\Delta}_*(\mathcal{V}) \subseteq H^{2n+1}(F\Gamma^n_{\mathbb{C}})$ is I.C.V.

Let $\mathcal{J}$ denote the set $\{1,2,\ldots,[n/2]\}$. With the notation of Chapter 5, let $\mathcal{V}'$ be the extension of $\mathcal{V}$ by $\mathcal{J}$; that is, $y_I c_J$ is in $\mathcal{V}'$ if $\deg y_{i_1} c_J = 2n + 1$ and $\{i_2,\ldots,i_s\} \subseteq \mathcal{J}$. Applying Theorem 5.11 gives our first result:

10.6 THEOREM. The set $\tilde{h}^\# \circ \zeta(\mathcal{V}') \subseteq \pi^*(B\Gamma^n_{\mathbb{C}})$ is I.C.V.

As a corollary, we obtain the following result on the non-triviality and variability of the secondary classes in $H^*(F\Gamma^n_{\mathbb{C}})$.

10.7 THEOREM. The set $\tilde{\Delta}_*(\mathcal{V}') \subseteq H^*(F\Gamma^n_{\mathbb{C}})$ is I.C.V. In particular, the map $\tilde{\Delta}_*: \langle \mathcal{V}' \rangle \to H^*(F\Gamma^n_{\mathbb{C}})$ is injective.

It is interesting to note that the secondary classes $\tilde{\Delta}_*(y_I c_J)$ which are shown to be non-zero in Theorem 10.7 are all variable classes. There does not exist an example in the literature of an integrable complex foliation with a non-trivial rigid secondary class. However, if it can be shown that $F\Gamma^n_{\mathbb{C}}$ is $(n+1)$-connected, then the non-triviality of a rigid class would follow as in Proposition 8.3.

Finally, we draw from Theorem 10.6 our final set of theorems. These are in the same spirit as the work of Haefliger in [27], but the care taken in the proof of Propositions 5.13 and 5.22 gives much sharper results.

Let $s\mathcal{X}'^* \subseteq \pi^*(I(G\ell(n,\mathbb{C}))_n)$ be the set corresponding to the Hall basis of the free, graded Lie algebra $\mathcal{L}'$ generated by the set $\mathcal{Y}' = s^{-1}\mathcal{V}'^*$. Let v_m denote the number of elements in $s\mathcal{X}'^*$ of degree m. As a consequence of Proposition 5.13, we then have:

10.8 THEOREM

(a) The set $\tilde{h}^{\#}(s\mathcal{X}'^*) \subseteq \pi^*(B\Gamma^n_{\mathbb{C}})$ is I.C.V.

(b) For each $m > 0$, there is an epimorphism of graded abelian groups $\pi_m(B\Gamma^n_{\mathbb{C}}) \to \mathbb{C}^{v_m}$.

Let $\mathcal{V}'$ be given by $\{z_1,\ldots,z_d\}$, where z_j has degree n_j for $1 \leq j \leq d$. Let Y be the wedge of spheres defined by $Y = \bigvee_{j=1}^{d} S^{n_j}$. With the notation of Proposition 5.22, the following result follows from Theorem 10.6:

10.9 THEOREM. There exists an uncountable set $\mathcal{Q}$ and maps $f_\alpha : Y \to B\Gamma^n_{\mathbb{C}}$ for $\alpha \in \mathcal{Q}$ such that the direct sum of maps

$$\bigoplus_{\alpha \in \mathcal{A}} (f_\alpha)_\#: \bigoplus_{\alpha \in \mathcal{A}} \pi_*(Y) \otimes \mathbb{Q} \to \pi_*(B\Gamma^n_{\mathbb{C}}) \otimes \mathbb{Q}$$

is injective.

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