APPENDIX A

The $\bar\partial$-Operator

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The purpose of this Appendix is to discuss the conclusion of the foliation index theorem in the context of foliations whose leaves are two-dimensional. Such foliations provide a class of reasonably concrete examples; while they are certainly not completely representative of the wide range of foliations to which the theorem applies, they are sufficiently complicated to warrant special attention, and possess the smallest leaf dimension for which the leaves have interesting topology. There is another, more fundamental reason for studying these foliations: given any leafwise C^∞-Riemannian metric on a two-dimensional foliation $\mathcal{F}$, there is a corresponding complex-analytic structure on leaves making $\mathcal{F}$ into a leafwise complex analytic foliation. Thus, two-dimensional foliations automatically possess a Teichmüller space, and for each point in this space of complex structures, there is an associated Dirac operator along the leaves. The foliation index theorem then assumes the role of a Riemann–Roch Theorem for these complex structures.

We begin in Section A1 with a discussion of the average Euler characteristic of Phillips–Sullivan, which is the prototype for the topological index character of the foliation index theorems for surfaces. In Section A2, the index theorem is reformulated for the $\bar\partial$-operator along the leaves of a leafwise-complex foliation. The Teichmüller spaces for two-dimensional foliations are discussed in Section A3, and a few remarks about their properties are given. In Section A4, some homotopy questions concerning the K-theory of the symbols of leafwise elliptic operators are discussed, with regard to the determination of all possible topological indices for a fixed foliation. Finally Section A5 describes *some* of the "standard" foliations by surfaces, especially of three-manifolds, and the calculation of the foliation indices for them.

The reader will observe that this Appendix concentrates upon topological aspects of the foliation index theorem and serves as an elaboration upon Connes' example of a foliation by complex lines on the four-manifold $\mathbb{C}/\Lambda_1 \times \mathbb{C}/\Lambda_2$ described in Section A3. A key point of this example is that the meaning of the analytic index along the leaves can also be explicitly described in terms of functions with prescribed zeros-and-poles and a growth condition. For the foliations we consider, such an explicit description of the analytic index is much harder to describe, and would take us too far afield, but must be considered an interesting open problem, especially with regards to the Riemann–Roch nature of the foliation index theorem.

A1. Average Euler Characteristic

The index theorem for the de Rham complex of a compact even-dimensional manifold, M, yields the Chern–Gauss–Bonnet formula for its Euler characteristic, which is equal to the alternating sum of the Betti numbers of M. In a likewise fashion, it was shown in Chapter VIII that the foliation index theorem for the tangential de Rham complex of a foliated space yields an alternating sum of "Betti measures". When the transverse measure ν has a special form, i.e., it is defined by an averaging sequence, the d-foliation index can also be interpreted as the ν-average Euler characteristic of the leaves. We examine this latter concept more closely, for it provides a prototype for the calculation of the topological index in the general foliation index theorem. First, recall the integrated form of Theorem 8.7 for the Euler characteristic:

Theorem A1.1 (d-Foliation Index Theorem). *Let ν be a transverse invariant measure for a foliation $\mathcal{F}$ of a foliated space X, with $C_\nu \in H_p^\tau(X)$ the associated Ruelle–Sullivan homology class of ν. Let d be the de Rham operator on the tangential de Rham complex of $\mathcal{F}$. Assume the tangent bundle FX is oriented, with associated Euler form $e^\tau(X)$. Then*

$$\chi(\mathcal{F}, \nu) \equiv \int_X \iota_d \cdot d\nu = \int_X e^\tau(X)\, d\nu = \langle e^\tau(X), C_\nu \rangle . \tag{A1.2}$$

The left-hand side of (A1.2) is interpreted in Chapter VIII as the alternating sum of the ν-dimensions of the L^2-harmonic forms on the leaves of $\mathcal{F}$. To give a geometric interpretation of the right-hand side of (A1.2), we require that ν be the limit of discrete regular measures:

Definition A1.3. An *averaging sequence* [Goodman and Plante 1979] for $\mathcal{F}$ is a sequence of compact subsets $\{L_j \mid j = 1, 2, \dots\}$, where each L_j is a submanifold with boundary of some leaf of $\mathcal{F}$, and

$$\frac{\operatorname{vol} \partial L_j}{\operatorname{vol} L_j} \to 0.$$

(The sets $\{L_j\}$ may belong to differing leaves as j varies, and we are assuming a Riemannian metric on FX has been chosen and fixed.)

The sequence $\{L_j\}$ is *regular* if the submanifolds ∂L_j of X have bounded geometry, meaning that there is a uniform bound on the sectional curvatures, the injectivity radii and the second fundamental forms of the ∂L_j.

For X compact, the measure ν_L associated to an averaging sequence is defined, on a tangential measure λ, by the rule

$$\int_X \lambda \, d\nu_L = \lim_{j\to\infty} \frac{1}{\operatorname{vol} L_j} \int_{L_j} \lambda,$$

where, if necessary, we pass to a subsequence of the $\{L_j\}$ for which the integrals converge in a weak-$*$ topology. The closed current associated to ν_L determines an asymptotic homology class denoted by $C_L \in H_p(X; \mathbb{R})$.

We say a transverse invariant measure ν is *regular* if $\nu = \nu_L$ for some regular averaging sequence $\{L_j \mid j = 1, 2, \dots\}$.

Not all invariant transverse measures arise from averaging sequences, but there are many examples where they do, the primary case being foliations with growth restrictions on the leaves. Choose a Riemannian metric on FX. Its restriction to a leaf $L \subset X$ of $\mathcal{F}$ defines a distance function and volume form on L. Pick a base point $x \in L$ and let $g(r, x)$ be the volume of the ball of radius r centered at x. We say L has

polynomial growth of degree $\le n$ if $\displaystyle\limsup_{r\to\infty} \frac{g(r,x)}{r^n} < \infty$;

subexponential growth if $\displaystyle\limsup_{r\to\infty} \frac{1}{r} \log g(r,x) = 0$;

nonexponential growth if $\displaystyle\liminf_{r\to\infty} \frac{1}{r} \log g(r,x) = 0$;

exponential growth if $\displaystyle\liminf_{r\to\infty} \frac{1}{r} \log g(r,x) > 0$.

For X compact, the growth type of L is independent of the choice of metric on FX and the basepoint x.

For a leaf L with nonexponential growth, there is a sequence of radii $r_j \to \infty$ for which the balls L_j of radius r_j centered at X form an averaging sequence [Plante 1975]. In this case, all of the sets L_j are contained in the same leaf L.

For X compact and the foliation of class C^2, these sets L_j can be modified to make them regular as well.

For a foliation $\mathcal{F}$ with even-dimensional leaves and a regular measure ν, the d-Index Theorem becomes

$$\chi(\mathcal{F}, \nu) = \lim_{j\to\infty} \frac{1}{\operatorname{vol} L_j} \int_{L_j} e^{\tau}(X).$$

By the Gauss–Bonnet theorem,

$$\int_{L_j} e^{\tau}(X) = e(L_j) + \int_{\partial L_j} \epsilon_j,$$

where $e(L_j)$ is the Euler characteristic of L_j and ϵ_j is a correction term depending on the Riemannian geometry of ∂L_j. The assumption that the submanifolds $\{\partial L_j\}$ have uniformly bounded geometry implies there is a uniform estimate

$$\left| \int_{\partial L_j} \epsilon_j \right| \leq K \cdot \operatorname{vol} \partial L_j.$$

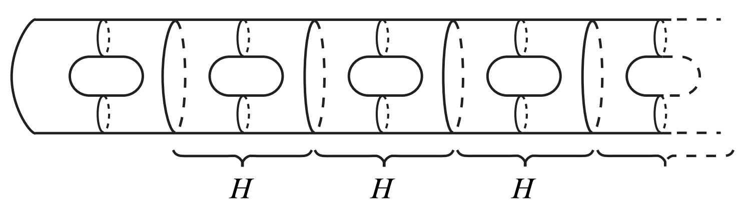
Therefore, in the limit we have

$$\chi(\mathcal{F}, \nu) = \lim_{j\to\infty} \frac{e(L_j)}{\operatorname{vol} L_j} \tag{A1.4}$$

and the right side of (A1.4) is called the *average Euler characteristic* of the averaging sequence $\{L_j\}$. Phillips and Sullivan [1981] and Cantwell and Conlon [1977] use this invariant of a noncompact Riemannian manifold to give examples of quasi-isometry types of manifolds which cannot be realized as leaves of foliations of a manifold X with $H_p(X, \mathbb{R}) = 0$.

