Topology, Vol. 20, pp. 365–387, 1981
Printed in Great Britain.

0040-9383/81/040365-23$02.00/0

DUAL HOMOTOPY INVARIANTS OF G-FOLIATIONS

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(*Received* 10 *December* 1979)

§1. INTRODUCTION

THIS PAPER develops a theory of dual homotopy invariants for G-foliations using the theory of minimal models. As an application of the theory which is constructed, we are able to extend the results of Heitsch on the independent variation of the secondary classes of foliations. A foliation with a non-zero rigid class is also shown to exist, based on an example of Schweitzer and Whitman.

For Riemannian foliations, all of the indecomposable secondary classes are shown to be linearly independent in $H^*(FR\Gamma^q)$. The work of Lazarov and Pasternack is used to show that all of the possibly variable indecomposable secondary classes are independently variable in $H^*(FR\Gamma^q)$.

The third type of G-foliations considered are those with an integrable complex structure on their normal bundles. The results of Baum and Bott are used to establish that many of the secondary classes for these foliations are independently variable.

For each of the three types of G-foliations considered, namely real, Riemannian and complex, it is shown that the homotopy groups of the corresponding classifying space $B\Gamma_G{}^q$ admit epimorphisms $\pi_n(B\Gamma_G{}^q)\to\mathbb{R}^{v_n}$, where $\{v_n\}$ is a sequence depending on q and G, but which in general has a subsequence tending to infinity.

If a manifold M is simply connected, then the invariants of a G-foliation $\mathcal{F}$ on M which we produce are functions on the homotopy groups of M. They can be viewed as generalizations of several other constructions of foliation invariants in the literature: There is a natural relation with the Chern–Simons invariants [9]. A means for producing such invariants was introduced by Haefliger in [18]. The various residue theorems for a G-foliation with singularities at a discrete set of points [2, 32 and 38] are special cases of this theory, where the residue is obtained by evaluating a dual homotopy class on the boundary of a disc about a singular point. For a Riemannian or complex foliation which is defined by a submersion [25], the secondary classes of the foliation are exactly cohomological representations of some of the dual homotopy invariants.

The general theory of the invariants is developed in §2. We begin by showing that the algebra homotopy class of the truncated Chern–Weil homomorphism $h(\omega)$ is a G-foliation invariant (Theorem 2.11); in fact, it is a universal invariant from which many other invariants of the foliation can be derived [23]. Applying the dual homotopy functor to $h(\omega)$ yields a characteristic map $h^{\#}$: $\pi^*(I(G)_l)\to\pi^*(M)$ from the infinite-dimensional vector space $\pi^*(I(G)_l)$ to the (pseudo) dual homotopy of the manifold (Theorem 2.12).

Section 3 relates the dual homotopy invariants of a foliation with its secondary classes (Theorem 3.1). In particular, this relationship shows that a secondary class is non-zero if its corresponding dual homotopy class is non-zero.

Section 4 presents a theorem of Andrews and Arkowitz [1] which enables us to evaluate the dual homotopy invariants (Theorem 4.4). The formula in Theorem 4.4 is clearly recognizable as corresponding to the evaluation formula for residues. Section 5 contains several useful results on the rational homotopy of an n-connected space.

Note that Definition 5.3 gives the definition of an independently variable (I.C.V.) set of invariants.

Finally, the techniques and constructions of §2–§5 are used to analyze the homotopy and cohomology algebras of the classifying spaces of real, Riemannian and complex foliations in §6–§8. There is a common theme behind the results we obtain: Let $B\Gamma_G{}^q$ denote the classifying space of G-foliations. Then the greater the connectivity of the classifying map of the normal bundle $\nu: B\Gamma_G{}^q \to BG$, the more non-zero Whitehead products there are in $\pi_*(B\Gamma_G{}^q)$. These non-zero homotopy classes correspond to non-zero dual homotopy classes in $\pi^*(B\Gamma_G{}^q)$, and to non-zero secondary classes in $H^*(F\Gamma_G{}^q)$.

Section 6 considers real foliations, where it is shown that there is a non-zero rigid class in $H^*(F\Gamma^q)$ corresponding to a particular Whitehead product in $\pi_*(B\Gamma^q)$, (Corollary 6.4). We show that the independent variability results of Heitsch in [21] imply that there are many independent families of Whitehead products in $\pi_*(B\Gamma^q)$, (Proposition 6.12). We conclude that the corresponding secondary classes are independently variable in $H^*(F\Gamma^q)$, giving an extension of the results of Heitsch (Remark 6.14 and Theorem 6.15).

If follows easily from our techniques, and Theorem 7.6, that all of the indecomposable secondary classes of Riemannian foliations are linearly independent in $H^*(FR\Gamma^q)$, (Theorem 7.2). Making use of the results of Lazarov and Pasternack[32], all of the indecomposable variable secondary classes are shown to vary in $H^*(FR\Gamma^q)$, (Corollary 7.4).

For complex foliations, an extension of the results of Baum and Bott[2] is given, showing that many of the variable secondary classes are independently variable in $H^*(F\Gamma_{\mathbb{C}}{}^n)$, (Corollary 8.4).

Our approach to foliation invariants primarily yields information about the homotopy groups of the classifying space $B\Gamma_G{}^q$ as indicated above. There are many other results which follow from these techniques[23]. For example, the graded Lie algebra $\pi_*(B\Gamma_G{}^q)$ has an uncountable number of linearly independent, free Lie subalgebras (Propositions 6.16, 7.5 and 8.5).

§2. CONSTRUCTION OF THE DUAL HOMOTOPY INVARIANTS

All manifolds are assumed to be connected, C^∞, paracompact and Hausdorff. All topological spaces are assumed to have base points and maps between spaces to be continuous and preserve basepoints. We denote by $H^*(X)$ the singular cohomology with real coefficients of a space X. Unless otherwise noted, all algebras are commutative, differential graded (DG) algebras with an augmentation. We use $[x]$ to denote the greatest integer $\leq x$.

Let G be a closed subgroup of the general linear group $Gl(q, \mathbb{R})$ with a finite number of connected components. In this section we produce invariants of a G-foliation on a manifold M:

2.1 *Definition*[26]. A G-foliation $\mathcal{F}$ of codimension q on a manifold M is an integrable subbundle $\mathcal{F}$ of the tangent bundle TM such that:

(i) There is given a model manifold B of dimension q having a G-structure on the tangent bundle TB with associated principal G-bundle $P' \to B$.

(ii) The distribution $\mathcal{F}$ is locally defined by submersions into B such that the local transition functions preserve the G-structure on TB.

If the model manifold B can be taken to be $\mathbb{R}^q$ with the natural flat G-structure, then we say $\mathcal{F}$ is an *integrable* G-foliation[26].

For the orthogonal group $O(q)$, an $O(q)$-foliation is exactly a Riemannian foliation[31, 36]. An integrable $Gl(n, \mathbb{C})$-foliation, for $q = 2n$, corresponds to a complex foliation, one which is modeled on $\mathbb{C}^n$ with holomorphic transition functions.

The notion of equivalence between two G-foliations which we use is that of concordance:

2.2 *Definition.* Two G-foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of codimension q on M are said to be concordant if there is a G-foliation $\mathcal{F}$ of codimension q on $M \times \mathbb{R}$ such that the inclusions $i_t\colon M \to M \times \{t\} \subseteq M \times \mathbb{R}$ induce $\mathcal{F}_t = i_t^* \mathcal{F}$ for $t = 0, 1$.

Let $I(G)$ denote the DG-algebra of AdG-invariant polynomials on $\mathfrak{G}$, the Lie algebra of G. The differential in $I(G)$ is trivial, and for some set of even degree generators $\{c_1, \ldots, c_r\}$ we have $I(G) \cong \mathbb{R}[c_1, \ldots, c_r]$. We next define the truncated polynomial algebras which arise from $I(G)$.

2.3 *Definition.* For each positive integer l, set $I(G)_l = I(G)/$ideal generated by monomials of degree $> 2l$.

The construction of invariants for a G-foliation $\mathcal{F}$ on M is based on the properties of a Bott (or adapted) connection for $\mathcal{F}$, [5, 6 and 27]. These connections always exist; one construction goes as follows. Let the G-foliation $\mathcal{F}$ be modeled on B, with $P' \to B$ the associated principal G-bundle. Choose a G-connection ω' on P'. Let $\{U_\alpha | \alpha \in A\}$ be an open covering of M for which there are submersions $\phi_\alpha\colon U_\alpha \to B$ defining $\mathcal{F}$, and let $\pi\colon P \to M$ be the G-bundle they induce. Choose a partition of unity $\{\lambda_\alpha | \alpha \in A\}$ subordinate to $\{U_\alpha | \alpha \in A\}$. We define an adapted connection ω on P by setting $\omega = \sum_{\alpha \in \mathcal{A}} (\lambda_\alpha \circ \pi) \cdot \phi_\alpha^* \omega'$.

The Bott Vanishing Theorem [5, 6] implies that if ω is an adapted connection for $\mathcal{F}$, then the Chern–Weil homomorphism induces a DG-algebra map $h(\omega)\colon I(G)_q \to \Omega(M)$. Here, $\Omega(M)$ denotes the deRham algebra of M. If G is the orthogonal group $O(q)$, then Pasternack has shown[30] that $h(\omega)$ descends to $I(O(q))_{q'}$, where $q' = [q/2]$. For an integrable $Gl(n, \mathbb{C})$-foliation, there is a DG-algebra map $h(\omega)\colon I(Gl(n, \mathbb{C}))_n \to \Omega(M)$.

Therefore, a G-foliation $\mathcal{F}$ of codimension q on M induces an algebra map $h(\omega)\colon I(G)_l \to \Omega(M)$, where l is a positive integer depending only on G. This map depends on the choice of the adapted connection ω, but it is well-known that the induced map in cohomology $h(\omega)_*$ depends only on the isomorphism class of the normal bundle of $\mathcal{F}$. In particular, $h(\omega)_*$ depends only on the concordance class of $\mathcal{F}$, [6, 27]. The invariants we define result from the observation that the *algebra homotopy* class of $h(\omega)$ depends only on the concordance class of $\mathcal{F}$. To develop this idea we use Sullivan's theory of minimal models. General references for this theory are [8, 10, 33 and 42]; our technical reference will be [19].

Let $\{t, dt\}$ denote the DG-algebra $\mathbb{R}[t] \otimes \Lambda(dt)$, where $t \otimes 1$ has degree 0 and the differential is defined by setting $d(t \otimes 1) = 1 \otimes dt$. For any DG-algebra $\mathcal{B}$ and $r \in \mathbb{R}$, there is an evaluation homomorphism $e_r\colon \mathcal{B} \otimes \{t, dt\} \to \mathcal{B}$ defined by setting

$$e_r(b_1 \otimes p_1(t) + b_2 \otimes p_2(t)\, dt) = p_1(r) \cdot b_1.$$

2.4 *Definition.* Two homomorphisms $f_0, f_1\colon \mathcal{A} \to \mathcal{B}$ of DG-algebras are said to be algebra homotopic, denoted by $f_0 \underset{a}{\simeq} f_1$, if there is a DG-algebra homomorphism $F\colon \mathcal{A} \to \mathcal{B} \otimes \{t, dt\}$ such that $f_r = e_r \circ F$ for $r = 0, 1$.

Note that if $f_0 \underset{a}{\simeq} f_1$, then f_0 and f_1 are cochain homotopic [Lemme 11.1; 33], but the converse is false.

A graded algebra is *free* if it is isomorphic to the tensor product of an exterior algebra with a symmetric algebra. For any connected DG-algebra $\mathcal{M}$, let $\bar{\mathcal{M}}$ denote the augmentation ideal. A connected DG-algebra $\mathcal{M}$ is *minimal* if it has a filtration by DG-subalgebras

$$\mathbb{R} = \mathcal{M}_0 \subseteq \mathcal{M}_1 \subseteq \ldots \subseteq \mathcal{M}, \quad \bigcup_{j=1}^{\infty} \mathcal{M}_j = \mathcal{M}$$

which satisfies, for each $j > 0$, $d\bar{\mathcal{M}}_j \subseteq (\bar{\mathcal{M}}_{j-1})^2$ and $\mathcal{M}_j \cong \mathcal{M}_{j-1} \otimes \Lambda(V_j)$ for some graded vector space V_j. It follows that the algebra $\mathcal{M}$ is free and $d\bar{\mathcal{M}} \subseteq (\bar{\mathcal{M}})^2$.

