THE WEIL MEASURES OF FOLIATIONS
- A SURVEY AND SOME APPLICATIONS -

Steven Hurder

Department of Mathematics
Princeton University
Princeton, N.J. 08544

The primary classes of a vector bundle, by their very definition, measure the geometric obstruction to making the vector bundle into a product. For a smooth manifold M and a subbundle $F \quad TM$ which is tangent to a foliation $\mathcal{F}$ of M, there are additional characteristic classes, the secondary classes as defined by Bott-Haefliger [ ], Bernstein-Rosenfel'd [ ] or Kamber-Tondeur [ ]. However, the definition of these classes does nothing to illuminate their geometric meaning. One of the most interesting problems in foliation theory at present is to understand what geometric aspects of a foliation contribute to the values of the secondary classes. In this note we survey some of the progress on this problem, giving some new results announced here for the first time. For example, Theorem 7 implies that a foliation which is measurably a product foliation has trivial residual secondary classes.

A second aim of this paper is to discuss some of the properties of the Godbillon and higher Weil measures of a foliation. Recall the Godbillon measure was introduced by Duminy to solve Sullivan's Conjecture [ ]. The Weil measures were introduced and studied in Heitsch-Hurder [ ]; they generalize to all codimensions Duminy's approach, and include many more measures than just that arising from the Godbillon operator. The Weil measures are <u>integral</u> invariants of a foliation, much in the spirit of the C*-analyitic invariants of a foliation introduced by A. Connes [,]. It is conjectured that the Weil measures can be calculated from the C*-algebra of a foliation, and we indicate here some of the first steps towards interlacing the two theories.

Throughout, questions and problems which arise in the study of foliations using the Weil measures are given. These hopefully indicate the variety of applications for the Weil measures.

We denote by M a closed oriented smooth manifold of dimension m, by $\mathcal{F}$ a codimension q oriented foliation of M, and $Q = TM/F$ is the normal bundle of $\mathcal{F}$. The quotient $M/\mathcal{F}$ is the <u>leaf space</u> of $\mathcal{F}$, and $\pi: M \to M/\mathcal{F}$ is the canonical map. A set $B \subseteq M$ is <u>saturated</u> if $B = \pi^{-1}(\pi(B))$. For convenience, fix a Riemannian metric h on M with corresponding Lebesgue measure $\underline{m}$ on M. Set

$$\mathcal{B} = \{ B \subseteq M \mid B \text{ is saturated and } \underline{m}\text{-measurable} \} .$$

$\mathcal{B}$ is the measure algebra for the quotient $(M/\mathcal{F}, \pi_* \underline{m})$. The Weil measures are vector-valued measures on $\mathcal{B}$, and continous with respect to $\pi_* \underline{m}$.

§1. Weil Measures

All constructions of secondary invariants for a foliation start with Bott's observation that the natural parallelism of Q along the leaves of can be extended (by use of the metric h on M) to a smooth connection $\nabla(h)$ on Q, said to be basic [ ]. Following Kamber-Tondeur, we use $\nabla(h)$ to define a map of the relative Weil algebra $W(g\ell_q, O_q)$ into the deRham complex of M,

$$\Delta : \quad W(g\ell_q, O_q) \to A^*(M) \quad .$$

The secondary classes arise by observing that one can truncate the symmetric part of the LHS above in degrees above 2q , and the map Δ will descend to the quotient algebra by Bott's Vanishing Theorem. The cohomology of the truncated Weil algebra consists of the secondary classes.

