Contemporary Mathematics
Volume 70, 1988

# TOEPLITZ OPERATORS AND THE ETA INVARIANT: THE CASE OF $S^1$

RONALD G. DOUGLAS [1]

STEVEN HURDER [1]

JEROME KAMINKER [1]

## §1 Introduction

Let D be an elliptic operator on a closed manifold M. Since the index of D depends only on the principal symbol, it is natural to consider this as a primary invariant of the operator. In this paper we describe an example of a theory which can provide a way to define higher order invariants of elliptic operators. These invariants will depend on more than just the principal symbol of the operator. To obtain them we will embed D in a family of perturbations and study the cumulative effect of the variations. All of this is done in the

[1]Research supported in part by grants from the National Science Foundation

context of cyclic cohomology.

What is presented here is a special case of a general theory which will appear in longer papers. One aspect of it describes a relation between the $\eta$-invariant of a self-adjoint elliptic operator on a closed manifold and the index theory for Toeplitz operators along the leaves of an associated foliated manifold. The latter is described by a longitudinal cyclic cocycle and the former by what may be viewed as a renormalized transverse cyclic cocycle. [1],[4],[8] The special case we work out in detail is where the manifold is the circle $S^1$ and the operator is the Dirac operator $-i\frac{d}{d\theta}$. Despite the simplicity of the example, the steps we follow are similar to those one would use in the general framework.

The actual procedure is as follows. To the manifold $S^1$, the operator $D = -i\frac{d}{d\theta}$, and a representation $\alpha:\pi_1(S^1) \longrightarrow S^1$ we associate a foliated manifold, $T^2$, and an operator $\tilde{D}$ on it. The original operator D was elliptic, but $\tilde{D}$ will not be. Now $T^2$ is provided with two transverse foliations and $\tilde{D}$ is transversally elliptic with respect to one and longitudinally elliptic with respect to the other. Following Connes, we define the transverse cocycle for $\tilde{D}$

on the convolution algebra for one of the foliations, $C^{\infty}(T^2 x S^1)$, (this is described below). It is then "renormalized" to obtain a cocycle on the commutative algebra $C^{\infty}(T^2)$, which will yield the $\eta$-invariant. Next, the longitudinal cocycle is defined on $C^{\infty}(T^2)$. The pairing of this cocycle with K-theory gives the Toeplitz index theory along the leaves of the foliation. Finally, we reach the most important aspect of the theory--the proof that the longitudinal cocycle and the renormalized transverse cocycle are equal. After this is accomplished, one pairs with appropriate unitaries to obtain a new proof of the Atiyah-Patodi-Singer Index Theorem for flat bundles in this case. In the general case, [11], [12], we will study an arbitrary self-adjoint, $1^{st}$-order, elliptic operator on a closed manifold and $\alpha$ will be a unitary representation of $\pi_1(M)$. We close with some remarks on the general framework of higher order invariants.

§2 <u>The basic data</u>

Our basic operator will be $D = -i(\frac{d}{d\theta})$ on the manifold $S^1$. The associated foliated manifold is $T^2$. It is represented as $(\mathbb{R}/2\pi\mathbb{Z})x(\mathbb{R}/2\pi\mathbb{Z})$ with coordinates

$(\theta,\varphi)$. Choose an irrational number $\alpha$, $0<\alpha<1$, and represent $\pi_1(S^1) = \mathbf{Z}$ on $S^1$ by sending 1 to $\overline{2\pi\alpha} \in S^1 = \mathbb{R}/2\pi\mathbf{Z}$. Form the quotient $\mathbb{R}\times_\alpha S^1$. It is isomorphic to $T^2$ via the trivialization $\tau$ given by $\tau(<\theta,\overline{\varphi}>) = (\overline{\theta},\overline{\varphi+\alpha\theta})$. We work on $T^2$ with the foliation, denoted $\mathcal{F}_\alpha$, whose leaves are the images of $\mathbb{R}\times\{\varphi\}$. The graph of the foliation can be identified with $T^2\times\mathbb{R}$. The fibers of the principal bundle $pr_2 : T^2 \longrightarrow S^1$ provide $T^2$ with a second foliation $\mathcal{F}_T$ whose graph is $T^2\times S^1$.