A homomorphism $\phi: \mathcal{A} \to \mathcal{B}$ of DG-algebras is called a *weak isomorphism* if $\phi_*: H^*(\mathcal{A}) \to H^*(\mathcal{B})$ is an isomorphism.

2.5 *Definition.* Let $\mathcal{A}$ be a DG-algebra. A pair $(\mathcal{M}_{\mathcal{A}}, \phi)$ is a minimal model of $\mathcal{A}$ if $\mathcal{M}_{\mathcal{A}}$ is minimal and $\phi: \mathcal{M}_{\mathcal{A}} \to \mathcal{A}$ is a weak isomorphism.

One of the basic results is:

2.6 Theorem [19, 41]. *Let $\mathcal{A}$ be a DG-algebra and suppose $H^*(\mathcal{A})$ is a connected algebra. Then $\mathcal{A}$ has a minimal model. If $(\mathcal{M}_{\mathcal{A}}, \phi_0)$ and $(\mathcal{N}_{\mathcal{A}}, \phi_1)$ are two minimal models of $\mathcal{A}$, then there exists a DG-isomorphism $\psi: \mathcal{M}_{\mathcal{A}} \to \mathcal{N}_{\mathcal{A}}$ such that $\phi_0 \underset{a}{\simeq} \phi_1 \circ \psi$.*

For a connected manifold M, it follows that the deRham algebra $\Omega(M)$ has a minimal model which we denote by $\phi: \mathcal{M}_M \to \Omega(M)$. For each positive integer l, the truncated polynomial algebra $I(G)_l$ has a minimal model denoted by $\eta: \mathcal{M}(I(G)_l) \to I(G)_l$.

A second basic result about minimal algebras is:

2.7 Proposition [Theorem 10.1; 19]. *Given a diagram of DG-algebras*

$$\begin{array}{ccc} & & \mathcal{B} \\ & & \downarrow \psi \\ \mathcal{M} & \xrightarrow{\phi} & \mathcal{A} \end{array}$$

with $\mathcal{M}$ minimal and ψ a weak isomorphism, there is a lift $\tilde{\phi}: \mathcal{M} \to \mathcal{B}$ such that $\psi \circ \tilde{\phi} \underset{a}{\simeq} \phi$.

2.8 Corollary. *Given a diagram of DG-algebras*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\psi} & \mathcal{B} \\ \phi_{\mathcal{A}} \uparrow & & \uparrow \phi_{\mathcal{B}} \\ \mathcal{M}_{\mathcal{A}} & \dashrightarrow & \mathcal{M}_{\mathcal{B}} \end{array}$$

where $(\mathcal{M}_{\mathcal{A}} \phi_{\mathcal{A}})$ and $(\mathcal{M}_{\mathcal{B}} \phi_{\mathcal{B}})$ are minimal models, there is induced a map $\tilde{\psi}: \mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{B}}$ such that $\phi_{\mathcal{B}} \circ \tilde{\psi} \underset{a}{\simeq} \psi \circ \phi_{\mathcal{A}}$. Given two maps $\psi_0, \psi_1: \mathcal{A} \to \mathcal{B}$ inducing maps $\tilde{\psi}_0, \tilde{\psi}_1$, if $\psi_0 \underset{a}{\simeq} \psi_1$, then $\tilde{\psi}_0 \underset{a}{\simeq} \tilde{\psi}_1$.

There are two other technical results we need; the first depends essentially on the minimality of $\mathcal{M}$.

2.9 PROPOSITION [Proposition 5.14; 19]. *Let $\mathcal{M}$ be a minimal DG-algebra. Then $\underset{a}{\simeq}$ is an equivalence relation on the set of maps, for a fixed $\mathcal{A}$, $\{\phi : \mathcal{M} \to \mathcal{A}\}$.*

2.10 PROPOSITION [Proposition 5.15; 19]. *Let $\psi_0 \underset{a}{\simeq} \psi_1 : \mathcal{M} \to \mathcal{A}$, and suppose $\mathcal{M}$ is minimal. Then for any DG-algebra map $g : \mathcal{A} \to \mathcal{B}$, we have $g \circ \psi_0 \underset{a}{\simeq} g \circ \psi_1$.*

Our first result is:

2.11 THEOREM. *Let $\mathcal{F}$ be a G-foliation of codimension q on a manifold M. Let $h(\omega) \circ \eta : \mathcal{M}(I(G)_l) \to \Omega(M)$, where l is given by:*

(i) *for $G = SO(q)$ or $O(q)$, $l \geq [q/2]$*
(ii) *for $\mathcal{F}$ an integrable $G = Gl(n, \mathbb{C})$-foliation, $l \geq n$*
(iii) *for $G = Gl(q, \mathbb{R})$ or otherwise, $l \geq q$.*

Then the algebra homotopy class of $h(\omega) \circ \eta$ is well-defined and depends only on the concordance class of $\mathcal{F}$.

Proof. By Proposition 2.9, the algebra homotopy class of $h(\omega) \circ \eta$ is well-defined.

Given two adapted connections ω_0 and ω_1, we show that $h(\omega_0) \underset{a}{\simeq} h(\omega_1)$. This result is implicit in several constructions given for the secondary characteristic classes [4, 7, 28 and 34]. Let $p : M \times \mathbb{R} \to M$ be the projection onto the first factor, which induces a foliation on $M \times \mathbb{R}$. An adapted G-connection for this foliation is given by $\omega_t = t\omega_1 + (1-t)\omega_0$, where $t \in \mathbb{R}$. The curvature of ω_t is given by $\Omega_t = \mathrm{d}(\omega_t) + 1/2[\omega_t, \omega_t]$, and this expression involves only $\mathrm{d}t$, powers of t and elements of $\Omega(M)$. Therefore, the Chern–Weil homomorphism is given by the composition:

$$h(\omega_t) : I(G)_l \to \Omega(M) \otimes \{t, \mathrm{d}t\} \subseteq \Omega(M \times \mathbb{R}).$$

It is then clear that $e_j \circ h(\omega_t) = h(\omega_j)$ for $j = 0$ or 1, so that $h(\omega_0) \underset{a}{\simeq} h(\omega_1)$ as claimed. This implies $h(\omega_0) \circ \eta \underset{a}{\simeq} h(\omega_1) \circ \eta$.

Finally, assume that $\mathcal{F}_0$, $\mathcal{F}_1$ are concordant G-foliations on M; there is a G-foliation $\mathcal{F}$ on $M \times \mathbb{R}$ such that $i_j^* \mathcal{F} = \mathcal{F}_j$ for $j = 0, 1$. Let ω be an adapted connection for $\mathcal{F}$. For $j = 0, 1$ we have the diagram

$$\begin{array}{ccccc}
I(G)_l & \xrightarrow{h(\omega)} & \Omega(M \times \mathbb{R}) & \underset{p^*}{\overset{i_j^*}{\rightleftarrows}} & \Omega(M) \\
\uparrow{\scriptstyle \eta} & & & & \uparrow{\scriptstyle \phi} \\
\mathcal{M}(I(G)_l) & & \xrightarrow{\quad \psi \quad} & & \mathcal{M}_M
\end{array}$$

where ψ is chosen, by Corollary 2.8, so that $h(\omega) \circ \eta \underset{a}{\simeq} p^* \circ \phi \circ \psi$. Then by Proposition 2.10, we have $i_j^* \circ h(\omega) \circ \eta \underset{a}{\simeq} i_j^* \circ p^* \circ \phi \circ \psi = \phi \circ \psi$. By Proposition 2.9, setting $j = 0, 1$ gives

$i_0^* \circ h(\omega) \circ \eta \underset{a}{\simeq} i_1^* \circ h(\omega) \circ \eta$. The composition $i_j^* \circ h(\omega)$ is clearly the Chern–Weil homomorphism of the foliation $\mathcal{F}_j$. This completes the proof of the Theorem. ■

Given a G-foliation $\mathcal{F}$ on M, and some $l > 0$ depending on G, this Theorem and Corollary 2.8 imply there is a well-defined algebra homotopy class of maps $\mathcal{M}h\colon \mathcal{M}(I(G)_l) \to \mathcal{M}_M$.

For any DG-algebra $\mathcal{A}$, with $H^*(\mathcal{A})$ a connected algebra, let $(\mathcal{M}_{\mathcal{A}}, \phi)$ be the minimal model. The vector space $\pi^*(\mathcal{A}) \overset{\text{def}}{=\!=} \bar{\mathcal{M}}_{\mathcal{A}} / \bar{\mathcal{M}}_{\mathcal{A}}^2$ is called the *dual homotopy* of $\mathcal{A}$. If a map $\psi\colon \mathcal{A} \to \mathcal{B}$ of DG-algebras is given, by Proposition 2.8 there is induced $\mathcal{M}\psi\colon \mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{B}}$, which, passing to quotients, defines $\psi^{\#}\colon \pi^*(\mathcal{A}) \to \pi^*(\mathcal{B})$. It is a basic result that if $\pi^1(\mathcal{A}) = 0$, then the induced map $\psi^{\#}$ depends only on the algebra homotopy class of $\mathcal{M}\psi$. Combining this with Theorem 2.11, we have:

2.12 THEOREM. *Let $\mathcal{F}$ be a G-foliation of codimension q on a manifold M, and let l be as given in Theorem 2.11.*

(i) *The Chern–Weil homomorphism induces a well-defined map $h^{\#}\colon \pi^*(I(G)_l) \to \pi^*(M)$ which depends only on the concordance class of $\mathcal{F}$.*

(ii) *Let $f\colon N \to M$ be a smooth map of manifolds with f transverse to $\mathcal{F}$. Then there is a commutative diagram*

$$\begin{array}{ccc} & \overset{h^{\#}}{\nearrow} & \pi^*(N) \\ \pi^*(I(G)_l) & & \uparrow f^{\#} \\ & \underset{h^{\#}}{\searrow} & \pi^*(M) \end{array}$$

where $f^*\colon \Omega(M) \to \Omega(N)$ induces $f^{\#}$.

This theorem gives algebraic invariants in $\pi^*(M)$ of the foliation $\mathcal{F}$ on M. They are related to the topology of M by the fundamental theorem of minimal models [8, 10, 33 and 41]:

2.13 THEOREM (Sullivan). *Let M be a simply connected manifold. Then for each positive integer n there is a natural isomorphism*

$$\pi^n(M) \cong \operatorname{Hom}(\pi_n(M), \mathbb{R}).$$

Thus for a simply connected G-foliated manifold M, the invariants defined above are functions on the homotopy groups of M.

There is a natural extension of the above construction yielding invariants which are defined for a finite CW complex X with a singular G-foliation. Let $B\bar{\Gamma}_G{}^q$ be the classifying space of G-foliated microbundles [11, 17]. Suppose $f\colon X \to B\bar{\Gamma}_G{}^q$ defines a $\bar{\Gamma}_G{}^q$-structure on X. Then by a construction of Haefliger, there is an open manifold M, and an inclusion $X \to M$ inducing a homotopy equivalence, such that M has a G-foliation whose classifying map makes the diagram
$$\begin{array}{ccc} M & \longrightarrow & B\bar{\Gamma}_G{}^q \\ \uparrow & \overset{f}{\nearrow} & \\ X & & \end{array}$$
commute. We define the characteristic homomorphism of the pair (X, f) by $h^{\#}\colon \pi^*(I(G)_l) \to \pi^*(M) \cong \pi^*(X)$.

For a CW complex X which is not finite, we define $\pi^*(X) = \varprojlim_{\alpha \in \mathcal{A}} \pi^*(X_\alpha)$, where

$X_\alpha \subseteq X$ for $\alpha \in \mathscr{A}$ are the finite subcomplexes ordered by inclusion. Given a map $f\colon X \to B\tilde{\Gamma}_G{}^q$, Theorem 2.12 implies there is induced a characteristic map of (X, f). In particular, there is induced a universal characteristic map $\tilde{h}^\#\colon \pi^*(I(G)_l) \to \pi^*(B\tilde{\Gamma}_G{}^q)$. We remark that if X is one connected, then Theorem 2.13 extends to give a natural isomorphism $\pi^*(X) \cong \operatorname{Hom}(\pi_*(X), \mathbb{R})$.

