THE JACOBIAN COCYCLE OF A DISTAL GROUP ACTION

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ABSTRACT

In this note we study the Jacobian cocycle associated to a smooth, distal action of a discrete group on a compact manifold. The first result is that with a mild additional hypothesis (which is satisfied, e.g., by all equicontinuous actions) the Radon-Nikodyn cocycle of the action is arbitrarily close to a coboundary in the L^1-topology. If the group is Kazhdan, this implies the existence of an invariant measure. The second result is that for any smooth distal action, the Lyapunov exponents of the Jacobian cocycle are zero. We give two applications: If a Kazhdan group acts smoothly on a compact manifold, V, and the action is distal with the extra hypothesis, then there is a measurable field of Riemannian metrics on TV invariant under the group action. Thus, the group action is tangentially distal. Secondly, our cocycle results imply new vanishing theorems for the secondary classes associated to smooth group actions.

§1. Distal Actions with Good Convergence

Let V be a smooth compact Riemannian manifold of dimension q with volume element $d\bar{x}$ and total volume 1, and let m(X) denote the corresponding volume of a measurable subset $X \subset V$. Let Γ be a discrete group, and $\varphi: \Gamma \times V \longrightarrow V$ a C^1-action of Γ on V. We also use the notation γx for $\varphi(\gamma,x)$. The additive Radon-Nikodyn cocycle associated to φ and $d\bar{x}$ is denoted by

$$d\nu: \Gamma \times V \longrightarrow \mathbb{R};\ d\nu(\gamma,x) = \log \left| \frac{\gamma^*(d\bar{x})}{d\bar{x}}(x) \right|.$$

A cocycle $\psi: \Gamma \times V \longrightarrow \mathbb{R}$ is cohomologous to $d\nu$ if there is a measurable (transfer) function $g: V \longrightarrow \mathbb{R}$ so that $\psi(\gamma,x) = g(\gamma x) + d\nu(\gamma,x) - g(x)$ all $\gamma \in \Gamma$, a.e. $x \in V$. Equivalently, define an absolutely continuous density on V by $d\mu_g = \exp(g)\cdot d\bar{x}$, then

$$\psi(\gamma,x) = \log \left| \frac{\gamma^*(d\mu_g)}{d\mu_g}(x) \right| \qquad \text{a.e. } x \in V.$$

Let d_V denote the metric on V obtained from the Riemannian metric on TV. The action $\varphi: \Gamma \times V \longrightarrow V$ is *distal at* $x \in V$ if for all $y \in V$, $x \neq y$, $d(x,y) \overset{\text{def}}{=\!=} \inf_{\gamma\in\Gamma} d_V(\gamma x,\gamma y) > 0$. The action φ is *distal* (resp., *a.e. distal*) if φ is distal at every $x \in V$ (resp., a.e. $x \in V$).

For $x \in V$, $s > 0$ and $\delta > 0$, set $X(x,s) = \{y \in V \mid d(x,y) < s\}$ and define the bad set $B(x,s,\delta) = \{y \in X(x,s) \mid d_V(x,y) > \delta\}$. Note φ is distal at x if and only if $\bigcap_{s>0} X(x,s) = \emptyset$.

The action φ is *a.e. distal with good convergence* if for a.e. $x \in V$ and all $\epsilon,\delta > 0$ there exists $S(x,\delta,\epsilon) > 0$ so that for all $0 < s < S(x,\delta,\epsilon)$,

$$m(B(x,s,\delta)) < \epsilon \cdot m(X(x,s)).$$

Equivalently, $\lim_{s\to 0} \frac{m(B(x,s,\delta))}{m(X(x,s))} = 0$. This says that for all $\delta > 0$, for s sufficiently small, most of the d-ball of radius s is contained in the d_V-ball of radius δ.

The action φ is *equicontinuous* at x if for all $\delta > 0$, there exists $S(x,\delta) > 0$ so that $s < S(x,\delta)$ implies $B(x,s,\delta) = \varnothing$. Clearly, an a.e. equicontinuous action is a.e. distal with good convergence. For background on equicontinuous and distal actions, see the monograph [3].