Consider three examples of open two-manifolds [Phillips and Sullivan 1981] whose metric is defined by the given embedding in E^3. Each of the following, with their quasi-isometry class of metrics, can be realized as leaves of some foliation of some three-manifold, but the first two cannot be realized (with the given quasi-isometry class of metrics) as leaves in S^3.

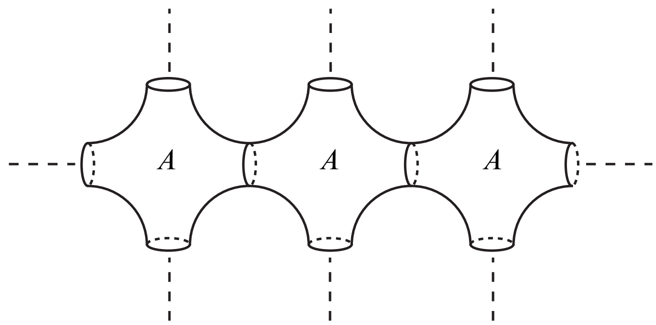
Jacob's ladder:

(A1.5) $L \sim$ 

The growth type of L is linear, and the average Euler characteristic of L is $1/\operatorname{vol} H$.

Infinite Jail Cell Window:

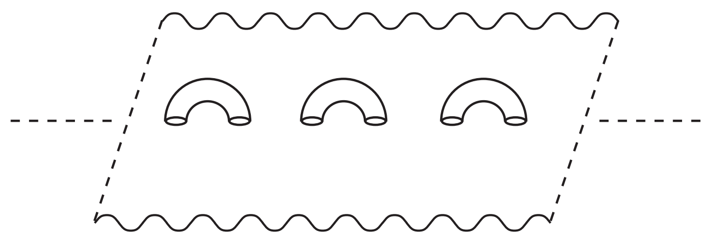
(A1.6) $L \sim$



The growth of L is quadratic, and the average Euler characteristic of L is $2/\operatorname{vol} A$.

Infinite Loch Ness Monster:

(A1.7) $L \sim$



The growth of L is quadratic, but the average Euler characteristic is zero.

The construction of the average Euler characteristic for surfaces suggests that a similar geometric interpretation can be given for the topological index of other differential operators. For the $\bar{\partial}$-operator of complex line foliations, this is indeed true, as discussed further in Section A3.

A2. The $\bar{\partial}$-Index Theorem and Riemann–Roch

We next examine in detail the meaning of the foliation index theorem for the tangential $\bar{\partial}$-operator. Let $\mathscr{F}$ be a foliation of a foliated space X and assume the leaves of $\mathscr{F}$ are complex manifolds whose complex structure varies continuously in X. That is, in Definition 2.1 (page 32), we assume that foliation charts $\{\varphi_x\}$ can be chosen for which the composition $t_y \circ \varphi_x^{-1}(\cdot, n)$ is holomorphic for all n, and $n \mapsto t_y \circ \varphi_x^{-1}(\cdot, n)$ is continuous in the space of holomorphic maps.

Let k be such that the dimension of the leaves of $\mathscr{F}$ is $p = 2k$.

A continuous vector bundle $E \to X$ is *holomorphic* if for each leaf $L \subset X$ with given complex structure, the restriction $E|_L \longrightarrow L$ is a holomorphic bundle. As before, FX is the tangent bundle to the leaves of $\mathscr{F}$, and this is holomorphic in the above sense. Let $A^{r,s} \to X$ be the bundle of smooth tensors of type (r, s):

$$A^{r,s} = \Lambda^{r,s}(F_{\mathbb{C}}X^*).$$

Given a holomorphic bundle E, let $A^{r,s} \otimes E$ denote the (r,s)-forms with coefficients in E. Assume that E has an Hermitian inner product, and then set

$$L^2(\mathcal{F}, E) = \bigoplus_{x \in X} L^2(L_x, E|_{L_x}),$$

where L_x is the leaf of $\mathcal{F}$ through x, $E|_{L_x}$ is the restriction of the Hermitian bundle E to L_x, and we then take the L^2-sections of E over L_x with respect to a Lebesgue measure on L_x inherited from a Riemannian metric on FX. Note that $L^2(\mathcal{F}, E)$ is in general neither a subspace nor a quotient of $L^2(X, E)$, the global L^2-sections of E over X.

For E a leafwise-holomorphic bundle, the leafwise $\bar\partial$-operator for $\mathcal{F}$ has a densely defined extension to

$$\bar\partial \otimes E : L^2(\mathcal{F}, A^{r,s} \otimes E) \to L^2(\mathcal{F}, A^{r,s+1} \otimes E)$$

which is tangentially elliptic. Let $\mathrm{Ker}_s(\bar\partial \otimes E)$ denote the kernel of

$$\bar\partial \otimes E : L^2(\mathcal{F}, A^{0,s} \otimes E) \to L^2(\mathcal{F}, A^{0,s+1} \otimes E).$$

An element $\omega \in \mathrm{Ker}_s(\bar\partial \otimes E)$ is a form whose restriction to each leaf L is a smooth form of type $(0,s)$ satisfying $\bar\partial(\omega|_L) = 0$. Furthermore, for each $s \geq 0$, $\mathrm{Ker}_s(\bar\partial \otimes E)$ is a locally finite-dimensional space over X (see Chapter I). For an invariant transverse measure ν, the total ν-density of the $(0,s)$-solutions ω to the equation $\bar\partial \otimes E(\omega) = 0$ is $\dim_\nu ker_s(\bar\partial \otimes E)$, and we set

$$\dim_\nu \mathrm{Ker}(\bar\partial \otimes E) = \sum_{s=0}^{k} \dim_\nu \mathrm{Ker}_s(\bar\partial \otimes E).$$

Similar arguments apply to the adjoint $\bar\partial^*$, and with the notation of Chapter IV we have

$$\int_X \iota_{\bar\partial \otimes E}\, d\nu = \dim_\nu \mathrm{Ker}(\bar\partial \otimes E) - \dim_\nu \mathrm{Ker}(\bar\partial^* \otimes E).$$

Theorem A2.1 ($\bar\partial$-Index Theorem). *Let ν be an invariant transverse measure for a complex foliation $\mathcal{F}$ of X, $C_\nu \in H^\tau_{2\kappa}(X;\mathbb{R})$ the associated Ruelle–Sullivan homology class, and* $\mathrm{Td}_\tau(X) = \mathrm{Td}(FX \otimes \mathbb{C})$ *the tangential Todd class for $\mathcal{F}$. Then*

(A2.2) $$\int_X \iota_{\bar\partial \otimes E}\, d\nu = \left\langle \mathrm{ch}(\bar\partial \otimes E)\, \mathrm{Td}_\tau(X), C_\nu \right\rangle.$$

The left-hand side of (A2.2) is identified with the *arithmetic genus* of $\mathcal{F}$ with coefficients in E,

$$\chi(\bar{\partial} \otimes E, \nu) = \sum_{i=0}^{\kappa} (-1)^i \dim_\nu H^i(\mathcal{F}; E),$$

where

$$H^i(\mathcal{F}, E) \equiv \mathrm{Ker}_i(\bar{\partial} \otimes E) / \mathrm{Ker}_i(\bar{\partial}^* \otimes E)$$

is a locally finite-dimensional space over X. The number $\dim_\nu H^i(\mathcal{F}; E)$ measures the density of this cohomology group in the support of ν, and generalizes the ν-Betti numbers of the operator d.

On the right-hand side of (A2.2), the term $\mathrm{ch}(\bar{\partial} \otimes E)$ is the Chern character of the K-theory class determined by the complex

$$A^{0,*} \otimes E.$$

There is a standard simplification of the cup product

$$\mathrm{ch}(\bar{\partial} \otimes E)\, \mathrm{Td}(FX \otimes \mathbb{C}),$$

which yields:

Corollary A2.3. $\chi(\bar{\partial} \otimes E, \nu) = \big\langle \mathrm{ch}(E)\, \mathrm{Td}_\tau(FX), C_\nu \big\rangle$.

