J. DIFFERENTIAL GEOMETRY
20 (1984) 291–309

SECONDARY CLASSES, WEIL MEASURES AND THE GEOMETRY OF FOLIATIONS

J. HEITSCH & S. HURDER

Introduction

One of the main problems in foliation theory is to understand how the topology of the leaves and the transversal geometry of a foliation influence the values of its differential invariants, especially the secondary characteristic classes. In this paper we use the Chern-Weil theory of characteristic classes to define a set of operators canonically associated to a C^2-foliation $\mathscr{F}$ on a manifold M. These are called the *Weil operators* as they capture the essence of the Weil approach to characteristic classes. The Weil operators determine the *residual* secondary classes, and the aim of this paper is to study the properties of these operators, especially their dependence on the geometry of a foliation, so as to gain a better understanding of how the secondary classes are related to geometry.

The outline of this paper is as follows. The Weil operators are defined in §1 and we describe their elementary properties. In §2 we restrict attention to a compact foliated manifold M and prove the existence of the Weil measures. Let $\mathscr{B} = \mathscr{B}(\mathscr{F})$ denote the Σ-algebra of measurable saturated sets in M, so $\mathscr{B}$ is the set of measurable subsets of the quotient $M/\mathscr{F}$. Theorem 2.1 shows that each Weil operator yields a vector-valued measure on $\mathscr{B}$. For a measurable saturated subset $B \in \mathscr{B}$ and a residual class $y_I c_J \in H^p(WO_n)$, this implies a localization theorem: *there is a well-defined restriction* $\Delta_*(y_I c_J)|B \in H^p(M)$. The Weil measures of B can be calculated from bounded transverse data specified in a neighborhood of B by Theorems 2.7 and 2.9, so the value of $\Delta_*(y_I c_J)|B$ can sometimes be determined just from the restriction $\mathscr{F}|B$. Theorems 3.1, 4.3 and 4.12 give geometric hypotheses on $\mathscr{F}|B$ which are sufficient to imply certain classes $\Delta_*(y_I c_J)|B = 0$. In particular, Corollary 4.4

Received June 27, 1983 and, in revised form, July 31, 1984. The first author was supported in part by National Science Foundation grant number MCS 83-01655, and the second by grant number MCS 82-01604.

generalizes Herman's vanishing theorem for the Godbillon-Vey class of a foliation of T^3 without holonomy [7].

The technical advantage to the Weil measures is that they depend only on first-order properties of $\mathscr{F}$ and so can be estimated using approximation techniques. In addition, they are directly related to the ergodic theory of $\mathscr{F}$. This is contrasted with the secondary classes, which have Chern form factors of second order, making them very difficult to estimate.

This work was inspired by the seminal paper of Duminy [6], and combines our extension of Duminy's approach to all codimensions with the results of [9]. For codimension one, Duminy defined the Godbillon operator and measure (Definition 1.5 and Corollary 2.2 below) and used them to settle a question raised by Moussu-Pelletier [15] and Sullivan [19]:

Theorem (*Duminy*). *Let $\mathscr{F}$ be a C^2-foliation of codimension-one on a compact manifold M. If the Godbillon-Vey class $\Delta_*(y_1c_1)$ is not zero, then the set of leaves of $\mathscr{F}$ with exponential growth has positive measure.*

Recent progress on the extension of this theorem to the Godbillon-Vey classes in all codimensions is given in [10]. Furthermore, a much broader generalization to the other residual secondary classes, involving amenability of $\mathscr{F}$ instead of nonexponential growth, is given in [11]. All of these results rely heavily on the use of the Godbillon and Weil measures.

The authors are grateful to G. Duminy for providing a preprint of this work and to L. Conlon for explaining his work to us. We are also indebted to P. Schweitzer for helpful remarks and to the referee for suggesting simplified proofs of Theorems 2.1 and 2.9. The first author was supported in part by the Institute for Advanced Study, whose support is gratefully acknowledged.

1. The Weil operators

Let $\mathscr{F}$ be a codimension n, C^2-foliation on a smooth manifold M. Let $A(M, \mathscr{F})$ denote the defining ideal for $\mathscr{F}$ in the deRham complex $A(M)$ of M. If the normal bundle $Q \to M$ of $\mathscr{F}$ is orientable, then there is a nonvanishing n-form ω on M whose kernel defines $\mathscr{F}$, and $A(M, \mathscr{F})$ consists of the p-forms ϕ on M, $p \geqslant n$, which have a factorization $\phi = \hat{\phi} \wedge \omega$ for some $(p-n)$-form $\hat{\phi}$. For Q nonorientable, ω is only locally defined and we require that $\phi \in A(M, \mathscr{F})$ have a local factorization as $\hat{\phi} \wedge \omega$. The integrability of $\mathscr{F}$ implies $d\omega = \eta \wedge \omega$ for some 1-form η, which implies $A(M, \mathscr{F})$ is a differential ideal.

Definition 1.1. $H^*(M, \mathscr{F}) = H^*(A(M, \mathscr{F}), d)$.

Given closed forms $\psi \in A(M)$ and $\phi \in A(M, \mathscr{F})$ we set $[\psi]\cdot[\phi] = [\psi \wedge \phi] \in H^*(M, \mathscr{F})$, making $H^*(M, \mathscr{F})$ into a module over $H^*(M)$. The structure of the module $H^*(M, \mathscr{F})$ is almost completely unknown, except that it may be infinite-dimensional. For example, if $\mathscr{F}$ is defined by a closed n-form ω, then

$$H^n(M, \mathscr{F}) = \{\text{smooth functions on } M, \text{ constant along the leaves}\}.$$

For $\mathscr{F}$ with a dense leaf, $H^n(M, \mathscr{F}) = R$. For $\mathscr{F}$ defined by a fibration $\pi: M \to X^n$, $H^n(M, \mathscr{F}) \cong C^\infty(X)$.

Next, we briefly recall the construction of the secondary classes of $\mathscr{F}$. Complete details can be found in [1], [3] and [13]. Let ∇^b be a basic connection on the normal bundle $Q \to M$, and let r denote a Riemannian metric on Q with associated torsion-free connection ∇^r. For any Chern monomial $c_J = c_1^{j_1} \cdots c_n^{j_n}$ of degree $2l$ on the Lie algebra gl_n, let $c_J(\nabla^b) \in A^{2l}(M)$ be the closed form obtained by applying c_J to the curvature matrix of ∇^b. If degree $c_J = 2n$, then $c_J(\nabla^b) \in A^{2n}(M, \mathscr{F})$. If degree $c_J > 2n$, then the form $c_J(\nabla^b)$ is identically zero, which is the strong form of the Bott Vanishing Theorem [1].

We also define forms $\bar{y}_i \in A^{2i-1}(M)$ by

$$\bar{y}_i = \Delta_{c_i}(\nabla^b, \nabla^r) = \int_0^1 i(\partial/\partial t) c_i(\nabla^t)\, dt,$$

where $\nabla^t = (1-t)\nabla^b + t\nabla^r$ is the connection on Q interpolating between ∇^b and ∇^r. For i odd we have $d\bar{y}_i = c_i(\nabla^b)$. Define a complex

$$WO_n = \Lambda(y_1, y_3, \cdots, y_{n'}) \otimes R[c_1, \cdots, c_n]_n,$$

where n' is the greatest odd integer $\leqslant n$, and the second factor is the graded polynomial algebra generated by the Chern polynomials, truncated in degrees above $2n$. The differential is defined by $d(y_i \otimes 1) = 1 \otimes c_i$ and $d(1 \otimes c_j) = 0$. Let Δ: $WO_n \to A^*(M)$ be the map of differential algebras, defined on the generators by $\Delta(y_i \otimes 1) = \bar{y}_i$ and $\Delta(1 \otimes c_j) = c_j(\nabla^b)$.

Proposition 1.2. *The induced map on cohomology $\Delta_* H^*(WO_n) \to H^*(M)$ is independent of the choice of connection ∇^b and metric r.*

The image of Δ_* consists of the characteristic classes of $\mathscr{F}$. The *residual* secondary classes are those $\Delta_*(y_I c_J)$ with degree $c_J = 2n$. Let $H_n^*(WO_n)$ denote the subspace of $H^*(WO_n)$ spanned by the classes $y_I c_J$ with degree $c_J = 2n$.