Theorem 1.1. *Let $\Gamma \times V \longrightarrow V$ be a smooth action which is a.e. distal with good convergence. For each finite set $\Delta \subset \Gamma$, there exists a constant K_Δ so that for all $\epsilon > 0$, there is a cocycle ψ_ϵ cohomologous to $d\nu$ and a set $X(\epsilon) \subset V$ such that*

(a) $m(V - X(\epsilon)) < \epsilon$

(b) $|\psi_\epsilon(\gamma,x)| < \epsilon$ *for all* $\gamma \in \Delta$, $x \in X(\epsilon)$

(c) $|\psi_\epsilon(\gamma,x)| < K_\Delta$ *for all* $\gamma \in \Delta$, *any* $x \in V$.

A cocycle $\varphi: \Gamma \times V \longrightarrow G$, with values in a Polish group G, is said to be in the L^p-closure of the coboundaries if for each finite set $\Delta \subset \Gamma$ and $\epsilon > 0$, there exists a cohomologous cocycle ψ_ϵ so that for all $\gamma \in \Delta$,

$$\int_V |\psi_\epsilon(\gamma,x)|_G^p \, d\bar{x} < \epsilon$$

where $|g|_G$ is the distance from $g \in G$ to the identity. Problem 25 of [6] asks which geometric cocycles are in the L^p-closure of the coboundaries. An immediate corollary of Theorem 1.1 provides a partial answer:

Corollary 1.2. *Let $\varphi: \Gamma \times V \longrightarrow V$ be a smooth a.e. distal action with good convergence. Then the Radon-Nikodym cocycle of φ is in the L^p-closure of the coboundaries for all $p > 0$.*

Let $\mathcal{H}$ be the Hilbert space of 1/2-densities on V with finite L^2-norm, and inner product $(f,g) = \int_V f \cdot \bar{g}$. A smooth action of Γ on V induces a unitary representation of Γ on $\mathcal{H}$, denoted by $\rho(\varphi): \Gamma \longrightarrow U(\mathcal{H})$.

Corollary 1.3. *For $\varphi: \Gamma \times V \longrightarrow V$ an a.e. distal smooth action with good*

convergence, the representation $\rho(\varphi)$ weakly contains the identity representation.

Proof. The representation $\rho(\varphi)$ weakly contains the identity precisely when there exists a sequence of norm one densities $\{h_n\} \subset \mathcal{H}$ such that for all $\gamma \in \Gamma$, $\lim_{n\to\infty} \langle \rho(\varphi)(\gamma)h_n, h_n\rangle = 1$. In the proof of Theorem 1, we will exhibit a sequence $\{\omega_n\}$ of norm 1 full densities on V which are L^1-almost invariant. Then let $h_n = \sqrt{\omega_n}$ be the corresponding 1/2-densities; these are also almost invariant, which proves the claim. □

A group Γ is Kahzdan if every unitary representation of Γ which weakly contains the identity, leaves fixed a non-zero vector [14].

Corollary 1.4. *Let φ: $\Gamma \times V \longrightarrow V$ be a smooth, ergodic distal action with good convergence. If Γ is a Kazhdan group, then there is a finite measure on V equivalent to $d\bar{x}$, and which is invariant under the Γ-action.*

Proof. By (1.3), the representation $\rho(\varphi)$ weakly contains the identity, so there is an invariant 1/2 density on V whose square is an invariant measure on V. The assumption that φ is ergodic implies the invariant measure is equivalent to Lebesgue measure. □

The next result sharpens Theorem 4.3 of [4]:

Corollary 1.5. *Let φ: $\Gamma \times V \longrightarrow V$ be an a.e. distal C^2-action with good convergence on a compact manifold V. Then the Godbillon-Vey invariants of the action φ are all zero.*

Proof. The Godbillon-Vey invariants are defined in Bott [2], Hurder [7] or Hurder-Katok [8]. To prove the corollary, use the cocycles ψ_ϵ given in Theorem 1.1 to calculate the Godbillon measure, exactly as in Theorem 4.1 of [7]. The L^1-convergence to zero of ψ_ϵ shows the Godbillon-measure is zero, hence all of the Godbillon-Vey classes are zero, as discussed in §2 of [7].