Proof. Use the splitting principle and the multiplicativity of the Chern and Todd characters. For details, see [Shanahan 1978]. □

Corollary A2.3 is exactly the classical Riemann–Roch Theorem in the context of foliations. The arithmetical genus $\chi(\bar{\partial} \otimes E, \nu)$ is the ν-density of the alternating sum of the dimensions of the $\bar{\partial}$-closed L^2-forms on the leaves of $\mathcal{F}$. The right-hand side is a topological invariant of E, FX and C_ν. For a given measure ν, one can hope to choose the bundle E so that $\chi(\bar{\partial} \otimes E, \nu) \neq 0$, guaranteeing the existence for ν-a.e. leaf L of $\mathcal{F}$ of $\bar{\partial}$-closed L^2-forms on L with coefficients in E.

A3. Foliations by Surfaces (Complex Lines or $k = 1$)

Let X be a compact foliated space with foliation $\mathcal{F}$ having leaves of dimension $p = 2$. For example, we may take $X = M$ to be a smooth manifold and assume TM admits a 2-plane subbundle F. Then by [Thurston 1976], F is homotopic to a bundle FM which is tangent to a smooth foliation of M by surfaces.

Lemma A3.1. *Let $\mathcal{F}$ be a two-dimensional foliation of X with FX orientable. Then every Riemannian metric g on FX canonically determines a continuous complex structure on the leaves of $\mathcal{F}$. That is, the pair $(\mathcal{F}, g)$ determines a complex foliation of X.*

Proof. Define a J-operator J_g on FX to be rotation by $+\pi/2$ with respect to the given metric g and the orientation. For each leaf L, the structure $J_g|L$ is integrable as the leaf is two-dimensional hence uniquely defines a complex structure on L. Furthermore, by the parametrized Riemann mapping theorem [Ahlfors 1966], there exist foliation charts for $\mathcal{F}$ with each $t' \circ \varphi_x(\cdot, n)$ holomorphic and continuous in the variable ν. □

We remark that if $\mathcal{F}$ has a given complex structure, J, then a metric g can be defined on FX for which $J_g = J$. Thus, the construction of Lemma A3.1 yields all possible complex structures on $\mathcal{F}$. This suggests the definition of the Teichmüller space of a two-dimensional foliation $\mathcal{F}$ of a space X. We say two metrics g and g' on FX are *holomorphically equivalent* if there is a homeomorphism $\phi : X \to X$ mapping the leaves of $\mathcal{F}$ smoothly onto themselves, and $\phi^* g'$ is conformally equivalent to g. We say that g and g' are *measurably holomorphically equivalent* if there is a measurable automorphism ϕ of X mapping leaves of $\mathcal{F}$ smoothly onto leaves of $\mathcal{F}$, and $\phi^*(g')$ is conformally equivalent to g by a measurable conformal factor on X.

Definition A3.2. *Teichmüller space* $T(X, \mathcal{F})$ is the set of holomorphic equivalence classes of metrics on FX. The *measurable Teichmüller space* $T_m(X, \mathcal{F})$ is the subset of $T(X, \mathcal{F})$ consisting of measurably holomorphic equivalence classes.

When $\mathcal{F}$ consists of one leaf, this reduces to the usual Teichmüller space of a surface. When $\mathcal{F}$ is defined by a fibration $X \to Y$ with fibre a surface Σ, let $T(\Sigma)$ be the Teichmüller space of Σ, then

$$T(X, \mathcal{F}) = C^0(Y, T(\Sigma))$$

is infinite-dimensional. The more interesting question is to study $T(X, \mathcal{F})$ for an ergodic foliation $\mathcal{F}$. There are constructions of foliated manifolds, due to E. Ghys, which show that $T(X, \mathcal{F})$ can be infinite-dimensional, even for $\mathcal{F}$ ergodic [Ghys 1997; 1999].

Related to this is a problem first posed by J. Cantwell and L. Conlon: when does there exist a metric on FX for which every leaf has constant negative curvature? A complete solution is given for codimension-one, proper foliations [Cantwell and Conlon 1989], as well as for leaves in Markov exceptional minimal sets [Cantwell and Conlon 1991]. There is also a more general problem, which is to “uniformize” $\mathcal{F}$ — that is, to find a metric on the leaves such that the curvature is constant. This problem was solved by Alberto Candel [1993] in the case all leaves are hyperbolic, or in the case they are spherical. Ghys [1997; 1999] gives a discussion and survey of the more general case of leaves of mixed type. As an analogue of the Phillips–Sullivan Theorem in Section A1, one can ask if given a surface Σ with complex structure J_Σ, and given a compact manifold X, does there exist a foliation $\mathcal{F}$ of X and $[g] \in T(X, \mathcal{F})$ with Σ a leaf of $\mathcal{F}$ so that the complex structure induced on Σ by $[g]$ coincides with J_Σ? The average Euler characteristic of Σ still provides an obstruction to solving this problem, when Σ has nonexponential growth type, but the additional requirement that Σ have a prescribed complex structure should force other obstructions to arise. This would be especially interesting to understand for Σ of exponential growth-type, where no obstructions are presently known.

We now turn to consideration of the $\bar{\partial}$-Index Theorem for a foliation by complex lines, and derive an analogue of the average Euler characteristic.

Lemma A3.3. *Let $\mathcal{F}$ be a complex line foliation of X. Then*

$$\chi(\bar{\partial} \otimes E, \nu) = \langle c_1(E), C_\nu \rangle + \tfrac{1}{2}\, \chi(\mathcal{F}, \nu). \tag{A3.4}$$

Proof. The degree-two component of $\mathrm{ch}_\tau(E)\,\mathrm{Td}_\tau(X)$ is

$$c_1(E) + \tfrac{1}{2}\, c_1(FX). \qquad \square$$

Our goal is to give a geometric interpretation of the term $\langle c_1(E), C_\nu \rangle$ in (A3.4) similar to the average Euler characteristic.

Let $\kappa \to \mathbb{C}\mathrm{P}^N$ be the canonical bundle over the complex projective N-space. For large N, there exists a tangentially smooth map

$$f_E : X \to \mathbb{C}\mathrm{P}^N \qquad \text{with} \qquad f_E^* \kappa = E.$$

(We say that f_E classifies E.) Let $H \subset \mathbb{C}\mathrm{P}^N$ be a hypersurface dual to the first Chern class $c_1 \in H^2(\mathbb{C}\mathrm{P}^N)$ of κ. For convenience, we now assume X is a C^1 manifold and $\mathcal{F}$ is also C^1. The complex structure on $\mathcal{F}$ orients its leaves, and the complex structure on $\mathbb{C}P^N$ orients the normal bundle to H. A connection on $\kappa \to \mathbb{C}\mathrm{P}^N$ pulls back under the f_E to a connection on $E \to X$, so $f_E^*(c_1) = c_1(E)$ holds both for cohomology classes and on the level of forms. Furthermore, a C^1-perturbation of f_E results in a C^0-perturbation of the form $c_1(E)$.

Given a regular averaging sequence $\{L_j\}$, for each $j \geq 1$ choose a C^1-perturbation f_j of f_E so that $f_j(L_j)$ is transverse to H, and $f_j^*(c_1)$ converges uniformly to $c_1(E)$. We say a point $x \in L_j \cap f_j^{-1}(H)$ is a *zero* of E if $f_j(L_j)$ is positively oriented at $f_j(x)$, and a *pole* if the orientation is reversed. Let $Z(L_j)$ and $P(L_j)$ denote the corresponding set of zeros and poles in L_j. Then elementary geometry shows that

$$\int_{L_j} c_1(E) = \#Z(L_j) - \#P(L_j) + \epsilon_j,$$

where the error term ϵ_j is proportional to $\mathrm{vol}\, \partial L_j$. This uses that $\{\partial L_j\}$ has uniformly bounded geometry. Combined with Lemma A3.3, we obtain:

Proposition A3.5. *Let X be a C^1 manifold and assume $\mathcal{F}$ is a C^1-holomorphic foliation by surfaces. For $\nu = \nu_L$ given by a regular averaging sequence $\{L_j\}$,*

$$\chi(\bar\partial \otimes E, \nu) = \lim_{j\to\infty} \frac{\#Z(L_j)}{\operatorname{vol} L_j} - \lim_{j\to\infty} \frac{\#P(L_j)}{\operatorname{vol} L_j} + \frac{1}{2}\chi(\mathcal{F}, \mu)$$

$$= \text{(average density of zeros of E)} \\ -\text{(average density of poles of E)} + \tfrac{1}{2}\text{(average Euler char)}.$$