If the normal bundle $Q \to M$ has a framing, denoted by s, then there are additional secondary classes for the pair $(\mathscr{F}, s)$. Let ∇^s be a connection on Q for which s is parallel. Define $\bar{y}_i = \Delta_{c_i}(\nabla^b, \nabla^s)$ as before, and set

$$W_n = \Lambda(y_1, y_2, \cdots, y_n) \otimes \mathbf{R}[c_1, \cdots, c_n]_n$$

with $d(y_i \otimes 1) = 1 \otimes c_i$ and $d(1 \otimes c_i) = 0$ for $1 \leqslant i \leqslant n$. There is a map Δ^s_*: $H^*(W_n) \to H^*(M)$ which now depends only on the homotopy class of s. For convenience we sometimes abuse notation and write Δ_* for Δ^s_*.

Lemma 1.3. *For each c_J of degree $2n$ and each $y_I = y_{i_1} \wedge \cdots \wedge y_{i_s} \otimes 1 \in WO_n$, where $y_I = 1$ is possible, there is a well-defined class $[\Delta(y_I c_J)] \in H^*(M, \mathscr{F})$.*

Lemma 1.4. *Let $Q \to M$ be trivial with framing s. For each c_J of degree $2n$ and each $y_I \in W_n$, there is a well-defined class $[\Delta^s(y_I c_J)] \in H^*(M, \mathscr{F})$ which depends only on the homotopy class of s.*

Proof of Lemmas 1.3 *and* 1.4. It was noted above that $d\Delta(y_I c_J) = 0$ and $\Delta(y_I c_J) \in A^*(M, \mathscr{F})$ so only the independence of the choice of connections must be shown. We prove that $[\Delta(c_J)] \in H^{2n}(M, \mathscr{F})$ is well defined, and then note the other cases follow similarly, using the methods of [1]. Let $\nabla^{b'}$ be another basic connection on Q and set $\nabla^t = (1-t)\nabla^b + t\nabla^{b'}$. Then

$$c_J(\nabla^{b'}) - c_J(\nabla^b) = d\Delta_{c_J}(\nabla^{b'}, \nabla^b) = d\int_0^1 i(\partial/\partial t) c_J(\nabla^t)\, dt.$$

Basic connections form a convex set, so ∇^t is basic on $M \times R$ for the codimension n foliation induced from $\mathscr{F}$, and hence $\int_0^1 i(\partial/\partial t) \cdot c_J(\nabla^t)\, dt \in A^{2n-1}(M, \mathscr{F})$. q.e.d.

Let $I = \Gamma(M, Q^*) \wedge A(M)$ be the ideal in $A(M)$ of forms whose restriction to $\mathscr{F}$ is zero. Then $I^n = A(M, \mathscr{F})$, $I^{n+1} = 0$. There is a spectral sequence $E_r^{p,q}(A(M), I)$ associated to the filtration of $A(M)$ by the powers of I, which generalizes the Leray-Hirsch spectral sequence of a fibration and has been considered by many authors. Kamber and Tondeur show in [12] that Δ induces a multiplicative map of WO_n into $E_2^{p,q}(A(M), I)$. Observing that $H^p(M, \mathscr{F}) \cong E_2^{n,p-n}(A(M), I)$ yields an alternate proof of Lemmas 1.3 and 1.4.

For notational convenience we identify $H^*(\mathrm{gl}_n, O_n) \cong \Lambda(y_1, y_3, \cdots, y_{n'})$ and $H^*(\mathrm{gl}_n) \cong \Lambda(y_1, y_2, \cdots, y_n)$. For $y \in H^p(\mathrm{gl}_n, O_n)$ set $\Delta(y) = \bar{y}$, a p-form on M.

For each $p \geqslant 0$ define a map χ: $H^p(\mathrm{gl}_n, O_n) \to \mathrm{Hom}(H^*(M, \mathscr{F}), H^{*+p}(M))$, where, for y of degree p and closed $\phi \in A(M, \mathscr{F})$, $\chi(y)[\phi] = [\bar{y} \wedge \phi]$. Observe $d(\bar{y} \wedge \phi) = d\bar{y} \wedge \phi = 0$ as $d\bar{y} \wedge \omega = 0$. It is straightforward to check that the cohomology class $[\bar{y} \wedge \phi] \in H^{*+p}(M)$ is independent of the choice of basic connection ∇^b, metric r on Q and representative ϕ of $[\phi]$. Thus, the map χ is well defined and depends only on $\mathscr{F}$. Extend χ to all of $H^*(\mathrm{gl}_n, O_n)$ by linearity.

Defintion 1.5. For each $y \in H^*(\mathrm{gl}_n, O_n)$, the functional $\chi(y)$ is the *Weil operator* associated to y. Let

$$y_1 = \frac{-1}{2\pi}\mathrm{tr} \in H^1(\mathrm{gl}_n, O_n)$$

be the normalized trace class. The *Godbillon operator* is the map (cf. [6])

$$g = -2\pi \cdot \chi(y_1): H^*(M, \mathscr{F}) \to H^{*+1}(M).$$

Let $H^*(M)$ have the vector space topology. We define a topology on $H^*(M, \mathscr{F})$ for which all $\chi(y)$ are continuous. Give $A(M, \mathscr{F})$ the compact-open C^∞-topology, and let $Z^p(M, \mathscr{F}) \subset A^p(M, \mathscr{F})$ denote the closed subspace of cocycles. Let $\overline{B^p(M, \mathscr{F})}$ denote the closure of the image $B^p(M, \mathscr{F})$ of d: $A^{p-1}(M, \mathscr{F}) \to Z^p(M, \mathscr{F})$. The quotient $\overline{H^p}(M, \mathscr{F}) = Z^p(M, \mathscr{F})/\overline{B_p(M, \mathscr{F})}$ is a topological vector space, and the quotient map $H^p(M, \mathscr{F}) \to \overline{H^p}(M, \mathscr{F})$ induces a topology on $H^p(M, \mathscr{F})$. If $\phi \in Z^p(M, \mathscr{F})$ is the limit of forms $\{d\psi_i\} \subset B^p(M, \mathscr{F})$, then

$$\chi(y)[\phi] = [\bar{y} \wedge \phi] = \left[\bar{y} \wedge \lim_{i \to \infty} d\psi_i\right] = \lim_{i \to \infty} [d(\bar{y} \wedge \psi_i)] = 0.$$

Thus, there is induced a continuous map $\overline{\chi(y)}$: $\overline{H}^*(M, \mathscr{F}) \to H^*(M)$, which implies $\chi(y)$ is continuous on $H^*(M, \mathscr{F})$.

If $\mathscr{F}$ is defined by a *closed* decomposable n-form ω on M, then we say $\mathscr{F}$ is an SL_n-foliation and ω defines an invariant transverse measure for $\mathscr{F}$. There is an associated nonzero class $[\omega] \in H^n(M, \mathscr{F})$. The Weil operators applied to $[\omega]$ define a natural map

$$\chi_\omega(\_) = \chi(\_)[\omega]: H^*(\mathrm{gl}_n, O_n) \to H^{*+n}(M).$$

The classes in the image of χ_ω are a special case of the μ-classes studied in [9].

The measure $\chi(y_I)$ determines the values of all residual secondary classes $\Delta_*(y_{I'}c_J)$ for which $I \subset I'$, where $y_{I'}c_J \in WO_n$ or W_n. To see this, let $I'' = I' - I$ and observe Lemmas 1.3 and 1.4 imply $\Delta_*(y_{I'}c_J) = \pm\chi(y_I)[\Delta(y_{I''}c_J)]$. This yields immediately

Proposition 1.6. *If* $\chi(y_I) = 0$, *then all residual secondary classes* $\Delta_*(y_{I'}c_J)$ *with* $I \subset I'$ *are zero.*

Now assume M is a closed oriented m-manifold. For $y \in H^p(\mathrm{gl}_n, O_n)$ there is a continuous linear map $\chi(y)$: $H^{m-p}(M, \mathscr{F}) \to R$, defined by $\chi(y)[\phi] = \int_M \bar{y} \wedge \phi$. Poincaré duality for M then yields

Proposition 1.7. *For each* $y \in H^p(\mathrm{gl}_n, O_n)$, *the map* $\chi(y) \in \mathrm{Hom}_{\mathrm{cont}}(H^{m-p}(M, \mathscr{F}), R) \equiv H^{m-p}(M, \mathscr{F})^*$ *completely determines the operator* $\chi(y)$.