It is well known there is a differential subcomplex $WO_q \subseteq W(g\ell_q, O_q)_q$ carrying the cohomology, where

$$WO_q = \Lambda(y_1, y_3, \ldots, y_{n'}) \otimes R[c_1, \ldots, c_q]_q \; .$$

with $d(y_i) = c_i$ and degree $c_i = 2i$. A typical cocycle in the Vey basis for $H^*(WO_q)$ has the form $y_I c_J = y_{i_1} \cdots y_{i_r} \cdot c_i^{j_1} \cdots c_q^{j_q}$ with $i_1 < \cdots < i_r$,

$$i_1 + \sum_{\ell=1}^{q} \ell \cdot j_\ell > q \quad \text{and} \quad \text{degree } c_J = 2 \cdot \sum_{j=1}^{q} \ell \cdot j_\ell \leq 2q \; .$$

A class $y_I c_J$ is residual if degree $c_J = 2q$. We set

$$H^*(g\ell_q, O_q) = \Lambda(y_1, \ldots, y_{n'}) \otimes 1 \subseteq WO_q \quad .$$

A typical element in this exterior algebra is denoted by y.

The class $\Delta(y_i) \in A^{2i-1}(M)$ is not closed, but it restricts to each leaf $L \subseteq M$ of $\mathcal{F}$ to give a cocycle, whose class $\chi_L(y_i) \in H^*(L)$ is characteristic for the flat bundle $Q|L \to L$. It is fundamental that these forms $\Delta(y_i)$ carry the transverse dynamics of $\mathcal{F}$, and so one tries to extract all possible characteristic invariants from them. This is the idea of the Weil measures.

The Weil operators are defined first. Let ω be a q-form on M defining F, so ω restricts to a volume form on each leaf of $\mathcal{F}$. Here, we assume for convenience that F is orientable. For $r \geq q$ set

$$A^r(M,F) = A^{r-q}(M)\cdot\omega ,$$

which by the Frobenius theorem is a differential ideal. Its cohomology is denoted by $H^r(M,F)$. This space is a natural but virtually intractable invariant of F, and has been in the literature since at least the 1960's. By Bott's Vanishing Theorem we have $d\Delta(y)\cdot\omega = 0$, so exterior multiplication by the form $\Delta(y)$ defines an operator on $H^*(M,F)$. An operator of this type is implicit in Kamber-Tondeur [§6,] where they study the spectral sequence associated to the ideal $A(M,F)$. The actual definition of such operators appears first in Duminy [ ], with the Godbillon operator.

DEFINITION 1.

a) The Godbillon Operator

$$g = \chi(y_1) : H^{m-1}(M,F) \to H^m(M) \cong R$$

is given by $g[\phi] = \int_M \Delta(y_1)\cdot\phi$.

b) For each $y \in H^\ell(g\ell_q, O_q)$, the Weil Operator

$$\chi(y) : H^{m-\ell}(M,F) \to H^m(M) \cong R$$

is given by $\chi(y)[\phi] = \int_M \Delta(y)\cdot\phi$.

PROPOSITION 1. [ ] Each $c_J \in WO_q$ with degree $c_J = 2q$ determines a well-defined class $[c_J] \in H^{2q}(M,F)$.

The residual secondary classes are obtained by applying the Weil operators to the $[c_J]$. For example, the Godbillon-Vey class of $\mathcal{F}$ results from applying g to $[c_1^q]$. Implicit in this formal reworking of the construction of the residual secondary classes is a fundamental observation of Duminy: To evaluate a residual secondary class $\Delta_*(y_I c_J)$, one choice of basic connection can be made to calculate the operator $\chi(y_I)$, and another to calculate the class $[c_J]$. This freedom of choice is a very powerful tool.