The classifying space of *integrable* G-foliations[11] is denoted by $B\Gamma_G{}^q$. For any CW complex X, and map $f\colon X \to B\Gamma_G{}^q$, there is a characteristic map of the pair (X, f), whose construction is identical to the above.

§3. RELATION BETWEEN THE INVARIANTS AND CHARACTERISTIC CLASSES

In this section we will give the structure of the vector space $\pi^*(I(G)_l)$, and in doing so, relate the dual homotopy invariants to the secondary characteristic classes of the foliation. We also relate the dual homotopy invariants to a construction of homotopy functionals given by Haefliger[18]. We consider a fixed codimension q G-foliation $\mathscr{F}$ on M and a fixed integer l as in Theorem 2.11.

Let $P \xrightarrow{\pi} M$ be the principal G-bundle associated with the foliation, and choose an adapted connection ω. We make the additional assumption that for some closed subgroup $H \subseteq G$, with the pair (G, H) reductive, the bundle P has an H-reduction $M \xrightarrow{s} P/H$. The subgroup H is often the trivial group $\{e\}$ or a maximal compact subgroup of G.

We will use the following notation: $I_l \overset{\text{def}}{=\!=} I(G)_l$ has algebra generators $\{c_1, \ldots, c_r\}$ with $\deg c_j \le \deg c_k \le 2l$ for $j \le k \le r$.

Let $\Lambda\mathfrak{G}^*$ be the exterior algebra of left-invariant forms on G.

Let $\langle y_1, \ldots, y_r\rangle \subseteq \Lambda\mathfrak{G}^*$ be the G-*invariant* primitives, with y_j the suspension of c_j. Let $\bar{\mathscr{P}} \subseteq \langle y_1, \ldots, y_r\rangle$ be the subspace spanned by the suspensions of the elements of degree $\le 2l$ in the kernel of the restriction homomorphism $I(G) \to I(H)$. Choosing a new algebra basis of $I(G)_l$ if necessary, we can assume that $\bar{\mathscr{P}}$ has a basis $\{y_{i_1}, \ldots, y_{i_s}\} \subseteq \{y_1, \ldots, y_r\}$.

Let $A(G, H)_l = \Lambda\bar{\mathscr{P}} \otimes I_l$, with differential defined by $d(y_{i_\alpha} \otimes 1) = 1 \otimes c_{i_\alpha}$ and $d(1 \otimes c_j) = 0$. For $H = \{e\}$, set $A(G)_l = A(G, \{e\})_l$. In the notation of [7], we have $A(Gl(q, \mathbb{R}), O(q))_q = WO_q$, $A(Gl(q, \mathbb{R}))_q = W_q$ and $A(Gl(n, \mathbb{C}))_n = W_n$.

Let $W(\mathfrak{G}, H)_l = (\Lambda\mathfrak{G}^* \otimes I(\mathfrak{G})_l)^H$ be the truncated Weil algebra defined in [26, 27]. There is a natural inclusion $A(G, H)_l \subseteq W(\mathfrak{G}, H)_l$ which, for the standard triples

$$(G, H, l) = \begin{cases} (Gl(q, \mathbb{R}),\ \{e\}, q) \\ (Gl(q, \mathbb{R}),\ O(q),\ q), \\ (Gl(n, \mathbb{C}), \{e\}, n) \end{cases}$$

induces an isomorphism in cohomology. (When $G = Gl(n, \mathbb{C})$, the ground field is assumed to be $\mathbb{C}$.) For many triples, however, the induced map will not be onto: e.g.

$$(O(q), \{e\}, l \le q) \text{ or } (SO(q), \{e\}, l < 2[q/2]).$$

The Chern–Weil construction gives a DG-algebra homomorphism

$k(\omega)$: $W(\mathfrak{G}, H)_l \to \Omega(P/H)$. Set $\Delta = s^* \circ k(\omega)$; then there is a commutative diagram

$$\begin{array}{ccc} H^*(A(G,H)_l) & \xrightarrow{\Delta_*} & H^*_{DR}(M) \\ & \searrow \quad \nearrow{\scriptstyle \Delta_*} & \\ & H^*(W(\mathfrak{G},H)_l) & \end{array}$$

The maps Δ_* depend only on the concordance class of the foliation, and the elements in the image of Δ_* are called the secondary characteristic classes of the foliation. These classes are related to the previously defined dual homotopy invariants. Let $\mathcal{H}$: $H^*(M) \to \text{Hom}(\pi_*(M), \mathbf{R})$ be the dual Hurewicz homomorphism; let the algebraic dual Hurewicz map $\bar{\mathcal{H}}^*$: $H^*(M) \cong H^*(\mathcal{M}_M) \to \pi^*(M)$ be induced by the quotient map $\bar{\mathcal{M}}_M \to \bar{\mathcal{M}}_M / \bar{\mathcal{M}}_M^2$. Then we have:

3.1 THEOREM. *There exists a vector space map ζ so that for any G-foliated manifold M, the following diagram commutes*:

$$\begin{array}{ccc} \pi^*(I(G)_l) & \xrightarrow{h^\#} & \pi^*(M) \\ {\scriptstyle \zeta}\uparrow & & \uparrow{\scriptstyle \bar{\mathcal{H}}^*} \\ H^*(A(G,H)_l) & \xrightarrow{\Delta_*} & H^*(M) \end{array}$$

If M is simply connected, then we can identify $\pi^*(M)$ with the dual homotopy of M and replace $\bar{\mathcal{H}}^*$ with $\mathcal{H}^*$.

Proof. Let η: $\mathcal{M}(I_l) \to I_l$ be the minimal model. For each $c_j \in I_l$, there is a unique homogeneous $x_j \in \mathcal{M}(I_l)$ with $\eta(x_j) = c_j$. Define a differential on the free algebra $\mathcal{M}(I_l) \otimes \Lambda \bar{\mathcal{P}}$, extending the differential on $\mathcal{M}(I_l)$, by setting $d(1 \otimes y_{i_\alpha}) = x_{i_\alpha} \otimes 1$ for $1 \le \alpha \le s$. The map $\hat{\eta} = \eta \otimes \text{id}$: $\mathcal{M}(I_l) \otimes \Lambda \bar{\mathcal{P}} \to A(G, H)_l$ is a weak isomorphism. Let $\Lambda \bar{\mathcal{P}}$ be the free exterior algebra with trivial differential. Then there is a commutative diagram of DG-algebras

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_l & \longrightarrow & A(G,H) & & \\ & & {\scriptstyle \eta}\uparrow & & \uparrow{\scriptstyle \hat{\eta}} & & \\ 0 & \longrightarrow & \mathcal{M}(I_l) & \to & \mathcal{M}(I_l) \otimes \bar{\mathcal{P}} & \longrightarrow & \Lambda \bar{\mathcal{P}}. \end{array}$$

In the terminology of [19], this is a Λ-minimal Λ-model of the inclusion $I_l \to A(G, H)_l$ and gives a long-exact sequence

$$\to \pi^{n-1}(\Lambda \bar{\mathcal{P}}) \xrightarrow{\delta} \pi^n(I_l) \to \pi^n(A(G,H)_l) \to \pi^n(\Lambda \bar{\mathcal{P}}) \xrightarrow{\delta}.$$

The map δ is easily seen to be injective; identifying the image of δ with the span of $\{c_{i_1}, \dots, c_{i_s}\}$, there are short-exact sequences

$$0 \to \langle c_{i_1}, \dots, c_{i_s} \rangle^n \to \pi^n(I_l) \overset{b}{\leftrightarrows} \pi^n(A(G,H)_l) \to 0. \tag{3.2}$$

Choose a splitting b of (3.2); then we have a commutative diagram:

$$\begin{array}{ccc} \pi^*(I_l) & \searrow^{h^\#} & \\ \uparrow b & & \\ \pi^*(A(G,H)_l) & \xrightarrow{\Delta^\#} & \pi^*(M) \\ \bar{\mathscr{H}}^* \uparrow & & \uparrow \mathscr{H}^* \\ H^*(A(G,H)_l) & \xrightarrow{\Delta_*} & H^*(M). \end{array} \tag{3.3}$$

The conditions of the Theorem are then satisfied for $\zeta = b \circ \bar{\mathscr{H}}^*$. ■

Note that in the above proof, it was shown there is an injection $b: \pi^*(A(G,H)_l) \to \pi^*(I_l)$. This is one reason why we consider $\pi^*(I_l)$ and not the seemingly more natural $\pi^*(A(G,H)_l)$.

An important consequence of the exact sequence (3.2) is:

3.4 Proposition. *There is a natural isomorphism*

$$\pi^n(I(G)_l) \cong \pi^n(A(G)_l) \oplus \langle c_1, \dots, c_r \rangle^n.$$

Proof. We have $\bar{\mathscr{P}} = \langle y_1, \dots, y_r \rangle$ when $H = \{e\}$. The sequence (3.2) has a unique splitting b because $\pi^n(A(G)_l) = 0$ for $n \le 2l$ (and $\deg c_r \le 2l$). ■

This Proposition completely determines the vector spaces $\pi^n(I(G)_l)$ for all n, as the spaces $\pi^n(A(G)_l)$ can be determined explicitly; under our assumptions, the result of Vey[14] implies there is a subalgebra of cocycles $Z_l \subseteq A(G)_l$ for which the inclusion is a weak isomorphism. Further, all products in Z_l are zero. Therefore, if $\{z_1, \dots, z_d\}$ is a basis of Z_l with $\deg z_j = n_j$, then

$$\pi^*(A(G)_l) \cong \pi^*(Z_l) \cong \pi^*\Big(\bigvee_{j=1}^{d} S^{n_j}\Big) \cong \mathrm{Hom}\,(\mathbb{L}(z_1, \dots, z_d), \mathbb{R}),$$

where $\mathbb{L}(z_1, \dots, z_d)$ is the free graded Lie algebra[18] on the generators $\{z_1, \dots, z_d\}$. These remarks imply that there is an inclusion $H^*(A(G)_l) \to \pi^*(A(G)_l)$. Thus, Proposition 3.4 implies:

3.5 Proposition. *The map $\zeta: H^*(A(G)_l) \to \pi^*(I(G)_l)$ is injective.*

3.6 Corollary. *Let $G = Gl(q, \mathbb{R})$ or $Gl(q, \mathbb{C})$. Then there is a commutative diagram, with ζ injective*:

$$\begin{array}{ccc} \pi^*(I(G)_q) & \xrightarrow{h^\#} & \pi^*(M) \\ \zeta \uparrow & & \uparrow \mathscr{H}^* \\ H^*(W(\mathfrak{G}, e)_q) & \xrightarrow{\Delta_*} & H^*(M) \end{array}$$

For an arbitrary reductive pair $H \subseteq G$, the inclusion $A(G,H)_q \xrightarrow{j} W(\mathfrak{G}, H)_q$ is not necessarily a weak isomorphism. For example, if the restriction $i^*: I(G) \to I(H)$ is not onto, then $H^*(W(\mathfrak{G}, H)_q)$ contains a factor $I(H)/\{\text{image } i^*\}$ in the cokernel of j_*[29]. These additional characteristic classes in $H^*(W(\mathfrak{G}, H)_q)$ correspond to the additional geometric data given by the H-reduction of $P \to M$ which is used in defining Δ_*, but not in defining $h^\#$.

An explicit construction of ζ is of interest and is useful in examples. With the notation of Theorem 3.1, recall $\eta(x_j) = c_j$ for $j = 1, \ldots, r$. If $J = (j_1, \ldots, j_r)$ is a sequence of non-negative integers, we use the notation $x_J = x_1^{j_1} \cdots x_r^{j_r}$, and let $|J| = 1/2 \cdot \deg x_J$.

In $\mathcal{M}(I_l)$, inductively choose elements:

For each i, J with $1 \le i \le r$, $|J| \le l$ and $\deg x_i x_J > 2l$, let $u_{i,J} \in \mathcal{M}(I_l)$ satisfy $du_{i,J} = -x_i x_J$.

