GROWTH OF LEAVES AND SECONDARY INVARIANTS OF FOLIATIONS

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The secondary classes of a foliation are global differential invariants which are obstructions to it being concordant to a foliation with simpler structure: for example, to one with trivial secondary classes. An intringuing unsolved problem is to find an interpretation of the secondary classes in terms of the global geometry of the given foliation. In this note, we relate the differential invariants of a codimension q, C^2-foliation with the growth type of its leaves. Our work suggests that if the leaves with exponential growth are sparse in the sense of Lebegue measure, then the secondary classes must vanish.

Given a compact riemannian manifold (M,g) and immersed submanifold $L \subseteq M$, the growth type of L in the relativized metric is well defined and independent of g. (See [6].). The growth function of L at a point $x \in L$ is

$$gr_L(r,x) = \text{volume } D(x,r)$$

where $D(x,r)$ is the ball of radius r in L centered at x. If there is a polynomial $p(r)$ with $gr_L(r,x) \leq p(r)\ \forall\, r \geq 0$, then L has polynomial growth. If $\liminf\limits_{r \to \infty} \frac{1}{r} \log(gr_L(r,x)) = 0$ then L has subexponential growth. Otherwise, we say L has exponential growth. A foliation has subexponential growth if every leaf has subexponential growth. It was conjectured by Sullivan that the Godbillon-Vey class, $gv(\mathcal{F})$, of a codimension one, C^2-foliation $\mathcal{F}$ with subexponential growth on a compact manifold must vanish (p. 247, [9]).

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A C^2-foliation $\mathcal{F}$ which is almost without holonomy on a compact manifold M has subexponential growth and $gv(\mathcal{F}) = 0$ [5]. For a C^2-foliation $\mathcal{F}$ with subexponential growth which is transverse to a fibration of M, it is proven in [2] that $gv(\mathcal{F}) = 0$. Thus, Sullivan's conjecture in codimension one is very close to being established [10]. Our purpose is to consider the case of foliations with codimension greater than one.

For higher codimensions, the secondary classes of $\mathcal{F}$ are given by a map [1,4] $\Delta_* : H^*(WO_q) \longrightarrow H^*(M)$. Also, for each leaf $L \subseteq M$, there are leaf classes [8] defined for the flat normal bundle to the leaf, and given by $\chi_L^* : H^*(g\ell_q, O_q) \longrightarrow H^*(L)$.

THEOREM 1. <u>Let $\mathcal{F}$ be a codimension q, transversely analytic foliation on a compact manifold M. Assume that M fibers smoothly as $F^q \to M \xrightarrow{\pi} B$ and $\mathcal{F}$ is transverse to the fibers of π.</u>

(a) <u>If the leaves of $\mathcal{F}$ with growth of degree $\geq m+1$ have measure zero, then all secondary classes for $\mathcal{F}$ of degree $> q+m$ vanish.</u>

(b) <u>If the leaves of $\mathcal{F}$ with exponential growth have measure zero, then all leaf classes of $\mathcal{F}$ vanish, except possibly for the generator</u> $y_1 \in H^1(g\ell_q, O_q)$.

This theorem gives the first known restrictions on the behavior of the leaves in an analytic foliation of codimension ≥ 2 with non-trivial secondary classes.

The assumption of analyticity in Theorem 1 can be replaced with a transverse regularity condition; e.g., if $\mathcal{F}$ is a hyperbolic, C^2-foliation, then the theorem is valid [3].

Given $\mathcal{F}$ as in Theorem 1, it is well known that the foliation determines a map $h : \pi_1(B) \longrightarrow \mathrm{Diff}^{\omega}(F) \subseteq \mathrm{Diff}^2(F)$. The image Γ of h is called the total holonomy of $\mathcal{F}$. If the action of Γ on F is sufficiently regular, then the growth conditions can be relaxed:

THEOREM 2. Let $\mathcal{F}$, M , Γ be as above. Suppose Γ contains a subgroup of finite index $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \subseteq G \subseteq \mathrm{Diff}^2 F$, where G is a finite-dimensional, connected Lie group. If the leaves of $\mathcal{F}$ with exponential growth have measure zero then all of the secondary classes of $\mathcal{F}$ vanish, except possibly $gv(\mathcal{F})$.

The foliations considered in Theorem 2 are the prototypes for the construction of examples with non-trivial secondary classes. Thus, all such examples must have a dense set of leaves with exponential growth.

A foliation has uniformly bounded growth if there is a function $f(r)$ such that for each leaf L , the growth function satisfies $gr_L(r,x) \leq f(r)$.

THEOREM 3. Let $\mathcal{F}$ be a C^2-foliation of a (possibly non-compact) manifold.

(a) If $\mathcal{F}$ is a riemannian foliation with subexponential growth, then all leaf classes vanish.

(b) If $\mathcal{F}$ has uniformly bounded subexponential growth, then all leaf classes vanish, except possibly y_1 .

For a foliation with all leaves compact (growth of degree 0), the first leaf class y_1 always vanishes [3]. The class y_1 can be non-trivial for a foliation on the torus (growth of degree 1). If a higher leaf class is non-zero, then the linear holonomy of the leaf must have exponential growth, and one conjectures that some leaf must have exponential growth as well. Theorem 4 points to a global geometric interpretation of the leaf classes of a foliation.

SKETCH OF PROOFS OF THEOREMS

The transverse analyticity assumption in Theorems 1 and 2 implies that the growth of the total holonomy Γ is dominated by that of the leaves. If $\mathcal{F}$ has subexponential growth, then Γ does, and the linear holonomy of each leaf must also. Thus, the linear holonomy of each leaf is almost nilpotent. A flat bundle with nilpotent structure group has trivial Euler class (as is well known), and the other $H^*(g\ell_q, O_q)$-classes also vanish (except y_1). This proves 1b. If Γ has polynomial growth of degree at most m, then $H^n(\Gamma; \mathbb{R}) = 0$ for $n > m$. Using standard classifying space techniques, the secondary classes of $\mathcal{F}$ in degrees $> q+m$ are seen to vanish, proving 1a.

In Theorem 2, by passing to a finite covering we can assume that $\Gamma \subseteq G$ is contained in a connected, solvable subgroup N. By somewhat standard techniques we are reduced to considering a foliation of the type $N \times_{\Gamma} F$. This is diffeomorphic to the foliation on $N/\Gamma \times F$ with leaves given by the diagonal action of N. Such a foliation has a reduction to a solvable transverse structure group, and the secondary classes consequently vanish.

The proof of Theorem 3 also depends on showing the linear holonomy of each leaf is almost nilpotent. The details of the proofs of the above results will appear in [3].

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