THE GODBILLON-VEY CLASS FOR ANALYTIC FOLIATIONS OF CODIMENSION ONE

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ABSTRACT

We prove that if a transversely analytic foliation of codimension one on a compact manifold has non-vanishing Godbillon-Vey class, then there is an open saturated set of leaves with exponential growth.

§1. MAIN THEOREM

The geometry of codimension one, C^2-foliations on a compact manifold has been greatly clarified by the recent works of many people. A notion of particular significance in these works is the growth of the leaves of a foliation. Growth also has relevance to the problem of relating the Godbillon-Vey class $GV(F) \in H^3(M)$ to the global geometry of a foliation F on M. Several important results have been obtained on this problem: For a C^2-foliation of codimension one which is almost without holonomy, $GV(F) = 0$, [3,8]. For a C^2-foliation transverse to a circle fibration of M and having all leaves of subexponential growth, $GV(F) = 0$, [4].

Our aim in this note is to show:

THEOREM 1. *Let F be a transversely analytic foliation of codimension one on a compact manifold* M. *If* GV(F) *does not vanish, then there is a non-empty, saturated open subset of* M *consisting of leaves with exponential growth.*

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This theorem confirms the analytic case of a conjecture made by Sullivan [p. 247; 13] that if all leaves of a C^2-foliation F have subexponential growth, then $GV(F) = 0$.

For transversely analytic foliations of higher codimension which are transverse to a fibration $F \longrightarrow M \longrightarrow B$, a generalization of the above theorem is given in [6].

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§2. SOME BASIC NOTIONS

Let F be a codimension one foliation on a compact riemannian manifold M. Let $L \subseteq M$ be the immersion of a leaf of F in M. The metric on M induces a metric on L, and we denote the corresponding distance function on L by $d(\, , \,)$. The <u>growth</u> of the leaf L (with respect to the metric on M and a point $x \in L$) is the function

$$g_L(r) = \text{vol } D(r,x)$$

where $D(r,x) = \{y \in L \mid d(y,x) \leq r\}$ and volume is calculated in the induced metric.

Given two functions $f(r)$, $g(r)$ we say f dominates g if there are positive constants A,B,C such that

$$f(r) \geq A \cdot g(Br+C) \quad \text{for all } r \geq 0.$$

Dominance is denoted by $f \succeq g$, and forms an equivalence relation. For a leaf L of F, the dominance class of $g_L(r)$ is independent of the choice of basepoint $x \in L$ and the metric on M, [7,10]. The <u>growth class</u> of L is the class of the function $g_L(r)$. Thus, for a compact manifold M the growth class of a leaf is a geometric invariant. We say a leaf L has <u>exponential growth</u> if $g_L(r) \geq \exp(r)$, and <u>polynomial growth</u> if for some n, $g_L(r) \leq r^n$.

Given a finitely generated group G, there is also a notion of the growth of G. Let Γ be a finite generating set of G which is symmetric: if $g \in \Gamma$ then $g^{-1} \in \Gamma$, and $e \in \Gamma$. Set $\Gamma^1 = \Gamma$, and inductively define $\Gamma^n = \Gamma \cdot \Gamma^{n-1}$. For a set S, let $|S|$ denote the cardinality of S. Then define a function

$$g(n) = |\Gamma^n| \quad .$$

The dominance class of $g(n)$ is independent of the choice of Γ, so we have a well-defined growth class associated to the group G. For more details, see Milnor [7], Bass [1] or Plante [10].

Another geometric idea needed is the holonomy group of a leaf [5], [12]. For a closed path $\gamma\colon [0,1] \longrightarrow L$ and a transverse embedding $\varepsilon\colon (\mathbb{R},0) \longrightarrow (M,L)$ with $\varepsilon(0) = \gamma(0) = \gamma(1) = x$, the foliation in a neighborhood of L determines a germ of a diffeomorphism $h_\gamma\colon (\mathbb{R},0) \longrightarrow (\mathbb{R},0)$. Then h_γ depends only on the homotopy class of γ, giving a map $h\colon \pi_1(L,x) \longrightarrow G^2(\mathbb{R},0)$. Here, $G^r(\mathbb{R},0)$ denotes the group of germs of C^r-diffeomorphisms. The image of h depends up to conjugation on the choice of transversal ε; for $\mathcal{F}$ transversely analytic and ε chosen to be transversely analytic, we then have $G_1 = \text{image } h \subseteq G^\omega(\mathbb{R},0)$.