One of the most useful techniques to come out of the study of Sullivan's Conjecture in codimension one was that the Godbillon-Vey class can be localized to open saturated subsets of M. See [ ] for versions of this result. Duminy made this all very explicit by defining the Godbillon measure on the Σ-algebra generated by the <u>open</u> saturated sets in M. In Heitsch-Hurder [ ] the following complete result is proven:

THEOREM 1. Let $B \in \mathcal{B}$ be a measurable saturated subset of M.
a) For each $\ell > 0$ there is a well-defined restriction to B

$$\chi_B : H^{\ell}(g\ell_q, O_q) \rightarrow \mathrm{Hom}(H^{m-\ell}(M,F), R)$$

where $\chi_B(y)[\phi] = \int_B \Delta(y)\cdot\phi$.

b) χ is continuous with respect to $\underline{m}$: If $\underline{m}(B) = 0$, then $\chi_B = 0$.

c) $\chi(y)$ is additive on $\mathcal{B}$: For B, B', B" $\in \mathcal{B}$ with $B = B' \cup B''$ and $\underline{m}(B' \cap B'') = 0$, then $\chi_B = \chi_{B'} + \chi_{B''}$.

Thus for each $y \in H^{\ell}(g\ell_q, O_q)$ we get a measure on $\mathcal{B}$ with values in the topological dual to $H^{m-\ell}(M,F)$. Following Duminy we call $g = \chi(y_1)$ the Godbillon measure of $\mathcal{F}$.

The point of Theorem 1 is that the residual secondary classes are obtained from the Weil operators, and these in turn can be decomposed according to the ergodic components of $\mathcal{F}$. So our problem becomes:

QUESTION 1. Given a measurable saturated set B and $y \in H^{\ell}(g\ell_q, O_q)$, what geometric properties of $\mathcal{F}$ influence the values (or even just the norm) of the linear functional $\chi_B(y)$?

The most profound result to date is:

THEOREM 2. (Duminy [ ]) Let $\mathcal{F}$ be a codimension one foliation of a compact manifold. If $g \neq 0$ then $\mathcal{F}$ contains a resilient leaf.

A leaf is <u>resilient</u> if it has an element of contracting holonomy, and the leaf itself intersects the domain of this contraction. Transversally, a resilient leaf contains a Cantor set, and the leaf must have exponential growth so as a consequence one obtains Sullivan's Conjecture:

COROLLARY 1. If the Godbillon-Vey class of a codimension one foliation of a compact manifold M is non-zero, then the foliation contains a leaf of exponential growth.

This has recently been generalized to open manifolds by J. Cantwell and L. Conlon:

THEOREM 3. (Cantwell-Conlon [ ]) If the Godbillon-Vey class of a codimension one foliation $\mathcal{F}$ is non-zero, then $\mathcal{F}$ contains a resilient leaf.

James Heitsch and the author looked for higher codimension versions of these theorems. We note two such which appear in [ ].

THEOREM 4. (Heitsch-Hurder [ ]) Assume the holonomy groupoid of $\mathcal{F}$ is equicontinuous in its action on the transverse space of $\mathcal{F}$. Then the Godbillon measure $g = 0$.

The conditions of Theorem 4 are satisfied when $\mathcal{F}$ admits a transverse continuous distance function which is holonomy invariant. This is the case for a foliation defined by a group action which is topologically conjugate to group of isometries.

THEOREM 5. (Heitsch-Hurder [ ]) Suppose $\mathcal{F}$ admits a transverse invariant measure μ which is isotropic. Then $g_B = 0$ for all $B \subseteq$ support μ.

A measure is isotropic if its mass is (infinitesimally) equally distributed in all directions. An absolutely continuous measure with bounded coefficients with respect to the metric measure is isotropic.

QUESTION 2. If $\mathcal{F}$ admits a transverse invariant measure μ and $B \subseteq \mathrm{supp}(\mu)$, then does $g_B = 0$?

It is an enticing observation that in all known examples of foliations where the residual secondary classes or Weil measures have been calculated, the geometric conditions which are key to evaluating the classes are topologically invariant. This leads one to ask

QUESTION 3. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be C^2 foliations on M_0 and M_1, respectively. Assume there is a homeomorphism $f : M_0 \to M_1$ which sends $\mathcal{F}_0$ to $\mathcal{F}_1$.

a) Does $f^*\Delta_{\mathcal{F}_1}(y_I c_J) = \Delta_{\mathcal{F}_0}(y_I c_J)$ when degree $c_J = 2q$?

b) Are the Weil measures quasi-invariant under f ? That is, if for $\mathcal{F}_0$ we have $\chi_B(y) = 0$ with $B \subseteq M_0$, then does $\chi_{f(B)}(y) = 0$ also ?