Consider the $1^{st}$-order, self-adjoint, differential operator $\tilde{D} = -i(\frac{\partial}{\partial\theta} + \alpha\frac{\partial}{\partial\varphi})$ on $C^\infty(T^2)$. It is obtained by lifting D to $\mathbb{R}$, extending to $\mathbb{R}\times S^1$ in the natural way, then descending to the quotient. One then uses the trivialization $\tau$ to transfer the operator to $T^2$. It is longitudinally elliptic with respect to $\mathcal{F}_\alpha$, in the sense that $\sigma(D)|(T^*\mathcal{F}_\alpha)$ is an isomorphism, where $T^*\mathcal{F}_\alpha$ denotes the cotangent bundle to $\mathcal{F}_\alpha$. On the other hand, it is transversally elliptic to $\mathcal{F}_T$ in the sense that its symbol is invariant under the action of $S^1$ and is invertible when restricted to the normal bundle to $\mathcal{F}_T$. Thus, we may apply both longitudinal and transverse index theory to this operator. We will show how it is related to the Atiyah-Patodi-Singer Index Theorem for flat bundles, [2], for this particular situation. In fact, the transverse cocycle is related to spectral data for the operator, while the longitudinal cocycle yields a topological formula involving secondary classes.

## §3 The transverse cocycle

Let $C^\infty(\mathcal{F}_T)$ denote the smooth convolution algebra for the holonomy groupoid of $\mathcal{F}_T$. We follow Connes [8] in constructing the transverse cocycle $C_D^{\pitchfork} \in Z_\lambda^1(C^\infty(\mathcal{F}_T)) = Z_\lambda^1(C^\infty(T^2 x S^1))$ associated with $\tilde{D} = -i(\frac{\partial}{\partial\theta} + \alpha\frac{\partial}{\partial\varphi})$. There is a representation of $C^\infty(T^2 x S^1)$ on $L^2(T^2)$ given by

$$k*\xi(\theta,\varphi) = \int k(\theta,\mu,\varphi)\xi(\theta,\mu)d\mu$$

where $k \in C^\infty(T^2 x S^1)$ and $\xi \in L^2(T^2)$, which extends to $C^*(T^2,\mathcal{F}_T)$. Let $\tilde{P}:L^2(T^2) \longrightarrow L^2(T^2)$ be the positive projection for the operator $\tilde{D}$, and let $F = 2\tilde{P} - 1$ denote the associated symmetry. The cyclic cocycle $C_D^{\pitchfork}$ is defined by

$$C_D^{\pitchfork}(\varphi_1,\varphi_2) = \frac{1}{4}\,\mathrm{Tr}(F[F,\varphi_1][F,\varphi_2]).$$

We shall compute it explicitly on certain elements. Let $k_{a,b,c} = e^{i(a\theta+b\varphi+c\delta)} \in C^\infty(T^2 x S^1)$ and $e_{m,n} = e^{i(m\theta+n\varphi)} \in L^2(T^2)$. Then we have

$$k_{a,b,c}*e_{m,n} = e_{a+m,c} \quad \text{if } b = -n, \text{ and } 0 \text{ otherwise},$$

and

$$F(e_{m,n}) = \sigma(m+\alpha n)e_{m,n}$$

where $\sigma(x) = \begin{cases} 1 & x\geq 0 \\ -1 & x<0 \end{cases}$.

Let $x = [x] + \{x\}$, where $0 \leq \{x\} < 1$, and $[x]$ is the greatest integer $\leq x$.

A direct computation yields the following result.

Proposition 3.1:

$$C_D^{\pitchfork}(k_{a,b,c}, k_{a',b',c'}) = \begin{cases} [-\alpha c]-[a+\alpha b] & \text{if } a'=-a,\ b'=-c,\ c'=-b \\ 0 & \text{otherwise} \end{cases}$$

Proof: We evaluate $F[F,k_{a,b,c}][F,k_{a',b',c'}]$ on $e_{m,n}$ and sum the diagonal terms to obtain the trace. Note that $[F,k_{a,b,c}]e_{m,n} = (\sigma(a+m+\alpha c)-\sigma(m+\alpha n))e_{a+m,c}$ if $b = -n$ and is 0 otherwise. Thus,

$$C_D^{\pitchfork}(k_{a,b,c}, k_{a'b'c'}) = \frac{1}{4}\mathrm{Tr}(F[F,k_{a,b,c}][F,k_{a',b',c'}])$$

$$= -\frac{1}{4}\sum_{m} \sigma(m+\alpha c)(\sigma(m+\alpha c)-\sigma(m-(a+\alpha b)))^2$$

since the only way there can be non-zero diagonal terms is if $a = a'$, $b' = -n = -c$, and $b = -c'$.