Consider the case of a foliation of a 3-manifold X by surfaces. Let $\{\gamma_1, \dots, \gamma_d\}$ be a collection of d embedded closed curves in X which are transverse to $\mathcal{F}$, and $\{n_1, \dots, n_d\}$ a collection of nonzero integers. This data defines a complex line bundle $E \to X$, and for a leaf L the restriction $E|_L$ is associated to the divisor

$$\sum_{i=1}^{d} n_i \cdot (\gamma_i \cap L).$$

Let ν be an invariant transverse measure. Then Proposition A3.5 takes on the more precise form:

Proposition A3.6.

$$\dim_\nu H^0(\mathcal{F}; E) - \dim_\nu H^1(\mathcal{F}; E) = \sum_{i=1}^{d} n_i \cdot \nu(\gamma_i) + \frac{1}{2}\chi(\mathcal{F}, \nu).$$

Proof. $c_1(E)$ is dual to the 1-cycle $\sum_{i=1}^{d} n_i \cdot \gamma_i$. □

If $\nu = \nu_L$ is defined by an averaging sequence $\{L_j\}$, then $\nu(\gamma_i)$ is precisely the limit density of $(\gamma_i \cap L_j)$ in L_j, so Proposition A3.6 relates the ν-dimension of L^2-harmonic forms on the leaves of $\mathcal{F}$ with the average density of the zeros and poles of E. This is exactly what a Riemann–Roch Theorem should do. The latitude in choosing E for a given $\mathcal{F}$ means one can often ensure that either $H^0(\mathcal{F}; E)$, the L^2-meromorphic functions on the leaves of $\mathcal{F}$ with order at least $\sum n_i \cdot \gamma_i$, or the corresponding space of meromorphic 1-forms $H^1(\mathcal{F}; E)$ has positive ν-density. This type of result is of greatest interest when the complex structures of the leaves of $\mathcal{F}$ can be prescribed in advance, as in Example A3.7 below.

For a complex line foliation $\mathcal{F}$ of an n-manifold X, given closed oriented submanifolds $\{V_1, \dots, V_d\}$ of codimension 2 in X transverse to $\mathcal{F}$, and integers $\{n_1, \dots, n_d\}$, there is a holomorphic line bundle $E \to X$ corresponding to the divisor $\sum_{i=1}^{d} n_i \cdot V_i$. The existence of such closed transversals V_i to $\mathcal{F}$, and more generally of holomorphic vector bundles $E \to X$, is usually hard to ascertain. However, there is one geometric context in which such V_i always exists in multitude, the foliations given as in (2.2) of Chapter II. We briefly recall their construction.

Let Y be a compact oriented manifold of dimension $n-2$, $\sum_g$ a surface of genus g, and $\rho : \Gamma_g \to \mathrm{Diff}(Y)$ a representation of the fundamental group $\Gamma_g = \kappa_1(\Sigma_g)$. The quotient manifold

$$M = (\tilde{\Sigma} \times Y)/\Gamma_g$$

has a natural 2-dimensional foliation transverse to the fibres of

$$\kappa : M \to \Sigma_g.$$

The leaves of $\mathcal{F}$ are coverings of Σ_g associated to the isotropy groups of ρ, and inherit complex structures from Σ_g. The d-index theorem for $\mathcal{F}$ can be deduced from Atiyah's L^2-index theorem for coverings [Atiyah 1976]. For the $\bar{\partial}$-index theorem, this is no longer the case. Also, note that the Teichmüller spaces of this class of foliations always has dimension at least that of Σ_g, as every metric on $T\Sigma_g$ lifts to a metric on FM. However, they need not have the same dimension, and $T(M, \mathcal{F})$ or $T_m(M, \mathcal{F})$ provide a very interesting geometric "invariant" of the representation ρ of Γ_g on Y.

For each point $x \in \Sigma_g$, the fibre $\pi^{-1}(x) \subset M$ is a closed orientable transversal to $\mathcal{F}$. To obtain further transversals, we assume the fibration $M \to \Sigma_g$ is trivial, so there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\cong} & \Sigma_g \times Y \\ & {\scriptstyle \pi}\searrow \quad \swarrow & \\ & \Sigma_g & \end{array}$$

Note that the foliation $\tilde{\mathcal{F}}$ on $\Sigma_g \times Y$ induced from its identification with M will not, in general, be the product foliation. A transversal to $\mathcal{F}$ corresponds to a transversal to $\tilde{\mathcal{F}}$, and the latter can often be found explicitly.

Example A3.7. Consider the example described in [Connes 1982]. Here, $\Sigma_1 = \mathbb{C}/\Gamma_1$ is a complex torus, as is $Y = \mathbb{C}/\Gamma_2$, for lattices Γ_1 and Γ_2 in $\mathbb{C}$. Let Γ_1 act by translations on $\mathbb{C}/\Gamma_2$, and form

$$\begin{array}{c} M = (\mathbb{C} \times \mathbb{C}/\Gamma_2)/\Gamma_1 \cong (\mathbb{C}/\Gamma_1) \times (\mathbb{C}/\Gamma_2) \\ \Big\downarrow {\scriptstyle \kappa} \\ \mathbb{C}/\Gamma_1 \end{array}$$

Connes takes $V_1 = 0 \times \mathbb{C}/\Gamma_2$ and $V_2 = \mathbb{C}/\Gamma_1 \times 0$ as the transversals in $\Sigma_1 \times Y$. Neither V_1 nor V_2 is homotopic to a fibre $\kappa^{-1}(x)$ so the $\bar{\partial}$-index theorem for E associated to the divisor $V_1 - V_2$ is not derivable from the L^2-index theorem for coverings. For ν the Euclidean volume on $\mathbb{C}/\Gamma_2$, Connes remarks that

$$\chi(\bar{\partial} \otimes E, \nu) = \text{ density } \Gamma_2 \ - \ \text{density } \Gamma_1,$$

so the dimension of the space of L^2-harmonic functions on almost every leaf $\mathbb{C} \subset M$ with divisor $\mathbb{C} \cap (V_1 - V_2)$ is governed by the density of the lattices Γ_1 and Γ_2. Again, this is exactly the role of a Riemann–Roch Theorem, where for foliations the degree of a divisor is replaced with the average density of the divisor.

A4. Geometric K-Theories

The examples described at the end of Section A3 for the $\bar{\partial}$-operator suggest that to obtain analytical results from the foliation index theorem, it is useful to understand the possible topological indices of leafwise elliptic operators. In the examples above, the ν-topological indices are varied by making choices of "divisors" which pair nontrivially with the foliation cycle C_ν. As a consequence, various spaces of meromorphic forms are shown to be nontrivial. To obtain similar results for a general foliation, $\mathcal{F}$, it is useful to determine the range of topological indices of leafwise elliptic operators for $\mathcal{F}$. In this section, we briefly describe the formal "calculation" of these indices in terms of K-groups of foliation groupoids. In some cases, these abstract results can be explicitly calculated, giving very useful information. The reader is referred to the literature for more detailed discussions. One other point is that the foliation index theorem equates the analytic index with the evaluation of a foliation cycle on a K-theory class; these evaluations can be much easier to make, than to fully determine the topological K-theory of the foliation. In this section, and in Section A5, we will examine more carefully the values of the topological index paired with a foliation current for some classes of foliations.

Recall from Definition 2.20 of Chapter II the *holonomy groupoid*, or *graph*, $G(X)$ associated to the foliated space X. A point in $G(X)$ is an equivalence class $[\gamma_{xy}]$ of paths $\gamma : [0,1] \to X$ with $\gamma(0) = x$, $\gamma(1) = y$ and γ remains on the same leaf for all t. Two paths are identified if they have the same holonomy. $G(X)$ is a topological groupoid with the multiplication defined by concatenation of paths.