§2. Operators on a Canonical Hilbert Bundle

The Weil measures involve the integration of forms on M over saturated sets. It is natural to ask how broad a class of geometric data can be used to define these measures. The best answer at present uses the notion of a measurable field of Hilbert bundles over M which arises in A. Connes' construction of the C*-algebra and von Neumann algebra of the foliation [, ,]. The complex we need appears in his Index formulation of the average Euler class of F. We make some preliminiary steps in the study of how the foliation K-theory $K^*(M,F)$ determines the Weil measures of $\mathcal{F}$.

Let $R \subseteq M \times M$ be the equivalence relation of $\mathcal{F}$. Denote the projection $p(x,y) = x$ by $p : R \to M$. Then $L_x = p^{-1}(x)$ is the leaf through x.

The metric h on M induces a metric on each leaf L_x, and it is well-known that the quasi-isometry class of the induced metric is independent of h. We let $L^p_{(2)}(x)$ denote the space of p-forms on L_x which have bounded $L_{(2)}$- norm with respect to the induced metric. Define a measurable field of Hilbert spaces over M by setting

$$L^p_{(2)}(M) = \bigcup_{x \in M} L^p_{(2)}(x)$$

Given measurable sections $\sigma, \tau : M \to L^p_{(2)}(M)$ their inner product is

$$\langle\sigma,\tau\rangle = \int_M \langle\sigma(x),\tau(x)\rangle \, d\underline{m} \quad .$$

Then let

$$S^p = \{ \text{measurable sections of } L^p_{(2)}(M) \text{ with continuous norms} \}.$$

d_F denote the leaf exterior derivative

$$Z^p(F) \;=\; \{\, \omega \in S^p \text{ with } d_F\omega = 0 \,\}$$

$$B^p(F) \;=\; \{\, \omega \in S^{p-1} \text{ with } d_F\omega \in S^p \,\}$$

$$H^p_{(2)}(F) \;=\; Z^p(F)/B^p(F) \quad .$$

A typical element $[\sigma] \in H^p_{(2)}(F)$ is a measurable family of forms over M : for each $x \in M$, $[\sigma(x)]$ is a class of closed $L_{(2)}$-forms on the leaf L_x.

The graph of a foliation G (see Haefliger [ ] for example) admits a map to the relation R, and the fiber over a point $(x,y) \in R$ is the holonomy covering of the leaf L containing x and y. Note that Connes considers the pull-back of the bundles above to the graph. We do not need the C*-module structure of the section spaces, and so we can stick with the simpler complexes defined over R. For this survey we need $H^p_{(2)}(F)$ in order to define a new pseudonorm on the forms $\Delta(y)$. First note

PROPOSITION 2. For each $y \in H^\ell(g\ell_q, O_q)$, exterior multiplication by $p^*\Delta(y)$ defines a bounded linear operator

$$e(y) : H^p_{(2)}(F) \;\to\; H^{p+\ell}_{(2)}(F).$$

DEFINITION 2. The e-pseudonorm on $H^\ell(g\ell_q, O_q)$ is

$$|y|_e \;=\; \operatorname*{ess.\,sup.}_{x \in M} \; |e(y)_x|$$

where $e(y)_x$ is the exterior multiplication on $H^p_{(2)}(x)$.

QUESTION 4. Does $|y|_e$ dominate the norms of the functionals $\chi_B(y)$ for $B \in \mathcal{B}$?