2. Properties of the Weil measures

For the remainder of this paper, M is closed oriented m-manifold and $\mathscr{F}$ is a fixed codimension n foliation on M. Choose a Riemannian metric h on TM, which defines an embedding $Q \to TM$ as the space of vectors perpendicular to

$\mathscr{F}$. For each $x \in M$ and $l \geqslant 0$, h defines a norm on the spaces $\Lambda^l T_x M$, $\Lambda^l T_x^* M$ and $\Lambda^l Q_x^*$, all denoted by $\| \, \|_x$. For a measurable form ψ on M we take $\|\psi\| = \sup_{x \in M} \|\psi\|_x$ and ψ is bounded if $\|\psi\| < \infty$. Let $\mathbf{m}$ denote the Lebesgue measure on M associated to the volume form of h.

Let π: $M \to M/\mathscr{F}$ denote the map onto the (generally non-Hausdorff) quotient space of $\mathscr{F}$. A set $B \subset M$ is *saturated* if it is the union of leaves of $\mathscr{F}$, or equivalently $B = \pi^{-1}(\pi B)$. Let $\mathscr{B} = \mathscr{B}(\mathscr{F})$ denote the Σ-algebra of $\mathbf{m}$-measurable saturated subsets of M. Then $\mathscr{B}$ is isomorphic to the Σ-algebra of measurable sets for the quotient measure space $(M/\mathscr{F}, \pi_* \mathbf{m})$. Let $B^\infty(M/\mathscr{F})$ denote the algebra of essentially bounded functions on M, which are measurable relative to $\mathscr{B}$, modulo the subalgebra of functions which are almost everywhere zero. The foliation $\mathscr{F}$ is *ergodic* if and only if $B^\infty(M/\mathscr{F}) = R$.

Theorem 2.1. (a) *Let $B \subset M$ be a saturated measurable subset. For each positive integer p, there is a well-defined linear map*

$$\chi_B \colon H^p(\mathrm{gl}_n, O_n) \to H^{m-p}(M, \mathscr{F})^*.$$

(b) *For $B = M$ and $y \in H^p(\mathrm{gl}_n, O_n)$, $\chi_M(y) = \chi(y)$.*

(c) *χ is continuous with respect to* $\mathbf{m}$: *If* $\mathbf{m}(B) = 0$, *then* $\chi_B = 0$.

(d) *χ is countably additive on $\mathscr{B}$.*

Corollary 2.2. *For each $y \in H^p(\mathrm{gl}_n, O_n)$, $\chi(y)$ defines an $H^{m-p}(M, \mathscr{F})^*$-valued countably additive measure on $\mathscr{B}$ which is continuous with respect to $\pi_* \mathbf{m}$ on $M/\mathscr{F}$.*

For $p = 1$, we set $g = -2\pi \cdot \chi(y_1)$ and following Duminy [6] call this the *Godbillon measure* for $\mathscr{F}$. For y of degree $p \geqslant 1$, $\chi(y)$ is called the *Weil measure* associated to y. The main problem is to determine what properties of the geometry of $\mathscr{F}$ the Weil measures "measure".

Corollary 2.3. *There is a bilinear pairing*

$$R \colon B^\infty(M/\mathscr{F}) \times H_n^*(WO_n) \to H^*(M)$$

such that $R(1, \ ) = \Delta_*(\ )$.

Proof. We first define R on the space of step functions in $B^\infty(M, \mathscr{F})$. Recall that f: $M \to R$ is a step function if there is a countable collection of disjoint sets $\{B_i | i = 1, 2, \cdots\} \subset \mathscr{B}$ and a bounded sequence of real numbers $\{a_1, a_2, \cdots\}$ so that $f = \sum_{i=1}^\infty a_i e_{B_i}$, where e_{B_i}: $M \to R$ is the characteristic function for B_i. Given $y_I c_J \in H_n^l(WO_n)$ and $[\psi] \in H^{m-l}(M)$, set

$$\langle R(f, y_I c_J) \cup [\psi], [M] \rangle = \sum_{i=1}^{\infty} a_i \cdot \chi_{B_i}(y_I)[\Delta(c_J) \wedge \psi].$$

The expression $\chi_B(y_I)[\Delta(c_J) \wedge \psi]$ is uniformly bounded for $B \in \mathscr{B}$ and continuous with respect to $\pi_* \mathbf{m}$, so the sum on the right is finite and thus determines $R(f, y_I c_J)$ by Poincaré duality as we let $[\psi]$ run through $H^{m-l}(M)$. Continuity of the measure $\chi(y_I)$ with respect to $\pi_* \mathbf{m}$ implies $R(\_\_, y_I c_J)$ extends to the L^∞-completion of the step functions in $B^\infty(M/\mathscr{F})$, which is all of this space.

Definition 2.4. Given $B \in \mathscr{B}$ and $y_I c_J \in H^l_n(WO_n)$, the localization of $\Delta_*(y_I c_J)$ to B is the class

$$\Delta_*(y_I c_J)|B = R(e_B, y_I c_J) \in H^l(M).$$

Proof of Theorem 2.1. Given $y \in H^p(\mathrm{gl}_n, O_n)$, $B \in \mathscr{B}$ and $[\phi] \in H^{m-p}(M, \mathscr{F})$ set

$$\chi_B(y)[\phi] = \int_B \bar{y} \wedge \phi.$$

We first show $\chi_B(y)[\phi]$ is well defined.

Lemma 2.5. *Let ∇^b and $\nabla^{b'}$ be basic connections on Q, and ∇^r and $\nabla^{r'}$ metric connections on Q. Then*

$$\Delta_{c_i}(\nabla^{b'}, \nabla^{r'}) - \Delta_{c_i}(\nabla^b, \nabla^r) = d\nu_i + \omega_i,$$

where $\omega_i \in \Gamma(M, Q^* \wedge \Lambda^{2i-2}T^*M)$.

Proof. Let π: $M \times R \to M$ denote the projection and set $\nabla^t = t \cdot \nabla^{b'} + (1 - t)\nabla^b$, which is a basic connection on $\pi^*Q \to M \times R$. Let $r(t)$ be a smooth metric on π^*Q such that $r(t) = r'$ for t near 1 and $r(t) = r$ for t near 0, and let $\nabla^{r(t)}$ denote the associated torsion-free connection on π^*Q. The lemma then follows by applying Theorem 3.10 of [1] to the form $\Delta_{c_i}(\nabla^t, \nabla^{r(t)})$ on $M \times I$.

To prove $\chi_B(y)[\phi]$ is well defined, it suffices by Lemma 2.5 to show $\int_B d\tau = 0$ whenever $\tau \in A^{m-1}(M, \mathscr{F})$. We will prove a more general statement than this. Say a form τ on M is *measurable* if the coefficients of τ in every smooth coordinate neighborhood on M are measurable, and τ is *bounded* if there is a finite covering of M by smooth coordinate charts such that the coefficients of τ in these charts are bounded. Say τ is *leafwise smooth* if for each leaf $L \subset M$, the restricted map $\tau|L$: $\Lambda^p TM|L \to R$ is smooth, where τ is a p-form. For τ leafwise smooth, suppose it can be expressed locally as $\tau = \hat{\tau} \wedge \theta$, where θ is an n-form defining $\mathscr{F}$ locally. We then say that τ has maximal transverse rank. By applying exterior differentiation to τ only in leaf directions, we obtain a well-defined form $d_{\mathscr{F}}\tau$, which is equal to $d\tau$ when τ is smooth. For a leafwise smooth function f on M, we define $d_{\mathscr{F}}f$ to be the exterior derivative of f along leaves composed with the projection $TM \to T\mathscr{F}$ determined by the metric h on TM.

Proposition 2.6 (*Leafwise Stokes' Theorem*). *Let τ be a bounded measurable $(m-1)$-form on M with maximal transverse rank, and assume τ is leafwise smooth and $d_{\mathscr{F}}\tau$ is bounded. Then for all $B \in \mathscr{B}$,*

$$\int_B d_{\mathscr{F}}\tau = 0.$$

Proof. Let $I_a = (-a, a)$ be the open interval and set $I_a^m = I_a \times \cdots \times I_a$, m-copies. A *foliation chart* (U, f, g) for $\mathscr{F}$ is a surjective coordinate chart $f\colon U \to I_a^m$, where $U \subset M$ is open and the composition $g\colon U \xrightarrow{f} I_a^m = I_a^{m-n} \times I_a^n \to I_a^n$ maps the connected components of the leaves of $\mathscr{F}|U$ onto the points of I_a^n. A chart (U, f, g) onto I_a^n is *regular* if there is a foliation chart $(\tilde{U}, \tilde{f}, \tilde{g})$, where $U \subset \tilde{U}$, $\tilde{f}\colon \tilde{U} \to I_b^m$ for $b > a$ and $\tilde{f}|U = f$. We can always assume a foliation chart is onto $I^m \equiv I_1^m$, and if regular, has an extension onto I_2^m.

