Let $F$ be a $C^2$-foliation of codimension $q$ on a compact manifold without boundary. If each leaf of $F$ is a compact submanifold of $M$, then $F$ is a compact foliation. The purpose of this paper is to show that the Godbillon-Vey class of a compact foliation vanishes, as must all of the 'residuable' secondary classes

$$H^*_q(WO_q) = \{ y, c_j \in H^*(WO_q) : \deg c_j = 2q \}.$$ 

**Theorem.** Let $F$ be a compact, $C^2$-foliation on a closed manifold $M$. For each $y, c_j \in H^*_q(WO_q)$ the secondary class $\Delta_*(y, c_j) \in H^*(M)$ of $F$ is zero. In particular, the Godbillon-Vey class $gv(F) = \Delta_*(y, c_j)$ is zero.

In codimension one, remarkable progress has been made on the problem of relating the Godbillon-Vey class of a general foliation $F$ with the geometry of $F$, especially the rates of growth of the leaves [3, 4, 11]. The solution of the Sullivan conjecture by Duminy [4] implies that the leaves of $F$ must have exponential growth if $gv(F) \neq 0$. For higher codimensions, the secondary classes in $H^*_q(WO_q)$ appear to be the best candidates for admitting an extension of the codimension one results. The above theorem asserts that the classes from $H^*_q(WO_q)$ vanish if all leaves of $F$ have growth of degree zero. The terminology 'residuable' is used for an element $y, c_j$ in $H^*_q(WO_q)$ because the evaluation of the class $\Delta_*(y, c_j) \in H^*(M)$ can be obtained using an integral formula which is often expressible as a generalized residue along the leaves of $F$ (for example, see [8]). The reduction to an integral formula is the key to the proof of the theorem.

It is well known that a compact foliation of codimension one with orientable normal bundle is defined by a submersion $M \to S^1$ [7]. For a codimension two compact foliation, the beautiful study of compact foliations by Edwards, Millet and Sullivan [5] showed that $M$ has a covering by saturated open sets $U_1, \ldots, U_n$ for which $F|_{U_i}$ has the property that there is a finite cover $\tilde{U}_i$ of $U_i$ such that the lifted foliation $\tilde{F}|_{\tilde{U}_i}$ on $\tilde{U}_i$ is defined by a submersion onto the disc $D^2$. A metric on $D^2$ pulls back to a metric on the normal bundle of $F$ which is invariant under the holonomy of $F$. Thus, $F$ is a riemannian foliation and all of its secondary classes vanish. For $q > 2$, a compact foliation need not be riemannian, as evidenced by the examples of Sullivan and others, even if $F$ is analytic [13]. These examples show that the geometry of a compact foliation of higher codimension can be very complicated. In spite of this, the restrictions on the holonomy of $F$ imposed by the condition that all leaves are compact leads to the conjecture that all of the secondary classes of $F$ must vanish [10, Question 1.22].

We give now the idea of the proof of the theorem. Passing to a double cover of $M$ if necessary, we can assume that $M$ is orientable. Multiplying the form on $M$
representing our secondary class by a closed form, we see that it is enough to show that an integral over $M$ vanishes. We use the Epstein filtration $[5, 6]$ of the bad set of $F$ to decompose this integral into a sum of integrals over saturated sets $Y_a = T_a \times L_a$ on which $F$ is the product foliation. We then show that the secondary class decomposes into a corresponding product, and the integral of one factor over a leaf $L_a$ is proportional to a leaf class of $F$ for $L_a$. The leaf classes of a compact foliation are identically zero, and so the integrals over the sets $Y_a$ all vanish.

Preparatory lemmas are given in §§1 and 2. The proof in a special case is given in §3, while in §4 we consider the general integral over the bad set.

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1. The bad set of a compact foliation

Let $F$ be a compact foliation on $M$, and fix a riemannian metric on $M$. Each leaf $L$ of $F$ has an induced metric, and therefore a volume form and a total volume $\text{vol}(L)$. Define a function on $M$ by assigning to a point $x$ the number $\text{vol}(L_x)$, where $L_x$ is the leaf through $x$. For details on the properties of the function $x \mapsto \text{vol}(L_x)$, the reader is referred to the concise summary given in Proposition 4.1 of [5]. Note that the function $\text{vol}(L_x)$ is continuous at $x$ precisely when $L_x$ has no holonomy.

