Homotopy invariants of foliations
by
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1. In this note we propose to study the homotopy groups of \( B_q^G \), the classifying space of \( G \)-foliated microbundles [H1]. A foliation \( F \) on a manifold \( X \) is a \( G \)-foliation if it is defined by local submersions into a \( q \)-dimensional model manifold \( B \), such that the local transition functions preserve a \( G \)-structure on \( B \).

With respect to an adapted connection \( \omega \), the Chern-Weil homomorphism defines a map
\[ h(\omega): I(G)_q \longrightarrow A^*(X), \]
where \( I(G)_q \) is the ring of invariant polynomials on \( G \) modulo the ideal of elements of degree \( > 2\ell \). The index \( \ell \) depends on \( G \) and whether the \( G \)-structure is integrable; it can always be taken \( \leq q \) [B].

For any commutative DG-algebra \( A \), we let \( \varphi: M_A \longrightarrow A \) be a minimal model [S]. Let \( \overline{M}_A \) denote the augmentation ideal; the quotient \( \pi^*(A) \overset{\text{def}}{=} \overline{M}_A/M_A^2 \) is called the dual homotopy of \( A \). For any (semi-simplicial) manifold \( X \), we set \( \pi^*(X) = \pi^*(A^*(X)) \) [D], [S], where \( A^*(X) \) is the deRham algebra in the sense of Sullivan-Dupont. If \( X \) is 1-connected and of finite rational type, then there is a natural isomorphism \( \pi^*(X) \cong \text{Hom}(\pi_*(X), \mathbb{R}) \). For a DG-algebra \( A \), whose cohomology is of finite type, we define
\[ \pi_*(A) = \text{Hom}_{\mathbb{R}}(\pi^*(A), \mathbb{R}). \]

Let \( X \) be a \( G \)-foliated manifold. The following result is proved in [Hu1]

1.1. THEOREM. The Chern-Weil homomorphism induces a map \( h^\#: \pi^*(I(G)_q) \longrightarrow \pi^*(X) \)
which depends only on the concordance class of the foliation.

If \( X \) is of finite rational type, the map \( h^\# \) induces by transposition a mapping
\[ h^\#: \Pi(X) \longrightarrow \Pi(I(G)_q), \]

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where \( \Pi(X) = s^{-1}(\pi_\ast(X) \otimes \mathbb{R}) \) and \( \Pi(A) = s^{-1}\pi_\ast(A) \) are the desuspended graded homotopy groups. It is known [B-L] that the functor \( \Pi \) has values in the category of graded Lie algebras. As \( h(\omega): I(G) \rightarrow \mathcal{A}(X) \) is a homomorphism of DG-algebras, \( h_\# \) is a homomorphism of graded Lie algebras. This construction extends to \( G \)-microbundles in an obvious way and therefore defines a homomorphism of Lie algebras

\[
h_\#: \Pi(BF^q_G) \rightarrow \Pi(I(G)_\lambda).
\]

If \( f:X \rightarrow BF^q_G \) denotes the classifying map of the \( G \)-foliation \( F \) on \( X \), the diagram

\[
\begin{array}{ccc}
\Pi(X) & \xrightarrow{\Pi(h)} & \Pi(I(G)_\lambda) \\
\downarrow{f_\#} & & \downarrow{h_\#} \\
\Pi(BF^q_G) & \xrightarrow{\Pi(h_\#)} & \Pi(I(G)_\lambda)
\end{array}
\]

is commutative. It is the purpose of this note to determine the Lie algebra structure of \( \Pi(I(G)_\lambda) \) (Section 2) and to detect elements in the image of \( h_\# \) via appropriate choices of \( (X,f) \) (Section 4). In 3 we study the relationship of \( h_\# \) with the characteristic homomorphism \( \Delta_\ast \) for \( G \)-foliations [K-T 1], [K-T 2].