For each $I = (i_1, \ldots, i_s)$, J with $1 \le i_1, \ldots, i_s \le r$, $|J| \le l$ and $\deg x_{i_m} x_J > 2l$ for $1 \le m \le s$, let $u_{I,J} \in \mathcal{M}(I_l)$ satisfy

$$du_{I,J} = \sum_{m=1}^{s} (-1)^m x_{i_m} u_{I_m, J} \tag{3.7}$$

where $I_m = (i_1, \ldots, \hat{i}_m, \ldots, i_s)$. For $I = \emptyset$, we define $u_{I,J} = x_J$; then (3.7) is valid for $u_{i,J}$.

The sum on the r.h.s. of (3.7) is a cocycle, and $\mathcal{M}(I_l) \xrightarrow{\eta} I_l$ is a weak isomorphism, so it is possible to choose some $u_{I,J}$ satisfying (3.7). Notice that if $u_{I,J}$ and $u'_{I,J}$ are two different choices, then $u_{I,J} - u'_{I,J}$ is a cocycle, must be exact, and is therefore decomposable. Thus, the choice of $u_{I,J}$ is unique modulo $\bar{\mathcal{M}}(I_l)^2$.

The set of elements $y_I c_J = y_{i_1} \cdots y_{i_s} \otimes c_1^{j_1} \cdots c_r^{j_r} \in A(G, H)_l$ with $1 \le i_1, \ldots, i_s \le r$, $|J| \le l$ and $\deg x_{i_m} x_J > 2l$ for each $1 \le m \le s$ are cocycles. A subset of these will form a basis of $H^*(A(G, H)_l)$[14], so it will suffice to define ζ on the cohomology classes $y_I c_J$ of this type.

3.8 Proposition. *The map* $\zeta: H^*(A(G, H)_l) \to \pi^*(I(G)_l)$ *is given by* $\zeta: y_I c_J - u_{I,J}$, *where we identify* $u_{I,J}$ *with its image in* $\pi^*(I(G)_l)$.

Proof. See [23].

This algebraic result has an interesting consequence. The class $\Delta_*(y_I c_J) \in H^*_{DR}(M)$ is represented by the cocycle $\Delta(y_I c_J) \in \Omega(M)$; the class $h^\#(u_{I,J}) \in \pi^n(M)$ corresponds to an element of $\operatorname{Hom}(\pi_n(M), \mathbb{R})$, and we have, by Theorem 3.1:

3.9 Proposition. *Let* $f: S^n \to M$ *be a* C^∞ *map determining an element* $\alpha \in \pi_n(M)$. *Then*

$$h^\#(u_{I,J})(\alpha) = \pm \int_{S^n} f^*(\Delta(y_I c_J)). \tag{3.10}$$

The integrand in (3.10) contains the form $f^*\Delta(y_I)$, which is defined using a section $M \xrightarrow{s} P/H$, while the functional $h^\#(u_{I,J})$ is defined using only the Chern forms $\{\Delta(c_i) \in \Omega(M)\}$ as data. The integral can also be defined in a way that is independent of s. Notice that $d(f^*\Delta(y_i)) = f^*\Delta(c_i)$; this is the only property of the forms $f^*\Delta(y_i)$ that is needed. Let τ_j be any form on S^n such that $d\tau_j = f^*\Delta(c_j)$; one can show the integral in (3.10) is also given by

$$h^\#(u_{I,J})(\alpha) = \pm \int_{S^n} \tau_{i_1} \cdots \tau_{i_s} f^*\Delta(c_J). \tag{3.11}$$

Homotopy periods of the type given by the r.h.s. of (3.11) were introduced by Haefliger in [18]. When c_J is indecomposable, a method for the evaluation of the integral in (3.11) is also given. For c_J decomposable, one can use techniques similar to the Residue Theorem of Schweitzer and Whitman[38] to evaluate the periods. In this form, the dual homotopy invariants in the image of $\zeta: H^*(A(G,H)_q) \to \pi^*(I(G)_q)$ have appeared in the literature. The algebraic construction given for $h^\#: \pi^*(I(G)_l) \to \pi^*(M)$ has the advantage of producing the invariants in a unified way, dealing efficiently with any group G and truncation index l. It is also clear from the definition of $h^\#$ that "reductions of the normal G-bundle" do not play a role in defining the dual homotopy invariants, though such a reduction is essential in defining the secondary characteristic classes.

It was originally suggested by Halperin that one should consider the minimal model of $\Delta: Z_l \subseteq A(G)_l \to \Omega(M)$, where Z_l is the subalgebra, with all products zero, weakly isomorphic to $A(G)_l$. A modification of the proof of Theorem 3.1 shows the algebra homotopy class of $\Delta|_{Z_l}$ is a well-defined concordance invariant, and one can define a characteristic homomorphism $\Delta^\#: \pi^*(Z_l) \to \pi^*(M)$. The space of invariants $\pi^*(Z_l)$ is dual to the free graded Lie algebra $\mathbb{L}(z_1, \ldots, z_d)$, which is infinite dimensional. Thus, $h^\#$ was introduced with the idea of producing a simpler set of invariants. We saw in Proposition 3.4, however, that $\pi^*(I(G)_l)$ is more complicated that $\pi^*(Z_l)$, rather than less. The advantage of using $h^\#$ lies elsewhere; it is the ability to compute the image of $h^\#$ for many examples. Some of the main results of this paper are contained in the sections of examples, 6–8.

§4. A METHOD FOR EVALUATING THE INVARIANTS

Let M be a simply connected manifold with a G-foliation. Given $u \in \pi^n(I(G)_l)$, we want a method for determining when $h^\#(u) \in \pi^n(M)$ is non-zero. A theorem establishing a partial answer (actually, the most complete general answer possible) has been given by Andrews and Arkowitz[1]. We give a statement of this theorem as it applies to $h^\#(u)$. When M is homotopic to a wedge of spheres (or is a formal space), and the classifying map $g: M \to B\Gamma_G{}^q$ is well-chosen, the image of $h^\#: \pi^*(I(G)_l) \to \pi^*(M)$ can be completely determined.

Let $u \in \mathcal{M}^n(I(G)_l)$ be indecomposable so that the class $u \in \pi^n(I(G)_l)$ is non-zero, and $h^\#(u) \in \pi^n(M)$ is a (possibly non-zero) function on $\pi_n(M)$.

4.1 *Definition.* Let $\mathcal{A}$ be a DG-algebra and let $\bar{\mathcal{A}}^p$ be the algebra generated by the p-fold products. An element $a \in \mathcal{A}$ is said to have order p if $\mathrm{d}a \in \bar{\mathcal{A}}^p$ but $\mathrm{d}a \notin \bar{\mathcal{A}}^{p+1}$.

Suppose $\mathcal{M}h(u) \in \mathcal{M}_M{}^n$ has order p. The value of $h^\#(u)$ will be determined on those $\alpha \in \pi_n(M)$ which are Whitehead products of order $s \leq p$:

4.2 *Definition*[1, 37]. An element $\alpha \in \pi_n(M)$ is an sth order Whitehead product of type $(\alpha_1, \ldots, \alpha_s)$, and we write $\alpha = [\alpha_1, \ldots, \alpha_s]$, if the data in (i)–(iii) below are given, for which (iv) holds:

(i) There is a partition $n_1 + \cdots + n_s = n + 1$.

(ii) There are maps $f_j: S^{n_j} \to M$ representing $\alpha_j \in \pi_{n_j}(M)$ for $1 \leq j \leq s$.

Define $P = S^{n_1} \times \cdots \times S^{n_s}$ and let $X = S^{n_1} \vee \cdots \vee S^{n_s} \subseteq P$ be the wedge of the factors. Choose a point $p_1 \in P$ not in X and set $T = P - \{p_1\}$. In the terminology of [1], T is homotopic to the fat wedge $T(S^{n_1}, \ldots, S^{n_s})$; in particular, for $s = 2$ we have $T \simeq S^{n_1} \vee S^{n_2}$.

(iii) The map $f_1 \vee \ldots \vee f_s: X \to M$ extends to a map $f: T \to M$.

Let $D \subseteq P$ be a neighborhood of p_1 diffeomorphic to the closed unit disc in $\mathbf{R}^{n+1}$.

(iv) α is represented by the composition $f: S^n = \partial D \subseteq T \xrightarrow{f} M$.

The obstruction to extending f to a map $\bar{f}: P \to M$ is exactly $\alpha \in \pi_n(M)$.

4.3 *Remark.* Let $f: T \to M$ be a smooth map representing a Whitehead product α as in the definition above. Let M have a G-foliation $\mathcal{F}$ and assume that f is transverse to $\mathcal{F}$; then there is an induced foliation on $P = T \cup \{p_1\}$ with singularity at p_1. Assume that $n+1 = \dim T$ is even. For $G = Gl(q, \mathbb{C})$, $Gl(q, \mathbb{R})$ or $O(q)$, a method has been given by several authors, Baum and Bott[2], Schweitzer and Whitman[33] and Lazarov and Pasternack[32], for the respective groups, assigning to each $c_I c_J \in I(G)$ of degree $n+1$ a residue at $\{p_1\}$ of the foliation on P.

On the other hand, there is a characteristic map $h^\#$ for the foliation induced on T. For each $c_I c_J \in I(G)$ of degree $n+1$, there is an element $u_{I,J} \in \pi^n(I(G)_l)$, giving a functional $h^\#(u_{I,J}) \in \pi^n(T)$. A canonical homotopy class $\gamma \in \pi_n(T)$ is determined by the inclusion $S^n = \partial D \subseteq T$. One consequence of Proposition 3.9 is that the residue of $c_I c_J$ at $\{p_1\}$ corresponds to $h^\#(u_{I,J})(\gamma)$, and Theorem 4.4 yields, as a special case, the evaluation formulas that have been given for this residue.

To state Theorem 4.4, we need the following notation: Let $\{\eta_1, \eta_2, \ldots,\}$ form an algebra basis of $\bar{\mathcal{M}}_M$. Let $u \in \mathcal{M}(I(G)_l)$ be an indecomposable element and suppose $\mathcal{M}h(u) \in \bar{\mathcal{M}}_M$ has order p. For any $s \leq p$ we can write:

$$\mathrm{d}\mathcal{M}h(u) = \sum_{I \in \mathcal{I}} \lambda_I \eta_I + \beta$$

where $\mathcal{I}$ is a set of indices $I = (i_1, \ldots, i_s)$, $\lambda_I \in \mathbb{R}$ for each I, $\eta_I = \eta_{i_1} \cdots \eta_{i_s}$ and $\beta \in \bar{\mathcal{M}}^{s+1}$. Each λ_I will be zero unless $s = p$.

Let $\alpha = [\alpha_1, \ldots, \alpha_s] \in \pi_n(M)$ be an sth order Whitehead product, where $\alpha_k \in \pi_*(M)$ has degree n_k. Each $\eta_i \in \bar{\mathcal{M}}_M$ induces a function on $\pi_*(M)$, and we define for each $I \in \mathcal{I}$ a matrix $A^I_{jk} = \eta_{i_j}(\alpha_k)$ for $1 \leq j, k \leq s$. Note that $A^I_{jk} = 0$ unless $\deg \eta_{i_j} = n_k$.

4.4 Theorem. *Let $s \leq p$, with notation as above. Then*

$$h^\#(u)([\alpha_1, \ldots, \alpha_s]) = (-1)^N \sum_{I \in \mathcal{I}} \lambda_I \kappa(A^I)$$

where

$$N = \sum_{1 \leq i < j \leq s} n_i n_j,$$

$$\kappa(A^I) = \sum_{\sigma} (-1)^{\epsilon(\sigma)} A^I_{1\sigma(1)}, \ldots, A^I_{s\sigma(s)},$$

the sum over the permutations σ of $\{1, \ldots, s\}$ and

$$\epsilon(\sigma) = \sum_{i=1}^{s} \sum_{i \leq j < \sigma^{-1}(i)} n_i n_j$$

or $\epsilon(\sigma) = 0$ if this sum is empty.

Proof. This follows directly from Theorem 5.4 of [1].