Choose a finite covering of M by foliation charts $\{(U_i, f_i, g_i)|1 \leqslant i \leqslant d\}$ so that in the local coordinates determined by each f_i, the coefficients of τ and $d_{\mathscr{F}}\tau$ are bounded on I^m. Let $\{\lambda_i|1 \leqslant i \leqslant d\}$ be a subordinate partition of unity. Then $\tau = \sum_{i=1}^d \tau_i$, where $\tau_i = \lambda_i \cdot \tau$ and τ_i has compact support in U_i. Then $\int_B d_{\mathscr{F}}\tau = \sum_{i=1}^d \int_B d_{\mathscr{F}}\tau_i$, so we can assume τ has compact support in some U_i. Let $d\bar{x}$ denote the Euclidean volume form on R^n restricted to I^n, and let $\omega_i = g_i^* d\bar{x}$ be the closed n-form on U_i. By assumption we can write $\tau = \hat{\tau} \wedge \omega_i$ for $\hat{\tau}$ a bounded measurable $(m-n-1)$-form on U_i which is leafwise smooth and has compact support in U_i. Then $d_{\mathscr{F}}\tau = d_{\mathscr{F}}\hat{\tau} \wedge \omega_i$ as $d_{\mathscr{F}}\omega_i = 0$. Since $\hat{\tau}$ is bounded we have for $B_i = g_i(B \cap U_i)$,

$$\begin{aligned} \int_B d_{\mathscr{F}}\tau &= \int_{x \in B_i} \left\{ \int_{I_x^{m-n}} d_{\mathscr{F}}\hat{\tau} \right\} \cdot \omega_i \\ &= \int_{x \in B_i} \left\{ \int_{\partial I_x^{m-n}} \hat{\tau} \right\} \wedge d\bar{x} \\ &= \int_{B_i} 0 \cdot d\bar{x} = 0 \quad \text{as } \hat{\tau}|\partial I_x^{m-n} \equiv 0. \end{aligned}$$

Proposition 2.6 is proved.

For part (a) of Theorem 2.1 we need only note that if $\tau \in A^{m-1}(M, \mathscr{F})$, then τ satisfies the conditions of Proposition 2.6 and $d\tau = d_{\mathscr{F}}\tau$. Parts (b) and (c) of the theorem now follow immediately. For (d), observe that if $\{B_i | i = 1, 2, \cdots\} \subset \mathscr{B}$ is a countable disjoint collection, then for $B = \bigcup_{i=1}^\infty B_i$, we have

$$\chi_B(y_I)[\phi] = \int_B \bar{y} \wedge \phi = \sum_{i=1}^\infty \int_{B_i} \bar{y} \wedge \phi = \sum_{i=1}^\infty \chi_{B_i}(y)[\phi].$$

Theorem 2.1 is now proved.

The localization result in Definition 2.4 has been previously observed for the restriction of the Godbillon-Vey class to *open* saturated sets in codimension one, and this plays an important role in the results of [4], [6], [14] and [16]. In these papers, B was required to be open because their proofs of localization used the structure theory of open saturated sets in codimension one.

A decisive advantage of the Weil measures is that they can be calculated locally: $\chi_B(y)$ depends only on the linear part of the normal Γ-cocycle to $\mathcal{F}$ restricted to B. This is the content of the next two results, which generalize Lemma 2 of [6]. Let $v \in \Gamma(M, \Lambda^n Q)$ be an n-vector field on M with $\|v\|_x = 1$ for all $x \in M$. For the next result, we assume that Q is orientable which implies that such an n-vector v exists.

Theorem 2.7. *Let $B \in \mathcal{B}$. Let ρ be an n-form defined in an open neighborhood $U \subset M$ of B such that ρ defines $\mathcal{F}|U$ and the 1-form*

$$\eta = \frac{(-1)^n}{\rho(v)} \cdot i(v)\, d\rho$$

has bounded norm on B. For B open, $U = B$ is allowed. Then the Godbillon measure of B can be calculated using η:

$$g_B[\phi] = \int_B \eta \wedge \phi \quad \textit{for all } [\phi] \in H^{m-1}(M, \mathcal{F}).$$

Proof. Let θ be the n-form on M defining $\mathcal{F}$ and satisfying $\theta(v) \equiv 1$. Set

$$\bar{y}_1 = \frac{-1}{2\pi} \cdot (-1)^n \cdot i(v)\, d\theta.$$

Then $d\theta = -2\pi \cdot \bar{y}_1 \wedge \theta$ and it is well known that $\bar{y}_1 = \Delta(y_1)$ for some basic connection ∇^b on $Q \to M$ (cf. [13, pp. 155–159]) so $\bar{y}_1$ can be used to calculate g_B.

Define a C^2-function $f\colon U \to R$ by requiring $\rho = \exp f \cdot \theta$ on U, and note

$$\begin{aligned}\eta &= \frac{(-1)^n}{\rho(v)} \cdot i(v)\, d\rho \\ &= \frac{(-1)^n}{\exp f} \cdot i(v)\{d(\exp f) \wedge \theta + \exp f \wedge d\theta\} \\ &= (-1)^n \cdot i(v)(df \wedge \theta) - 2\pi \cdot \bar{y}_i.\end{aligned}$$

Both $\bar{y}_1$ and η are bounded on B, so $i(v)(df \wedge \theta)$ must be bounded on B. Noting that $(-1)^n \cdot i(v)(df \wedge \theta) \wedge \phi = d_{\mathcal{F}}(f \wedge \phi)$ for $\phi \in Z^{m-1}(M, \mathcal{F})$, we have

$$g_B[\phi] = -2\pi \int_B \bar{y}_1 \wedge \phi = \int_B \eta \wedge \phi - \int_B d_{\mathcal{F}}(f \wedge \phi).$$

The idea is to use Proposition 2.6 to conclude $\int_B d_{\mathscr{F}}(f \wedge \phi) = 0$, but $f \wedge \phi$ need not be bounded on B. To circumvent this, we employ a trick due to Duminy.

Lemma 2.8. *For all $N > 0$ there exists a smooth function $f_N\colon U \to R$ such that*

(a) $|f_N(x)| \leqslant N$ *for all* $x \in U$.

(b) $\|d_{\mathscr{F}} f_N\|_x \leqslant \|d_{\mathscr{F}} f\|_x$ *for all* $x \in U$.

(c) Support$(f - f_N) \to \phi$ *as* $N \to \infty$.

Proof. For $N > 0$ choose a smooth function $\xi_N\colon R \to R$ such that

$$\xi_N(s) = \begin{cases} N & \text{for } s \geqslant N + 1, \\ s & \text{for } 1 - N \leqslant s \leqslant N - 1, \\ -N & \text{for } x \leqslant -N - 1 \end{cases}$$

and with $|\xi'_N(x)| \leqslant 1$ for all s. Set $f_N = \xi_N \circ f$. Then $d_{\mathscr{F}} f_N = \xi'_N \circ f \cdot d_{\mathscr{F}} f$, so $\|d_{\mathscr{F}} f_N\|_x \leqslant 1 \cdot \|d_{\mathscr{F}} f\|_x$ which implies (b). Parts (a) and (c) are then clear.

By Lemma 2.8, for each N the form $f_N \wedge \phi$ satisfies Proposition 2.6 so $\int_B d_{\mathscr{F}}(f_N \wedge \phi) = 0$. Define $\rho_N = \exp f_N \cdot \theta$ with corresponding

$$\eta^N = (-1)^n \cdot i(v)(d_{\mathscr{F}} f_N \wedge \theta) - 2\pi \cdot \bar{y}_1.$$

Observe that $\eta^N \to \eta$ pointwise on U and

$$\|\eta^N\|_x \leqslant \|d_{\mathscr{F}} f_N\|_x + 2\pi\|\bar{y}_1\| \leqslant \|d_{\mathscr{F}} f\| + 2\pi \cdot \|\bar{y}_1\|$$

is uniformly bounded on B. By the Dominated Convergence Theorem,

$$g_B[\phi] = -2\pi \int_B \bar{y}_1 \wedge \phi = \lim_{N\to\infty} \int_B \eta^N \wedge \phi = \int_B \lim_{N\to\infty} \eta^N \wedge \phi = \int_B \eta \wedge \phi.$$

Lemma 2.8 is proved.