§2. Exponents of Distal Actions

Let E be a measurable framing of TV; so for $x \in V$, $E(x)$ is an ordered n-tuple of vectors in T_xV which form a basis, and depend measurably on x. One way to obtain such E is to choose a set $V_0 \subset V$ of measure zero whose complement $(V - V_0) \cong \mathbb{R}^q$. Then E on $V - V_0$ is the natural framing of $\mathbb{R}^q$, and on $TV|_{V_0}$ the framing is arbitrary. For $\gamma \in \Gamma$ and $x \in V$, the differential $\varphi(\gamma)_*: T_xV \longrightarrow T_{\varphi(\gamma,x)}V$ can be expressed as a matrix via the framings $E(x)$ and $E(\varphi(\gamma,x))$. The yields a measurable mapping $J\varphi(\gamma): V \longrightarrow GL(q,\mathbb{R})$, and the joint map $J\varphi: \Gamma \times V \longrightarrow GL(q,\mathbb{R})$ is the Jacobian cocycle of φ (with respect to E).

For $\gamma \in \Gamma$, define the *upper Lyapunov exponent* of γ at $x \in V$ to be

$$\chi^+(\gamma,x) = \limsup_{n\to\infty} \frac{1}{n} \log \|J\varphi(\gamma^n,x)\|$$

where $\|A\|$ denotes the multiplicative matrix norm of $A \in GL(q,\mathbb{R}) \subset \mathrm{End}(\mathbb{R}^q)$. For a discussion of these exponents, see [9], [10], [11].

Theorem 2.1. *Let $\varphi: \Gamma \times V \longrightarrow V$ be a distal C^1-action on a compact manifold V. Then for all $\gamma \in \Gamma$ and $x \in V$, the exponent $\chi^+(\gamma,x) = 0$.*

There is an a.e.-version of this theorem, valid when we are given an invariant finite measure.

Theorem 2.2. *Let $\varphi: \Gamma \times V \longrightarrow V$ be an a.e. distal C^1-action on a manifold V. Suppose that φ preserves a finite measure on V which is equivalent to Lebesgue measure. Then for all $\gamma \in \Gamma$ and a.e. $x \in V$, the exponent $\chi^+(\gamma,x) = 0$.*

Corollary 2.3. *Let $\varphi: \Gamma \times V \longrightarrow V$ be as in (2.2) and assume that Γ is Kazhdan. Then there exists a measurable field of Riemannian metrics on TV which is invariant under the action φ. In particular, the action of Γ is tangentially distal.*

Combining Corollaries 1.4 and 2.3 we obtain the main result of this note:

Corollary 2.4. *Let φ: $\Gamma \times V \longrightarrow V$ be an ergodic, a.e.-distal C^1-action with good convergence of a Kazhdan group Γ on a compact manifold V. Then φ preserves a measurable field of Riemannian metrics on TV, so φ acts tangentially distal. In particular, an equicontinuous action of a Kazhdan group is always tangentially distal.*

The conclusion of (2.3) and (2.4) is that the Jacobian cocycle $J\varphi$ is cohomologous to a cocycle with values in O(q), where q = dimension V. This, and ergodicity of the action, forces restrictions on the dimension q vis-a-vis Γ (cf. [12], [14]).

It is an open problem whether the invariant Riemannian metric in (2.3) or (2.4) must be smooth, or at least continuous.

Note that if Γ is not Kazhdan, then Corollary 2.4 is false: there exist equicontinuous actions of Z on the circle which admit no invariant measure equivalent to Lebesgue measure [1] (see also the Introduction to [5]).

Proof of (2.3). The action φ leaves invariant a finite measure and by (2.2) has zero exponents for a.e. $x \in V$. We can then apply the superrigidity theorem, as in Theorem 2.13 of Zimmer [12], to conclude that $J\varphi$ is cohomologous to a cocycle with values in the orthogonal group O_q. As explained in §2 of [12], this implies that $J\varphi$ leaves invariant a measurable field of Riemannian metrics on TV. □

As an application of (2.3) or (2.4) we have:

Corollary 2.4. *Let φ: $\Gamma \times V \longrightarrow V$ be as in (2.3) or (2.4), and assume the action is C^2. Then all of the residual secondary classes of the action are zero.*

Proof. The residual secondary classes are defined in [4], [8], based on ideas in Bott [2]. Combining (2.3) or (2.4) with Theorem 3.17 of [8], we see that all of the Weil measures of the action are zero. Thus, all of the residual secondary invariants also vanish. □

For a further discussion on the tangential properties of distal group actions, the reader is refered to the papers of Zimmer [12], [13], the text [14], and Section 6 of the MSRI problem session [6].

