On the Homotopy and Cohomology of the Classifying Space of Riemannian Foliations

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CLASSIFYING SPACE
OF RIEMANNIAN FOLIATIONS

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ABSTRACT. Let $G$ be a closed subgroup of the general linear group. Let $BT_G$ be the
classifying space for $G$-foliated microbundles of rank $q$. (The $G$-foliation is not
assumed to be integrable.) The homotopy fiber $F\Gamma Y$ of the classifying map $\nu$:
$BT_G \to BG$ is shown to be $(q - 1)$-connected. For the orthogonal group, this
implies $FR\Gamma Y$ is $(q - 1)$-connected. The indecomposable classes in $H^*(RW_\alpha)$
therefore are mapped to linearly independent classes in $H^*(FR\Gamma Y)$; the indecomposable
variable classes are mapped to independently variable classes. Related
results on the homotopy groups $\pi_*(FR\Gamma Y)$ also follow.

1. The main theorem. Let $BR\Gamma Y$ be the Haefliger classifying space for Riemann-
nian foliations, $BO(q)$ the classifying space for $O(q)$-bundles and $\nu$:
$BR\Gamma Y \to BO(q)$ the map classifying the normal bundle of the universal $R\Gamma Y$-structure on
$BR\Gamma Y$ [3]. Let $FRI Y$ be the homotopy theoretic fiber of $\nu$. $H^*(\cdot)$ will denote singular
cohomology with real coefficients. In this note we show

THEOREM 1.1. $FR\Gamma Y$ is $(q - 1)$-connected.

This implies there is a section of $\nu$ over the $q$-skeleton of $BO(q)$, so $\nu^*$:
$H^q(BO(q)) \to H^q(BR\Gamma Y)$ is injective. On the other hand, the vanishing Theorem of
J. Pasternack [9] implies $\nu^*$: $H^{q+1}(BO(q)) \to H^{q+1}(BR\Gamma Y)$ is the zero map. The-
orem 1.1 is therefore the best result possible for $q = 4k + 3$. For other $q$, it would be
interesting to know whether $FR\Gamma Y$ has higher connectivity.

Theorem 1.1 is a special case of a more general result. Let $G \subseteq GL(q, \mathbb{R})$ be a
closed subgroup. A foliation on a manifold $M$ is said to be a $G$-foliation [1], [7] if
there is given

(i) a model manifold $B$ of dimension $q$ with a $G$-structure on $TB$,

(ii) an open covering $\{U_a\}$ of $M$ and local submersions $\phi_a$: $U_a \to B$ defining the
foliation such that the transition functions $\gamma_{ab}$ are local $G$-morphisms of $B$.

A $G$-foliation is integrable if it is modeled on $\mathbb{R}^q$ with the flat $G$-structure.

A classifying space for $G$-foliations is constructed as follows: Let $\mathfrak{G}(G, \mathbb{R}^q)$
denote the total space of the sheaf of local $C^\infty$-sections of the bundle $\mathbb{R}^q \times
GL(q, \mathbb{R})/G \to \mathbb{R}^q$. This is a (non-Hausdorff) $C^\infty$-manifold, with a canonical $G$-
structure. Let $\mathfrak{G}_G$ be the pseudogroup of all local, $C^\infty$, $G$-diffeomorphisms of

Received by the editors October 18, 1979 and, in revised form, February 22, 1980.
Key words and phrases. Classifying spaces, foliations, characteristic classes, minimal models.
1This paper is part of the author's doctoral thesis. The author would like to thank his advisor, Franz
Kamber, for his guidance and encouragement.
Let $\mathcal{R}(G, \mathbb{R}^q)$ and let $\tilde{G}_G^q$ be its associated topological groupoid [1, §2], [3]. Let $B\tilde{G}_G^q$ be the Haefliger classifying space of $\tilde{G}_G^q$-structures. For $G = O(q)$, we have $BR\Gamma^q = B\tilde{G}_G^q$ and, in general, $B\tilde{G}_G^q$ is the classifying space of $G$-fibrations.

There is a natural map $\nu: B\tilde{G}_G^q \to BG$ classifying the normal bundle of the $\tilde{G}_G^q$-structure on $B\tilde{G}_G^q$. Let $F\tilde{G}_G^q$ be the homotopy theoretic fiber of $\nu$.