Recall that h is the fixed metric on TM, and let $\bar{h}$ denote the metric on $Q \to M$ induced by h. Then h and $\bar{h}$ define fiberwise metrics and norms on all tensor algebra bundles associated to Q and T, which we will again denote by $\|\ \|_x$ for $x \in M$. Let ∇^h denote the connection on Q associated to $\bar{h}$.

Theorem 2.9. *Let $B \in \mathscr{B}$. Given an open neighborhood $U \subset M$ of B, let $\nabla^{b'}$ be a basic connection for $\mathscr{F}|U$ on $Q|U \to U$, and let r be a Riemannian metric on $Q|U$ with connection ∇^r. Suppose there exists an upper bound K for*

(a) $\|r\|_x$ *and* $\|r^{-1}\|_x$ *for all* $x \in B$;

(b) *the partial derivatives of r in the leaf directions on B:* $\|\nabla^h_v r\|_x \leqslant K$ *for all* $x \in B$ *and all* $v \in T_x\mathscr{F}$ *with* $\|v\|_x = 1$.

Then for all $y \in H^*(\mathrm{gl}_n, O_n)$,

$$\chi_B(y)[\phi] = \int_B \bar{y}' \wedge \phi,$$

where $\bar{y}'$ is the representative form for y given by the product of forms $\Delta_{c_i}(\nabla^{b'}, \nabla^r)$ on U.

Proof. It suffices to show that each $\chi_B(y_i)$ has this local representation. Let $[\phi] \in H^{m+1-2i}(M, \mathscr{F})$. On $U \times I$ define $\nabla^t = (1-t)\nabla^b + t \cdot \nabla^{b'}$, a basic connection for $\mathscr{F}|_U \times I$. Choose a smooth path $r(t)$ of metrics on Q from $\bar{h}$ to r so that $r(t)$ is bounded on B and has bounded leafwise partial derivatives on B for all t. Let $\nabla^{r(t)}$ be the metric connection on $Q|_U \to U \times I$ associated to $r(t)$. Define on U the $(2i-2)$-form

$$\tau = \int_0^1 i(\partial/\partial t)\Delta_{c_i}(\nabla^t, \nabla^{r(t)})\, dt.$$

By Theorem 3.10 of [1], $d(\tau \wedge \phi) = d\tau \wedge \phi = \Delta_{c_i}(\nabla^{b'}, \nabla^h) \wedge \phi - \Delta_{c_i}(\nabla^b, \nabla^r) \wedge \phi$, so Theorem 2.9 follows from Proposition 2.6 if we show $\tau \wedge \phi$ is bounded on B. Since $\phi = \hat{\phi} \wedge \omega$ on M, it suffices to show τ is bounded in leaf directions. This is equivalent to showing that for all leaves $L \subset B$, $\tau|L$ is a bounded form. First note that $\nabla^t|L = \nabla^b|L$ as all basic connections have the same restrictions to $Q|L$, so τ depends only on ∇^b and $\nabla^{r(t)}|L$. For each t, the class $\Delta_{c_i}(\nabla^b, \nabla^{r(t)})|L$ is a closed form, a leaf class, which depends only on the leafwise partial derivatives on $r(t)$, by the well-known formula (5.74) of [13]. Thus, $\Delta_{c_i}(\nabla^b, \nabla^{r(t)})|L$ is bounded for all t and $L \subset B$, so $\tau|L$ is the integral of forms with a bound independent of L, hence bounded.

3. Compact foliations

We say $\mathscr{F}$ is *compact* if every leaf in M is compact, and given $B \in \mathscr{B}$ we say $\mathscr{F}|B$ is compact if every leaf in B is compact. The dynamics of a compact foliation are relatively tame, and thus one expects its secondary classes to vanish; this is known to hold for the residual classes [8].

Theorem 3.1. *Let $B \in \mathscr{B}$ and suppose $\mathscr{F}|B$ is compact. Then for all $y \in H^*(\mathrm{gl}_n, O_n)$, $\chi_B(y) = 0$.*

Proof. Let $y \in H^p(\mathrm{gl}_n, O_n)$ and $[\phi] \in H^{m-p}(M, \mathscr{F})$ be given. To evaluate $\chi_B(y)[\phi] = \int_B \bar{y} \wedge \phi$ we follow the outline of the proof given for the residual classes in [8]. First, the *Epstein filtration* of B is a countable partition $B = \bigcup_{\alpha \in \mathfrak{A}} B_\alpha$, where $\{B_\alpha | \alpha \in \mathfrak{A}\} \subset \mathscr{B}$ and for each α, $\mathscr{F}|B_\alpha$ has no holonomy. By deleting a set of measure zero from each B_α, we can also assume that each leaf $L \subset B_\alpha$ has trivial linear holonomy in M. It will suffice to show $\int_{B_\alpha} \bar{y} \wedge \phi = 0$ for each $\alpha \in \mathfrak{A}$.

The quotient $T_\alpha = B_\alpha/\mathscr{F}$ is a Hausdorff space and has a standard Lebesgue measure θ_α inherited from $\mathbf{m}$ on M. One can lift θ_α on T_α back to an invariant transverse measure ν_α on $B_{\alpha'}$ which has a smooth extension to an n-form ω_α

defined in an open neighborhood of B_α in M. For the construction of ω_α see §4 of [8]. The closed form ϕ then factors in a neighborhood of B_α as $\phi = \hat{\phi}_\alpha \wedge \omega_\alpha$ and

$$(3.2) \qquad \int_{B_\alpha} \bar{y} \wedge \phi = \int_{B_\alpha} (\bar{y} \wedge \hat{\phi}_\alpha) \wedge \omega_\alpha = \int_{T_\alpha} \left\{ \int_{L \subset B_\alpha} \bar{y} \wedge \hat{\phi}_\alpha \right\} \cdot \theta_\alpha.$$

For each $L \subset B_\alpha$ the restriction $\bar{y}|L$ is a closed form representing the cohomology class $\chi_L(y)$, where χ_L: $H^p(\mathrm{gl}_n, O_n) \to H^p(L)$ is the leaf characteristic map defined by the flat bundle $Q|L \to L$ associated to the linear holonomy of L. As the linear holonomy of L is trivial, χ_L is the zero map so $\bar{y}|L$ must be exact. A simple check shows that $\hat{\phi}_\alpha|L$ is a closed form, hence $\bar{y} \wedge \hat{\phi}_\alpha|L$ is exact. As L is compact, each $\int_L \bar{y} \wedge \hat{\phi}_\alpha = 0$ and the integrand in (3.2) identically vanishes, proving the theorem.

Corollary 3.3. *Let $\mathcal{F}$ be a compact foliation. Then all Weil measures of $\mathcal{F}$ are zero.*

Proof. Each $B \in \mathcal{B}$ satisfies the hypothesis of Theorem 3.1.

Corollary 3.4. *Let $B \in \mathcal{B}$ with $\mathcal{F}|B$ compact. Then for all $y_I c_J \in H_n^*(WO_n)$ the restriction $\Delta_*(y_I c_J)|B = 0$.*

4. The Godbillon measure

There are special techniques available for analyzing the Godbillon measure which do not seem to have counterparts for the higher degree Weil measures. These are based on the observation that g_B measures the obstruction to putting an almost invariant absolutely continuous transverse measure on B. More precisely, suppose a sequence of defining forms $\{\omega_n\}$ for $\mathcal{F}$ near B is given such that the corresponding sequence of 1-forms $\{\eta^n\}$ tends to zero on B. Then g_B must be zero, regardless of whether or not the forms $\{\omega_n\}$ converge to a nonsingular measure on B. This principle is behind the results of [5] and [6] for codimension-one and is the idea of Proposition 4.1 below. We use Proposition 4.1 to show that for $\mathcal{F}$ equicontinuous, or for $\mathcal{F}$ admitting an isotropic invariant transverse measure, the Godbillon-Vey classes of $\mathcal{F}$ are zero.