The idea of the proof of Theorem 1 is to relate the growth of the leaves of $\mathcal{F}$ with the growth of the holonomy groups of the leaves. This is not usually possible, but for transversely analytic foliations of codimension one, we show that polynomial growth of the leaves implies polynomial growth of the holonomy groups.

§3. A REDUCTION OF THE THEOREM

In the following, $\mathcal{F}$ will denote a codimension one transversely analtyic foliation on a compact manifold M. It was shown by Tsuchiya [Theorem 5; 14] that the leaves of polynomial growth of $\mathcal{F}$ form a closed set in M. Further, a leaf which does not have polynomial growth must have exponential growth [Theorem 6.13; 2]. Therefore, if $\mathcal{F}$ does not have an open, saturated set of leaves with exponential growth, then all leaves have polynomial growth. To prove Theorem 1, it suffices to show $GV(\mathcal{F}) = 0$ if all leaves of $\mathcal{F}$ have polynomial growth. We need:

PROPOSITION 1. *Let $\mathcal{F}$ be a transversely analytic foliation of codimension one which is transversely oriented. Let $L \subseteq M$ be a leaf of $\mathcal{F}$. If the set of leaves with exponential growth has measure zero in a neighborhood of L, then the holonomy group G_L of L is abelian.*

Theorem 1 is a consequence of this proposition. We can assume $\mathcal{F}$ is transversely oriented by passing to a double covering if necessary. If there is no non-empty open set of leaves with exponential growth, then all leaves have polynomial growth and, by Proposition 1, abelian holonomy. We now use a theorem of Tsuchiya [15,9] that if $\mathcal{F}$ is a C^2-foliation of a compact manifold such that all leaves have polynomial growth and abelian holonomy then $GV(\mathcal{F}) = 0$.

§4. PROOF OF PROPOSITION 1.

Fix a leaf $L \subseteq M$ and basepoint $x \in L$. Set G = image $h: \pi_1(L,x) \longrightarrow G^{\omega}(\mathbb{R},0)$. For $\overline{f},\overline{g} \in G$ with $\overline{f} \neq \overline{e} \neq \overline{g}$, let $\langle \overline{f},\overline{g} \rangle$ denote the subgroup they generate. We will show $\langle \overline{f},\overline{g} \rangle$ is abelian. By Theorem 5.2 of Plante-Thurston [11], if $\langle \overline{f},\overline{g} \rangle$ has subexponential growth, then $\langle \overline{f},\overline{g} \rangle$ is abelian. We will show that if $\langle \overline{f},\overline{g} \rangle$ has exponential growth, then in a neighborhood of L almost all leaves have exponential growth. This will establish the proposition and the theorem.

Let f,g be diffeomorphisms representing the germs $\bar{f},\bar{g}$. Replacing f,g by their inverses if necessary, we can assume that for some $\delta > 0$, $f,g\colon [0,\delta] \longrightarrow [0,\delta]$. The analyticity implies we can write

$$f(x) = a_1 x + a_m x^m + \ldots$$

$$g(x) = b_1 x + b_n x^n + \ldots$$

where $0 < a_1, b_1 \leq 1$ and if $a_1 = 1$ then $a_m < 0$; if $b_1 = 1$ then $b_n < 0$. Further, we can assume $a_1 \leq b_1$, and if $a_1 = b_1 = 1$ then $m \leq n$, and if $m = n$ then $a_m \leq b_m$. Finally, if equality holds above ($a_1 = b_1 < 1$, or if $a_1 = b_1 = 1$, $m = n$ and $a_m = b_m$) then set $\lambda = f^2$; otherwise, set $\lambda = f$.