**Theorem 1.1**. $F\tilde{G}_G^q$ is $(q - 1)$-connected.

For $G = O(q)$ we recover Theorem 1.1.

There is also a classifying space for integrable $G$-fibrations, denoted by $B\Gamma^q_G$. We let $F\Gamma^q_G$ denote the homotopy theoretic fiber of $\nu: B\Gamma^q_G \to BG$. When $G = SL(q, \mathbb{R})$, one can show $B\tilde{G}_G^q \cong B\Gamma^q_G$ [1, Remark 4.2], recovering from Theorem 1.1' Haefliger’s result that $F\Gamma^q_G$ is $(q - 1)$-connected.

2. Applications. In this section, we give some consequences of Theorem 1.1'. The proofs of the propositions stated use Sullivan's theory of minimal models [10], and are given in [5].

Theorem 1.1 implies there are many nontrivial Whitehead products in $\pi_\ast(B\Gamma^q)$ and that many of the secondary characteristic classes map injectively into $H^\ast(F\Gamma^q)$, To be precise, let $q' = [q/2]$ and $W(\mathfrak{so}(q))_{q'}$ denote the truncated Weil algebra for the orthogonal Lie algebra $\mathfrak{so}(q)$ [7]. The Chern-Weil construction gives a characteristic map $\Delta_\ast: H^\ast(W(\mathfrak{so}(q))_{q'}) \to H^\ast(F\Gamma^q)$. Let $k = [q/4] + 1$ and $m = [(q - 1)/2]$. The set of invariants factors as

$$H^\ast(W(\mathfrak{so}(q))_{q'}) \cong A \otimes \Lambda(y_k, \ldots, y_m),$$

where $A$ is an algebra with all products zero and the second factor is the exterior algebra on generators $y_j$ of degree $4j - 1$. The algebra $A$ has an explicit basis, given by $1$ and the cocycles $y_jp_j \in W(\mathfrak{so}(q))_{q'}$ where

$$y_{i_1} \cdots y_{i_k}p_{j_1} \cdots p_{j_{k-1}}$$

$1 < i_1 < \cdots < i_k < k$ and $l < i_j \iff j_l = 0$, and $\deg p_{i_j}p_j > q$, $\deg p_j < q$.

For $q$ even, additional cocycles involving the Euler class must be added to this list [8]. A basis element $y_jp_j \in A$ is said to be variable if $\deg y_jp_j = 2q' + 1$.

Let $V \subseteq H^\ast(W(\mathfrak{so}(q))_{q'})$ be the subspace given by the direct sum

$$V = A \otimes 1 \oplus 1 \otimes \Lambda(y_k, \ldots, y_m).$$

**Proposition 2.1.** $\Delta_\ast: V \to H^\ast(F\Gamma^q)$ is injective, and the variable basis elements in $V$ are mapped to independently variable classes in $H^\ast(F\Gamma^q)$.

The first statement follows from Theorem 1.1 and the results of F. Kamber and Ph. Tonendeur [6, Theorem 6.52]. The variability follows from the examples of C. Lazarov and J. Pasternack [8, Theorem 3.6] combined with Theorem 1.1. Details can be found in [5].

Similar results concerning the homotopy of $F\Gamma^q$ can be shown. Set $\pi^\ast(F\Gamma^q) = \text{Hom}(\pi_\ast(F\Gamma^q), \mathbb{R})$. Let $\langle y_k, \ldots, y_m \rangle$ denote the real vector space spanned by $\{y_k, \ldots, y_m\}$. In [5], a vector-space map $h^\ast: H^\ast(W(\mathfrak{so}(q))_{q'}) \to \pi^\ast(F\Gamma^q)$ is defined, for which
PROPOSITION 2.2. $h^g \circ \xi : A \oplus \langle y_k, \ldots, y_m \rangle \to \pi^*(FRI_M)$ is injective and the variable basis elements of $A$ are mapped to independently variable classes.

For any commutative cochain algebra $\mathcal{C}$ there is a vector space $\pi^*(\mathcal{C})$, the dual homotopy of $\mathcal{C}$, constructed by choosing a minimal model $\mathfrak{M} \to \mathcal{C}$, and setting $\pi^*(\mathcal{C}) = \mathfrak{M}^* / (\mathfrak{M}_+^{\mathcal{C}} \cdot \mathfrak{M}_+^{\mathcal{C}})$ [10]. The algebra $A$ has trivial products and differential, so for $q = 4, 6$ or $\geq 8$ the vector space $\pi^*(A)$ is of finite type but not finite dimensional. There is induced a map $\Delta^q : \pi^*(A) \to \pi^*(FRI_M)$, extending $h^g \circ \xi$, for which we have [5]

PROPOSITION 2.3. $\Delta^q : \pi^*(A) \oplus \langle y_k, \ldots, y_m \rangle \to \pi^*(FRI_M)$ is injective and the variable classes are mapped to independently variable classes.