**Definition 1.1.** The bad set $X_1$ of $F$ is the union of the leaves with holonomy:

$$X_1 = \{ x_0 \in M \mid x \mapsto \text{vol}(L_x) \text{ is not continuous at } x_0 \}.$$

The bad set is a saturated, closed and nowhere dense subset of $M$. The Lebesgue measure $\mu(X_1)$ need not be zero, and this forces us to introduce a nice decomposition of $X_1$. The Epstein filtration of the bad set is a decreasing collection $\{ X_a \mid a \in \mathcal{A} \}$ of closed subsets of $X_1$ on whose successive differences the function $\text{vol}(L_x)$ is continuous. See [5, §6] or [6, §6] for complete details. The indexing set $\mathcal{A}$ consists of the ordinals, and $X_\beta = \emptyset$ for some countable successor ordinal $\beta \in \mathcal{A}$. For each $\beta > \alpha$, $X_\beta$ is a saturated, closed and nowhere dense subset of $X_\alpha$.

For each $\alpha \in \mathcal{A}$, set $Y_\alpha = X_\alpha - X_{\alpha+1}$, and let $Y_0 = M - X_1$. Then $\text{vol}(L_x)$ is a continuous function when restricted to $Y_\alpha$, whence $F|_{Y_\alpha}$ has no holonomy [5, Proposition 4.1]. The set $Y_\alpha$ is relatively open in the closed set $X_\alpha$. We observe that $Y_\alpha$ has an exhaustion by closed, saturated subsets; this will be used in §4.

**Lemma 1.2.** For $\alpha \in \mathcal{A}$ and given $\varepsilon > 0$, there is a closed, saturated subset $K \subseteq X_\alpha$ with $K \cap X_{\alpha+1} = \emptyset$ and $\mu(Y_\alpha - K) < \varepsilon$.

**Proof.** Let $W \subseteq M$ be open with $X_{\alpha+1} \subseteq W$ and $\mu(W - X_{\alpha+1}) < \varepsilon$. Let $K$ denote the $F$-saturation of the closed set $Z = (M - W) \cap X_\alpha$. Since $F|_{Y_\alpha}$ is without holonomy, by [6, Proposition §8] the quotient map $\pi : Y_\alpha \to Y_\alpha/F$ is open and proper. The set $\pi(Z)$ is compact and $K = \pi^{-1}(\pi(Z))$, so that $K$ is compact and hence closed in $M$. Both $X_\alpha$ and $X_{\alpha+1}$ are saturated, and so $K \subseteq Y_\alpha$. Finally, $X_\alpha - K \subseteq W$ implies that

$$\mu(Y_\alpha - K) = \mu(X_\alpha - X_{\alpha+1} - K) \leq \mu(W - X_{\alpha+1}) < \varepsilon.$$
2. Leaf and secondary classes

Let $\mathcal{A}'(M)$ denote the deRham algebra of $M$. The choice of a Bott connection for $\mathcal{F}$ defines a differential algebra map $\Delta : W_0 \rightarrow \mathcal{A}'(M)$. The algebra $W_0$ is the product of an exterior algebra with a truncated polynomial algebra

$$W_0 \cong \wedge (y_1, y_3, \ldots, y_{2(q+1)/2-1}) \otimes \mathbb{R}[c_1, \ldots, c_q],$$

where $\deg y_i = 2i - j$, $\deg c_i = 2i$ and the differential is determined by $dy_i = c_i$ and $dc_i = 0$. The construction of $\Delta$ and its properties are described in detail in the foundational paper of Bott [1]; see also [2]. The image of the map in cohomology, $\Delta_* : H^*(W_0) \rightarrow H^*(M)$, consists of the secondary classes of $\mathcal{F}$.