§3. Proof of Theorem 1.1

For $s > 0$ and d as in §1, set $X(s) = \{(x,y) \mid d(x,y) < s\} = \bigcup_{x \in V} X(x,s)$. Note that d is the decreasing limit of continuous functions, hence d is measurable, as are the functions $y \longmapsto d(x,y)$ for all $x \in V$. Therefore, the sets $X(s)$ and $X(x,s)$ are all measurable. For each integer $n > 0$, let $k_n : V \times V \longrightarrow \{0,1\}$ be the characteristic function for the set $X(1/n)$. Then k_n is measurable, and invariant under the diagonal action of Γ on $V \times V$.

Recall that $d\bar{x}$ is the Riemannian volume density on V. For each $\gamma \in \Gamma$, let

$$D(\gamma,x) = \exp\{d\nu(\gamma,x)\} = \left| \frac{\gamma^*(d\bar{x})}{d\bar{x}}(x) \right|$$

denote the volume expansion of γ at x. Then set

$$I(\gamma) = \min_{x \in V} D(\gamma,x); \quad S(\gamma) = \max_{x \in V} D(\gamma,x).$$

For each integer $n > 0$, define a measurable function

$$f_n(y) = \int_V k_n(y,x)\, d\bar{x}$$

and a measurable density on V

$$\omega_n = f_n(x)^{-1} \cdot d\bar{x}.$$

Lemma 3.1. *For each $\gamma \in \Gamma$, $\epsilon > 0$ and a.e. $x \in V$, there exists an integer $N(\gamma,\epsilon,x)$ so that for all $n > N(\gamma,\epsilon,x)$,*

$$|\log\{\gamma^*\omega_n(Z)/\omega_n(Z)\}| < \epsilon$$

where $Z \in \Lambda^q T_x V$ is the unit volume q-vector. Furthermore, for all $n > 0$ the term above is bounded by a constant $K(\gamma)$.

Proof. $\dfrac{\gamma^* \omega_n(Z)}{\omega_n(Z)} = \dfrac{f_n(x) \cdot D(\gamma, x)}{f_n(\gamma x)}$, and we estimate the denominator. Choose $\epsilon' > 0$ so that $|\log(1 \pm \epsilon')| < \epsilon$, and then choose $\delta > 0$ and an integer $N(\gamma, \delta, x)$ so that:

(3.2) For all $y \in V$ with $d_V(x,y) < \delta$,

$$|D(\gamma,y) - D(\gamma,x)| < \frac{1}{2} \cdot \epsilon' \cdot I(\gamma)$$

(3.3) $\delta \cdot S(\gamma) < \frac{1}{2} \cdot \epsilon' \cdot I(\gamma)$

(3.4) For all $n > N(x,\delta)$,

$$m(B(x,\tfrac{1}{n},\delta)) < \delta \cdot m(X(x,1/n)).$$

By the assumption that Γ acts a.e. distally with good convergence, for a.e. $x \in V$ we can satisfy the last condition (3.4). Then calculate:

$$f_n(\gamma x) = \int_V k_n(\gamma x,y)\, d\bar{y} = \int_V k_n(x,y)\, D(\gamma,y)\, d\bar{y},$$

$$|f_n(\gamma x) - D(\gamma,x) f_n(x)| \leqslant \int_V k_n(x,y)\, |D(\gamma,y) - D(\gamma,x)|\, d\bar{y}$$

(3.5) $$\leqslant \frac{1}{2} \cdot \epsilon' \cdot I(\gamma) \cdot \int_{V - B(x,1/n,\delta)} k_n(x,y)\, d\bar{y} + S(\gamma) \cdot \int_{B(x,1/n,\delta)} k_n(x,y)\, d\bar{y}$$

$$\leqslant \frac{1}{2} \cdot \epsilon' \cdot I(\gamma) \cdot f_n(x) + S(\gamma) \cdot \delta \cdot f_n(x)$$

$$\leqslant \epsilon' \cdot D(\gamma,x) \cdot f_n(x)$$

where we use (3.4) to estimate

$$\int_{B(x,1/n,\delta)} k_n(x,y)\, d\bar{y} = m(B(x,1/n,\delta)) \leq \delta \cdot m(X(x,1/n)) = \delta \cdot f_n(x).$$