We assume that $Q \to M$ is orientable. Choose a finite covering $\{(U_i, f_i, g_i)|i = 1, \cdots, d\}$ of M by regular foliation charts with extensions $\tilde{f}_i$: $\tilde{U}_i \to I_2^m$. For each $1 \leqslant i \leqslant d$ set $T_i = I^n$ and $\tilde{T}_i = I_2^n$, then set $T = \bigcup_{i=1}^d T_i$ and $\tilde{T} = \bigcup_{i=1}^d \tilde{T}_i$, the disjoint unions of open sets. Define an immersion $\tilde{h}$: $\tilde{T} \to M$, where for $x \in \tilde{T}_i$, $\tilde{h}(x) = \tilde{f}_i^{-1}(0 \times \{x\})$. We say (i, j) is *admissible* if $U_{ij} \equiv U_i \cap U_j \neq \varnothing$. For (i, j) admissible, set $\tilde{T}_{ij} = \tilde{g}_i(\tilde{U}_{ij}) \subset \tilde{T}_i$ and define γ_{ij}: $\tilde{T}_{ij} \to \tilde{T}_{ji}$ by $\gamma_{ij}(x) = \tilde{g}_j \circ \tilde{g}_i^{-1}(x)$. Since Q is orientable, we can assume each γ_{ij} is orientation preserving with respect to the standard orientation on R^n. Let $d\bar{x}$ denote the

Euclidean volume form on R^n and also its restrictions to I^n and I_2^n. Let $\bar{e}$ denote the n-vector field on R^n such that $d\bar{x}(\bar{e}) = 1$. For all $x \in \tilde{T}_{ij}$, the Jacobian of γ_{ij} is denoted $|\gamma_{ij}|_x = \gamma_{ij}^* d\bar{x}(\bar{e})_x$ which is positive by assumption. Finally, let $\bar{T}_i$ denote the closure of T_i in $\tilde{T}_i$.

For two sequences $\{a_n | n = 1, 2, \cdots\}$ and $\{b_n | n = 1, 2, \cdots\}$ we write $a_n \sim b_n$ if $\lim_{n\to\infty}(a_n/b_n) = 1$.

Proposition 4.1. *Let $B \in \mathscr{B}$. Suppose there exists a collection $\{\bar{\omega}_j^n | j = 1, \cdots, d\}_{n=1,2,\ldots}$, where $\bar{\omega}_j^n$ is a volume form defined on an open neighborhood V_j of $g_j(U_j \cap B)$ in T_j such that for all (i, j) admissible,*

$$\gamma_{ij}^* \bar{\omega}_j^n(\bar{e})_y \sim \bar{\omega}_i^n(\bar{e})_y$$

uniformly for all $y \in g_i(U_{ij} \cap B)$. Then $g_B = 0$.

Proof. On $W_i = g_i^{-1}(V_i)$ set $\omega_i^n = g_i^* \bar{\omega}_i^n$. Define $f_{ij}^n\colon W_i \cap W_j \to R$ by the rule $g_i^* \gamma_{ij}^* \bar{\omega}_j^n = \exp f_{ij}^n \cdot \omega_i^n$. Choose a partition of unity $\{\lambda_i | 1 \leqslant i \leqslant d\}$ subordinate to the cover $\{U_i\}$, and on W_i set

$$\omega^n = \exp\left(\sum_{j=1}^{d} \lambda_j f_{ij}^n\right) \cdot \omega_i^n.$$

An easy check shows ω^n is a well-defined n-form on an open neighborhood of B in M, which defines $\mathscr{F}$ near B. Calculating η^n using ω^n gives, for $\phi \in A^{m-1}(M, \mathscr{F})$,

$$\eta^n \wedge \phi = \left(\sum_{j=1}^{d} d\lambda_j \cdot f_{ij}^n\right) \wedge \phi.$$

For $y \in g_i(U_{ij} \cap B)$,

$$\lim_{n\to\infty} f_{ij}^n(y) = \lim_{n\to\infty} \log \frac{\gamma_{ij}^* \bar{\omega}_j^n(\bar{e})_y}{\bar{\omega}_i^n(\bar{e})_y} = 0.$$

The convergence is uniform, so $\eta^n \wedge \phi$ converges uniformly to zero on B. In particular, $\eta^n \wedge \phi$ is eventually bounded on B, so by Theorem 2.7 we have $g_B[\phi] = \lim_{n\to\infty} \int_B \eta^n \wedge \phi = 0$.

Definition 4.2. For $B \in \mathscr{B}$, $\mathscr{F}$ is *equicontinuous on B* if there is a covering of M by regular foliation charts such that there is a continuous metric d: $\tilde{T} \times \tilde{T} \to R^+$, where the metric topology of d on $\tilde{T}$ is standard and such that

$$d(x, y) = d(\gamma_{ij}(x), \gamma_{ij}(y)) \quad \text{for all } x, y \in \tilde{g}_i(\tilde{U}_{ij} \cap B).$$

If this holds for $B = M$, then we say $\mathscr{F}$ is equicontinuous. Intuitively, $\mathscr{F}$ is equicontinuous on B if two leaves in B which are close at some point remain relatively close always.

Theorem 4.3. *Assume $\mathcal{F}$ is equicontinuous on B. Then $g_B = 0$ and all Godbillon-Vey classes of $\mathcal{F}$ vanish when restricted to B.*

Theorem 4.3 applies in particular to foliated twisted products for which equicontinuity has a standard interpretation. Let X^n and Y^{m-n} be closed orientable manifolds and suppose a representation ρ: $\Gamma = \pi_1(Y) \to \operatorname{Diff} X$ defines a C^∞-action of Γ on X. Then Γ also acts freely on the universal cover $\tilde{Y}$ of Y, and the product foliation on $\tilde{Y} \times X$ descends to a codimension n foliation $\mathcal{F}$ on $M = (\tilde{Y} \times X)/\Gamma$. Then $\mathcal{F}$ is equicontinuous if and only if the action of Γ on X is equicontinuous, or if there is a continuous invariant (standard) distance function d: $X \times X \to R^+$. Thus Theorem 4.3 yields the following extension to higher codimensions of Herman's vanishing theorem [7] for foliations of T^3 defined by an equicontinuous action of $Z^2 = \pi_1(T^2)$ on S^1:

Corollary 4.4. *Let $M = (\tilde{Y} \times X)/\Gamma$ with $\mathcal{F}$ defined as above. Suppose Γ acts equicontinuously on X. Then g vanishes on $\mathcal{B}$ and all generalized Godbillon-Vey classes of $\mathcal{F}$ are zero.*

Theorem 4.3 will follow from Lemma 4.7 and Theorem 4.8 below.

Definition 4.5. Given $\varepsilon > 0$, a *kernel function K for $\mathcal{F}$ with ε-support* is a set of nonnegative continuous maps $\{K_i$: $\tilde{T}_i \times \tilde{T}_i \to R^+ | 1 \leqslant i \leqslant d\}$ such that

(a) $K_i(x, y) = K_i(y, x)$.

(b) For each $y \in \overline{T}_i$, $0 < \int_{\tilde{T}_i} K_i(x, y)\, d\overline{x} < \infty$.

(c) The support of K_i on $\overline{T}_i \times \overline{T}_i$ is contained in an ε-neighborhood of the diagonal $\Delta \subset \overline{T}_i \times \overline{T}_i$.

Definition 4.6. Let $B \in \mathcal{B}$. A kernel K for $\mathcal{F}$ is (δ, ε)-invariant on B if:

(a) K has ε-support.

(b) For (i, j) admissible there exists λ_{ij}: $\tilde{T}_{ij} \times \tilde{T}_{ij} \to R$ such that

$$K_j \circ (\gamma_{ij} \times \gamma_{ij})(x, y) = \lambda_{ij}(x, y) \cdot K_i(x, y)$$

for $(x, y) \in \tilde{T}_{ij} \times \tilde{T}_{ij}$ and $|\lambda_{ij}(x, y) - 1| < \delta$ for $x, y \in \tilde{g}_i(U_{ij} \cap B)$.

We say K is *invariant on B* if $\lambda_{ij}(x, y) = 1$ for $x, y \in \tilde{g}_i(U_{ij} \cap B)$.

Lemma 4.7. *Given $B \in \mathcal{B}$, if $\mathcal{F}$ is equicontinuous on B then for all $\varepsilon > 0$ and $\delta > 0$ there exists a kernel K for $\mathcal{F}$ which is (δ, ε)-invariant on B.*

Proof. Let d: $\tilde{T} \times \tilde{T} \to R$ be a continuous distance function which is invariant on B. For each positive integer n choose a monotone smooth function ϕ_n: $R \to [0, 1]$ with

$$\phi_n(x) = \begin{cases} 1 & \text{for } x \leqslant 1/n, \\ 0 & \text{for } x \geqslant 2/n. \end{cases}$$

Set $K^n(x, y) = \phi_n(d(x, y))$. Then K^n is a continuous kernel on $\tilde{T}$ which is invariant on B, and the support of K^n tends uniformly on compact sets to the diagonal of $\tilde{T} \times \tilde{T}$. Thus for some n, K^n will be (δ, ε)-invariant on B.