Next recall the construction of the leaf classes for a leaf $L \subseteq M$. Let $Q = TM/\mathcal{F}$ be the normal bundle to $\mathcal{F}$ and let $\Gamma(M, Q^*) \subseteq \mathcal{A}'(M)$ be the space of 1-forms which annihilate $\mathcal{F}$. The crucial property of the map $\Delta$ is that, for all $i$, a consequence of constructing $\Delta$ using a Bott connection. For the leaf $L$, this implies that the form $\Delta(c_i)$ vanishes when restricted to $L$, and therefore each $\Delta(y_i)|_L$ is a closed form. We remark that this vanishing corresponds to the observation that the restricted bundle $Q|_L \rightarrow L$ has a natural flat structure obtained by restricting the Bott connection on $Q$. The curvature of a flat connection is zero, and hence $\Delta(c_i)|_L = 0$.

For an index $I = (i_1, \ldots, i_s)$ with $1 \leq i_1 < \ldots < i_s \leq q$ and all $i_i$ odd, set $y_I = y_{i_1} \cdots y_{i_s} \in W_0$. The form $\Delta(y_I) \in \mathcal{A}'(M)$ is not closed in general, but the above remarks imply that $\Delta(y_I)|_L$ is closed and determines a class in $H^*(L)$. The restriction of $\Delta$ thus defines an algebra map

$$\chi_L : (\mathfrak{gl}_q, O_q) \rightarrow H^*(L)$$

where we identify the relative Lie algebra cohomology $H^*(\mathfrak{gl}_q, O_q)$ with the exterior algebra in $W_0$. The image of $\chi_L$ consists of the leaf classes of $\mathcal{F}$ for $L$. The map $\chi_L$ is an invariant of the germ of $\mathcal{F}$ about $L$, and in fact depends only on the flat bundle structure of $Q|_L \rightarrow L$. To be precise, recall that the foliation in a neighborhood of $L$ determines the linear holonomy $dh : \pi_1(L, x) \rightarrow \text{Gl}_q\mathbb{R}$ where $x \in L$. The flat structure on $Q|_L$ is classified by the induced map $B(dh) : L \rightarrow B\text{Gl}_q\mathbb{R}$ where $\text{Gl}_q\mathbb{R}$ has the discrete topology.

In the study of the leaf classes by Shulman and Tischler [12] (see also [9, Chapter 6]) the following relationship is proven.

**Proposition 2.1.** There is a commutative diagram

$$\begin{array}{ccc} H^*(\mathfrak{gl}_q, O_q) & \xrightarrow{\chi_L} & H^*(L) \\ \text{VE} \downarrow & & \downarrow B(dh)^* \\
H^*(B\text{Gl}_q\mathbb{R}) & & \\
\end{array}$$

where $\text{VE}$ is the Van Est map defining the continuous cohomology of $\text{Gl}_q\mathbb{R}$.
**Corollary 2.2.** If the linear holonomy of a leaf $L$ is trivial, then all leaf classes $\chi_L(y_t)$ are zero in $H^*(L)$.

Actually, a stronger form of 2.2 can be shown which is relevant to compact foliations. Recall that a matrix $A \in \text{GL}_q \mathbb{R}$ is unipotent if all eigenvalues of $A$ have modulus 1. The linear holonomy of a leaf $L$ is said to be unipotent if for all $y \in \pi_1(L, x)$ the matrix $dh(y)$ is unipotent.

**Proposition 2.3.** If the linear holonomy of a leaf $L$ has a solvable subgroup of finite index, then all leaf classes from $H^m(\text{gl}_q, \Omega_q)$ vanish for $m > 1$. If the linear holonomy is unipotent, then all leaf classes for $L$ vanish.