The previous estimate then yields

$$\left| \log \left\{ \frac{f_n(\gamma x)}{f_n(x) D(\gamma, x)} \right\} \right| \leq |\log (1 \pm \epsilon')| < \epsilon.$$

To finish the proof of the lemma, note first that $d(x,y) \leq d_V(x,y)$ for all (x,y), so far all x and n,

$$m(X(x,1/n)) \geq m(\{y \in V \mid d_V(x,y) < \tfrac{1}{n}\}) \stackrel{\text{def}}{=\!=} v(x,\tfrac{1}{n}).$$

The function $v(x,1/n)$ is continuous in x, and $m(V) = 1$, so

$$0 < v(x,1/n) \leq f_n(x) \leq 1.$$

We define

$$K(\gamma) = \max_{x \in V} \{ |\log v(x,1/n)| + |d\nu(\gamma,x)| + |\log v(\gamma x,\tfrac{1}{n})| \},$$

and the bound follows from the inequality $|\log f_n(x) + \log D(\gamma,x) - \log f_n(\gamma x)| \leq K(\gamma)$. □

Lemma 3.1 is essentially a pointwise version of Theorem 1.1. To finish the proof, we remove the dependence on x. First, given $\epsilon > 0$ the continuity of $D(\gamma,x)$ implies there exists $\delta(\gamma,\epsilon) > 0$ so that (2.2) and (2.3) hold for $\delta = \delta(\gamma,\epsilon)$ and all $x \in V$.

Next define the good set

$$G(\delta,n) = \{x \in V \mid \frac{m(B(x,1/n,\delta))}{m(X(x,1/n))} < \delta\}.$$

Our hypothesis implies that $m(V - \bigcup_{n>0} G(\delta,n)) = 0$ for all $\delta > 0$. Thus, given $\epsilon,\delta > 0$ there exists $N(\epsilon,\delta)$ so that $n > N(\epsilon,\delta)$ implies $m(G(\delta,n)) > 1 - \epsilon$.

Given $\epsilon > 0$ and a finite subset $\Delta \subset \Gamma$, set $\delta = \min_{\gamma\in\Delta} \delta(\gamma,\epsilon)$; $K_\Delta = \max_{\gamma\in\Delta} K(\gamma)$ and for a choice of $n > N(\epsilon,\delta)$, $X(\epsilon) = G(\delta,n)$. Define a transfer function by $g_n(x) = -\log f_n(x)$, and the cocycle ψ_ϵ by

$$\psi_\epsilon(\gamma,x) = g_n(\gamma x) + d\nu(\gamma,x) - g_n(x).$$

Then $m(V - X(\epsilon)) < \epsilon$, and for $x \in X(\epsilon)$ and all $\gamma \in \Delta$, the proof of Lemma 3.1 shows that $|\psi_\epsilon(\gamma,x)| < \epsilon$. For $x \notin X(\epsilon)$, we also showed $|\psi_\epsilon(\gamma,x)| < K(\gamma) \leq K_\Delta$. □

§4. Proofs of Theorems 2.1 and 2.2

The proofs of Theorems 2.1 and 2.2 are based on the use of the stable manifold theory of Pesin [11] (see also Katok [10]) for non-uniformly hyperbolic actions. The key result of Pesin Theory we need is that for $\gamma \in \Gamma$, given a measure μ on V which is $\varphi(\gamma)$-invariant, for μ-a.e. $x \in V$ the action of $\varphi(\gamma)$ is regular at x. This implies that if $\chi^+(\gamma,x) > 0$ for a regular point x, then there is a stable manifold L_x through x on which $\varphi(\gamma^{-1})$ acts as a strict contraction. In particular, this means x is not a distal point.