Also associated to the foliated manifold $(X, \mathcal{F})$ is a groupoid $\Gamma(X)$, constructed in [Haefliger 1984]. The groupoid $\Gamma(X)$ coincides with one of the restricted groupoids $G_N^N(X)$ of Chapter II. Let $\{U_\alpha\}$ be a locally finite open cover of X by foliation charts such that $U_\alpha \cap U_\beta$ is contractible if nonempty. For each α, there is given a diffeomorphism

$$\phi_\alpha : U_\alpha \to \mathbb{R}^p \times \mathbb{R}^q$$

sending the leaves of $\mathcal{F}|_{U_\alpha}$ to $\mathbb{R}^p \times pt$. Define a transversal

$$T_\alpha = \phi_\alpha^{-1}(\{0\} \times \mathbb{R}^q) \subset U_\alpha$$

for each α. By a judicious choice of the $\{U_\alpha\}$, we can assume the $\{T_\alpha\}$ are pairwise disjoint; see [Hilsum and Skandalis 1983]. Then set $N = \bigcup T_\alpha$, an embedded open q-submanifold of X. It is an easy exercise to show $G_N^N(X)$ coincides with the Haefliger groupoid $\Gamma(X)$ constructed from the foliation charts $\{\phi_\alpha : U_\alpha \to \mathbb{R}^{p+q}\}$.

The inclusion $N \times N \subset X \times X$ induces an inclusion of topological groupoids $\Gamma(X) \subset G(X)$. The cofibre of the inclusion is modeled on the trivial groupoid $R^p \times R^p$, where all pairs (x, y) are morphisms. One thus expects the above inclusion to be an equivalence, and Haefliger [1984] shows that this is indeed so:

Theorem A4.1 (Haefliger). *The inclusion $\Gamma(X) \subset G(X)$ is a Morita equivalence of categories.*

For any topological groupoid $\mathcal{G}$, there is a classifying space $B\mathcal{G}$ of $\mathcal{G}$ structures, which is constructed using a modification of the Milnor join construction [Haefliger 1971; Milnor 1956]. Applying this to $G(X)$ yields the space $BG(X)$ which is fundamental for foliation K-theory; see [Connes 1982, Chapter 9]. Applying the B-construction to $\Gamma(X)$, we obtain a space $B\Gamma(X)$ which is fundamental for the characteristic class theory of $\mathcal{F}$.

Corollary. *Let $(X, \mathcal{F})$ be a foliated space. The inclusion $\Gamma(X) \subset G(X)$ induces a homotopy equivalence $B\Gamma(X) \simeq BG(X)$.*

Thus, the topological invariants of $B\Gamma(X)$ and $BG(X)$ agree. Note the open contractible covering $\{U_\alpha\}$ of X defines a natural continuous map $X \to B\Gamma(X)$. If all leaves of $\mathcal{F}$ are contractible, this inclusion is a homotopy equivalence, so that the topological type of $BG(X)$ is the same as X. By placing weaker restrictions on the topological types of the leaves of $\mathcal{F}$, one can more generally deduce that the inclusion is an N-equivalence on homotopy groups; see [Haefliger 1984]. For the generic foliation, however, one expects that the space $BG(X)$ will have a distinct topological type from X, probably more complicated.

The space $B\Gamma(X)$ can be studied from a "universal viewpoint" by introducing the Haefliger classifying spaces. For the class of transversally C^r-differentiable foliations of codimension q, Haefliger defines a space $B\Gamma_q^{(r)}$, and there is a universal map

$$i_X : B\Gamma(X) \to B\Gamma_q^{(r)}.$$

The cohomology groups of $B\Gamma_q^{(r)}$ then define universal classes which pull back to $B\Gamma(X)$ via $(i_X)^*$. The nontriviality of $(i_X)^*$ is then a statement about both the topology of $B\Gamma(X)$ and the inclusion i_X. A short digression will describe the situation for C^∞ foliations.

Let $B\Gamma_q$ be the universal classifying space of codimension q C^∞-foliations. (It is important to specify the transverse differentiability of $\mathcal{F}$, as the topology of $B\Gamma_q$ depends strongly on how much differentiability is required.) The composition

$$f_{\mathcal{F}} : X \to B\Gamma(X) \to B\Gamma_q,$$

or more precisely its homotopy class, was introduced by Haefliger in order to "classify" the C^∞-foliations on a given X. The classification is modulo an equivalence relation which turns out to be *concordance* for X compact, and *integrable homotopy* for X open; see [Haefliger 1971].

For $B\Gamma(X)$, the principal invariants are the characteristic classes. Recall the definition of the differential graded algebra

$$WO_q = \Lambda(h_1, h_3, \ldots, h_{q'}) \otimes \mathbb{R}[c_1, c_2, \ldots, c_q]_{2q},$$

where the subscript $2q$ indicates that this is a truncated polynomial algebra, truncated in degrees greater than $2q$, and q' is the greatest odd integer not exceeding q. The differential is determined by $d(h_i \otimes 1) = 1 \otimes c_i$ and $d(1 \otimes c_i) = 0$. The monomials $\wedge c_J = h_{1_1} \wedge \cdots \wedge h_{i_\ell} \wedge c_1^{j_1} \cdots c_q^{j_q}$, where

$$\text{(A4.2)} \qquad 1_1 < \cdots < i_\ell, \quad |J| = j_1 + 2j_2 + \cdots + q j_q \leq q, \quad i_1 + |J| > q$$

are closed, and they span the cohomology $H^*(WO_q)$ in degrees greater than $2q$. The *Vey basis* is a subset of these [Bott and Haefliger 1972; Kamber and Tondeur 1974; 1975; Lawson 1974].

A foliation $\mathcal{F}$ on M determines a map of differential algebras into the de Rham complex of M, $\Delta_{\mathcal{F}}: WO_q \to \Omega^*(M)$. The induced map in cohomology,

$$\Delta^*_{\mathcal{F}}: H^*(WO_q) \to H^*(M),$$

depends only the integrable homotopy class of $\mathcal{F}$. The secondary classes of $\mathcal{F}$ are spanned by the images $\Delta^*_{\mathcal{F}}(h_I \wedge c_J)$ for $h_I \wedge c_J$ satisfying (A4.2).

The construction of the map $\Delta_{\mathcal{F}}$ is functorial, so there exists a universal map

$$\tilde{\Delta}_* : H^*(WO_q) \to H^*(B\Gamma_q)$$

(see [Lawson 1977]), and for given $\mathcal{F}$ on X we obtain its secondary classes via

$$\Delta_* = f^*_{\mathcal{F}} \circ \tilde{\Delta}_* : H^*(WO_q) \to H^*(X).$$

We next describe how the topology of $BG(X)$ is related to the topological indices of leafwise elliptic operators of $\mathcal{F}$. For $\mathcal{F}$ a C^1-foliation of a manifold X, the groupoid $G(X)$ has a natural map to $\mathrm{GL}(q, \mathbb{R})$ obtained by taking the Jacobian matrix of the holonomy along a path $[\gamma_{xy}]$. This induces a map

$$BG(X) \to B\,\mathrm{GL}(q, \mathbb{R}),$$

which defines a rank q vector bundle $\xi \to BG(X)$ whose pullback to X under $X \to B\Gamma(X) \to BG(X)$ is the normal bundle to $\mathcal{F}$. The ξ-twisted K-theory of

$BG(X)$ is defined as

$$K_*^\xi(BG(X)) \equiv K_*(B(\xi), S(\xi)),$$

where $B(\xi)$ is the unit disc subbundle of $\xi \to BG(X)$, and $S(\xi)$ is the unit sphere bundle.

Connes and Skandalis [1984] construct a map

$$\mathrm{Ind}_t : K_*^\xi(BG(X)) \to K_*(C_r^*(X)),$$

which they call the topological index map, via an essentially topological procedure that converts a vector bundle or unitary over $BG(X)$ into an idempotent or invertible element over $C_r^*(X)$. Let F_1^*X denote the unit cotangent bundle to $\mathcal{F}$ over X. Then there is a natural map of K-theories,

$$b : K^1(F_1^*X) \to K_0^\xi(BG(X)),$$

obtained from the exact sequence for the pair $(B(\xi), S(\xi))$. If $\mathcal{F}$ admits a transverse invariant measure ν, there is a linear functional ϕ_ν on $K_0(C_r^*(X))$ (Proposition 6.23), and the composition $\phi_\nu \circ \mathrm{Ind}_t \circ b = \mathrm{Ind}_\nu^t$, the topological measured index. That is, for D a leafwise operator with symbol class $u = [\sigma_D] \in K^1(F_1^*X)$.

$$\phi_\nu \circ \mathrm{Ind}_t \circ b(u) = \big\langle \mathrm{ch}(D)\,\mathrm{Td}_\tau(X), C_\nu \big\rangle.$$