**Proof.** We can put $dh$ into standard form (for example, see [14, Proposition 3.2]) and so we can assume that the image of $dh$ is contained in a closed subgroup $H \subseteq \text{GL}_q \mathbb{R}$, where $H$ is either unipotent or solvable. For the Lie algebra $\mathfrak{h}$ of $H$, the induced map $H^*(\text{gl}_q, \Omega_q) \to H^*(\mathfrak{h})$ is then zero for $H$ unipotent, or zero when $* > 1$ for $H$ solvable. The composition $B(dh)^* \circ \text{VE}$ can be factored

$$H^*(\text{gl}_q, \Omega_q) \longrightarrow H^*(\mathfrak{h}) \longrightarrow H^*(BH^3) \longrightarrow H^*(L)$$

and hence is zero for $H$ unipotent, or vanishes in degrees greater than one for $H$ solvable. The composition is equal to $\chi_L$ by Proposition 2.1, and Proposition 2.3 follows.

For a compact foliation the linear holonomy of each leaf $L$ is unipotent (see Lemma 4.5 below). Thus, for every leaf $L$ of a compact foliation, all leaf classes vanish.

3. **Proof of the theorem**

Let $\mathcal{F}$ be a compact foliation on the compact $m$-manifold $M$ without boundary. By passing to a two or four-fold covering of $M$, we can assume that both $M$ and the normal bundle $Q$ of $\mathcal{F}$ are orientable. For $y_t, c_j \in H^*(\Omega_q)$ with $\text{deg} c_j = 2q$, we must show that $\Delta_*(y_t, c_j) \in H^*(M)$ is zero. It suffices to show, by Poincaré duality, that for every closed $(m-n)$-form $\phi$ the integral $\int_M \phi \cdot \Delta(y_t, c_j) = 0$. In §1 a countable decomposition of $M$ associated to the Epstein filtration of $X_1$ was introduced: $M = \bigcup_{s} Y_s$. It is enough to prove that the integral over each $Y_s$ is zero. We treat the technically simpler case of $Y_0 = M - X_1$ in this section, and the case of a general $Y_s$ in the next. If $X_1$ has measure zero in $M$, then the theorem is equivalent to the statement that the integral over $Y_0$ vanishes. For example, if $\mathcal{F}$ is transversely analytic, or if every leaf in $X_1$ has non-trivial linear holonomy, then $\mu(X_1) = 0$.

For convenience, set $Y = Y_0$. Recall that $Y$ is an open saturated set with every leaf compact and $\mathcal{F}|_Y$ has no holonomy. The quotient space $T = Y/\mathcal{F}$ is therefore an open, smooth Hausdorff manifold with $\mathcal{F}|_Y$ defined by the submersion $\pi : Y \to T$ [5, §8]. Choose a volume form $\tilde{\omega}$ on $T$. Then $\omega = \pi^*\tilde{\omega}$ is a transverse invariant volume form for $\mathcal{F}$ on $Y$, satisfying $d\omega = 0$ and $i(v)\omega = 0$ for all vector fields $v$ on $Y$ tangent to $\mathcal{F}$. 
Give $M$ a riemannian metric, which determines an embedding $Q \subseteq TM$ orthogonal to $\mathcal{F}$. The exterior power bundle $\Lambda^q Q$ is orientable and we choose a section $z \in \Gamma(Y, \Lambda^q Q)$ over $Y$ satisfying $\omega(z) = 1$. Define a $q$-form on $Y$

$$\hat{c}_j = i(z)\Delta(c_j).$$

Locally, there are vector fields $z_1, \ldots, z_q$ which frame $Q$ with $z = z_1 \cdots z_q$; then for vector fields $y_1, \ldots, y_q$ we have $\hat{c}_j(y_1, \ldots, y_q) = c_j(z_1, \ldots, z_q, y_1, \ldots, y_q)$. The assumption that $c_j$ has degree $2g$ implies that the form $\Delta(c_j)$ belongs to $\Gamma(M, \Lambda^q Q^*) \cdot \mathfrak{A}^q(M)$, whence $\Delta(c_j) = \omega \cdot \hat{c}_j$ on $Y$. The form $\hat{c}_j$ is not closed, but we observe that

$$0 = d\Delta(c_j) = d(\omega \cdot \hat{c}_j) = d\omega \cdot \hat{c}_j \pm \omega \cdot d\hat{c}_j = \pm \omega \cdot d\hat{c}_j.$$

This implies that $d\hat{c}_j$ vanishes when restricted to a leaf $L \subseteq Y$, and so $\hat{c}_j|_L \in \mathfrak{A}^q(L)$ is a closed form.