2. The structure of \( \Pi(I(G)_\lambda) \)

In this section we determine the structure of the graded Lie algebra \( \Pi(I(G)_\lambda) \). Let \( G \) be a reductive Lie group. In order to simplify the following discussion, we will assume that \( G \) is connected in which case \( I(G) \cong \mathbb{R}[c_1, \ldots, c_\tau] \) is a polynomial algebra generated by the characteristic classes \( c_j \) of even degree. As before, we denote by

\[
I_\lambda = I(G)_\lambda = \mathbb{R}[c_1, \ldots, c_\tau]/(\psi(c_1, \ldots, c_\tau) | \deg \psi > 2\lambda)
\]

the truncated polynomial algebra, where \( c_1, \ldots, c_\tau \) denote the generators of degree \( \leq 2\lambda \).
Let $A_L = \Lambda P(2L) \otimes I_L$ be the DG-algebra introduced in section 3. The inclusion $0 \longrightarrow I_L \longrightarrow A_L$ dualizes to give an epimorphism of DG-coalgebras, $A_L^* \longrightarrow I_L^* \longrightarrow 0$. Applying Quillen's $L$ construction [Q], [B-L], we get an exact sequence of free DG-Lie algebras

\[(2.1)\quad 0 \longrightarrow \ker j^* \longrightarrow L(A_L^*) \longrightarrow L(I_L^*) \longrightarrow 0,\]

where $L(C) = \mathbb{L}(s^{-1}C)$ is the free DG-Lie algebra generated by a suspended reduced DG-coalgebra $C$ [B-L], [N-M]. Passing to cohomology we get an exact sequence

\[(2.2)\quad 0 \longrightarrow H_*(L(A_L^*)) \longrightarrow H_*(L(I_L^*)) \xrightarrow{\delta} H_{*-1}(\ker j^*) \longrightarrow 0\]

2.3. THEOREM. There is an extension of graded Lie algebras

\[0 \longrightarrow \Pi(A_L^*) \longrightarrow \Pi(I_L^*) \longrightarrow P_{(2L)}^* \longrightarrow 0,
\]

where $P_{(2L)}^*$ is an abelian Lie algebra and $\Pi(A_L^*) \cong L(H^*(A_L^*))$ is a free Lie algebra.

We remark that the Lie algebra structure of the extension $\Pi(I_L^*)$ is uniquely determined by the induced representation of $P_{(2L)}^*$ in the Lie algebra of outer derivations of $\Pi(A_L^*)$. This follows from the fact that the free Lie algebra $\Pi(A_L^*)$ has trivial center and from the general theory of extensions of Lie algebras (compare [HQ] for the ungraded case). The proof of this theorem, culminating in the determination of this induced action, will occupy the rest of this section.

First note that by taking a $\Lambda$-minimal model of the KS extension

\[0 \longrightarrow I_L \longrightarrow A_L \longrightarrow \Lambda P(2L) \longrightarrow 0\quad [Ha],\]

there is a long exact dual homotopy sequence with injective coboundary $\delta^*$ and therefore a short exact sequence

\[(2.4)\quad 0 \longrightarrow \delta^* P_{(2L)} \longrightarrow \kappa^*(I_L) \longrightarrow \kappa^*(A_L) \longrightarrow 0.\]
Dualizing this sequence gives the exact sequence of the theorem. The elements of $\mathcal{P}(2\mathcal{L})^*$ all have odd degree, so as a Lie algebra this must be abelian. We want to analyze how the elements in $\mathcal{P}(2\mathcal{L})^*$ act on the image of $\Pi(A_\mathcal{L})$, and this will show that $\Pi(A_\mathcal{L})$ is an ideal in $\Pi(I_\mathcal{L})$.

The algebra $A_\mathcal{L}$ admits a subalgebra $Z_\mathcal{L} \subset A_\mathcal{L}$ with trivial differential and products, which induces an isomorphism in cohomology $Z_\mathcal{L} \cong H^*(A_\mathcal{L})$ [K-T3]. Therefore $A_\mathcal{L}$ is biformal and we have isomorphisms

$$L(Z_\mathcal{L}^*) \cong H^*(L(Z_\mathcal{L}^*)) \xrightarrow{\cong} H^*(L(A_\mathcal{L}^*)) \cong \Pi(A_\mathcal{L}).$$

It follows that $\Pi(A_\mathcal{L})$ is a free graded Lie algebra generated by $s^{-1}Z_\mathcal{L}^*$.