Connes and Skandalis also construct a direct map,

$$\mathrm{Ind}_a : K^1(F_1^*X) \to K_0(C_r^*(X)),$$

which they call the analytic index homomorphism, by associating to an invertible u the index projection operator over $C_r^*(X)$ of a zero-order leafwise elliptic operator whose symbol class is u. Also,

$$\mathrm{Ind}_\nu^a \equiv \phi_\nu \circ \mathrm{Ind}_a(u)$$

is the analytic index of this operator, calculated using the dimension function associated to ν. They then proved in [Connes and Skandalis 1984]:

Theorem A4.3 (Connes–Skandalis General Foliation Index Theorem). *For any foliation $\mathcal{F}$, there is an equality of maps*

$$\mathrm{Ind}_a = \mathrm{Ind}_t \circ b : K^1(F_1^*X) \to K_0(C_r^*(X)).$$

Note that Theorem A4.3 makes sense even when $\mathcal{F}$ possesses no invariant measures. If there is an invariant measure, ν, then by the above remarks, the theorem implies the ν-measured foliation index theorem proved in Chapters 7 and 8. Note also that this formulation of the index theorem shows that the possible range of the analytic traces of leafwise operators, with respect to a given invariant measure ν, are contained in the image of the map

$$\phi_\nu \circ \mathrm{Ind}_t : K_0^\xi(BG(X)) \to \mathbb{R}.$$

This is the meaning of the earlier statement that the topology of $BG(X)$ dictates the possible analytic indices of leafwise operators, and motivates the study of $BG(X)$. In fact, Connes has conjectured that this space has K-theory isomorphic to that of $C_r^*(X)$.

Conjecture A4.4. *Suppose that all holonomy groups of* $\mathcal{F}$ *are torsion-free. Then* Ind_t *is an isomorphism.*[1]

It is known that Conjecture A4.4 is true if $\mathcal{F}$ is defined by a free action of a simply connected solvable Lie group on X; see [Connes 1982]. Also, for flows on the 2-torus and for certain "Reeb foliations" of three-manifolds, the work of Torpe [1985] and Penington [1983] shows that conjecture A4.4 holds.

Given a foliated manifold X with both FX and TX orientable, a natural problem, related to Conjecture A4.4, is to determine to what extent the composition

$$K_*(X) \xrightarrow[\cong]{\text{Thom}} K_*^{\xi}(X) \longrightarrow K_*^{\xi}(BG(X)) \xrightarrow{\mathrm{Ind}_t} K_*(C_r^*(X))$$

is an isomorphism. We describe three quite general results on this, and then show that the $\bar\partial$-Index Theorem also sheds some light on this problem in particular cases.

Let G be a connected Lie group. A locally free action of G on X is *almost free* if given $g \in G$ with fixed point $x \in X$, either $g = \mathrm{id}$ or the germ of the action of g near x is nontrivial. If $\mathcal{F}$ is defined by an almost free action of G on X, then $G(X) \cong X \times G$. If G is also contractible, then $X \to BG(X)$ is a homotopy equivalence.

Theorem A4.5 [Connes 1982]. *Let* $\mathcal{F}$ *be defined by an almost free action of a simply connected solvable Lie group* G *on* X*. Then there is a natural isomorphism* $K_*(X) \cong K_*(C_r^*(X))$.

For $B^p = \Gamma\backslash G/K$ a locally symmetric space of rank one with negative sectional curvatures, there is a natural action of the lattice Γ on the sphere at infinity ($\cong S^{p-1}$) of the universal cover G/K. The manifold

$$M = (G/K \times S^{p-1})/\Gamma$$

can be identified with the unit tangent bundle T^1B. The foliation of M of codimension $q = p-1$ defined in Chapter II corresponds here with the Anosov (= weak stable) foliation of T^1B.

Theorem A4.6 [Takai 1986]. *The index yields an isomorphism*

$$K_*(M) \cong K_*(C_r^*(M)).$$

[1] We have been told that this conjecture is still open as of 2004.

For B a surface of genus ≥ 2, this result is due to Connes [1982, Chapter 12].

The third result deals with the characteristic classes of C^∞-foliations. Recall from above that each class $[z] \in H^*(WO_q)$ defines a linear functional $\Delta_*[z]$ on $H_*(X)$. Connes [1986] has shown that $[z]$ also defines a linear functional on $K_*(C_r^*(X))$, and these functionals are natural with respect to the map

$$H_*(X) \to K_*(C_r^*(X)).$$

From this one concludes:

Theorem A4.7 [Connes 1986]. *Suppose there exist*

$$[z] \in H^*(WO_n) \qquad \textit{and} \qquad [u] \in H_*(X)$$

such that $\Delta_*[z]([u])$ does not vanish. Then $[u]$ is mapped to a nontrivial class in $K_*(C_r^*(X))$.

Theorem A4.7 shows that the characteristic classes of $\mathcal{F}$ can be used to prove certain classes in $H_*(X)$ inject into $K_*(C_r^*(X))$.

After these generalities, we consider foliations of three-manifolds with an invariant measure ν given, and study the ν-topological index, $\mathrm{Ind}_\nu^t(u)$, for $u \in K_1(X)$, which calculates the composition

$$K_1(X) \to K_0^\xi(X) \cong K^1(F_1^*X) \xrightarrow{\mathrm{Ind}_\nu^a} \mathbb{R}.$$

First, here is a general statement for such foliations. Recall that a simple closed curve in γ in X transverse to $\mathcal{F}$ determines a complex line bundle E_γ over $\mathcal{F}$ with divisor $[\gamma]$. Take $\bar\partial$ along leaves and form $\bar\partial \otimes E_\gamma$; this gives a map

$$H_1(X;Z) \to K^1(F_1^*X)$$
$$[\gamma] \mapsto [\bar\partial \otimes E_\gamma]$$

and composing with Ind_a yields a map

$$\mathrm{Ind} : H_1(X;Z) \to K_0(C_r^*(X)).$$

Proposition A4.8. *Let $\mathcal{F}$ be a codimension-one C^1-foliation of a compact three-manifold X. Assume both TX and FX are orientable.*

(a) *Suppose ν is an invariant transverse measure with $C_\nu \neq 0$ in $H_2(X;\mathbb{R})$, and the support of ν does not consist of isolated toral leaves. (A toral leaf L is isolated if no closed transverse curve to $\mathcal{F}$ intersects L.) Then there exists a holomorphic line bundle $E \to X$ such that* $\mathrm{Ind}_\nu(\bar\partial \otimes E)$, *and thus* $\mathrm{Ind}(\bar\partial \otimes E) \in K_0(C_r^*(X))$, *are nonzero.*

(b) *Let $\{\nu_1, \dots, \nu_d\}$ be a collection of invariant transverse measures such that the associated currents $\{C_1, \dots, C_d\} \subset H_2(X;\mathbb{R})$ are linearly independent when evaluated on closed transversals to $\mathcal{F}$. Then there exist* *holomorphic line bundles $E_1, \ldots, E_d$ over X such that the elements*

$$\{\mathrm{Ind}(\bar{\partial} \otimes E_i) \mid i = 1, \ldots, d\} \subset K_0(C_r^*(X))$$

are linearly independent.

For example, it is not hard to show that if $\mathcal{F}$ has a dense leaf and the currents $\{C_1, \ldots, C_d\} \subset H_2(X; \mathbb{R})$ of part (b) are independent, then they are independent on closed transversals. Define $H(\Lambda) \subset H_2(X; \mathbb{R})$ to be the subspace spanned by the currents associated to the invariant measures for $\mathcal{F}$.

Corollary A4.9. *If $\mathcal{F}$ has a dense leaf, there is an inclusion*

$$H(\Lambda) \subset K_0(C_r^*(X)) \otimes \mathbb{R}.$$

Proof of Proposition A4.8. First assume there is a closed transverse curve γ to $\mathcal{F}$ which intersects the support of ν. Then $\nu(\gamma) \neq 0$, and we define $E = E_{n \cdot \gamma}$ and use (A3.4) to calculate

$$\mathrm{Ind}_\nu(\bar{\partial} \otimes E_{n \cdot \gamma}) \neq 0$$

for all but at most one value of n. If no such curve γ exists, then the support of ν must consist of compact leaves. One can show these leaves must be tori which are isolated and this contradicts the hypothesis that there is a nonisolated toral leaf in the support of ν. This proves (a). The proof of (b) is similar. □

A5. Examples of Complex Foliations of Three-Manifolds

The geometry of foliations on three-manifolds has been intensively studied. In this section, we select four classes of these foliations for study, and consider the $\bar{\partial}$-index theorem for each. Let M be a compact oriented Riemannian three-manifold. Then M admits a nonvanishing vector field, and this vector field is homotopic to the normal field of some codimension one foliation of M. Moreover, M even has uncountably many codimension one foliations which are distinct up to diffeomorphism and concordance; see [Thurston 1974]. This abundance of foliations on three-manifolds makes their study especially appealing.

