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★**Global analysis on foliated spaces.**

With appendices by S. Hurder, Moore, Schochet and Robert J. Zimmer.
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Let (M, F) be a foliated manifold. As explained by A. Connes [in *Algèbres d'opérateurs* (Les Plans-sur-Bex, 1978), 19–143, Lecture Notes in Math., 725, Springer, Berlin, 1979; MR 81g:46090] and Connes and G. Skandalis [Publ. Res. Inst. Math. Sci. **20** (1984), no. 6, 1139–1183; MR 87h:58209], the usual index theorem for elliptic pseudodifferential operators (PDOs) may be extended to PDOs which are elliptic along the leaves of (M, F). In this book, the authors give a nice and self-contained exposition of the measured index theorem for foliations, and some related topics.

The first chapter introduces the notion of "locally traceable operator". Let $A \cong L^\infty(X, \mu)$ be an abelian von Neumann algebra on a Hilbert space H, and let T be a positive operator on H. The measure μ_T defined on the measurable subsets E of X by $\mu_T(E) = \mathrm{Tr}(\mathbf{1}_E T \mathbf{1}_E)$ may sometimes be finite on each member of some exhaustion of X by increasing Borel sets. In this case, T is said to be locally traceable, with local trace μ_T. The authors give a systematic treatment of this notion of (a not necessarily positive) locally traceable operator, which allows one to introduce the local index ι_D of an elliptic operator D on a noncompact manifold. In particular, the restriction to each leaf of (M, F) of a tangentially elliptic differential operator D has a local index. The authors discuss also several situations outside the realm of foliations in which the use of locally traceable operators may be useful.

The second chapter gives some basic material about foliations, including the concept of holonomy. It contains original smoothing results allowing the authors to assume systematically that all the needed bundles over the leaf space are smooth in the leaf direction. It also introduces the holonomy groupoid of a foliation, which is necessary to define the leaf-space of (M, F) as the reduced C^*-algebra $C^*(M, F)$ introduced by Connes. To study $C^*(M, F)$, it is sometimes useful to replace it by the Morita equivalent C^*-algebra $C^*(G_N^N)$ associated to the restricted (discrete) groupoid G_N^N. The structure of G_N^N is described in several examples, including the case of foliated bundles.

The third chapter discusses the de Rham tangential cohomology groups $H_\tau^*(M)$ of (M, F). The difference from other works on tangential cohomology (such as the works of Kamber and Tondeur or of Zimmer) appears in the assumption that the forms are assumed to be continuous in the transverse direction. To compute the tangential cohomology groups of (M, F), the authors give a lot of tools, including the Mayer-Vietoris sequence, the Künneth isomorphism, and the Thom isomorphism theorem. Finally, they indicate the definition of a tangential homology theory.

Chapter IV deals with transverse measures. It is shown, in the context of measurable and topological groupoids, that tangential measures on a foliation (M, F) may be integrated against transverse measures to give measures on the ambient manifold. This procedure, applied to continuous Radon tangential measures, gives the construction of the Ruelle-Sullivan current. If D is a tangentially elliptic differential operator on (M, F), the construction associates a measure on M to any transverse measure, by integration of the local index on each leaf. Also, the invariant transverse measures are related to invariant measures on a complete transversal. Finally, a Riesz representation theorem is given, which identifies the finite invariant transverse measures on (M, F) with the group $H_{\mathrm{cont}}(H_\tau^{\dim(F)}(M), \mathbf{R})$.

Chapter V is devoted to the systematic Chern-Weil development of tangential characteristic classes. This leads to the definition of the tangential Chern classes, Pontryagin classes, Euler class and Todd genus. These classes are constructed at the level of forms, so that the topological index is a uniquely defined form, when a tangential Riemannian connection is fixed.

Chapter VI introduces the (reduced) C^*-algebra $C^*(M, F)$ associated with the foliation (M, F), which plays the role of the leaf-space M/F. The main basic properties of $C^*(M, F)$ are presented, including the Hilsum-Skandalis stability theorem, which show how to compute $C^*(M, F)$ from a complete transversal. It is proved that any invariant transverse measure ν gives rise to a trace Φ_ν on $C^*(M, F)$, and the authors show how to compute the associated von Neumann algebra by using a complete transversal. Finally, a partial Chern character from $K_0(C^*(V, F))$ to $\overline{H}_\tau^p(M)$ is constructed.

The first part of Chapter VII introduces all the necessary material concerning PDOs which are elliptic along the leaves of a foliation. The reader who is not familiar with this theory will find in this chapter all that is needed for a discussion of the measured index theorem for foliations: smoothing tangential operators, tangential PDOs, the principal symbol of such PDOs, construction of the PDO algebra, etc. The second part of this chapter establishes the McKean-Singer formula for an elliptic tangential PDO D: $\mathrm{Ind}_\nu(D) = \Phi_\nu^s[\exp(-t\hat{D})]$ $(t > 0)$, where $\hat{D}$ is the usual associated selfadjoint superoperator and Φ_ν^s is the supertrace associated with ν. Then, using an asymptotic expansion of $\Phi_\nu^s[\exp(-t\hat{D})]$ as $t \to 0$, it is proved that $\mathrm{Ind}_\nu(D) = \int \omega_D(g, E)\, d\nu$, where $\omega_D(g, E)$ is a tangentially smooth p-form $(p = \dim F)$ which depends on the bundle E of D and on the tangential Riemannian metric g.

The last chapter shows that the analytical index ι_D and the topological index ι_D^{top}, viewed as tangential measures, give rise to the same element in $H_\tau^p(M)$ (measured index theorem for foliations). The coincidence of these two elements is proved for tangential twisted signature operators, and then extended by topological arguments to any tangentially elliptic PDO.

The book contains also three interesting appendices. The first one, by Hurder, gives applications of the index theorem to foliations whose leaves have a complex structure. The second appendix, by the authors and Zimmer, gives some applications to the existence of square-integrable harmonic forms on certain noncompact manifolds. The third one, by Zimmer, gives applications to the existence of tangential metrics with positive scalar curvature along the leaves.

In conclusion, this book gives a very nice and complete introduction to the measured index theorem for foliations and some of its applications.

Reviewed by *Thierry Fack*