The following proposition gives our final remark on the homotopy of $FRI_M$. The proof is obvious, using minimal models.

PROPOSITION 2.4. Let $X$ be an $n$-connected space, $n \geq 1$. Then the rational Hurewicz map $\mathcal{C} : \pi_m(X) \otimes \mathbb{Q} \to H_m(X; \mathbb{Q})$ is an isomorphism for $m < 2n$ and an epimorphism for $m = 2n + 1$.

COROLLARY 2.5. $\mathcal{C} : \pi_m(FRI_M) \otimes \mathbb{Q} \to H_m(FRI_M; \mathbb{Q})$ is an isomorphism for $m < 2q - 2$ and an epimorphism for $m = 2q - 1$.

3. Proof of Theorem 1.1'. Let $X \subseteq \mathbb{R}^q$ be an open subset homotopic to $S^n$. When $n$ is zero, we consider $S^0$ to consist of a single point. Then $\pi_n(FRI_M) \cong [X, FRI_M]$, the set of homotopy classes of maps $f : X \to FRI_M$. By the Gromov-Phillips-Haefliger Theorem [2], there is a bijection between $[X, FRI_M]$ and the set of integrable homotopy classes of $G$-foliations on $X$ with trivial $G$-structure. We will show two such foliations on $X$ are integrably homotopic.

Recall that two codimension $q$ $G$-foliations $\mathcal{F}_0$, $\mathcal{F}_1$ on $X$ are integrably homotopic if there is a codimension $q$ $G$-foliation $\mathcal{F}$ on $X \times [0, 1]$ such that the slices $i_t : X \times \{t\} \to X \times [0, 1]$ are transverse to $\mathcal{F}$ for all $t$, and induce $\mathcal{F}_0$ for $t = 0, 1$.

Fix an integer $n$ with $0 < n < q$. Let $(\theta, r) \in \mathbb{R}^{n+1}$ be polar coordinates, with $\theta \in S^n$ and $r \in \mathbb{R}$. For any $a, b \in \mathbb{R}$ with $0 < a < b$, define

$$B(a, b) = \{(\theta, r) \in \mathbb{R}^{n+1} | a < r < b\} \times \mathbb{R}^{q-n-1}.$$  

Set $X = B(0, 1)$; then $X \subseteq \mathbb{R}^q$ is open and homotopic to $S^n$.

A codimension $q$ $G$-foliation on $X$ must be the point foliation with a $G$-structure on the tangent bundle $TX$. The tangent bundle is trivial, so the $G$-structure is characterized by a smooth map $\alpha : X \to Y$, where $Y$ is the coset space $Gl(q, \mathbb{R})/G$. We denote by $(X, \alpha)$ the $G$-foliation on $X$ with characteristic map $\alpha$. The $G$-structure on $(X, \alpha)$ is trivial if $\alpha$ is homotopic to the constant map with image the identity coset of $Y$. For two $G$-foliations $(X, \alpha_0)$ and $(X, \alpha_1)$ with trivial $G$-structures, it is apparent that $\alpha_0$ and $\alpha_1$ are homotopic.

To prove the theorem, it will suffice to show that if $\alpha_0$ and $\alpha_1$ are homotopic, then there is an integrable homotopy from $(X, \alpha_0)$ to $(X, \alpha_1)$. To do this, we will construct three integrable homotopies, on $X \times [0, 1]$, $X \times [1, 2]$ and $X \times [2, 3]$ which combine to give the desired integrable homotopy.
**Step 1.** Choose a monotone, $C^\infty$-function

\[ \phi : [0, 1] \to [1/2, 1] \quad \text{with} \quad \phi(t) = \begin{cases} 1 & \text{for } t < 1/4, \\ 1/2 & \text{for } t \geq 3/4. \end{cases} \]

Define $H : X \times [0, 1] \to X$ by

\[ H_t(\theta, r, v) = (\theta, \phi(t) \cdot (r - 1/2) + 1/2, v). \]

For each $t$, $H_t$ is a submersion; $H_0$ is the identity and $H_1$ maps $X$ to a subannulus of $X$. Also, $H_t$ is constant with respect to $t$ for $t$ near 0 or 1.