Now consider

$$\int_Y \phi \cdot \Delta(y_j c_j) = \int_Y \phi \cdot \Delta(y_j) \cdot \omega \cdot \hat{c}_j = (-1)^q \int_x \left\{ \int_T \phi \cdot \Delta(y_j) \cdot \hat{c}_j|_L \right\} \cdot \omega, \quad (3.1)$$

where the expression in (3.1) is the integral over the compact fibers $\{L_x | x \in T\}$ of the fibration $\pi : Y \to T$, and the integrand factors because $\omega$ is a basic form on $Y$ of degree $q = \dim T$. The foliation $\mathcal{F}|_Y$ has no holonomy, and so for all leaves $L_x \subseteq Y$ the leaf class $\chi_{L_x}(y_j) \in H^{n-2q}(L)$ is zero. Since $\hat{c}_j|_{L_x}$ is a closed form, and $L_x$ is closed, each integral $\int_{L_x} \phi \cdot \Delta(y_j) \cdot \hat{c}_j = 0$. The expression in (3.1) thus vanishes, and this proves our claim.

4. When the bad set has positive measure

The bad set $X_1$ is closed and nowhere dense in $M$; thus if the Lebesgue measure $\mu(X_1) > 0$, then the transverse structure of $X_1$, and hence of each $Y_x = X_x - X_{x+1}$, is very complicated. We must show that $\int_{Y_x} \phi \cdot \Delta(y_j c_j) = 0$ for each $x$, and special care is needed to make the techniques of §3 extend. We fix $x$. The set $Y_x$ can be exhausted by closed saturated sets $K \subseteq Y_x$ by Lemma 1.2, and so it is enough to show that the integral over such a $K$ vanishes. The difficulty in extending the formula 3.1 is that the quotient space $T = K/\mathcal{F}$ is compact and Hausdorff but has no interior, and so we cannot use $T$ to define a transverse invariant volume form for $\mathcal{F}$ on $K$. To circumvent this, we reduce the integral over $K$ to a finite sum of integrals over compact saturated subsets $K_i \subseteq K$. Each set $K_i$ is chosen to have an open neighborhood $U_i$ on which there is a transverse volume form $\omega_i$ which is invariant when restricted to $K_i$. The existence of such an $\omega_i$ is then sufficient to make the method of §3 work for the integral over $K_i$.

For the reader's convenience and to fix our notation, we give the definition of a foliation chart.
Definition 4.1. A foliation chart \((U, \phi)\) for \(\mathcal{F}\) centered at \(x \in M\) consists of an open neighborhood \(U \subseteq M\) of \(x\) and a diffeomorphism

\[
\phi = (\psi, f): U \to D^{m-q} \times D^q \subseteq \mathbb{R}^m
\]

such that \(\phi(x) = 0\), and the second factor defines \(\mathcal{F}|_U\) as the level sets of \(f: U \to D^q \subseteq \mathbb{R}^q\). For each \(y \in U\), the set

\[
D_y = \phi^{-1}(\psi(y) \times D^q) \subseteq U
\]

is the transversal to \(\mathcal{F}\) through \(y\) associated to \((U, \phi)\).

Since \(Q\) is orientable, we choose an orientation, and require also that the local map \(f_*: Q|_U \to T\mathbb{R}^q\) be orientation preserving, where \(\mathbb{R}^q\) has the standard orientation.

The decomposition \(\{K_1, \ldots, K_r\}\) of \(K\) will be defined after some preliminary constructions. Given a leaf \(L \subseteq K\), choose a base point \(x \in L\). Since \(L\) is compact, we can choose a finite set of open foliation charts \(\{(V_j, \phi_j) | j = 1, \ldots, p\}\) with \(V_j\) centered at \(y_j \in L\), \(y_1 = x\), and \(L \subseteq \bigcup_{j=1}^p V_j \equiv V\). Let \(\beta_j\) be a path in \(L\) from \(x\) to \(y_j\). Recall that \(D_j\) is the transverse disc for \((V_j, \phi_j)\) centered at \(y_j\). By shrinking \(V_j\) in the transverse direction if necessary, we can assume that the holonomy along \(\beta_j\) defines a diffeomorphism into, denoted by \(\gamma_{1j} : D_x \to D_{y_j}\).