Note that if order $\alpha = s < p =$ order $\mathcal{M}h(u)$, then $h^\#(u)$ $(\alpha) = 0$. No implication may be made about the value of $h^\#(u)$ on Whitehead products of order $>p$([1], p. 975).

Theorem 4.4 relates the homotopy period $h^{\#}(u)(\alpha)$ to the homotopy periods $\eta_{i_j}(\alpha_k)$ of lower degree. In examples, these lower degree periods are specified and we use this Theorem to conclude $h^{\#}(u)(\alpha) \neq 0$.

Two special cases of Theorem 4.4 are often used:

Let $du = \lambda\eta_1 \ldots \eta_s + \beta$ and let $\alpha = [\alpha_1, \ldots, \alpha_s]$ with at most one $n_k = \deg \alpha_k$ odd. Then

$$h^{\#}(u)(\alpha) = \lambda \sum_{\sigma} A_{1\sigma(1)} \ldots A_{s\sigma(s)}. \tag{4.5}$$

Let $du = \lambda\eta_1\eta_2 + \beta$ and let $\alpha = [\alpha_1, \alpha_2]$. Then

$$h^{\#}(u)(\alpha) = \lambda(A_{12}A_{21} + (-1)^{n_1 n_2} A_{11}A_{22}). \tag{4.6}$$

§5. THE RATIONAL HUREWICZ THEOREM

The following Theorem and its consequences will be used in the next section:

5.1 THEOREM. *Let X be an n-connected space.*

(i) *The rational Hurewicz homomorphism* $\mathcal{H}: \pi_m(X)\otimes\mathbb{Q} \to H_m(X; \mathbb{Q})$ *is an isomorphism for* $m \leq 2n$ *and an epimorphism for* $m = 2n+1$.

(ii) *If* $H^m(X; \mathbb{Q})$ *is finite dimensional for* $m \leq 2n+1$, *then the dual map* $\mathcal{H}^*: H^m(X; \mathbb{Q}) \to \mathrm{Hom}(\pi_m(X), \mathbb{Q})$ is an isomorphism for $m \leq 2n$ and an inclusion for $m = 2n+1$.

The proof of this can be found in [Proposition 3.8; 3] or [Theorem 18.3; 35].

Let $e: H^m(X; \mathbb{Q}) \to \mathrm{Hom}(H_m(X; \mathbb{Q}), \mathbb{Q})$ be the evaluation map. If X is not of finite type, then we have:

5.2 COROLLARY. *Let X be n-connected. Then* $\mathcal{H}^*$ *is surjective for* $m \leq 2n$, *and for* $m = 2n+1$, *rank* $\mathcal{H}^*$ = *rank e.*

Proof. This follows immediately, using the Universal Coefficient Theorem.

Let $\{z_1, \ldots, z_d\} \subseteq H^m(X)$ be a given subset. Using the evaluation map e, there is induced a homomorphism $z: H_m(X; Z) \to \mathbb{R}^d$, defined by: for $c \in H_m(X; Z)$, set $z(c) = (e(z_1)(c), \ldots, e(z_d)(c)) \in \mathbb{R}^d$.

5.3 *Definition.* A set $\{z_1, \ldots, z_d\} \subseteq H^m(X)$ is said to be independently continuously variable (I.C.V.) if the map $z: H_m(X; Z) \to \mathbb{R}^d$ is onto.

For any connected topological space X there is a map $\pi^*(X) \to \mathrm{Hom}(\pi_*(X), \mathbb{R})$, so that an element u in $\pi^m(X)$ can be considered as a linear functional on $\pi_m(X)$. For a given subset $\{u_1, \ldots, u_d\} \subseteq \pi^m(X)$ we define a homomorphism $u: \pi_m(X) \to \mathbb{R}^d$ as follows: for $\gamma \in \pi_m(X)$, set $u(\gamma) = (u_1(\gamma), \ldots, u_d(\gamma)) \in \mathbb{R}^d$.

5.4 *Definition.* A set $\{u_1, \ldots, u_d\} \subseteq \pi^m(X)$ is said to be I.C.V. if the map $u: \pi_m(X) \to \mathbb{R}^d$ is onto.

5.5 *Remark.* In the next section we need the following application of Theorem 5.1: we show a cohomology I.C.V. set can give rise to a dual homotopy I.C.V. set. Let $F\Gamma^q$ denote the classifying space of $Gl(q, \mathbb{R})$-foliations with trivial normal bundle. For odd $q \geq 3$, Heitsch has shown in [21] that there exists an I.C.V. set $\tilde{\Delta}_*(\mathcal{V}) \subseteq H^{2q+1}(F\Gamma^q)$. What is essential is that he exhibits a manifold M with a family of foliations classified by maps $\{f_\alpha: M \to F\Gamma^q | \alpha \in \mathcal{A}\}$ such that $\tilde{\Delta}_*(\mathcal{V})$ is onto $\mathbb{R}^d$ when restricted to the additive subgroup of $H_{2q+1}(F\Gamma^q; Z)$ generated by the images $(f_\alpha)_* H_{2q+1}(M; Z)$ for $\alpha \in \mathcal{A}$.

Give M the structure of a CW complex, and then collapse the $(q+1)$-skeleton of M to a basepoint to obtain a $(q+1)$-connected space X. It is known that $F\Gamma^q$ is $(q+1)$-connected (see Theorem 6.1) so each map f_α descends to a map $\bar{f}_\alpha: X \to F\Gamma^q$. We also have ismorphisms

$$\pi_{2q+1}(X) \otimes \mathbb{Q} \cong H_{2q+1}(X; \mathbb{Q}) \cong H_{2q+1}(M; \mathbb{Q}),$$

the first being a consequence of Theorem 5.1. It follows that $\tilde{\Delta}_*(\mathcal{V})$ is onto $\mathbb{R}^d$ when restricted to the additive subgroup of $H_{2q+1}(F\Gamma^q; Z)$ generated by the images $\mathcal{H} \circ (\bar{f}_\alpha)_\# \pi_{2q+1}(X)$ for $\alpha \in \mathcal{A}$. Hence, the set $\mathcal{H}^* \circ \tilde{\Delta}_*(\mathcal{V}) \subseteq \pi^{2q+1}(F\Gamma^q)$ is I.C.V., and by Theorem 3.1 the set $\tilde{h}^\# \circ \zeta(\mathcal{V})$ is also I.C.V.

§6. APPLICATIONS: $B\Gamma^q$ AND $F\Gamma^q$

Let $B\Gamma^q$ be the classifying space of $Gl(q, \mathbb{R})$-foliations with $\nu: B\Gamma^q \to BO(q)$ the classifying map of the normal bundle [17]. We denote by $F\Gamma^q$ the homotopy theoretic fiber of ν. Using the notation of Bott–Haefliger [7], we let W_q denote the algebra $A(Gl(q, \mathbb{R}))_q$.

To proceed with our constructions, we need the following fundamental result of Thurston and Mather [43]:

6.1 THEOREM. *The space $F\Gamma^q$ is $(q+1)$-connected.*

The first application of this theorem is to construct a map $\tilde{g}_j: S^{4j} \to B\Gamma^q$ detecting the jth Pontrjagin class p_j. The rational homotopy of $BO(q)$ can be computed using the minimal model of its universal covering $BSO(q)$. Recall there are algebra isomorphisms

$$H^*(BSO(q)) \cong \begin{cases} \mathbb{R}[p_1, \ldots, p_r], & q = 2r+1 \\ \mathbb{R}[p_1, \ldots, p_{r-1}, e_r], & q = 2r \end{cases},$$

where p_j has degree $4j$ and e_r has degree $2r$, [35]. This implies $\pi^*(BSO(q))$ has a vector space basis with elements in degrees: $0, 4, \ldots, 4r$ for $q = 2r+1$, and $0, 4, \ldots, 4(r-1)$, $2r$ for $q = 2r$. Further, for each Pontrjagin class p_j there is a map $g_j: S^{4j} \to BSO(q)$ with $g_j^*(p_j) \neq O$. Each map g_j also defines a map $g_j: S^{4j} \to BO(q)$ for which $g_j^*(p_j) \neq 0$. Setting $m = [(q+2)/4]$, Theorem 6.1 implies there is a lift $\tilde{g}_j: S^{4j} \to B\Gamma^q$ of g_j for each $1 \leq j \leq m$. The maps $\{\tilde{g}_1, \ldots, \tilde{g}_m\}$ will be used in the proofs of subsequent propositions.

The next lemma is used to translate results on $\pi^*(B\Gamma^q)$ to results on $H^*(F\Gamma^q)$.

6.2 LEMMA. *For $z \in H^n(W_q)$, suppose that $\tilde{h}^\# \circ \zeta(z) \in \pi^n(B\Gamma^q)$ is non-zero (resp. variable). Then $\Delta_*(z) \in H^n(F\Gamma^q)$ is non-zero (resp. variable).*

Proof. Let $f: S^n \to B\Gamma^q$ satisfy $\tilde{h}^\# \circ \zeta(z)([f]) \neq 0$. By assumption, n is greater than $2q$ so the composition $\nu \circ f: S^n \to BO(q)$ is rationally homotopic to a constant. Replacing f by an integral multiple if necessary, we can assume $\nu \circ f \simeq 0$. Therefore, Theorem

3.1 implies $\Delta_*(z) \in H^n(S^n)$ is non-zero. It is clear that $\Delta_*(z)$ is proportional to $h^\#\circ\zeta(z) \in \pi^n(S^n)$. Thus, if $\tilde{h}^\#\circ\zeta(z)$ is variable then $\tilde{\Delta}_*(z)$ will be also. ■

Equivalently, Lemma 6.2 states that there is a commutative diagram

$$\begin{array}{ccc} \pi^*(I(Gl(q,\mathbb{R}))_q) & \xrightarrow{\tilde{h}^\#} & \pi^*(B\Gamma^q) \\ {\scriptstyle \zeta}\uparrow & & \uparrow{\scriptstyle \mathscr{H}^*} \\ H^*(W_q) & \xrightarrow{\tilde{\Delta}_*} & H^*(F\Gamma^q). \end{array}$$

Let $q = 4m-2$ or $4m-1$ with $q \geq 3$. The map $\tilde{g}_m: S^{4m} \to B\Gamma^q$ defined above satisfies $(\nu\circ\tilde{g}_m)^*(p_m) \neq 0$. Using residue techniques, Schweitzer and Whitman showed that the Whitehead product $[\tilde{g}_m, \tilde{g}_m] \in \pi_{8m-1}(B\Gamma^q)\otimes\mathbb{R}$ is non-trivial[38]. In our framework, this is given by:

6.3 Proposition. *For q as above, the dual homotopy class $\tilde{h}^\#\circ\zeta(y_{2m}c_{2m}) \in \pi^{8m-1}(B\Gamma^q)$ is non-trivial.*

6.4 Corollary. *For q as above, the class $\tilde{\Delta}_*(y_{2m}c_{2m}) \in H^{8m-1}(F\Gamma^q)$ is non-trivial. When q is even, this is a rigid class.*

Proof. Let $f: S^{8m-1} \to S^{4m}$ represent the Whitehead product $[\text{id}, \text{id}] \in \pi_{8m-1}(S^{4m})$, where id: $S^{4m} \to S^{4m}$ is the identity mapping. Setting $\zeta(y_{2m}c_{2m}) = u_{2m,J}$, we have $du_{2m,J} = -x_{2m}^2$ in $\mathcal{M}(I(Gl(q,\mathbb{R}))_q)$. For the Γ^q-structure on S^{4m} determined by $\tilde{g}_m$, we evaluate $h^\#(u_{2m,J}) \in \pi^{8m-1}(S^{4m})$ using formula (4.6):

$$\begin{aligned} h^\#(u_{2m,J})([f]) &= -2\{h^\#(x_{2m})([\text{id}])\}^2 \\ &= -2\left\{\int_{S^{4m}} (\nu\circ\tilde{g}_m)^*(p_m)\right\}^2 \\ &\neq 0. \end{aligned}$$

■

Because $\pi_*(S^{4m})\otimes\mathbb{R}$ is spanned by the element [id] and the second order Whitehead product $[f]$, we can use formula (4.6) to evaluate $h^\#\circ\zeta$ completely. In fact, if $y_Ic_J \in H^*(W_q)$ is a standard basis element other that $y_{2m}c_{2m}$, then $h^\#\circ\zeta(y_Ic_J)$ is zero. It is only necessary to evaluate this class on the element $[f]$, and by (4.6) this will give zero unless $d\zeta(y_Ic_J) = \pm x_{2m}^2 + \text{other terms}$. The only basis cocycle satisfying this condition is $y_{2m}c_{2m}$. Since $H^*(S^{8m-1}) \to \pi^*(S^{8m-1})$ is injective, we also have $\Delta_*(y_Ic_J) \in H^*(S^{8m-1})$ is zero, showing that $\tilde{\Delta}_*(y_{2m}c_{2m}) \in H^{8m-1}(F\Gamma^q)$ is independent of the other secondary classes.