Define a $G$-structure on $X$ by $\alpha_0 = \alpha_0 \circ H_1^*: X \times \{1\} \to Y$. Then the submersion $H : X \times I \to (X, \alpha_0)$ defines a $G$-foliation on $X \times [0, 1]$ which is an integrable homotopy from $(X, \alpha_0)$ to $(X, \alpha_0')$.

**Step 2.** Define $H'' : X \times [2, 3] \to X$ by $H''_t = H_{3-t}$. Define a $G$-structure on $X$ by setting $\alpha'_1 = \alpha_1 \circ H''_2$. Then the submersion $H'' : X \times [2, 3] \to (X, \alpha_1)$ defines a $G$-foliation which is an integrable homotopy from $(X, \alpha'_1)$ to $(X, \alpha_1)$.

**Step 3.** We next produce an integrable homotopy from $(X, \alpha_0')$ to $(X, \alpha_2)$ by constructing a $G$-foliation $(X, \alpha)$ and a submersion $H' : X \times [1, 2] \to (X, \alpha)$ so that $\alpha' = \alpha \circ H'_1$ and $\alpha'_1 = \alpha \circ H'_2$.

Define functions $f_0$ and $f_1$ as follows

\[
\begin{align*}
 f_0 : B(5/8, 1) &\to B(0, 3/4) \quad \text{by} \quad f_0(\theta, r, v) = (\theta, 2r - 5/4, v), \\
 f_1 : B(0, 3/8) &\to B(1/4, 1) \quad \text{by} \quad f_1(\theta, r, v) = (\theta, 2r + 1/4, v).
\end{align*}
\]

Note that $f_0$ maps $B(3/4, 1)$ to the image of $H_1$ and $f_1$ maps $B(0, 1/4)$ to the image of $H''_2$.

There are inclusions

\[
\begin{align*}
i_0 : S^n \times \{3/4\} \times \mathbb{R}^{n-q} &\subseteq B(5/8, 1), \\
i_1 : S^n \times \{1/4\} \times \mathbb{R}^{n-q} &\subseteq B(3/8)
\end{align*}
\]

and the composites $\alpha_0 \circ f_0 \circ i_0$ and $\alpha_1 \circ f_1 \circ i_1$ are homotopic by assumption. Therefore, there exists a smooth extension

\[
\tilde{\alpha} : S^n \times [1/4, 3/4] \times \mathbb{R}^{n-q} \to B(1/4, 3/4) \to Y
\]

of $\alpha_0 \circ f_0 \circ i_0 \cup \alpha_1 \circ f_1 \circ i_1$. We define a smooth map $\alpha : X \to Y$ by

\[
\alpha = \begin{cases}
\alpha_0 \circ f_0 & \text{on } B(3/4, 1), \\
\tilde{\alpha} & \text{on } B(1/4, 3/4), \\
\alpha_1 \circ f_1 & \text{on } B(0, 1/4).
\end{cases}
\]

Finally, we construct the submersion $H' : X \times [1, 2] \to X$. Choose a monotone, $C^\infty$-function $\varphi : [1, 2] \to [0, 3]$ with

\[
\varphi(t) = \begin{cases}
3 & \text{for } t < 5/4, \\
0 & \text{for } t > 7/4.
\end{cases}
\]

Then $H'$ at time $t$ is given by

\[
H'_t(\theta, r, v) = (\theta, 1/4(r + \varphi(t)), v).
\]

The map $H'$ has the effect of sliding the image of $X$ from image $H_1$ to image $H''_2$ as $t$ varies from 1 to 2.
Let $X \times [1, 2]$ have the $G$-structure defined by the submersion $H': X \times [1, 2] \to (X, \alpha)$. This gives an integrable homotopy from $(X, \alpha \circ H_1')$ to $(X, \alpha \circ H_2')$. A straightforward check shows that $f_0 \circ H_1' = H_1$ and $f_1 \circ H_2' = H_2'$. This implies $\alpha_0' = \alpha \circ H_1'$ and $\alpha_1' = \alpha \circ H_2'$, which finishes Step 3 and the proof of Theorem 1.1'.

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