There are exactly two simply connected solvable Lie groups of dimension two, the abelian group R^2 and the solvable affine group on the line.

$$A^2 = \left\{ \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \;\middle|\; x > 0 \right\} \subset \mathrm{SL}(2, \mathbb{R}).$$

A locally free action of R^2 or A^2 on a three-manifold M defines a codimension one foliation with very special properties. The foliations defined by an action of R^2 have been completely classified: see the next two pages. For $\pi_1 M$ solvable, the locally free actions of A^2 on M have been classified in [Ghys and Sergiescu 1980] and [Plante 1975]; see pages 245–247. For $\pi_1 M$ not solvable, some restrictions on the possible A^2-actions are known.

Note that Connes' Theorem A4.5 applies only when $\mathcal{F}$ is defined by an almost free action of R^2 or A^2. This assumption does not always hold in the following examples, so we must use the geometry of $\mathcal{F}$ to help calculate the image of the index map.

Throughout, M will denote a closed, oriented Riemannian three-manifold and $\mathcal{F}$ an oriented two-dimensional foliation of M.

Locally free $\mathbb{R}^2$-actions. Let $a \in \mathrm{SL}(2, \mathbb{Z})$, which defines a diffeomorphism $\phi_a : T^2 \to T^2$, and a torus bundle over S^1 by setting

$$M_a = T^2 \times \mathbb{R}/(x, t) \sim (\phi_a(x), t+1).$$

Theorem A5.1 [Rosenberg et al. 1970]. *Suppose M admits a locally free action of R^2. Then M is diffeomorphic to M_a for some $a \in \mathrm{SL}(2, \mathbb{Z})$.*

For $\mathcal{F}$ defined by an R^2-action. $\pi_1 M$ is solvable by Theorem A5.1 and $\mathcal{F}$ has no Reeb components. The foliated three-manifolds with $\pi_1 M$ solvable and no Reeb components have been completely classified by Plante [1979, 4.1]: note that only his cases II, III or V are possible for an R^2-action).

For $\pi_1 M$ solvable, there is also a classification of the invariant measures for any $\mathcal{F}$ on M:

Theorem A5.2 (Plante–Thurston). *If $\pi_1 M$ is solvable and $\mathcal{F}$ is transversally oriented, the space $H(\Lambda) \subset H_2(M)$ of foliation cycles has real dimension* 1.

For $\mathcal{F}$ defined by an R^2-action, this implies there is a unique nontrivial projective class of cycles in $H_2(M)$ which arise from invariant transverse measures. Fix such an invariant measure ν.

For the d-index theorem, evaluation on C_ν yields the average Euler characteristic of the leaves in the support of ν. These leaves are covered by R^2, hence have average Euler characteristics zero, and T_ν annihilates the class $\mathrm{Ind}(d)$.

For the operator $\bar\partial$, we use formula (A3.4) to construct holomorphic bundles over M for which $T_\nu \circ \mathrm{Ind}(\bar\partial \otimes E) \neq 0$. The number of such bundles is controlled by the *period mapping* of ν. This is a homomorphism $P_\nu : H_1(M; \mathbb{Z}) \to \mathbb{R}$ defined as $P_\nu(\alpha) = \nu(\gamma)$, where γ is a simple closed curve representing the homology class α. The rank of its image is called the *rank* of $(\mathcal{F}, \nu)$, denoted by $r(\mathcal{F})$. Note that $1 \leq r(\mathcal{F}) \leq 3$.

Proposition A5.3. *The elements* $\mathrm{Ind}(\bar\partial \otimes E) \in K_0(C_r^*(X))$, *for $E \to M$ a holomorphic line bundle, generate a subgroup with rank at least $r(\mathcal{F})$.*

Proof. For each $\alpha \in \pi_1 M$ with $P_\nu(\alpha) \neq 0$, choose a simple closed curve γ in M representing α and transverse to $\mathcal{F}$. This is possible by Theorem A5.1 and the known structure of R^2-actions. Then take $E = E_\gamma$ as in Section A3 to obtain

$$T_\nu \circ \mathrm{Ind}(\bar\partial \otimes E) = \langle \mathrm{ch}(E), C_\nu \rangle = \nu(\gamma) = P_\nu(\gamma).$$

This shows the map T_ν is onto the image of P_ν. □

It is easy to see that $r(\mathcal{F}) = 3$ if and only if $\mathcal{F}$ is a foliation by planes. This coincides with the R^2-action being free, and then one knows by Theorem A4.5 that

$$\alpha \mapsto \operatorname{Ind}(\bar{\partial} \otimes E_\gamma)$$

is an isomorphism from $H_2(M; Z)$ onto the summand of $K_0(C_r^*(M))$ corresponding to the image of $H_2(M; \mathbb{Z}) \subset K_0(M) \cong K_0(C_r^*(M))$.

An $\mathbb{R}^2$-action on a nilmanifold. Let N_3 be the nilpotent group of strictly triangular matrices in $\mathrm{GL}(3, \mathbb{R})$:

$$N_3 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \text{ such that } a, b, c \in \mathbb{R} \right\}.$$

For each integer $n > 0$, define a lattice subgroup

$$\Gamma_n = \left\{ \begin{pmatrix} 1 & p & r/n \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} \text{ such that } p, q, r \in \mathbb{Z} \right\}.$$

Then $M = N_3/\Gamma_n$ is a compact oriented three-manifold, and the subgroup

$$R^2 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

acts almost freely on M via left translations. Also note M is a circle bundle over T^2, and $H_2(M; \mathbb{R}) \cong \mathbb{R}^2$. By Theorem A4.5, the index map is an isomorphism, so $K_0(C_r^*(M)) \cong \mathbb{Z}^3$. The curve representing the homology class of

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \in \pi_1 M$$

is transverse to $\mathcal{F}$ and $P_\nu(\alpha) \neq 0$ for a transverse measure ν with $C_\nu \neq 0$. However, $\operatorname{Ind}(\bar{\partial} \otimes E_\gamma)$ cannot detect the contribution to $K_0(C_r^*(M))$ from the curve defined by a fibre of $M \to T^2$.

Foliations without holonomy. If for every leaf L of a foliation, $\mathcal{F}$, the holonomy along each closed loop in L is trivial, then we say $\mathcal{F}$ is without holonomy. In codimension-one, such foliations can be effectively classified up to topological equivalence. We discuss this for the case of C^2-foliations. By Sacksteder's Theorem [Lawson 1977], a codimension-one, C^2-foliation without holonomy of a compact manifold admits a transverse invariant measure ν whose support is all of M. Moreover, there is foliation-preserving homeomorphism between M and a model foliated space,

$$X = (\widetilde{B} \times S^1)/\Gamma,$$

where Γ is the fundamental group of a compact manifold B, $\widetilde{B}$ is its universal cover with Γ acting via deck translations, and Γ acts on S^1 via a representation

$$\exp(2\pi i \rho) : \Gamma \to \mathrm{SO}(2),$$

for $\rho : \Gamma \to \mathbb{R}$. The foliation of X by sheets $\widetilde{B} \times \{\theta\}$ has a canonical invariant measure, $d\theta$, and ν corresponds to $d\theta$ under the homeomorphism. Since the index invariants are topological, in this case we can assume that M is one of these models. For a three-manifold this implies $B = \Sigma_g$ for Σ_g a surface of genus $g \geq 1$. The case $g = 1$ is a special case of examples (A5.1) above.

Let Λ denote the abelian subgroup of $\mathbb{R}$ which is the image of ρ. Denote by $r(\mathcal{F})$ the rank of Λ. It is an easy geometrical exercise to see that the group Λ agrees with the image of the evaluation map

$$[d\theta] : H_1(M; Z) \to \mathbb{R}.$$

Moreover, there exists simple closed curves $\{\gamma_1, \dots, \gamma_r\}$ in M transverse to $\mathcal{F}$ for which $\{P_\nu(\gamma_i)\}$ yields a $\mathbb{Z}$-basis for Γ. Form the holomorphic bundles $\{E_i\}$ corresponding to the $\{\gamma_i\}$, then the set $\{\mathrm{Ind}(\bar{\partial} \otimes E_i)\}$ generates a free subgroup of rank r in $K_0(C_r^*(M))$. Since $H_2(BG(M); \mathbb{R})$ has rank r, this implies

Proposition A5.4. *The index map*

$$K_0^\tau(BG(M)) \to K_0(C_r^*(M))$$

is a monomorphism.