We note that one can use the $(q+1)$-connectivity of $F\Gamma^q$ to also show that some elements $\tilde{h}\circ\zeta(y_Ic_J)$ in $\pi^*(B\Gamma^q)$, for c_J decomposable, are non-trivial when evaluated on higher order Whitehead products in $\pi_*(B\Gamma^q)$,[23]. This implies the corresponding secondary classes in $H^*(F\Gamma^q)$ are non-trivial.

The most general examples of non-zero dual homotopy classes in $\pi^*(B\Gamma^q)$ are deduced from the known results on $H^*(F\Gamma^q)$. For $q \geq 3$ Heitsch has shown that many of the variable classes in a basis of $H^*(W_q)$ are mapped to independently variable classes in $H^*(F\Gamma^q)$. For our purposes, we use just Theorem 6.3 of [21].

6.5 THEOREM. *For $q \geq 3$ and odd, there is a set $\mathcal{V} \subseteq H^{2q+1}(W_q)$, containing at least two elements, such that $\tilde{\Delta}_*(\mathcal{V}) \subseteq H^{2q+1}(F\Gamma^q)$ is I.C.V. Further, the set $\tilde{\Delta}_*(\mathcal{V})$ varies independently for a family of foliations on a fixed manifold.*

An exact description of the set $\mathcal{V}$ can be found in the paper [21].
More generally, the following result was announced by Fuks [12, 13]:

6.6 THEOREM. *The composition*

$$H^*(W_q) \xrightarrow{\tilde{\Delta}_*} H^*(F\Gamma^q) \xrightarrow{e} \mathrm{Hom}\,(H_*(F\Gamma^q; Z), \mathbf{R})$$

is injective, and the image of a basis of the variable classes in $H^*(W_q)$ is an I.C.V. set in $H^*(F\Gamma^q)$.

The next Proposition, which follows directly from Theorems 5.1 and 6.1, is the means by which we can apply the above information about $H^*(F\Gamma^q)$:

6.7 PROPOSITION. *The rational Hurewicz homomorphism*

$$\mathcal{H}: \pi_n(F\Gamma^q) \otimes \mathbf{Q} \to H_n(F\Gamma^q; \mathbf{Q})$$

is an isomorphism for $n \leq 2q+2$ and an epimorphism for $n = 2q+3$.

A cocycle $y_I c_J \in W_q$ is said to be *admissible* if:

$$1 \leq i_1 < \ldots < i_s \leq q,$$
$$|J| \leq q \text{ and } i_1 + |J| > q,$$
$$l < i_1 \text{ implies } j_l = 0.$$

The set Z_q of admissible cocycles is the *Vey basis* of $H^*(W_q)$, [14]. We say $y_I c_J$ has length s if $I = (i_1, \ldots, i_s)$. This gives a grading of Z_q by setting $Z_q(s) = \{y_I c_J \in Z_q$ with length $= s\}$, and induces a filtration $F^s Z_q = \bigoplus_{r=1}^{s} Z_q(r)$.

Now let $\mathcal{X}$ be the largest subset of $Z_q(1) \cap \{H^n(W_q) | n \leq 2q+3\}$ for which $\langle \mathcal{X} \rangle \xrightarrow{\Delta_*} H^*(F\Gamma^q) \to \mathrm{Hom}\,(H_*(F\Gamma^q), \mathbf{R})$ is injective, where $\langle - \rangle$ denotes the span in $H^*(W_q)$. Using Proposition 6.7 and Theorem 3.1, we then conclude:

6.8 COROLLARY. *The map $\tilde{h}^\# \circ \zeta: \langle \mathcal{X} \rangle \to \pi^*(B\Gamma^q)$ is injective.*

It is next shown that $\tilde{h}^\# \circ \zeta$ is injective on the span of a much larger set $\mathcal{X}'$ containing $\mathcal{X}$. Recall that $m = [(q+2)/4]$. We define

$$\mathcal{X}' = \{y_I c_J \in Z_q | y_{i_1} c_J \in \mathcal{X} \text{ and } (i_2, \ldots, i_s) \subseteq (2, 4, \ldots, 2m)\}.$$

Note that the filtration on Z_q induces one on $\mathcal{X}'$ for which $F^1 \mathcal{X}' = \mathcal{X}$.

6.9 PROPOSITION. *Let $\mathcal{X}'$ be as above. Then $\tilde{h}^\# \circ \zeta$: $\langle \mathcal{X}' \rangle \to \pi^*(B\Gamma^q)$ is injective.*

Letting d_n denote the number of elements in $\mathscr{X}'$ having degree n, this Proposition implies the vector space $\pi_n(B\Gamma^q)\otimes\mathbb{R}$ has dimension at least equal to d_n. We can also conclude, using Lemma 6.2, that the secondary classes in $\mathscr{X}'$ are all independent:

6.10 COROLLARY. *For $\mathscr{X}$ and $\mathscr{X}'$ as above, the map $\tilde{\Delta}_*\langle\mathscr{X}'\rangle\to H^*(F\Gamma^q)$ is injective.*

Proof of Proposition 6.9. For $s=1$, we are given that $\tilde{h}^\#\circ\zeta\colon\langle F^s\mathscr{X}'\rangle\to\pi^*(B\Gamma^q)$ is injective. Assume this is true for $s\geq 1$; we show it is for $s+1$. Suppose that $\sum_{\alpha=1}^{\mu}\lambda_\alpha y_{I^\alpha}c_{J^\alpha}\in\langle F^{s+1}\mathscr{X}^1\rangle$ of degree n is in the kernel of $\tilde{h}^\#\circ\zeta$, with each $\lambda_\alpha\neq 0$ and $y_{I^\alpha}c_{J^\alpha}\neq y_{I^\beta}c_{J^\beta}$ for $\alpha\neq\beta$. By the inductive hypothesis, some I^α must have length $s+1\geq 2$. Let i be the largest integer occurring in the sets $\{I^\alpha|\text{length } I^\alpha\geq 2\}$. Then i is even and $i\leq 2m$. Define an index set $\mathscr{A}=\{\alpha|i\in I^\alpha\}$.

Let $p=n-2i+1$, and let $g\colon S^p\to B\Gamma^q$ be any map. For the Whitehead product $[\tilde{g}_i,g]\colon S^n\to B\Gamma^q$, and using the notation $u_{I,J}=\zeta(y_Ic_J)$, we have by (4.6):

$$\begin{aligned}&\tilde{h}^\#\left(\sum_{\alpha=1}^{\mu}\lambda_\alpha u_{I^\alpha,J^\alpha}\right)([\tilde{g}_i,g])\\&=\sum_{\alpha\in\mathscr{A}}\lambda_\alpha\,\tilde{h}^\#(x_i)(\tilde{g}_i)\cdot\tilde{h}^\#(u_{I^\alpha-i,J^\alpha})(g)\\&=\tilde{h}^\#(x_i)(\tilde{g}_i)\cdot\tilde{h}^\#\left(\sum_{\alpha\in\mathscr{A}}\lambda_\alpha u_{I^\alpha-i,J^\alpha}\right)(g).\end{aligned}\tag{6.11}$$

Note that $I^\alpha=(i)$ implies $\deg u_{I^\alpha-i,J^\alpha}=\deg x_{J^\alpha}\leq 2q$. By the choice of i and $\mathscr{A}$, there exists an $\alpha\in\mathscr{A}$ with $I^\alpha\neq(i)$. Therefore $\deg u_{I^\alpha-i,J^\alpha}>2q$, which implies $I^\alpha\neq(i)$ for all $\alpha\in\mathscr{A}$, hence $\{y_{I^\alpha-i}c_{J^\alpha}|\alpha\in\mathscr{A}\}\subseteq F^s\mathscr{X}'$. By the inductive hypothesis, $\tilde{h}^\#(\sum_{\alpha\in\mathscr{A}}\lambda_\alpha u_{I^\alpha-i,J^\alpha})\neq 0$. Thus, there exists a map $g\colon S^p\to B\Gamma^q$ making the term in (6.11) non-zero, contrary to assumption. ∎

The techniques embodied in the proof above can be used to construct I.C.V. sets in $\pi^*(F\Gamma^q)$ as well. Let $\mathscr{V}$ be the largest subset of $Z_q\cap H^{2q+1}(W_q)$ for which $\tilde{h}^\#\circ\zeta(\mathscr{V})\subseteq\pi^*(F\Gamma^q)$ is an I.C.V. set. It follows that $\mathscr{V}\subseteq Z_q(1)$; we define a new set

$$\mathscr{V}'=\{y_Ic_J\in Z_q|y_{i_1}c_J\in\mathscr{V}\text{ and }(i_2,\ldots,i_s)\subseteq(2,4,\ldots,2m)\}.$$

6.12 PROPOSITION. *Let $\mathscr{V}$ be as above with $\tilde{h}^\#\circ\zeta(\mathscr{V})$ an I.C.V. set. Then the set $\tilde{h}^\#\circ\zeta(\mathscr{V}')\subseteq\pi^*(F\Gamma^q)$ is I.C.V.*

Theorem 3.1 yields the following corollary of this Proposition:

6.13 COROLLARY. *The set $\tilde{\Delta}_*(\mathscr{V}')\subseteq H^*(F\Gamma^q)$ is I.C.V. That is, the elements in $\mathscr{V}'$ are mapped to independently variable classes in $H^*(F\Gamma^q)$.*

Proof. The filtration of Z_q defined above induces a filtration $F^s\mathscr{V}'$ of $\mathscr{V}'$. We proceed by induction on the filtration degree s. It is given that $\tilde{h}^\#\circ\zeta(F^1\mathscr{V}')\subseteq\pi^{2q+1}(F\Gamma^q)$ is I.C.V. For $s\geq 1$ we assume $\tilde{h}^\#\circ\zeta(F^s\mathscr{V}')$ is I.C.V. and will show that the set $\tilde{h}^\#\circ\zeta(F^{s+1}\mathscr{V}')$ is I.C.V. Let V^n denote the elements in $F^{s+1}\mathscr{V}'$ of degree n and suppose that $V^n\cap\mathscr{V}'(s+1)$ is non-empty. We must show $\tilde{h}^\#\circ\zeta(V^n)\subseteq\pi^n(F\Gamma^q)$ is I.C.V.

Our method of proof is to consider $\tilde{h}^{\#}\circ\zeta(V^n)$ as a set of functionals on $\pi_n(B\Gamma^q)$ and then to establish that this set is I.C.V. on Whitehead products in $\pi_n(B\Gamma^q)$ which come from homotopy classes in $\pi_n(F\Gamma^q)$.

Let i be the largest integer occurring in the index sets $\{I|y_Ic_J \in V^n$ and length $I \geq 2\}$. Then i must be even with $i \leq 2m$. List the elements of V^n as $\{y_{I^\alpha}c_{J^\alpha}|1 \leq \alpha \leq d\}$ with $i \in I^\alpha$ for $\alpha \leq r$ and $i \not\in I^\alpha$ for $\alpha > r$. Since $n > 2q+1$, it follows that $I^\alpha \neq (i)$ for all α and thus $\{y_{I^\alpha - i}c_{J^\alpha}|\alpha \leq r\} \subseteq F^s\mathscr{V}'$.