The general question is whether one can assemble the leaf operators $e(y)$ to get the global operator $\chi_B(y)$. For example, this is true of foliations with all leaves compact, and is the essence of our proof that the secondary classes of such foliations are always zero. There is a known relation between the leaf class $[\Delta(y)] \in H^{\ell}(L_x)$ and the transverse geometry of $\mathcal{F}$.

THEOREM 6. (Hurder [ ]) Let $\mathcal{F}$ be a C^2- foliation on an open manifold M. If all leaves of $\mathcal{F}$ have subexponential growth, then all leaf classes $[\Delta(y)] \in H^{\ell}(L_x)$ are zero.

The proof of Theorem 6 gives no estimate of the norm of the form σ_x such that $d\sigma_x = \Delta(y)$. By the next proposition, the general form of Sullivan's Conjecture would follow if one could show the family $\{\sigma_x\}$ has bounded $L_{(2)}$-norm.

PROPOSITION 3. Suppose there exists a section $\sigma \in S^{p-1}$ such that $d_F\sigma = \Delta(y)|F$. Then the Weil measure $\chi(y) = 0$, as well as $e(y) = 0$.

The hypothesis is that for each $X \in M$ there is an $L_{(2)}$-form σ_x with $d\sigma_x = \Delta(y)|L_x$ and the choice of σ_x can be done measurably. However, we are not assuming that the restrictions $\Delta(y)|L_x$ are $L_{(2)}$.

QUESTION 5. If the linear functional $\chi_B(y)$ is non-zero, does this impose any restriction on the possible $L_{(2)}$-cohomology of the leaves in B ? If all leaves in B have the same $L_{(2)}$-cohomology, must $\chi_B(y) = 0$?

Observe that $e(y)$ depends only on the value of the restrictions $\Delta(y)|L_x$ which are characteristic for the flat bundles $Q|L_x \to L_x$. This suggests inquiring just how general the "metric" h can be and still obtain the operators $\Delta(y)$ and $e(y)$.

THEOREM 7. (Hurder [ ]) Let h_0 be a metric on $Q \to M$ which :

a) is measurable on M

b) restricts to a smooth metric on each leaf $L_x \quad M$

c) has bounded local oscillation on M. That is, there are constants $\delta > 0$ and K such that if x and y are on the same leaf L_x with the leaf distance in the induced metric $h|L_x$, $\mathrm{Dist}^h_{L_x}(x,y) < \delta$, then comparing the metric h_0 at x with h_0 at y using the local holonomy γ_{xy} we have

$$|\gamma^*_{xy} h_0(y)|_{h_0(x)} < K .$$

Then the Weil measures $\chi(y)$ and the leaf operators $e(y)$ can be calculated from the forms $\Delta_0(y)$ defined leafwise by means of the metric h_0.

The proof of Theorem 7 is rather technical, involving the study of how the forms $\Delta(y)$ operate on the space of forms as we vary the choice of the metric on Q. It is suspected, though we cannot prove it at present, that the assumption on bounded leaf geometry (condition c) is unnecessary.

The typical construction for such a metric h_0 proceeds by choosing a full transversal $T \subseteq M$ to the foliation, say by covering M with flow boxes and then choosing a section for each of these. So we can assume that T is the disjoint union of closed q-discs, $T = \bigcup_i T_i$. For pairs (i,j) we have the local holonomy γ_{ij} defined from a closed subset of T_i to a closed subset of T_j. On T choose a <u>measurable</u> metric $\tilde{h}_0$ for $Q|T$ which has $|\gamma^*_{ij}\tilde{h}_0|_{\tilde{h}_0} < K$ when this is defined. Then $\tilde{h}_0$ extends to a metric h_0 satisfying the conditions of the theorem.

Some corollaries now illustrate the power of being able to calculate the operators using data which is transversally measurable.

COROLLARY 2. If a foliation $\mathcal{F}$ admits a measurable section, then all of the the Weil measures and the leaf operators $e(y)$ vanish. In particular, the residual secondary classes of $\mathcal{F}$ are zero.