Set $\Gamma = \{f,g,f^{-1},g^{-1},e\}$, and define inductively $\Gamma^n = \Gamma\cdot\Gamma^{n-1}$, where $\Gamma^1 = \Gamma$. By assumption, $\langle f,g\rangle$ has exponential growth, so there is a constant $c > 1$ such that $|\Gamma^n| \geq c^n$ for all $n \geq 1$.

We will show there exists an $\varepsilon > 0$ such that the set $\Gamma^n\,\lambda^n$ acts a.e. effectively on [0,1]. More precisely, there is a countable set K such that for all $x \in [0,\varepsilon]\backslash K$, $|\Gamma^n\circ\lambda^n(x)| = |\Gamma^n\circ\lambda^n|$. Then for such x, we have

$$\begin{aligned} |\Gamma^{3n}(x)| &\geq |\Gamma^n\circ\lambda^n(x)| \\ &= |\Gamma^n\circ\lambda^n| \\ &= |\Gamma^n| \\ &\geq c^n . \end{aligned}$$

The orbit of x under the holonomy group G of L thus has exponential growth, and it is standard that the leaf through x has exponential growth. It follows that at most a countable number of leaves in a neighborhood of L do not have exponential growth.

It remains to exhibit ε and the set K. The choice of λ was made so that for x near 0, $f\circ\lambda$, $f^{-1}\circ\lambda$, $g\circ\lambda$ and $g^{-1}\circ\lambda$ are all decreasing functions. Let ε be a number such that $0 < \varepsilon < \delta$ and for all x with $0 \leq x \leq \varepsilon$,

$$\lambda(x) \leq f(x),\ g(x) \leq x$$

and

$$[0,\varepsilon] \subseteq \text{Domain } (\lambda^{-1}) \cap \text{Domain } (g^{-1}).$$

Consequently, for all $0 \leq y \leq \varepsilon$,

$$\lambda^{-1}(y) \geq f^{-1}(y), g^{-1}(y) \geq y.$$

We claim now that for all $n \geq 1$ and $h \in \Gamma^n$, the function $h\circ\lambda^n$ is defined on $[0,\varepsilon]$. More precisely, $h\circ\lambda^n$ is actually decreasing on $[0,\varepsilon]$. For $n = 1$, the choice of ε implies that $h\circ\lambda$ is defined and decreasing on $[0,\varepsilon]$. Now let $h \in \Gamma^n$ and write $h = h_1\circ h_2$ where $h_1 \in \Gamma$ and $h_2 \in \Gamma^{n-1}$. Inductively, assume $h_2\circ\lambda^{n-1}$ is defined and decreasing on $[0,\varepsilon]$. Since $\lambda([0,\varepsilon]) \subseteq [0,\varepsilon]$, we can write for $x \in [0,\varepsilon]$:

$$\begin{aligned} h\circ\lambda^n(x) &= h_1\circ h_2\circ\lambda^{n-1}(\lambda(x)) \\ &\leq h_1(\lambda(x)) = h_1\circ\lambda(x) \\ &\leq x. \end{aligned}$$

Now define

$$K_n = \{x \in [0,\varepsilon] \mid h \circ \lambda^n(x) = h' \circ \lambda^n(x) \quad \text{for some} \quad h,h' \in \Gamma^n \text{ with } h \neq h'\} .$$

For $h \neq h'$, the set of x with $h \circ \lambda^n(x) = h' \circ \lambda^n(x)$ is a finite set by the identity theorem. Therefore, the set K_n is finite, and the union $K = \bigcup_{n=1}^{\infty} K_n$ is countable. By definition, for every $x \in [0,\varepsilon] \setminus K$ we have $h \circ \lambda^n(x) \neq h' \circ \lambda^n(x)$ if $h \neq h'$. It follows that $|\Gamma^n \circ \lambda^n(x)| = |\Gamma^n \circ \lambda^n|$ and we are done.

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