The algebra $Z_\mathcal{L}$ and the isomorphism $Z_\mathcal{L} \cong H(A_\mathcal{L})$ have been described in [K-T1], [K-T3]. We use here a slightly different notation, which is more convenient in the present context. For ordered sequences $I = (i_1, \ldots, i_s)$, $J = (j_1, \ldots, j_m)$, the symbol $(I/J)$ is called admissible if it satisfies

$$\deg c_J \leq 2\mathcal{L}, \quad \sum_{\alpha=1}^m c_{i\alpha};$$

$$\deg c_{i_1} c_{i_m} > 2\mathcal{L};$$

$$i_1 \leq j_1.$$

For an admissible symbol $(I/J)$, the cochain

$$z(I/J) = y_I \otimes c_J = y_{i_1} \wedge \cdots \wedge y_{i_s} \otimes c_{j_1} \cdots c_{j_m} \in A_\mathcal{L}$$

is clearly a cocycle and the product of any two such cocycles = 0. The algebra $Z_\mathcal{L}$ is then given by the linear space spanned by 1 and the cocycles $z(I/J)$ for $(I/J)$ admissible.

Let $I = \mathbb{R}[c_1, \ldots, c_t]$; the canonical quotient map $I \rightarrow I_\mathcal{L}$ dualizes to an inclusion $L(I_\mathcal{L}) \subset L(I^*)$. Since $I^*$ has a canonical Hopf algebra structure and trivial differential, we find easily that

$$H^*(L(I^*)) \cong \mathcal{P}(2\mathcal{L})^* = \text{span}\{y_1, \ldots, y_\mathcal{L}\}.$$
where \( Y_j = s^{-1}c^*_j, j = 1, \ldots, t \). Hence all the cycles in \( L(I^*_\ell) \) of degree \( \geq 2\ell \) are boundaries. Thus all the cycles in \( L(I^*_\ell) \) of degree \( \geq 2\ell \) are boundaries in \( L(I^*_\ell) \). With this observation in mind, we produce explicitly a set of cycles in \( L(I^*_\ell) \), which will generate \( H(L(I^*_\ell)) \).

For any monomial \( c^*_K, K = (k_1, \ldots, k_m) \) in \( I \), we set \( Y_K = s^{-1}c^*_K \in I^*_\ell \).

The diagonal \( \Delta \) in \( I^*_\ell \) is given by

\[
\Delta(c^*_K) = c^*_K \otimes 1 + 1 \otimes c^*_K + \frac{1}{2} \sum_{(\alpha, \beta)} (c^*_\alpha \otimes c^*_\beta + c^*_\beta \otimes c^*_\alpha),
\]

where \((\alpha, \beta)\) runs over all ordered proper partitions of the set \( \{k_1, \ldots, k_m\} \). By definition, the differential \( d_L \) of \( L(I^*_\ell) \) is determined by the formula

\[
d_L Y_K = -\frac{1}{2} \sum_{(\alpha, \beta)} [Y_\alpha, Y_\beta] \in L(I^*_\ell).
\]

For an admissible symbol \((I/J)\) with \( I = (i) \), it follows that \( d_L Y_{(i, J)} \in L(I^*_\ell) \) is a cycle of degree \( \geq 2\ell \), and we define

\[
s^{-1}u^*_I = d_L Y_{(i, J)} = \frac{1}{2} \sum_{(\alpha, \beta)} [Y_\alpha, Y_\beta]
\]

If \((I/J)\) is an arbitrary admissible symbol, we set

\[
s^{-1}u^*_I = \text{ad}(Y^*_I s^{-1}u^*_J) s^{-1}u^*_I(1/J), s > 1,
\]

where \( \text{ad}(Y) = [Y, -] \) denotes the adjoint representation. Clearly the \( Y_I \) and the \( s^{-1}u^*_I(1/J) \) are cycles in \( L(I^*_\ell) \). Their corresponding homology classes in \( H_*(L(I^*_\ell)) \) are denoted by the same symbol. Observe that the elements \( s^{-1}u^*_I(1/J) \) correspond exactly to a minimal set of relations \( c^*_I c^*_J = 0 \) for the quotient algebra \( I^*_\ell \). For \( z^*_I(1/J) \in Z^*_\ell \), denote by \( z^*_I(1/J) \) the corresponding dual basis element of \( Z^*_\ell \).