Set \(C \equiv M - V\), a compact set. Let \(T = K/\mathcal{F}\); by the proof of Lemma 1.2, we know that \(\pi: K \to T\) is an open proper map. Thus, \(\pi(K \cap C)\) is compact in \(T\), and \(Z \equiv \pi^{-1}(\pi(K \cap C))\) is a closed saturated set. For technical reasons, we shrink the cover \(\{V_j\}\) once again by deleting the set \(Z\). Define open sets \(U_j \equiv V_j - Z\) for \(j = 1, \ldots, p\) and let \(U_L = \bigcup_{j=1}^p U_j\). Observe that \(K - Z\) is saturated and we have \(L \subseteq K - Z = K \cap U_L \subseteq U_L\).

The compact set \(K\) is covered by the open sets \(U_L\) for \(L \subseteq K\). Choose a finite subcover \(U_{L_1}, \ldots, U_{L_r}\). For each \(L_i\) let \(D_i\) be the transverse disc through a base point \(x_i \in L_i\), as in the above construction. Each leaf \(L \subseteq K\) must intersect some \(U_{L_i}\) and thus intersect the transverse disc \(D_i\). Recall that the quotient \(T = K/\mathcal{F}\) is a compact hausdorff space, and let \(\pi: K \to T\) be the quotient map. Then the sets \(\bar{D}_i = \pi(D_i \cap K), 1 \leq i \leq r\), form an open cover for \(T\). Choose a decomposition \(T = T_1 \cup \ldots \cup T_r\) with \(T_i \subseteq \bar{D}_i\) a closed subset of \(T\) and such that, for \(i \neq j\), \(T_i \cap T_j\) has measure zero as a subset of the transversal disc \(D_i\). We can now define the closed subsets of \(K\):

\[
K_i = \pi^{-1}(T_i), \quad 1 \leq i \leq r.
\]

It is immediate that \(K = K_1 \cup \ldots \cup K_r\) and \(\mu(K_i \cap K_j) = 0\) if \(i \neq j\). Note that \(K_i\) is contained in the open set \(U_{L_i}\).

The care taken above to construct the sets \(K_i\) and \(U_{L_i}\) seems to be necessary, because the set \(K \subseteq X_1\) has no \textit{a priori} restriction on its global topology. To make the constructions which follow, we need to localize to well-behaved pieces of \(K\). For the rest of this section, we fix \(i\) and set \(K = K_i, U = U_{L_i}\) and \(L = L_i\).

Lemma 4.2. There is a smooth \(q\)-form \(\omega\) on \(U\) which defines \(\mathcal{F}\) on \(U\) and satisfies \(\theta(v)\omega_x = 0\) for all \(x \in K\) and vector fields \(v\) on \(U\) tangent to \(\mathcal{F}\).
Here, $\theta(v)\omega_x$ denotes the Lie derivative of $\omega$ along $v$ evaluated at $x$.

Since $\omega(v) = 0$ for all $v$ tangent to $\mathcal{F}$, the Cartan formula $\theta(v) = i(v) \circ d + d \circ i(v)$ applied to the form $\omega$ in (4.2) yields the vanishing condition

$$i(v)d\omega_x = 0 \quad \text{for all } x \in K \text{ and } v \text{ tangent to } \mathcal{F}. \quad (4.3)$$