Proof: For a full transversal T as above, there is a measurable subset S T such that each leaf intersects S exactly once. Now use the holonomy of to translate the restriction $h_0|S$ to all of T. This will give a well defined metric almost everywhere on T which satisfies the conditions of Theorem 7, and which is holonomy invariant. By standard methods one then sees that $\chi(y) = 0$ and $e(y) = 0$.

COROLLARY 3. Assume the von Neumann algebra $\mathfrak{M}(M,F)$ of $\mathcal{F}$ is of type I and the set of leaves with non-trivial holonomy has measure zero. Then all Weil operators of $\mathcal{F}$ are zero. In particular, the residual secondary classes of $\mathcal{F}$ vanish.

Proof: Connes' shows in [ ] that a foliation satisfying these conditions admits a measurable section.

COROLLARY 4. If the von Neumann algebra of a transversally analytic foliation $\mathcal{F}$ has type I, then all residual secondary classes of $\mathcal{F}$ vanish.

QUESTION 6. What further implications for the structure of the von Neumann algebra of $\mathcal{F}$ does the non-vanishing of the Weil measures have?

CONJECTURE 1. If the Godbillon measure $g_B \neq 0$, then B supports a type III_1 factor in $\mathfrak{M}(M,F)$.

COROLLARY 5. The Weil measures $\chi(y)$ are quasi-invariant under C^1 foliation-preserving diffeomorphisms.

QUESTION 7. Under what conditions on $\mathcal{F}$ is it possible to choose the metric h_0 as in Theorem 7 so that the forms $\Delta(y)|F$ give a section in $Z^p(F)$? That is, when can we choose a measurable metric on T so that the leaf classes are in the $L_{(2)}$-cohomology of the leaves?

For a foliation with dense leaves, an injudicious choice of metric h_0 yields classes $\Delta(y)|L_x$ with infinite norm. But given that h_0 need only be measurable, it seems the answer will depend more on the holonomy of the leaves or other aspects of the transverse dynamics.

§3. Bounds on the Norms of the Weil Operators

When the codimension q is odd and $\ell_0 = \frac{q(q-1)}{2} - 1$, the basis element $y_{max} \in H^{\ell_0}(s\ell_q, SO_q)$ is the volume form on SL_q/SO_q .

Let σ GL_q/O_q be a linear simplex. Its projection along rays to SL_q/SO_q is called a straight simplex. For $q = 2$, a straight simplex is also a geodesic simplex in H^2 under the standard identification. It is a classical result that the geodesic n-simplices in H^n have bounded volume. In his thesis, R. Savage generalized this to the straight ℓ_0-simplices in SL_q/SO_q .

THEOREM 8. (Savage [ ]) There is a constant k_q such that all straight simplices of dimension ℓ_0 in SL_q/SO_q have volume bounded by k_q .

Savage uses this result to show the higher rank spaces $\Gamma \backslash SL_q/SO_q$ for Γ cocompact have non-zero Gromov norm.

We apply Theorem 8 to the case of flat $\mathrm{Diff}^{(2)}X$ bundles over the torus T^n. Let X be a compact oriented manifold of dimension q, and $\rho : Z^n \to \mathrm{Diff}^{(2)}X$ a representation. Then the quotient $M = R^n \times_{Z^n} X$ carries a natural codimension q foliation $\mathcal{F}$ transverse to the fibers of $M \to T^n$.

THEOREM 9. a) For $n = \ell_0 + 1 + q$ and degree $c_J = 2q$, the secondary classes $\Delta_*(y_1 y_{max} c_J) \in H^{n+q}(M)$ vanish.

b) For $n = \ell_0 + q$ and degree $c_J = 2q$, the secondary classes $\Delta(y_{max} c_J) = 0$.