2.13. Lemma. The injective homomorphism \( \Pi_1: \Pi(I^*_\ell) \rightarrow H(L(I^*_\ell)) \) of Theorem 2.3 is induced by the homomorphism of free DG-Lie algebras \( L(Z^*_\ell) \rightarrow L(I^*_\ell) \), which is determined on the generators by \( s^{-1}z^*_I(1/J) \rightarrow s^{-1}u^*_I(1/J) \) for \((I/J)\) admissible.
It follows that the homology classes \( s^{-1}u_{(I/J)} \) generate a free subalgebra in \( l(I^*_k) \). The formulas in the following Proposition have to be understood with the convention: Whenever a symbol \( (I'/J') \) is not admissible, the term in which the symbol occurs must be replaced by \( 0 \).

2.14. PROPOSITION. Let \( (I/J) \) be admissible and \( k \leq j \) in \( \{1, \ldots, t\} \). Then the following formulas hold in \( H_*(l(I^*_k)) \):

\[
(2.15) \quad \text{ad}(Y_k)(Y_j) = s^{-1}u_{(k/j)};
\]

\[
(2.16) \quad \text{ad}(Y_k)s^{-1}u_{(I/J)} = s^{-1}u_{(i_1 \cdots i_s j | J)}, \quad \text{for } k > i_s;
\]

\[
(2.17) \quad \text{ad}(Y_k)s^{-1}u_{(I/J)} = \sum_{\beta=1}^{s} (-1)^{s-\beta} \text{ad}(Y_{i_\beta}) \cdots \text{ad}(Y_{i_{\beta+1}})[s^{-1}u_{(k/i_\beta)}] s^{-1}u_{(i_1 \cdots i_{\beta-1} j | J)}
\]

\[
+ (-1)^{s-\alpha+1} s^{-1}u_{(i_1 \cdots i_{\alpha-1} k i_{\alpha} \cdots j_j | J)}, \quad \text{for } i_{\alpha-1} < k < i_{\alpha}, 1 < \alpha < s;
\]

\[
(2.18) \quad \text{ad}(Y_k)s^{-1}u_{(I/J)} = \sum_{\beta=1}^{s} (-1)^{s-\beta} \text{ad}(Y_{i_\beta}) \cdots \text{ad}(Y_{i_{\beta+1}})[s^{-1}u_{(k/i_\beta)}] s^{-1}u_{(i_1 \cdots i_{\beta-1} j | J)}
\]

\[
+ (-1)^{s} \text{ad}(Y_i) \cdots \text{ad}(Y_{i_{2}}) \text{ad}(Y_k)s^{-1}u_{(i_j | J)} , \quad \text{for } k < i;
\]

\[
(2.19) \quad \text{ad}(Y_k)s^{-1}u_{(I/J)} = \sum_{\beta=0}^{m} s^{-1}u_{(k, j_{\beta}/j_0 \cdots j_{\beta'}, \cdots j_m)}, \quad \text{for } k < j_0 = i.
\]

Together with corresponding formulas for \( k = i_\alpha, \alpha = 1, \ldots, s \), (2.15) to (2.19) completely determine the Lie algebra extension in Theorem 2.3 and hence the structure of \( H(I^*_k) = H(l(I^*_k)) \). They also show that the subalgebra \( l(Z^*_k) \subset H(l(I^*_k)) \) is an ideal. We denote by \( D(Y_k) \) the derivation on \( l(Z^*_k) \) induced by \( \text{ad}(Y_k) \), \( k = 1, \ldots, t \). (2.15) implies

\[
(2.20) \quad [D(Y_k), D(Y_j)] = D(Y_k) \circ D(Y_j) + D(Y_j) \circ D(Y_k) = \text{ad}(s^{-1}z_{(k/j)}) \cdot D(Y_k).
\]