Theorem 4.8. *Let $B \in \mathscr{B}$. Suppose that for all $\varepsilon > 0$ there exists a kernel K^{ε} for $\mathscr{F}$ which is $(\varepsilon, \varepsilon)$-invariant on B. Then $g_B = 0$.*

Proof. For each integer $n > 0$ choose K^n which is $(1/n, 1/n)$-invariant on B. For each $n > 0$ and $1 \leqslant i \leqslant d$ set

$$f_i^n(y) = \left\{ \int_{\tilde{T}_i} K_i^n(x, y)\, d\bar{x} \right\}^{-1} \quad \text{for } y \in \tilde{T}_i.$$

Lemma 4.9. *$f_j^n \circ \gamma_{ij}(y) \sim |\gamma_{ij}|_y^{-1} \cdot f_i^n(y)$ uniformly in $y \in g_i(B \cap \overline{U}_i)$.*

Proof. Because the support of K_i^n tends to $\Delta \subset \tilde{T}_i \times \tilde{T}_i$ and $\tilde{T}_{ij}$ is compact, there exists N such that for all $n > N$ and $y_0 \in \overline{T}_{ij}$, support $K_i^n(x, y_0) \subset \tilde{T}_{ij}$. So for $n > N$ we have

$$\begin{aligned}
\left[f_j^n \circ \gamma_{ij}(y) \right]^{-1} &= \int_{\tilde{T}_j} K_j^n\big(x, \gamma_{ij}(y)\big)\, d\bar{x} = \int_{\tilde{T}_i} K_j^n\big(\gamma_{ij}(x), \gamma_{ij}(y)\big) \cdot |\gamma_{ij}|_x\, d\bar{x} \\
&= \int_{\tilde{T}_i} \lambda_{ij}^n(x, y) \cdot K_i^n(x, y) \cdot |\gamma_{ij}|_x\, d\bar{x} \sim \int_{\tilde{T}_i} K_i^n(x, y) \cdot |\gamma_{ij}|_x\, d\bar{x} \\
&\sim |\gamma_{ij}|_y \cdot \int_{\tilde{T}_i} K_i^n(x, y)\, d\bar{x} = |\gamma_{ij}|_y \cdot \left[f_i^n(y) \right]^{-1}.
\end{aligned}$$

Lemma 4.9 is proved.

Choose positive smooth functions $\bar{f}_i^n$ on $\tilde{T}_i$ such that $\bar{f}_i^n \sim f_i^n$ uniformly on $\overline{T}_i$ for $1 \leqslant i \leqslant d$. Then set $\omega_i^n = \bar{f}_i^n \cdot d\bar{x}$ on $\tilde{T}_i$. Then uniformly for $y \in g_i(B \cap \overline{U}_i) \subset \overline{T}_i$ we have

$$\begin{aligned}
\gamma_{ij}^* \omega_j^n(\bar{e})_y &= \bar{f}_j^n \circ \gamma_{ij}(y) \cdot |\gamma_{ij}|_y \\
&\sim f_j^n \circ \gamma_{ij}(y) \cdot |\gamma_{ij}|_y \sim f_i^n(y) \sim \bar{f}_i^n(y) = \omega_i^n(\bar{e})_y.
\end{aligned}$$

Thus the collection $\{\omega_i^n\}$ satisfies Proposition 4.1 and $g_B = 0$. Theorem 4.8 is now proved.

A finite measure μ on $\tilde{T}$ is *good* if every open subset of $\tilde{T}$ has positive μ-measure. We say μ is *invariant* if $\gamma_{ij}^* \mu = \mu$ on $\tilde{T}_{ij}$ for each (i, j) admissible. More generally, for $B \in \mathscr{B}$ we say μ is invariant on B if $\mu(\gamma_{ij} C) = \mu(C)$ for all measurable $C \subset g_i(B \cap \tilde{U}_{ij})$.

Conjecture 4.10. *Let $B \in \mathscr{B}$ and suppose there is a good measure μ on $\tilde{T}$ which is invariant on B. Then $g_B = 0$.*

If μ is absolutely continuous in Conjecture 4.10 and $\mu(C) = 0$ implies $C \subset T$ has **m**-measure zero, then the conjecture follows from Corollary 3.8 of [10]. We introduce next a natural condition on good measures which is sufficient to prove (4.10). Note that an invariant measure on $\tilde{T}$ defines an invariant transverse measure for $\mathscr{F}$ as in [17] and [19].

For a measurable set $X \subset R^n$ let $\mathrm{vol}(X)$ denote the Euclidean volume of X. Given a smooth metric r on T, $y \in T$ and $\varepsilon > 0$ let $B(y, \varepsilon, r)$ denote the ball of r-radius ε in T centered at y. Given μ on T, the (ε, r)-density of μ at y is

$$D(y, \varepsilon, r) = \frac{\mu(B(y, \varepsilon, r))}{\mathrm{vol}(B(y, \varepsilon, r))}.$$

Definition 4.11. A good measure μ on T is *isotropic* at $y \in T$ if for any two smooth metrics r and r' on $\tilde{T}$,

$$\lim_{\varepsilon \to 0} \frac{D(y, \varepsilon, r')}{D(y, \varepsilon, r)} = 1.$$

We say μ is isotropic on B if the limit converges to 1 uniformly for $x \in g_i(B \cap \overline{U}_i)$.

Intuitively, μ is isotropic at y when its mass is infinitesimally uniformly distributed in all directions at y.

Theorem 4.12. *Let $B \in \mathscr{B}$. Suppose there exists a good measure μ on T, invariant on B and isotropic on B. Then $g_B = 0$.*

Proof. We are given isotropic good measures μ_i on $\tilde{T}_i$ for $1 \leqslant i \leqslant d$. By replacing each μ_i with the measure associated to $e_B \cdot d\mu_i$, where e_B is the characteristic function of $\tilde{g}_i(B \cap \tilde{U}_i)$, we can assume μ is invariant on M. For each positive integer n, choosing a monotone smooth function $\phi_n \colon R \to R$ such that

$$\phi_n(x) = \begin{cases} 1 & \text{for } x \leqslant 1/n, \\ 0 & \text{for } x \geqslant (n+1)/n^2. \end{cases}$$

Let $d_i \colon \tilde{T}_i \times \tilde{T}_i \to R$ be the Euclidean distance function and define a sequence of kernels on $\tilde{T}_i$ by $K_i^n(x, y) = \phi_n(d_i(x, y))$. Note K_i^n is smooth near $\overline{T}_i \times \overline{T}_i$ for n large. Define functions $\tilde{f}_i^n \colon \tilde{T}_i \to R$ by

$$\tilde{f}_i^n(y) = \frac{\int_{\tilde{T}_i} K_i^n(x, y) \cdot d\mu_i}{\int_{\tilde{T}_i} K_i^n(x, y) \cdot d\bar{x}}.$$

Then $\tilde{f}_i^n$ is positive on $g_i(B \cap \tilde{U}_i)$. Modify each $\tilde{f}_i^n$ to obtain f_i^n which is smooth and positive on all of $\tilde{T}_i$ and agrees with $\tilde{f}_i^n$ on $g_i(B \cap \overline{U}_i)$. Then set $\omega_i^n = f_i^n \cdot d\bar{x}$. The theorem now follows from

Lemma 4.13. $\gamma_{ij}^* \omega_j^n(\bar{e})_y \sim \omega_i^n(\bar{e})_y$ *uniformly for* $y \in g_i(B \cap \overline{U}_i)$.