Heitsch [ ] and Kamber-Tondeur [ ] have shown that for $X = S^q$ there are many flat Diff S^q bundles over base spaces $B = \Gamma \backslash GL_q/O_q$ such that $\Delta_*(y_1 y_{max} c_J) \neq 0$. Thus $H^n(BDiff^\delta S^q)$ is highly non-trivial, but the above shows these classes are never non-zero on cycles of the form $T^n \to BDiff^\delta S^q$. Note that $\Delta_*(y_{max} c_J)$ is a rigid class; unfortunately, there are no examples to show that this particular class can be non-zero.

The first application of Theorem 9, for codimension $q = 3$, shows that the variable secondary classes $\Delta_*(y_1 y_3 c_J)$ vanish for a foliated bundle over the torus T^9. It is still an open problem to show the Godbillon-Vey type classes $\Delta_*(y_1 c_J)$ must vanish in this case. Note this is a special case of the generalization of Sullivan's Conjecture.

The proof of Theorem 9 uses work of D. Ellis and R. Szczarba to write the class $\Delta(y_I c_J)$ as a semi-simplicial cochain, whose coefficients are then estimated with the aid of Savage's Theorem. The point of the estimate is that it shows the norm of the class $\Delta(y_1 y_{max} c_J)$ grows at the rate r^{q+1} as the representation ρ is raised to the r-th power. However, the cohomology class $\Delta_*(y_1 y_{max} c_J)$ grows at the rate r^{ℓ_0+1}, so it must be zero ! The proof is thus in the spirit of Herman's proof that the Godbillon-Vey class $\Delta_*(y_1 c_1) = 0$ for flat bundles over T^2.

Now consider a manifold Y and a representation $\rho : \Gamma = \pi_1(Y) \to \text{Diff } X$ Suppose that $\rho(\Gamma)$ leaves invariant a probability measure $\tilde{\mu}$ on X. Then the pair $(\rho, \tilde{\mu})$ determines a family of characteristic classes for the induced measured foliation on $M = Y \tilde{\times}_\Gamma X$, the μ-classes of [Proposition 4.4;]. These are denoted $\hat{\chi}_\mu(y) \in H^\ell(Y)$ for $y \in H^\ell(g\ell_q, O_q)$. Techniques similar to those of Gromov in [ ] are used to prove :

THEOREM 10. $\hat{\chi}_\mu(y_{max}) \in H^{\ell_0}(Y)$ is a bounded cohomology class. That is, there is a constant c_q such that for each class $Z \in H_{\ell_0}(Y)$

$$| < \hat{\chi}_\mu(y_{max}), Z > | \leq c_q \cdot k_q \, |Z|_G$$

where $|Z|_G$ is the Gromov norm on homology.

COROLLARY 6. If $H^*_{bd}(Y) = 0$ then the μ-class $\hat{\chi}_\mu(y_{max}) = 0$.

For example, $\pi_1(Y)$ amenable implies $H_{bd}(Y) = 0$. Thus $\hat{x}_\mu(y_{max})$ vanishes for such base spaces Y. This is analogous to the proof that the Euler class of a flat sphere bundle over Y vanishes if $\pi_1(Y)$ is amenable.

QUESTION 8. Does Savage's Theorem generalize? Is there a bound on the integral of an arbitrary $y \in H^\ell(s\ell_q, SO_q)$ over a straight simplex of dimension ℓ in SL_q/SO_q ? If so, this would then extend Theorems 9 and 10 to these classes also.

§4. Transversally Complete Foliations

A vector field v on M preserves F if for all vectorfields w tangent to $\mathcal{F}$, the Lie bracket $[v,w]$ is tangent to $\mathcal{F}$. A foliation is transversally complete if, for all $x \in M$ and for all $w_x \in T_x M$, there is a vector field w on M which preserves F and has $w|_x = w_x$. Molino has classified such foliations, with the main result:

THEOREM 11. (Molino [ ]) Let $\mathcal{F}$ be a transversally complete foliation on a compact manifold M. Then the closure of the leaves of $\mathcal{F}$ are the fibers of a locally trivial fibration $\pi : M \to X$. Furthermore, there exists a real Lie algebra g such that, on each fiber of π, $\mathcal{F}$ induces a foliation modelled on the Lie algebra g.