For each α, set $u_\alpha = \tilde{h}^{\#}\circ\zeta(y_{I^\alpha}c_{J^\alpha})$ and form the corresponding evaluation map $u = (u_1, \ldots, u_d)\colon \pi_n(B\Gamma^q) \to \mathbb{R}^d$. Let $p = n - 2i + 1$ and set $\Pi = \pi_p(F\Gamma^q)$. For the wedge of spheres $X = \bigvee_{\gamma\in\Pi} S_\gamma^n$, define a map $g\colon X \to B\Gamma^q$ as follows:

Let g restricted to the γth factor be given by the Whitehead product $[g_i, \gamma]\colon S_\gamma^n \to S^{2i} \vee S^p \to B\Gamma^q$. Note that g factors through $F\Gamma^q$ since $\nu_\#\colon \pi_*(B\Gamma^q) \to \pi_*(BO(q))$ is a Lie algebra homomorphism and $\nu_\#(\gamma) = 0$.

Consider the composition $\pi_n(X) \xrightarrow{g_\#} \pi_n(B\Gamma^q) \xrightarrow{u} \mathbb{R}^d$. From (4.6) and Proposition 3.8 we see that $u_\alpha \circ g_\# = 0$ for $\alpha > r$. Further, using (4.6), the fact that $\{y_{I^\alpha - i}c_{J^\alpha}|\alpha \leq r\} \subseteq F^s\mathscr{V}'$ and the inductive hypothesis we have that $(u_1, \ldots, u_r)\circ g_\#$ is onto $\mathbb{R}^r$. Therefore, the set $\{u_1, \ldots, u_r\} \subseteq \pi^n(B\Gamma^q)$ is I.C.V. and independent of the set $\{u_{r+1}, \ldots, u_d\}$. If the set $\{y_{I^\alpha}c_{J^\alpha}|\alpha > r\}$ is contained in $F^s\mathscr{V}'$, then we are done. Otherwise, choose a new maximal index $i \in I^\alpha$ for $\alpha > r$ and proceed as before. ■

6.14 *Remark.* Using Theorem 6.5 on the existance of a set $\mathscr{V} \subseteq H^{2q+1}(W_q)$ for which $\tilde{\Delta}_*(\mathscr{V})$ is I.C.V., and forming the extension $\mathscr{V}'$ as above, Remark 5.5 and Corollary 6.13 then imply that $\tilde{\Delta}_*(\mathscr{V}')$ consists of independently variable classes in $H^*(F\Gamma^q)$. Thus we obtain an alternate proof of some of the results in Theorem 6.6. We note that Heitsch has shown ([21], Theorem 6.12) that there exists a larger set $\bar{\mathscr{V}}$ than $\mathscr{V}$, with $\mathscr{V} \subseteq \bar{\mathscr{V}} \subseteq H^*(W_q)$, for which $\tilde{\Delta}_*(\bar{\mathscr{V}})$ is I.C.V. The set $\bar{\mathscr{V}}$ of Heitsch and our set $\mathscr{V}'$ have many elements in common but do not coincide. For example, the set $\mathscr{V}'$ contains all of the cocycles of the type $y_Ic_J = y_{i_1}\cdots y_{2m}c_J$ for $y_{i_1}c_J \in \mathscr{V}$ and $(i_2, \ldots, i_{s-1}) \subseteq (2, 4, \ldots, 2m-2)$, but many of these are not in the set $\bar{\mathscr{V}}$.

It is possible to combine the results of Corollary 6.13 and ([21], Theorem 6.12) by showing that the elements of $\mathscr{V}'$ vary independently of many of the elements of $\bar{\mathscr{V}}$. Given an index $I = (i_1, \ldots, i_s)$, set $I_1 = (i_2, \ldots, i_s)$. Then for any $y_Ic_J \in \bar{\mathscr{V}}$ with $I_1 \not\subseteq (2, 4, \ldots, 2m)$, the class $\tilde{\Delta}_*(y_Ic_J)$ varies independently of the set $\tilde{\Delta}_*(\mathscr{V}')$ in $H^*(F\Gamma^q)$. To see this, note that the proof of Proposition 6.12 actually shows that the set $\tilde{h}^{\#}\circ\zeta(\mathscr{V}')$ is independently variable on the iterated Whitehead products in $\pi_*(B\Gamma^q)$ formed from the elements of $\pi_{2q+1}(B\Gamma^q)$ and the maps $\{\tilde{g}_1, \ldots, \tilde{g}_m\}$. If $f\colon S^n \to B\Gamma^q$ represents one of these iterated Whitehead products, then for $y_Ic_J \in \bar{\mathscr{V}}$ with $I_1 \not\subseteq (2, 4, \ldots, 2m)$ it follows from Proposition 3.8 and Theorem 4.4 that $\tilde{h}^{\#}\circ\zeta(y_Ic_J)([f]) = 0$. By Theorem 3.1 we conclude that $\tilde{\Delta}_*(y_Ic_J)$ vanishes for a family of foliations on which the set $\tilde{\Delta}_*(\mathscr{V}')$ is independently variable. Thus, setting $\mathscr{V}'' \stackrel{\text{def}}{=\!=} \mathscr{V}' \cup \{y_Ic_J \in \bar{\mathscr{V}}|I_1 \not\subseteq (2, 4, \ldots, 2m)\}$ and combining the above remarks with Theorem 6.12 of [21], we obtain the following result on the variability of the secondary classes:

6.15 THEOREM. *For each odd integer $q \geq 3$, and for $\mathscr{V}''$ as defined above, the set $\tilde{\Delta}_*(\mathscr{V}'')$ in $H^*(F\Gamma^q)$ is I.C.V. In particular, the map $\tilde{\Delta}_*\colon \langle\mathscr{V}''\rangle \to H^*(F\Gamma^q)$ is injective.*

We conclude our consideration of $\pi_*(B\Gamma^q)$ by giving some consequences of the preceding results. Let $\mathscr{V}' = \{z_1, \ldots, z_N\} \subseteq Z_q$ be the largest set for which $\tilde{h}^{\#}\circ\zeta(\mathscr{V}') \subseteq \pi^*(B\Gamma^q)$ is I.C.V. Let n_j denote the degree of the element z_j, and form the wedge of spheres $Y = \bigvee_{j=1}^{N} S^{n_j}$. Let $\hat{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$, and let $\{k_\alpha | \alpha \in \mathscr{A}\}$ be a transcendence basis for $\mathbb{R}$ over $\hat{\mathbb{Q}}$. Note that the set $\mathscr{A}$ is uncountable. By the definition of an I.C.V. set, for each $\alpha \in \mathscr{A}$ we can choose a map $g_\alpha: Y \to B\Gamma^q$ such that the composition $\pi_*(S^{n_j}) \to \pi_*(Y) \xrightarrow{(g_\alpha)_\#} \pi_*(B\Gamma^q) \xrightarrow{\tilde{h}^\# \circ \zeta(z_i)} \mathbb{R}/\mathbb{Q}$ is onto the coset of k_α for $i = j$, and zero otherwise. It follows from the work of Haefliger in [18] that the map $(g_\alpha)_\#$ is injective on rational homotopy. Since $\pi_*(Y) \otimes \mathbb{Q}$ is a free, graded Lie algebra with a basis corresponding to the set $\mathscr{V}'$, this implies that $\pi_*(B\Gamma^q) \otimes \mathbb{Q}$ is non-trivial in arbitrarily high degrees if $\mathscr{V}'$ contains more than one element. The following propositions represent generalizations of this result; their proofs are given in [23]:

6.16 Proposition. *Let $\mathscr{V}' \subseteq H^*(W_q)$ be the largest set such that $\tilde{h}^\# \circ \zeta(\mathscr{V}') \subseteq \pi^*(B\Gamma^q)$ is I.C.V. Then with notation as above, it follows that the direct sum of maps*

$$\bigoplus_{\alpha \in \mathscr{A}} (g_\alpha)_\#: \bigoplus_{\alpha \in \mathscr{A}} \pi_*(Y) \otimes \mathbb{Q} \to \pi_*(B\Gamma^q) \otimes \mathbb{Q}$$

is injective.

In other words, if $\mathscr{V}'$ contains more than one element, then there are uncountably many linearly independent, free graded Lie algebras in $\pi_*(B\Gamma^q) \otimes \mathbb{Q}$.

6.17 Proposition. *For $\mathscr{V}'$ as above, there is an epimorphism of abelian groups*

$$\pi_*(B\Gamma^q) \to \pi_*(Y) \otimes \mathbb{R}.$$

The space $\pi_*(Y) \otimes \mathbb{R}$ is a free graded Lie algebra over $\mathbb{R}$, which will be infinite-dimensional if $\mathscr{V}'$ contains more than one element.

§7. APPLICATIONS: $BR\Gamma_+{}^q$ AND $FR\Gamma^q$

Let $BR\Gamma_+{}^q$ be the classifying space of $SO(q)$-foliations and let ν: $BR\Gamma_+{}^q \to BSO(q)$ be the classifying map of the normal bundle. Let $FR\Gamma^q$ be the homotopy theoretic fiber of ν.

Set $q' = [q/2]$, $p = [(q-1)/2]$ and $k = [q/4] + 1$. For this section, we adopt the notation $A_{q'} = A(SO(q))_{q'}$. We will show:

7.1 Theorem. *The characteristic homomorphism $\tilde{h}^\#$: $\pi^*(I(SO(q)_{q'}) \to \pi^*(BR\Gamma_+{}^q)$ is injective.*

The set of cohomology invariants defined for an $SO(q)$-foliation with trivial normal bundle is given by the algebra $H^*(W(\mathfrak{O}(q), e)_{q'}) \cong H^*(RW_q)$, and there is a universal map $\tilde{\Delta}_*$: $H^*(RW_q) \to H^*(FR\Gamma^q)$[26, 31]. This algebra is related to the complex $A_{q'}$ by $H^*(RW_q) \cong H^*(A_{q'}) \otimes \Lambda(y_k, \ldots, y_p)$, where y_j is the suspension of p_j.

7.2 Theorem. *The homomorphism $\tilde{\Delta}_*: H^*(A_{q'}) \oplus \Lambda(y_k, \ldots, y_p) \to H^*(FR\Gamma^q)$ is injective.*

7.3 Theorem. *The homomorphism $\tilde{h}^\# \circ \zeta$ maps a basis of the variable classes in $H^*(A_{q'})$ to an I.C.V. set in $\pi^*(FR\Gamma^q)$.*

Theorem 7.3 and Theorem 3.1 imply:

7.4 COROLLARY. *$\tilde{\Delta}_*$ maps a basis of the variable classes in $H^*(A_{q'})$ to an I.C.V. set in $H^*(FR\Gamma^q)$.*

Theorem 7.3 also has the following consequence. Let v_n be the dimension of the space of variable classes in $H^n(A_{q'})$. Form the wedge of spheres $Y = \bigvee_{n>0} (\bigvee_{j=1}^{v_n} S_j^n)$. Corresponding to Propositions 6.16 and 6.17 we have the following result, which is proved in [23]:

7.5 PROPOSITION. (i) *If $\sum_{n>0} v_n \geq 2$, then there is an uncountable set $\mathscr{A}$ and corresponding maps g_α: $Y \to BR\Gamma_+{}^q$ for $\alpha \in \mathscr{A}$ such that the direct sum of maps*

$$\bigoplus_{\alpha \in \mathscr{A}} (g_\alpha)_{\#}\colon \bigoplus_{\alpha \in \mathscr{A}} \pi_*(Y) \otimes \mathbb{Q} \to \pi_*(BR\Gamma_+{}^q) \otimes \mathbb{Q}$$

is injective. (ii) *There is an epimorphism of graded abelian groups*

$$\pi_*(BR\Gamma_+{}^q) \to \pi_*(Y) \otimes \mathbb{R}.$$

If q is of the form $4k-2$ or $4k-1$ and $q \geq 10$, then it will be seen in the proof of Theorem 7.3 that $\sum_{n>0} v_n \geq 2$. Proposition 7.5 then says there are uncountably many free Lie algebras in $\pi_*(BR\Gamma_+{}^q) \otimes \mathbb{Q}$. Further, the group $\pi_n(BR\Gamma_+{}^q)$ maps onto $\mathbb{R}^{d_n}$, where d_n is the dimension of $\pi_n(Y) \otimes \mathbb{R}$ and is positive for an infinite number of n.