Proof.

$$\gamma_{ij}^*\omega_j^n(\bar{e})_y = f_j^n \circ \gamma_{ij}(y) \cdot |\gamma_{ij}|_y$$

$$= \frac{\int_{\tilde{T}_j} K_j^n(x, \gamma_{ij}(y)) \cdot |\gamma_{ij}|_y \, d\mu_j(x)}{\int_{\tilde{T}_j} K_j^n(x, \gamma_{ij}(y)) \cdot d\bar{x}} \tag{4.14}$$

$$= \frac{\int_{\tilde{T}_i} K_j^n(\gamma_{ij}(x), \gamma_{ij}(y)) \cdot |\gamma_{ij}|_y \cdot d\mu_i(x)}{\int_{\tilde{T}_i} K_j^n(\gamma_{ij}(x), \gamma_{ij}(y)) \cdot |\gamma_{ij}|_x \cdot d\bar{x}},$$

since μ is invariant under γ_{ij}. Now

$$K_j^n(\gamma_{ij}(x), \gamma_{ij}(y)) = \phi_n \circ d_j(\gamma_{ij}(x), \gamma_{ij}(y))$$

and $d_j \circ \gamma_{ij} \times \gamma_{ij}$ is the distance function on $\tilde{T}_{ij}$ for the metric r' induced by γ_{ij} from the Euclidean metric r on $\tilde{T}_{ji}$. Then the continuity of $|\gamma_{ij}|_x$ and the choice of ϕ_n imply

$$\int_{\tilde{T}_i} K_j^n(\gamma_{ij}(x), \gamma_{ij}(y)) \cdot |\gamma_{ij}|_x \, d\bar{x} \sim |\gamma_{ij}|_y \cdot \int_{\tilde{T}_i} \phi_n \circ d_j(\gamma_{ij}(x), \gamma_{ij}(y)) \, d\bar{x}$$

$$\sim |\gamma_{ij}|_y \cdot \operatorname{vol} B(y, 1/n, r').$$

The numerator of (4.14) is similarly asymptotic to $|\gamma_{ij}|_y \cdot \mu_i(B(y, 1/n, r'))$, so

$$\gamma_{ij}^*\omega_j^n(\bar{e})_y \sim \frac{\mu_i(B(y, 1/n, r'))}{\operatorname{vol}(B(y, 1/n, r'))}$$

$$\sim \frac{\mu_i(B(y, 1/n, r))}{\operatorname{vol}(B(y, 1/n, r))} \sim \omega_i^n(\bar{e})_y.$$

5. Geometry of the Weil measures and open problems

Our last theorem states what is currently known about the dependence of the Godbillon-Vey classes on the geometry of $\mathscr{F}$ for arbitrary codimension.

Theorem 5.1. *Let $\mathscr{F}$ be a codimension n foliation of a closed manifold M. Suppose there exists a countable partition $\{B_\alpha | \alpha \in \mathfrak{A}\} \subset \mathscr{B}$ with $M = \bigcup_\alpha B_\alpha$ such that for each $\alpha \in \mathfrak{A}$ one of the following holds:*

(a) *$\mathscr{F}|B_\alpha$ is compact.*

(b) *$\mathscr{F}|B_\alpha$ has an isotropic good invariant measure.*

(c) *$\mathscr{F}|B_\alpha$ has an absolutely continuous invariant transverse measure μ with almost every leaf essential for μ.*

(d) *$\mathscr{F}|B_\alpha$ is equicontinuous.*

(e) *Almost every leaf* $L \subset B_\alpha$ *has subexponential growth* [10]. (*Recall this means that the growth function of a.e. leaf is dominated by* $\exp(\varepsilon r)$ *for every positive* ε.)

Then all Godbillon-Vey classes $\Delta_*(y_I c_J) \in H^{2n+1}(M)$ *and* $\Delta_*(y_1 y_I c_J) \in H^*(M)$ *for* $\mathscr{F}$ *are zero.*

For the residual secondary classes not covered by Theorem 5.1 our understanding of their dependence on the geometry of $\mathscr{F}$ is just beginning (cf. [11]). We conclude with several questions.

Question 5.2. Assume M has a continuous decomposition into saturated measurable sets. That is, there is a standard Borel measure space (X, μ) and a Borel map $X \to \mathscr{B}$ so that $(M, \mathbf{m}) = \int_X B_x d\mu(x)$. For each $B_x \in \mathscr{B}$, possibly of measure zero, is it possible to define $\Delta_*(y_I c_J)|B_x \in H^*(M)$ so that $\Delta_*(y_I c_J) = \int_X \Delta_*(y_I c_J)|B_x \, d\mu(x)$? What properties must such a "derivative" $\Delta_*(y_I c_J)|B_x$ satisfy?

Define a leaf $L \subset M$ to be *essential* if there is a sequence $\{B_n | n = 1, 2, \cdots\} \subset \mathscr{B}$ with $L \subseteq B_i$ for all i and $\lim_{i\to\infty} \mathbf{m}(B_i) = 0$, such that for some $y \in H^p(\mathrm{gl}_n, O_n)$ and $[\phi] \in H^{m-p}(M, \mathscr{F})$,

$$\liminf_{i\to\infty} \mathbf{m}(B_i)^{-1} \cdot \chi_{B_i}(y)[\phi] \equiv c(y, [\phi], L) > 0.$$

We say L is *singular* if $c(y, [\phi], L) = \infty$ for some y and $[\phi]$. Thus, L is respectively an essential or singular point for the measure $\chi(y)$ on $M/\mathscr{F}$.

Question 5.3. Can an essential leaf exist? A singular leaf? If so, how does $c(y, [\phi], L)$ depend upon the geometry of $\mathscr{F}$ near L? More generally, as we let y and $[\phi]$ vary, what are the measure theoretic isomorphism types of the measure spaces $(M/\mathscr{F}, \chi(y)[\phi])$ thus obtained?

Question 5.4. What geometric hypotheses on $\mathscr{F}|B$ are sufficient to imply $\chi_B(y) \neq 0$ for some y?

Question 5.5. Can the assumption μ is isotropic be removed from the hypotheses of Theoren 4.12? What implications does the existence of a good invariant measure for $\mathscr{F}|B$ have for the geometry of $\mathscr{F}$ in B?

Question 5.6. There are natural notions of bounded cohomology for groupoids, and the measures $\chi(y)$ are known to vanish on $B \in \mathscr{B}$ precisely under the same hypotheses which imply the bounded cohomology of $\mathscr{F}|B$ is zero. Does $\chi_B(y)$ define a bounded cohomology class on the groupoid homology of $\mathscr{F}|B$?

Question 5.7. Does an analogue of Proposition 4.1 hold for the higher degree Weil measures?

References

[1] R. Bott, *Lectures on characteristic classes and foliations*, Lecture Notes in Math. Vol. 279, Springer, Berlin, 1972, 1–94.

[2] ______, *On some formulas for the characteristic classes of group actions*, Lecture Notes in Math. Vol. 652, Springer, Berlin, 1978, 25–61.

[3] R. Bott & A. Haefliger, *On characteristic classes of* Γ*-foliations*, Bull. Amer. Math. Soc. **78** (1972) 1038–1044.

[4] J. Cantwell & L. Conlon, *A vanishing theorem for the Godbillon-Vey invariant*, preprint, 1981.

[5] ______, *The dynamics of open foliated manifolds and a vanishing theorem for the Godbillon-Vey class*, Advances in Math. **53** (1984) 1–27.

[6] G. Duminy, *L'invariant de Godbillon-Vey d'un feuilletage se localise dans les feuilles ressort*, preprint, 1982.

[7] M. R. Herman, *The Godbillon-Vey invariant of foliations by planes of* T^3, Lecture Notes in Math. Vol. 597, Springer, Berlin, 1977, 294–307.

[8] S. Hurder, *Vanishing of secondary classes for compact foliations*, J. London Math. Soc. (2) **28** (1983) 175–183.

[9] ______, *Global invariants for measured foliations*, Trans. Amer. Math. Soc. **280** (1983) 367–391.

[10] ______, *The Godbillon measure of amenable foliations*, preprint, 1984.

[11] S. Hurder & A. Katok, *Ergodic theory and Weil measures of foliations*, preprint, 1984.

[12] F. Kamber and P. Tondeur, *Characteristic invariants of foliated bundles*, Manuscripta Math. **11** (1974) 51–89.

[13] ______, *Foliated bundles and characteristic classes*, Lecture Notes in Math. 493, Springer, Berlin, 1975, 1–294.

[14] T. Mitzutani, S. Morita & T. Tsuboi, *The Godbillon-Vey classes of codimension-one foliations which are almost without holonomy*, Ann. of Math. **113** (1981) 515–527.

[15] R. Moussu & F. Pelletier, *Sur le Théorème de Poincare-Bendixson*, Ann. Inst. Fourier (Grenoble) **14** (1974) 138.

[16] T. Nishimori, *SRH-decompositions of codimension-one foliations and the Godbillon-Vey class*, Tôhoku Math. J. **32** (1980) 9–34.

[17] J. Plante, *Foliations with measure preserving holonomy*, Ann. of Math. **102** (1975) 327–361.

[18] P. Schweitzer (Editor), *Some problems in foliation theory and related areas*, Lecture Notes in Math. Vol. 652, Springer, Berlin, 1978, 240–252.

[19] D. Sullivan, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Invent. Math. **36** (1976) 225–255.

University of Illinois at Chicago
Mathematical Sciences Research Institute