\( D \) therefore induces a representation \( \tilde{D} \) of the abelian Lie algebra \( l^*_k \) in \( \text{Der}(l(Z^*_k)) / \text{IntDer} \). This is the representation canonically associated to the extension (2.3).
As an example, we describe \( \Pi(I_2) \) for \( I = I(\mathfrak{gl}(2)) = \mathbb{R}[c_1, c_2] \). We have \( I_2 = [c_1, c_2]/(c_1^3, c_1 c_2, c_2^2) \). The Lie algebra \( \Pi(I_2) \cong H(L(I_2^*)) \) is generated by \( Y_j = s^{-1}c_j^*, j = 1, 2 \) and \( s^{-1}u^{(1/11)} = [Y_1, Y^{(1,1)}_1], dY^{(1,1)}_1 = [Y_1, Y_1] \). The free subalgebra \( L(Z_2^*) \subset H(L(I_2^*)) \) is generated by \( s^{-1}u^{(1/11)} \) and the elements \( s^{-1}u^{(1/2)} = [Y_1, Y_2], s^{-1}u^{(2/2)} = [Y_2, Y_2], s^{-1}u^{(12/11)} = [Y_2, [Y_1, Y^{(1,1)}_1]] \) and \( s^{-1}u^{(12/2)} = [Y_2, [Y_1, Y_2]] \). The non-zero brackets in Prop. 2.14 are given by

\[
[Y_1, s^{-1}u^{(2/2)}] = -2s^{-1}u^{(12/2)},
\]

\[
[Y_1, s^{-1}u^{(12/11)}] = [s^{-1}u^{(1/2)}, s^{-1}u^{(1/11)}],
\]

\[
[Y_1, s^{-1}u^{(12/2)}] = [s^{-1}u^{(1/2)}, s^{-1}u^{(1/2)}],
\]

\[
[Y_2, s^{-1}u^{(12/11)}] = \frac{1}{2} [s^{-1}u^{(2/2)}, s^{-1}u^{(1/11)}]
\]

and

\[
[Y_2, s^{-1}u^{(12/2)}] = \frac{1}{2} [s^{-1}u^{(2/2)}, s^{-1}u^{(1/2)}].
\]

Theorem 2.3. has the following consequence

2.21. THEOREM. The minimal algebra \( M(I_\mathbb{L}) \) appears as an extension of DG-algebras

\[
0 \longrightarrow \text{Id}(c_1, \ldots, c_\mathbb{L}) \longrightarrow M(I_\mathbb{L}) \xrightarrow{M} M(A_\mathbb{L}) \longrightarrow 0
\]

\[
\text{II} \quad \text{II} \quad \text{II} \quad \text{II}
\]

\[
0 \longrightarrow \text{Id}(c_1, \ldots, c_\mathbb{L}) \longrightarrow \mathcal{C}^*(\Pi(I_\mathbb{L})) \longrightarrow \mathcal{C}^*(L(Z_\mathbb{L}^*)) \longrightarrow 0,
\]

where \( \mathcal{C}^*(I) \equiv S^*(sI) \) denotes the cochain complex of a graded Lie algebra. As \( A_\mathbb{L} \) is bi-formal, the isomorphism \( M(A_\mathbb{L}) \cong \mathcal{C}^*(L(Z_\mathbb{L}^*)) \) preserves differentials. For \( I_\mathbb{L} \), which is only formal, the cochain differential \( d_{\mathbb{C}} \) describes only the quadratic terms of \( d_M \).

The 1-cochains \( u_{(I/J)} \in M(I_\mathbb{L}) \), dual to the basis elements \( u_{(I/J)}^* \) described earlier, are mapped to the cocycles \( z_{(I/J)} \in Z_\mathbb{L} \subset A_\mathbb{L} \). The obvious relation in \( M(I_\mathbb{L}) \)
(2.23) \[ d_M^{i/J} = c_i c_J \]

shows that \( d_M \) has non-quadratic terms (for \( c_j \) decomposable); hence \( I_\ell \) is not coformal for \( \ell > 1 \). The minimal algebras of the form \( C(L(Z_\ell^\infty)) \) have the homotopy type of a finite wedge of spheres, and their study goes back to P. J. Hilton (compare e.g. [H1], [H2]). By contrast the minimal algebra \( M(I_\ell) \) appears to be quite complicated as far as the differential is concerned. Details of these constructions will appear elsewhere [K-T4].