Assuming Lemma 4.2, we finish the proof of the theorem. Choose a section $z$ of $\Lambda^q\mathcal{Q}$ over $U$ satisfying $\omega(z) = 1$, and then define $\hat{c}_j = i(z)\Delta(c_j)$ as in §3 so that $\Delta(c_j) = \omega \cdot \hat{c}_j$. On a neighborhood of a point $x \in K$, let $z_1, \ldots, z_q$ be a framing of $Q$ with $z = z_1 \cdots z_q$ and let $y_1, \ldots, y_{q+1}$ be vector fields tangent to $\mathcal{F}$. Since $\omega \cdot d\hat{c}_j = \pm d\omega \cdot \hat{c}_j$ we have at $x$ that

$$d\hat{c}_j(y_1, \ldots, y_{q+1}) = \omega \cdot d\hat{c}_j(z_1, \ldots, z_q, y_1, \ldots, y_{q+1}) = \pm d\omega \cdot \hat{c}_j(z_1, \ldots, z_q, y_1, \ldots, y_{q+1})$$

$$= \pm \sum_{i=1}^{q+1} (-1)^{q+i} \cdot d\omega(y_1, z_1, \ldots, z_q) \cdot \hat{c}_j(y_1, \ldots, \hat{y}_i, \ldots, y_{q+1}) = 0$$

because $i(y_i)d\omega_x = 0$ from (4.3). For a leaf $L \subseteq K$, this implies that $\hat{c}_j|_L$ is a closed form.

The $q$-form $\omega$ determines a transverse measure to $\mathcal{F}$ on $K$, and the property $\theta(v)\omega = 0$ on $K$ implies that the measure is invariant. Then by [5, Lemma, p. 25], we have a decomposition

$$\int_K \phi \cdot \Delta(y_j)c_j = \int_K \phi \cdot \Delta(y_j) \cdot \hat{c}_j \cdot \omega = \int_{x \in L} \left\{ \int_{L_x} (\phi \cdot \Delta(y_j) \cdot \hat{c}_j)|_{L_x} \right\} \cdot \omega. \quad (4.4)$$

The restriction $\phi \cdot \hat{c}_j|_L$ is a closed form, and Proposition 2.3 implies that $\Delta(y_j)|_L$ is exact for every leaf $L \subseteq K$. Therefore, for each $x \in L$, the integral $\int_{L_x} \phi \cdot \Delta(y_j) \cdot \hat{c}_j = 0$ since $L_x$ is a closed manifold. The integral in (4.4) thus is zero, as was to be shown.

We now prove Lemma 4.2. Let $\{(U_i, \phi_i) : 1 \leq i \leq p\}$ be the cover of $L$ by foliation charts as defined earlier with $U = \bigcup_{i=1}^p U_i$. Then $f_i : U_i \to \mathbb{R}^q \subseteq \mathbb{R}^q$ defines $\mathcal{F}$ on $U_i$, and $D_i = \phi_i^{-1}(0 \times \mathbb{R}^q)$ is a transversal to $\mathcal{F}$ in $U_i$. Set $T = K \cap D_1$, a compact set. Recall that this cover of $L$ was chosen so that, for each $i$, there is an open neighborhood $V$ of $T$ in $D_i$ and a diffeomorphism into, $\gamma_{1i} : V \to D_i$, the transition function from $f_1$ to $f_i$. Let $W$ be an open set with $T \subseteq W \subseteq \overline{W} \subseteq V$.

The standard volume form on $\mathbb{R}^q$ is denoted by $d\mu = dx_1 \cdots dx_q$. We begin by defining $\omega_i = f_i^* (d\mu)$, a closed $q$-form on $U_i$ which restricts to a volume form on $V \subseteq D_1 \subseteq U_1$. Let $V_i$ (respectively $W_i$) denote the image of $V$ (respectively $W$) under the diffeomorphism into,

$$V \xrightarrow{\gamma_{1i}} D_1 \xrightarrow{f_i} \mathbb{R}^q,$$

and let $\tilde{\omega}_{1i}$ denote the $q$-form on $V_i$ which is the push-forward of $\omega_1|_V$. We extend $\tilde{\omega}_{1i}$ to a volume form defined on all of $\mathbb{R}^q$: choose a partition of unity $\{\lambda_i, (1 - \lambda_i)\}$ for the cover $\{V_i, \mathbb{R}^q - W_i\}$ of $\mathbb{R}^q$, and set $\tilde{\omega}_i = \lambda_i \tilde{\omega}_{1i} + (1 - \lambda_i) \cdot d\mu$. Define $\omega_i = f_i^*(\tilde{\omega}_i)$, a
transverse invariant volume form for $\mathcal{F}$ on $U_i$. Note that the restriction of $\omega_i$ to the transversal $\phi_i^{-1}(0 \times W_i)$ agrees with the translation $(\gamma_i^{-1})^*(\omega_i|_W)$.