It is not hard to see that Molino's Theorem implies the transverse structure group of a transversally complete foliation reduces to $g + \underline{\underline{o}}_f$ for the purposes of calculating the trace invariants. Thus we have:

THEOREM 12.(Hurder) Let $\mathcal{F}$ be a transversally complete foliation of a compact manifold M.

a) All secondary classes of $\mathcal{F}$ are zero.

b) The operators $\chi(y)$ and $e(y)$ depend only on the g-foliations which induces on the fibers $\bar{L}$ of $\pi : M \to X$. In particular, if $\pi_1(\bar{L})$ has subexponential growth, then all operators $\chi(y)$ and $e(y)$ vanish.

Compare Theorem 12b to the result of Ghys [ ] which shows that must be Riemannian under similar circumstances.

If the secondary classes of $\mathcal{F}$ are non-trivial, one sees from this Theorem that the foliation $\mathcal{F}$ cannot be too homogeneous. Also, Theroem 12a is an extention of the vanishing results for G-foliations which Kamber-Tondeur give in [ ].

As all leaves in a transversally complete foliation are diffeomorphic, the above suggests:

QUESTION 9. If all leaves of $\mathcal{F}$ are diffeomorphic, do the secondary classes of $\mathcal{F}$ vanish? If some secondary class of $\mathcal{F}$ is non-zero, what can be said about the number of distinct diffeomorphism types of leaves of $\mathcal{F}$? Can they be finite in number?

§5. The Godbillon Invariant of a Diffeomorphism

Let $f : X \to X$ be a C^1-diffeomorphism of a compact manifold X^q. Choose a volume form ω on X.. Define div f(x) by $f^*\omega|_y = \text{div } f(y)\omega|_y$.

DEFINITION 3. For each $x \in X$, the Godbillon measure of f at x is

$$g_x(f) = \lim_{k\to\infty} \frac{1}{2k} \sum_{n=-k}^{n=k} \log \{\text{div } f(f^n(x))\}.$$

PROPOSITION 4. $g_x(f)$ is independent of the choice of ω, and depends only on the orbit class of x .

Proof: The suspension of f gives a codimension q foliation $\mathcal{F}$ of $M = R \underset{Z}{\times} X$. The choice of x defines a transverse invariant measure μ_x for $\mathcal{F}$. Note that μ_x depends only on the orbit of x under f. Then $g_x(f) = 2\pi\cdot\langle \chi_{\mu_x}(y_1), M\rangle$ is the first measure class for this foliation. The definition is derived using the formulae for the measure classes of Ellis-Szczarba [ ]. The form y_1 is calculated with respect to the volume ω, and it is standard that the resulting class is independent of ω.

PROPOSITION 5. $g_x(f) = 0$ if f preserves an isotropic probability measure μ with support $\mu = X$.

Proof: This follows using techniques similar to those in Theorem 4.10 of [ ].

The Godbillon measure appears to be a new invariant of C^1-diffeomorphisms. We introduce it because it would be interest~~e~~ing to know if $g_x(f)$ is a topological invariant of f. Clearly for x a fixed point of f, this is <u>not</u> the case. However, for x a dense orbit, it seems reasonable to ask whether $g_x(f)$ is invariant. A test case would be:

QUESTION 10. For $f : S^1 \to S^1$ a diffeomorphism with all orbits dense, must $g_x(f) = 0$?

Such diffeomorphisms are topologically conjugate to a rotation, for which $g_x(f) = 0$ is immediate.

QUESTION 11. For $f : X \to X$ a Morse-Smale diffeomorphism, is there a simple expression for $g_x(f)$ in terms of the behavior of f on the non-wandering set?

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