These foliations have been analyzed in further detail by Natsume [1985], where he shows that this map is also a surjection.

Solvable group actions. The locally free actions of A^2 on three-manifolds has been studied by many authors; see in particular [Ghys and Sergiescu 1980; Ghys 1984; 1985; 1993]:

Theorem A5.5. *Let $\pi_1 M$ be solvable and suppose A^2 acts on M. Then M is diffeomorphic to a torus bundle M_a over S^1, and the monodromy map $a \in$* $\mathrm{SL}(2, \mathbb{R})$ *has two distinct real eigenvalues.*

Theorem A5.6 [Ghys 1985]. *Suppose that A^2 acts locally free on M and preserves a smooth volume form. Then M is diffeomorphic to* $\mathrm{SL}(2, \mathbb{R})/\Gamma$ *for some cocompact lattice in the universal covering group*

$$\widetilde{\mathrm{SL}(2, \mathbb{R})},$$

and the action of A^2 on M is via left translations.

Proposition A5.7. *Suppose $H_1(M) = 0$ and A^2 acts locally freely on M. Then the action preserves a smooth volume form on M.*

Let us describe the foliation on $M_0 = T^2 \times \mathbb{R}/\phi_a$. Let $\bar{v} \in \mathbb{R}^2$ be an eigenvalue with eigenvalue $\lambda > 0$. The foliation of $\mathbb{R}^3$ by planes parallel to the span of $\{(\bar{v} \times 0), (\bar{0} \times 1)\}$ is invariant under the covering transformations of $\mathbb{R}^3 \to M_a$, so descends to a foliation $\mathcal{F}_\lambda$ on M_a. When $\lambda = 1$, the R^2-action on $\mathbb{R}^3$ defining the foliation there descends to an $\mathbb{R}^2$ action on M_a, defining $\mathcal{F}_\lambda$. When $\lambda \neq 1$, the leaves of $\mathcal{F}_\lambda$ are defined by an action of A^2 on M_a.

For A^2-actions on M with $\pi_1 M$ not solvable, it seems reasonable to conjecture they must have this form given in Theorem A5.6.

If the action of A^2 preserves a volume form on M^3, then $\mathcal{F}$ is transversally affine [Ghys and Sergiescu 1980], so there can be no invariant measures for $\mathcal{F}$. In this case Theorem 8.7 of Chapter VIII reveals no information about $K_0(C_r^*(M))$. However, one has Connes' Theorem A4.5 since the A^2-action is almost free. To give an illustration, let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a cocompact lattice, and set $M = \mathrm{SL}(2, \mathbb{R})/\Gamma$. The group A^2 acts via left translations and preserves a smooth volume form on M. Then

$$G(M) \cong M \times A^2,$$
$$K_0^\tau(BG(M)) \cong K^0(M),$$

and

$$\mathrm{Ind} : K_0^\tau(BG(M)) \to K_0(C_r^*(M))$$

is an isomorphism. Note the foliation on M admits $2g$ closed transversals $\{\gamma_1, \ldots, \gamma_{2g}\}$ which span $H_1(M)$. Form the corresponding bundles $E_i \to M$, and consider the classes $\{\mathrm{Ind}(\bar{\partial} \otimes E_i)\} \subset K_0(C_r^*(M))$. It is natural to ask whether these classes are linearly independent, and for a geometric proof if so.

Foliations with all leaves proper. A leaf $L \subset M$ is *proper* if it is locally closed in M. $\mathcal{F}$ is proper if every leaf is proper. The geometric theory of codimension-one proper foliations is highly developed [Cantwell and Conlon 1981; Hector and Hirsch 1986]. We recall a few general facts relevant to our discussion.

Theorem A5.8. *Let $\mathcal{F}$ be a proper foliation of arbitrary codimension. Then the quotient measure space $M/\mathcal{F}$, endowed with the Lebesgue measure from M, is a standard Borel space.*

Corollary A5.9. *Let $\mathcal{F}$ be a proper foliation of arbitrary codimension. Then any ergodic invariant transverse measure for $\mathcal{F}$ with finite total mass is supported on a compact leaf.*

Theorem A5.10. *For a codimension one proper foliation $\mathcal{F}$, all leaves of $\mathcal{F}$ have polynomial growth, and the closure of each leaf of $\mathcal{F}$ contains a compact leaf.*

Let $\mathcal{F}$ be a proper codimension-one foliation of M^3. Given a transverse invariant measure ν, we can assume without loss of generality that the support of ν is a compact leaf L. If L has genus at least 2, then there exists a closed transversal γ which intersects L, so $T_\nu \circ \operatorname{Ind}(\bar\partial \otimes E_{n\cdot\gamma}) \neq 0$ for all but at most one value of n. Thus, the class

$$[L] \in H_2(M; Z)$$

corresponds to a nontrivial class

$$\operatorname{Ind}(\bar\partial \otimes E_{n\cdot\gamma}) \in K_0(C_r^*(M)).$$

If L is a 2-torus, then it is difficult to tell whether the homology class of L is nonzero, and if so, whether it generates a nonzero class in $K_0(C_r^*(M))$. There is a geometric criterion which yields an answer.

Theorem A5.11 (Rummler–Sullivan). *Suppose M admits a metric for which each leaf of $\mathcal{F}$ is a minimal surface. Then every compact leaf of $\mathcal{F}$ has a closed transversal which intersects it.*

Corollary A5.12. *Suppose $\mathcal{F}$ is a proper and minimal foliation. For each ergodic invariant transverse measure ν, there is a holomorphic bundle $E_\nu \to M$ such that* $\operatorname{Ind}(\bar\partial \otimes E_\nu) \in K_0(C_r^*(M))$ *is nonzero, and* $\operatorname{Ind}_\nu(\bar\partial \otimes E_\nu) \neq 0$.

We cannot conclude from Corollary A5.12 that the elements

$$\{\operatorname{Ind}(\bar\partial \otimes E_\nu) \mid \nu \text{ ergodic}\}$$

are independent. (Consider the product foliation $\Sigma_g \times S^1$.) However, if M has a metric for which every leaf is geodesic submanifold, then there are as many independent classes in $K_0(C_r^*(M))$ as there are independent currents $C_\nu \in H_2(M; \mathbb{R})$.

The Reeb foliation of S^3 is another relevant example of a proper foliation. It is not minimal, and $K_0(C_r^*(M)) \cong \mathbb{Z}$ so the toral leaf does not contribute; see [Penington 1983; Torpe 1985].

Foliations with nonzero Godbillon–Vey class. There is exactly one characteristic class for codimension-one foliations (of differentiability at least C^2), the Godbillon–Vey class $GV \in H^3(M; \mathbb{R})$. Recall from Section A4 that GV defines linear functionals, also denoted by GV, on $K_*(M)$ and (noncanonically) on $K_*(C_r^*(M))$,[2] and these functionals agree under the map

$$K_*(M) \to K_*(C_r^*(M)).$$

If $GV \neq 0$ in $H^3(M)$, then there is a class $[u] \in K_*(C_r^*(M))$ on which GV is nontrivial. From this we conclude that the composition

$$H_3(M; Z) \to K_1^\xi(BG(M)) \to K_0(C_r^*(M))$$

is injective.

[2] The map $GV : K_*(C_r^*(M)) \to \mathbb{R}$ depends upon the choice of a smooth dense subalgebra of $C_r^*(M)$.

The information on $K_1(C_r^*(M))$ obtained from GV is about all one knows for these foliations $\mathcal{F}_\alpha$ on M, which underlines the need for better understanding of how the geometry of a foliation is related to the analytic invariants in $K_0(C_r^*(M))$.

Update 2004

Since the first edition of this book, there have been great advances in understanding the relationship between the foliation indices and the geometry of $\mathcal{F}$. We mention in particular the works of Hitoshi Moriyoshi, who has given a very explicit description of the Godbillon–Vey invariant as an analytic invariant [Moriyoshi 1994a; 1994b; 2002, Moriyoshi and Natsume 1996].

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