Given $\gamma \in \Gamma$, define an action of the integers on V by:

$$\Phi\colon Z \times V \longrightarrow V;\ \Phi(n,x) = \varphi(\gamma^n,x).$$

The Jacobian cocycle of Φ is the restriction of $J\varphi$:

$$J\Phi\colon Z \times V \longrightarrow GL(q,\mathbb{R});\ J\Phi(n,x) = J\varphi(\gamma^n,x),$$

and $\chi^+(\gamma,x) = \limsup_{n\to\infty} \frac{1}{n} \|J\Phi(n,x)\|$.

To prove Theorem 2.2, note that an invariant measure μ for the action φ is given. Now, μ-a.e. point $x \in V$ is regular for all $\gamma \in \Gamma$ in the sense of Pesin Theory. Suppose that there exists $\gamma \in \Gamma$ and a set Y with positive μ-measure for which $\chi^+(\gamma,y) > 0$, all $y \in Y$. Then μ-a.e. point y in Y is regular, hence μ-a.e. point in Y is not distal by the discussion above. It is given that μ is equivalent to Lebesgue measure, and a.e. point in V is distal. This contradiction implies that for all $\gamma \in \Gamma$, $\chi^+(\gamma,y) = 0$ for a.e. $y \in V$. □

The proof of Theorem 2.1 is more delicate, for no invariant measure for the action φ is given. To begin, assume that for some $\gamma \in \Gamma$ and $x \in V$, $\chi^+(\gamma,x) > 0$. Then form the action Φ of the integers on V as above. The set of probability measures on V invariant under Φ is non-empty by the Krylov-Bogolinbov method, so the task is to show there is a set Y of positive μ-measure, for some invariant measure μ, on which $\chi^+(\gamma,y) > 0$ for $y \in Y$. We will invoke Theorem 10.1 of [9], which requires a further estimate:

Lemma 4.1. *For any* $x \in V$ *and for all* $S,T > 0$,

$$\underset{\substack{m \geq S \\ n \geq T}}{\text{supremum}} \frac{1}{n} \log \|J\Phi(n,\Phi(m,x))\| \geq \chi^+(\gamma,x).$$

<u>**Proof**</u>. V is compact, so

$$M(\gamma) = \sup_{x \in V} \|\Phi(1,x)\| < \infty.$$

Then by multiplicativity of the matrix norm and the cocycle rule, $\|\Phi(n,x)\| \leq M(\gamma)^n$ for all $n > 0$. Fix $S,T > 0$. Given any $1 > \epsilon > 0$, choose $N > \frac{2}{\epsilon} \cdot M(\gamma) \cdot \max\{S,T\}$ such that

$$\frac{1}{N} \log \|J\Phi(N,x)\| > \chi^+(\gamma,x) - \epsilon/2.$$

Then by the cocycle law,

$$\frac{1}{N} \log \|J\Phi(N-S,\Phi(S,x))\| + \frac{1}{N} \log \|J\Phi(S,x)\|$$

$$\geq \frac{1}{N} \log \|J\Phi(N,x)\| > \chi^+(\gamma,x) - \epsilon/2,$$

and from the estimate $\frac{1}{N} \log \|J\Phi(S,x)\| < \epsilon/2$, we get $\frac{1}{N-S} \log \|J\Phi(N-S,\Phi(S,x))\| \geq \chi^+(\gamma,x) - \epsilon$. As ϵ can be chosen arbitrarily small and $N-S > R$, we get $\sup_{m>R} \frac{1}{m} \|J\Phi(m,\Phi(S,x))\| \geq \chi^+(\gamma,x)$. □

Suppose that $\chi^+(\gamma,x) > 0$. Let Ω be the ω-limit set of the forward orbit $\{\Phi(n,x) \mid n > 0\}$. Then Lemma 4.1, combined with Theorem 10.1 of [9] (see also Remark 10.1) implies that there is an ergodic measure μ, supported on Ω and invariant under the action Φ, and a set Y of μ-measure 1 for which $\chi^+(\gamma,y) > 0$ for all $y \in Y$. We apply the Pesin Theory to μ, to conclude μ-a.e. point in Y is regular, so μ-a.e. point in Y is not distal. This is a contradiction, forcing

$x^+(\gamma,x) = 0$ for all $\gamma \in \Gamma$ and $x \in V$. □

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