The proofs of Theorems 7.1, 7.2 and 7.3 depend upon

7.6 THEOREM. *$FR\Gamma^q$ is $(q-1)$-connected.*

The proof of this parallels the line of reasoning used by Haefliger in [16] to show $F\Gamma^q$ is $(q-1)$-connected. A geometric construction is used, in addition, to show that any two metrics on $(\mathbb{R}^{n+1} - \{0\}) \times \mathbb{R}^{q-n-1} \simeq S^n$ can be joined by an integrable homotopy; therefore, there is a unique homotopy class of maps $S^n \to FR\Gamma^q$ for $n < q$. Details are in [22].

Theorem 7.6 implies the map ν: $BR\Gamma_+{}^q \to BSO(q)$ is q-connected, so the vanishing theorem of Pasternack is exact[36]. Using Theorem 5.1, we also have:

7.7 COROLLARY. *$\mathscr{H}$: $\pi_n(FR\Gamma^q) \otimes \mathbb{Q} \to H_n(FR\Gamma^q; \mathbb{Q})$ is an isomorphism for $n \leq 2q-2$, and an epimorphism for $n = 2q-1$.*

Proof of Theorem 7.1. Let X be a q-dimensional, simply connected CW complex with a map f: $X \to BSO(q)$ inducing an algebra isomorphism f^*: $H^n(BSO(q)) \to H^n(X)$ for $n \leq q$. By the q-connectivity of ν, there is a lifting

$$\begin{array}{ccc} & & BR\Gamma_+{}^q \\ & \overset{\tilde{f}}{\nearrow} & \downarrow \nu \\ X & \xrightarrow{f} & BSO(q) \end{array}.$$

The Chern–Weil homomorphism h: $I(SO(q))_{q'} \to \Omega(X)$ is a weak isomorphism; therefore, the composition $\mathcal{M}(I(SO(q))_{q'}) \xrightarrow{\eta} I(SO(q))_{q'} \xrightarrow{h} \Omega(X)$ is a minimal model for $\Omega(X)$. Any two minimal models for $\Omega(X)$ are isomorphic; thus, the induced map $h^{\#}$: $\pi^*(I(SO(q))_{q'}) \to \pi^*(X)$ is an isomorphism. ∎

Proof of Theorem 7.2. Let $\tilde{f}$: $X \to BR\Gamma_+{}^q$ be the CW complex used in the proof of Theorem 7.1. Let ξ: $P \to X$ be the principal $SO(q)$-bundle induced by $f = \nu \circ \tilde{f}$. The composition $f \circ \xi \simeq 0$, hence by Theorem 3.1 there is a commutative diagram

$$\begin{array}{ccccc} \pi^*(I(SO(q))_{q'}) & & \xrightarrow{h_P{}^{\#}} & & \pi^*(P) \\ \uparrow & \searrow^{h_X{}^{\#}} & & \nearrow^{\xi^{\#}} & \uparrow \\ \zeta & & \pi^*(X) & & \mathcal{H}^* \\ H^*(A_{q'}) & & \xrightarrow{\Delta_*} & & H^*(P) \end{array} \tag{7.8}$$

The map $h_X{}^{\#}$ is an isomorphism; thus, ker $h_P{}^{\#} \cong$ ker $\xi^{\#} \cong \langle p_1, \ldots, p_{k-1} \rangle$ which implies the image of ζ intersects ker $h_P{}^{\#}$ trivially. Therefore ker $\Delta_* =$ ker $h_P{}^{\#} \circ \zeta$ is zero.

It remains to show that $\tilde{\Delta}_*\{\Lambda(y_k, \ldots, y_p)\}$ is independent of $\tilde{\Delta}_*\{H^*(A_{q'})\}$ in $H^*(FR\Gamma^q)$. First, consider the case $q = 2q'$. Kamber and Tondeur have shown ([25], Theorem 6.52) that $\tilde{\Delta}_*$ is injective when restricted to the ideal generated by $\Lambda(\chi e) \otimes \Lambda(y_k, \ldots, y_p)$ in $H^*(RW_q)$, where e is the Euler class and χ is the suspension of e. Suppose $y \in \Lambda^n(y_k, \ldots, y_p)$ is a non-zero element such that $\tilde{\Delta}_*(y) \in \Delta_*\{H^n(A_{q'})\}$. Because $e \cdot A_{q'} = \{0\}$, this implies $\tilde{\Delta}_*(y) \cdot \tilde{\Delta}_*(\chi e) = 0$, yielding a contradiction.

The case $q = 2q' + 1$ follows from the injectivity of $H^*(RW_q) \to H^*(RW_{q-1})$, with the understanding that $y_p \to \chi e$, and the straightforward:

7.9 Lemma. *For any q, there is a commutative diagram*:

$$\begin{array}{ccc} H^*(RW_{q-1}) & \xrightarrow{\tilde{\Delta}_*} & H^*(FR\Gamma^{q-1}) \\ \uparrow & & \uparrow \\ H^*(RW_q) & \xrightarrow{\tilde{\Delta}_*} & H^*(FR\Gamma^q). \end{array}$$

∎

Proof of Theorem 7.3. If $q = 4k - 4$ or $4k - 3$, then all classes in $H^*(RW_q)$ are rigid([31], Theorem 5.5). For $q = 2$ (resp. $q = 3$), the variable class χe (resp. y_1) in $H^3(RW_q)$ was shown to vary continuously for a family of foliations on S^3[31]. Therefore, $\tilde{h}^{\#} \circ \zeta(\chi e)$ (resp. $\tilde{h}^{\#} \circ \zeta(y_1)$) is variable.

The case $q = 4k - 1$ follows from the case $q = 4k - 2$ using the dual homotopy version of Lemma 7.9. Therefore, we need only show the case $q = 4k - 2 \geq 6$.

An element $y_I p_J \in A_{q'}$ is admissible if $i_1 < \ldots < i_s < k$; $|J| \leq q'$; $i_1 + |J| > q'$ and $l < i_1 \Rightarrow j_l = 0$. Let $\mathcal{X}$ be the set of admissible elements of degree $q + 1$. Then $\mathcal{X}$ is a basis of $H^{q+1}(A_{q'})$ and it was shown by Lazarov and Pasternack ([32] Theorem 3.3) that $\tilde{h} \circ \zeta(\mathcal{X})$ is an I.C.V. set in $\pi^*(FR\Gamma^q)$. Let $\mathcal{X}'$ be the set of admissible $y_I p_J$ with $y_{i_1} p_J \in \mathcal{X}$; then $\mathcal{X}'$ is a basis for the variable classes in $H^*(A_{q'})$. Using the $(q-1)$-connectivity of $FR\Gamma^q$, the proof of Proposition 6.12 carries over directly to show $\tilde{h}^{\#} \circ \zeta(\mathcal{X}')$ is an I.C.V. set. ∎

§8. APPLICATIONS: $B\Gamma_{\mathbb{C}}^n$ and $F\Gamma_{\mathbb{C}}^n$

Let $B\Gamma_{\mathbb{C}}^n$ be the classifying space of integrable $Gl(n, \mathbb{C})$-foliations of complex codimension n, and let $\nu: B\Gamma_{\mathbb{C}}^n \to BU(n)$ be the map classifying the normal bundle. Let $F\Gamma_{\mathbb{C}}^n$ denote the homotopy theoretic fiber of ν. In this section we will extend the results of Baum and Bott[2] on the homotopy and cohomology of $B\Gamma_{\mathbb{C}}^n$ and $F\Gamma_{\mathbb{C}}^n$. *All algebras are over the field* $\mathbb{C}$, *and* $H^*(X) = H^*(X; \mathbb{C})$. We use the notation $W_n \overset{\text{def}}{=\!=\!=} A(Gl(n, \mathbb{C}))_n$ of [7]. The following useful result was proved by Landweber:

8.1 THEOREM ([30], Theorem 1.6). *The space* $F\Gamma_{\mathbb{C}}^n$ *is* $(n-1)$*-connected.*

The space $BU(n)$ is simply connected, so this theorem implies $B\Gamma_{\mathbb{C}}^n$ is simply connected for $n > 1$. Haefliger has shown $F\Gamma_{\mathbb{C}}^1$ is simply connected[17], so $B\Gamma_{\mathbb{C}}^1$ is also simply connected. The minimal model of $BU(n)$ is the algebra $\mathbb{C}[c_1, \ldots, c_n] \cong H^*(BU(n))$. Therefore, for each $1 \le j \le n$ there is a map $g_j: S^{2j} \to BU(n)$ such that $g_j^*(c_j) \in H^{2j}(S^{2j})$ is non-zero. By Theorem 8.1, for each $1 \le j \le [n/2]$ there exists a lift $\tilde{g}_j: S^{2j} \to B\Gamma_{\mathbb{C}}^n$ of g_j. It follows that $(\nu \circ \tilde{g}_j)^*(c_j) = g_j^*(c_j) \neq 0$.

A cocycle $y_I c_J \in W_n$ is *admissible* if $i_1 < \ldots < i_s \le n$; $|J| \le n$; $i_1 + |J| > n$ and $l < i_1$ implies $j_l = 0$. Let $\mathscr{X}$ be the set of admissible cocycles of degree $2n+1$. Then $\mathscr{X}$ is the Vey basis (over $\mathbb{C}$) of $H^{2n+1}(W_n)$[14]. Baum and Bott have shown:

8.2 THEOREM[2]. *The set* $\tilde{h}^\# \circ \zeta(\mathscr{X}) \subseteq \pi^{2n+1}(F\Gamma_{\mathbb{C}}^n)$ *is I.C.V.*

In this context, a set is I.C.V. if the evaluation map $\pi_{2n+1}(F\Gamma_{\mathbb{C}}^n) \xrightarrow{z} \mathbb{C}^d$ is onto.

Using Theorem 8.2 and the remarks following Theorem 8.1, the proof of Proposition 6.12 carries over directly to give:

8.3 THEOREM. *Let* $\mathscr{X}'$ *be the set of admissible cocycles* $y_I c_J$ *satisfying* $y_{i1} c_J \in H^{2n+1}(W_n)$ *and* $i_1 < \ldots < i_s \le [n/2]$. *Then* $\tilde{h}^\# \circ \zeta(\mathscr{X}')$ *is an I.C.V. set in* $\pi^*(F\Gamma_{\mathbb{C}}^n)$.

8.4 COROLLARY. *For* $\mathscr{X}'$ *as above, the set* $\tilde{\Delta}_*(\mathscr{X}') \subseteq H^*(F\Gamma_{\mathbb{C}}^n)$ *is I.C.V. Let* $\langle \mathscr{X}' \rangle$ *denote the span of* $\mathscr{X}'$; *then* $\tilde{\Delta}_*: \langle \mathscr{X}' \rangle \to H^*(F\Gamma_{\mathbb{C}}^n)$ *is injective.*

Let v_m denote the number of elements in $\mathscr{X}'$ of degree m. Form the wedge of spheres $Y = \bigvee_{m>0} \left(\bigvee_{j=1}^{v_m} S_j^m \right)$. Using Proposition 8.3, the following proposition is proven in [23].

8.5 PROPOSITION. (i) *For* $n > 1$, *there is an uncountable set* $\mathscr{A}$ *and maps* $g_\alpha: Y \to B\Gamma_{\mathbb{C}}^n$ *for* $\alpha \in \mathscr{A}$ *such that the direct sum of maps*

$$\bigoplus_{\alpha \in \mathscr{A}} (g_\alpha)_\#: \bigoplus_{\alpha \in \mathscr{A}} \pi_*(Y) \otimes \mathbb{Q} \to \pi_*(B\Gamma_{\mathbb{C}}^n) \otimes \mathbb{Q}$$

is injective. (ii) *There is an epimorphism of graded abelian groups* $\pi_*(B\Gamma_{\mathbb{C}}^n) \to \pi_*(Y) \otimes \mathbb{C}$.

Acknowledgements—This paper is derived from the author's doctoral thesis. The author would like to thank his advisor, Franz Kamber, for many helpful conversations. Thanks are also due to Philippe Tondeur and Kuo-Tsai Chen for their suggestions and encouragement in the development of this work, and to Stephen Halperin and James Stasheff for their constructive comments.

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