3. The relationship of \( h^\# \) with \( \Delta_\ast \)

Let \( X \) have a \( G \)-foliation; we assume that the \( G \)-frame bundle \( F(Q) \) of the normal bundle admits an \( H \)-reduction, where \( H \subseteq G \) is closed and \( (G,H) \) is a reductive, \( CS \)-pair [K-T2]. Let \( P \subseteq H^\ast \) be the space of primitives:
\[ P = \text{span}(Y_1, \ldots, Y_r) \]
where \( Y_j \) is the cohomology suspension of \( c_j \). Let \( P = \tilde{P} \oplus \tilde{P} \) be a Samelson decomposition. Denote the image of the transgression mapping by
\[ V = \tau P = \tilde{V} \oplus \tilde{V} \subseteq I(g), \] so that \( \text{Ideal}(\tilde{V}) = \ker(\tilde{P} : I(g) \longrightarrow I(h)) \). We denote by \( V(2\ell) \), resp. \( V(2\ell) \) the subspace generated by the elements of degree \( > 2\ell \), resp. \( \leq 2\ell \). Similarly we decompose \( P = P(2\ell) \oplus P(2\ell) \).

The complex \( A_\ell \) used in section 2 is defined by \( A_\ell = \Lambda P(2\ell) \otimes I(e/\ell) \), with differential defined by the transgression. A similar relative complex is defined by \( \hat{A}_\ell = \hat{A}(2\ell) \otimes I(e/\ell) \). The relative Weil algebra of the pair then has cohomology [K-T 3]
\[ H^\ast(W(e, h)_\ell) \cong H^\ast(\Lambda P \otimes I(e/\ell) \otimes I(h)) \]
\[ = \Lambda \hat{P}(2\ell) \otimes H^\ast(\hat{A}(2\ell)) \otimes I(e/\ell) I(h). \]

The Chern-Weil theory gives a characteristic homomorphism for \( G \)- foliations [K-T,2]
\[ \Delta : W(g, h)_\ell \longrightarrow A^\ast(F(Q)/H) \longrightarrow A^\ast(X), \]
giving a commutative diagram of minimal models

\[
\begin{array}{ccc}
M(W(g,h)_{\lambda}) & \xrightarrow{M\Delta} & M(F(Q)/H) \\
\uparrow M J & & \downarrow M G \\
M(I(G)_{\lambda}) & \xrightarrow{M h} & M(X)
\end{array}
\]

(3.1)

We deduce two results from (3.1): First, there is a relationship between \(h^\#\) and \(\Delta_{\lambda}^*\), given by [Hul]

3.2. THEOREM. The diagram

\[
\begin{array}{ccc}
\pi^* (I(G)_{\lambda}) & \xrightarrow{h^\#} & \pi^* (X) \\
\uparrow \zeta & & \downarrow \pi^* \\
H^* (\hat{A}_{\lambda}) & \xrightarrow{\Delta_{\lambda}^*} & H^* (X) \\
\downarrow & & \downarrow \Delta_{\lambda}^* \\
H^* (W(g,h)_{\lambda})
\end{array}
\]

(3.3)

naturally commutes, where \(\pi^*\) is the dual Hurewicz map and \(\zeta\) is the inclusion mapping \(\nu(I/J) \longrightarrow u(I/J)\).

This result gives a new method for showing the non-triviality of \(\Delta_{\lambda}^*\): a class which is non-zero in the image of \(h^\# \circ \zeta\) is mapped to a non-zero class by \(\Delta_{\lambda}^*\). Conversely, the non-triviality of \(\Delta_{\lambda}^*\) for a given \(X\) can be used to show \(h^\#\) is non-trivial, if the map \(\pi^*\) is known. Section 4 will indicate what can be shown using these techniques.