Choose a partition of unity $\{\alpha_1, ..., \alpha_{p+1}\}$ subordinate to the open cover $\{U_1, ..., U_p, M - X\}$ of $M$, where $X = \bigcup_{i=1}^{p+1} \phi_i^{-1}(D^m \times W_i)$. Let $\omega_{p+1}$ be a $q$-form on $M - X$ which defines $\mathcal{F}$ and has the same orientation as $\omega_1$ on $U_1$. Set $\omega = \sum_{i=1}^{p+1} \alpha_i \omega_i$ and restrict to $U$ to get the $q$-form of 4.2. The assumption that the foliation charts are compatibly oriented implies that $\omega$ defines $\mathcal{F}$ on $U$.

It remains to show that $\theta(v)\omega_x = 0$ for $x \in K$ and $v$ a vector field tangent to $\mathcal{F}$. This is equivalent to proving that $\omega_x = (\omega_i)_x$ for all $x \in U_i \cap K$, as $\omega_i = f^*_i(\tilde{\omega}_i)$ is invariant under the flow $\Phi_i$ of $v$, and $\Phi_i$ preserves the fibers of $\pi: U_i \to \mathbb{R}^q$.

The function $\alpha_{p+1}$ vanishes on $X$, and so $\omega = \sum_{i=1}^{p} \alpha_i \omega_i$ on $K \subseteq X$. If we show that $(\omega_i)_x = (\omega_j)_x$ for $x \in K \cap U_i \cap U_j$, then we can conclude that $\omega_x = (\omega_i)_x$ as desired. Because $\omega_i$ and $\omega_j$ are pull-backs from $\mathbb{R}^q$, we need only show, for the transverse slice $V_x = \phi_i^{-1}(\psi_i(x) \times V_i)$, that the restrictions $\omega_i|_{V_x}$ and $\omega_j|_{V_x}$ agree at $x$. Both of these restricted forms are defined as the push-forward via $(\gamma_{ii}^{-1})^*$ and $(\gamma_{jj}^{-1})^*$ of $\omega_i|_V$, and so

$$\omega|_{V_x} = (\gamma_{ii}^{-1})^*\omega_i|_{V_x}.$$

At $x$, the action of $(\gamma_{ii}^{-1})^*$ on the $q$-forms $\Lambda^q Q_x$ is induced from the action of the linear holonomy along the leaf $L$ containing $x$. This action is trivial by the following result.

**Lemma 4.5.** Let $L$ be a leaf in a compact foliation. Then the linear holonomy of $L$ is unipotent, and if the normal bundle $Q$ restricted to $L$ is orientable, then every element has determinant one.

**Proof.** Let $dh : \pi_1(L, x) \to GL_q \mathbb{R}$ denote the linear holonomy of $L$. Assume that there exists $\gamma \in \pi_1(L, x)$ for which $dh(\gamma)$ has an eigenvalue of modulus not equal to 1; we can assume it to be less than one. By the stable manifold theorem, a local diffeomorphism representing the holonomy element $h(\gamma)$ has a stable contracting manifold of dimension at least one. This implies that there is a leaf of $\mathcal{F}$ asymptotic to $L$, which is impossible if all leaves are compact.

For each $\gamma$ the determinant of $dh(\gamma)$ is real with modulus one, and so must be $\pm 1$. For $Q|_L \to L$ orientable the only possibility is that $\det(dh(\gamma)) = 1$.

Under our assumptions, the linear holonomy of each leaf in $K$ has determinant 1; thus the action of $(\gamma_{ii}^{-1})^*$ on $\Lambda^q Q_x$ is trivial, and hence $\omega|_{V_x} = \omega|_{V_x}$ on $K \cap V_x$. This finishes the proof of the theorem.

**References**

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