Let \(FT^q_G\) be the classifying space of trivialized, \(G\)-foliated micro-bundles; let \(K \subseteq G\) be a maximal compact group. By the functoriality of \(h^\#\) and \(\Delta_{\lambda}^*\), dualizing (3.1) gives
3.4. THEOREM. Let $f: X \rightarrow \text{BR}_G^q$ classify a $G$-foliation on $X$. In a natural way, there are defined maps so that the diagram commutes:

\[
\begin{array}{ccc}
\Pi(\text{FT}_G^q) & \xrightarrow{\tilde{\Delta}_\#} & \Pi(\mathfrak{w}(g, e, G)) \\
\downarrow & & \downarrow \\
\Pi(\text{BT}_G^q) & \xrightarrow{\tilde{\Delta}_\#} & \Pi(\mathfrak{w}(g, k, G)) \\
\downarrow & & \downarrow \\
f_\# & \xleftarrow{\tilde{h}_\#} & j_\# \\
\Pi(X) & \xrightarrow{\tilde{h}_\#} & \Pi(\text{I}(G, G)) \\
\end{array}
\]

(3.5)

The cokernel of $j_\#$ is $\tilde{P}^*$; (3.5) forces $\tilde{h}_\#$ to have cokernel $\supset \tilde{P}^*$. The obvious question is whether equality holds: Does image $\tilde{h}_\#$ = image $j_\#$?

4. The homotopy of $\text{BR}_G^q$

For the three standard types of $G$-foliation, we indicate the extent to which $\tilde{h}_\#$ is known.

Let $G = G\ell(q, \mathbb{R})$. Mather and Thurston [T] have shown that $\nu: \text{BR}_G^q \rightarrow \text{BO}(q)$ is $(q + 2)$ connected. Therefore

(4.1) $\tilde{h}_\#$ maps onto $Y_{2j}$ for $4j \leq q + 2$.

(4.2) $\nu_*(\text{BR}_G^q) \otimes Q \rightarrow H_*(\text{BR}_G^q; Q)$ is an isomorphism (resp. onto) for $m \leq 2q + 2$ (resp. $m = 2q + 3$).

By (4.1) we see that $\tilde{h}_\#$ is onto $s_{(2m, 2m)}^1$, for $q = 4m - 2 > 3$ or $q = 4m - 1$. Theorem 3.2 implies $\Delta_*(Y_{2m}^m c_{2m}) \neq 0$ in $H^{6m-1}(\text{FT}_G^q)$. This is a rigid class for $q$ even. Other Whitehead products are similarly in the image of $\tilde{h}_\#$ [Hu1].

Many more results follow from the Theorems of Heitsch [He] or Fuks [F] on the variability of the classes in the image of $\Delta_\#$. 
Using (4.2) we conclude there is a surjection of \( \pi_{2q+1}(B\Gamma^q) \longrightarrow \mathbb{R}^d \), for some \( d > 0 \). For example, by Fuks we have \( \pi_{2q}(B\Gamma^q) \longrightarrow \pi_{2q}(I_q) \) is onto.

The homotopy of \( B\Gamma^q \) therefore maps onto a rather large Lie subalgebra of \( \Pi(I_q) \).

The \( 2q \) connectivity of \( \nu \) would imply \( \tilde{\mathfrak{h}}_\# \) is almost onto, within the restrictions of (3.5).

When \( G = \text{GL}(n,\mathbb{C}) \), the classes \( Y_1, \ldots, Y_s \) are in the image of \( \tilde{\mathfrak{h}}_\#: \Pi(B\Gamma^q_n) \longrightarrow \Pi(I_n) \), where \( s = \lfloor \sqrt{n} \rfloor \). Coupled with the Theorem of Baum and Bott [B-B], this shows \( \Pi(B\Gamma^q_n) \longrightarrow \Pi(I_n) \) is onto a much larger subalgebra than originally considered in [H3]. Further details are in [Hul].

When \( G = \text{SO}(q) \) the map \( \tilde{\mathfrak{h}}_\#: \Pi(B\Gamma^q) \longrightarrow \Pi(I_q') \), \( q' = [q/2] \), is onto, and complete variation occurs [Hu2]. In this case the Lie algebra \( \Pi(I_q') \) is injected into \( \Pi(B\Gamma^q) \). The variability of the classes implies there are uncountably many distinct ways of choosing a section \( \Pi(I_q) \longrightarrow \Pi(B\Gamma^q) \).
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