Note that $\sigma(m+\alpha c)-\sigma(m-(a+\alpha b)) = \begin{cases} 0 & m>-\alpha c,\ m>a+\alpha b \text{ or } m<-\alpha c,\ m<a+\alpha b \\ 2 & -\alpha c<m<a+\alpha b \\ -2 & a+\alpha b<m<-\alpha c \end{cases}$

which yields the desired result.∎

If $\varphi \in C^{\infty}(T^2 x S^1)$, then one can write $\varphi = \sum \lambda_{a,b,c} k_{a,b,c}$ where $\{\lambda_{a,b,c}\}$ is a rapidly decreasing sequence. It is directly checked that

$$C_D^{\phi}(\varphi_1,\varphi_2) = \sum \lambda^1_{a,b,c} \lambda^2_{-a,-c,-b}([-\alpha c]-[a+\alpha b])$$

is finite and satisfies the cocycle condition $bC_D^{\phi}(\varphi_1,\varphi_2,\varphi_3) = 0$ for $\varphi_i \in C^{\infty}(T^2 x S^1)$. Thus, we obtain

<u>Proposition 3.2</u>: $C_D^{\phi}(\varphi_1,\varphi_2)$ is a cyclic 1-cocycle on the smooth convolution algebra $C^{\infty}(T^2 x S^1)$.

This cocycle has an index theoretic interpretation in terms of spectral flow which will be discussed in §6.

## §4 The $\eta$-invariant and spectral flow

We review some of the basics of the $\eta$-invariant and relate it to the cocycle $C_D^{\phi}$ of §3. If D is a self-adjoint elliptic differential operator on a compact manifold then one defines $\eta(D,s) = \sum \text{sign}(\lambda_i)|\lambda_i|^{-s}$, the sum being over the non-zero eigenvalues. It is holomorphic for Re(s) large and has a meromorphic extension to the entire plane with 0 as a regular value. The number $\eta(D,0)$ has geometric, analytic and physical significance in several contexts, ([2],[11],[12],[15]), and is called the $\eta$-invariant of D. It is customary to modify it by setting $\xi(D) = \frac{1}{2}(\eta(D,0) + \dim(\ker D))$. If $D_t$, $0 \leq t \leq 1$, is a smooth path of operators, then $\xi(D_t)$ may not be continuous. The discontinuities occur when an eigenvalue crosses 0, since then $\xi$ jumps by $\pm 1$. Thus, the variation of $\xi$ breaks up into a continuous and discontinuous part. The continuous part can be obtained by the following procedure. If one projects $\xi$ to $\mathbb{R}/\mathbb{Z}$ the resulting function is smooth and its derivative can be lifted smoothly to $\mathbb{R}$. Set $\mathrm{Eta}(D_t) = \int_0^1 \frac{d}{dt}(\xi(D_t))dt + \xi(D_0)$. In general the result will depend on the path between $D_0$ and $D_1$, but if the path runs through elliptic differential operators of the same order and the same principal symbol then $\mathrm{Eta}(D_t)$ depends only on the endpoints and it will be denoted by $\mathrm{Eta}(D_1,D_0)$, or some variant. The discontinuous part of the variation of $\xi$ is called spectral flow and is denoted by $\mathrm{Sf}(D_t)$ or $\mathrm{Sf}(D_1,D_0)$, as above. It is the number of eigenvalues which pass through 0 during the deformation.

Consider now our particular case. The operator $\tilde{D} = -i(\frac{\partial}{\partial\theta} + \alpha\frac{\partial}{\partial\varphi})$ is invariant under the action of $S^1$ on $T^2$, so it preserves the summands in the decomposition $L^2(T^2) \cong \bigoplus_\rho (L^2(S^1)\otimes V_\rho)$, where $\rho$ runs through the irreducible representations of $S^1$. Let $E_{\rho\alpha}$ denote the line bundle $\mathbb{R}\times_{\rho\alpha}\mathbb{C} \longrightarrow S^1$ provided with its natural flat connection, $\nabla_{\rho\alpha}$. Let $\tilde{D}_\rho$ denote $\tilde{D}|L^2(S^1)\otimes V_\rho$. There is a commutative diagram

$$\begin{array}{ccc}
L^2(S^1)\otimes V_\rho & \xrightarrow{\quad \tilde{D}_\rho \quad} & L^2(S^1)\otimes V_\rho \\
\tau \big\downarrow & & \big\downarrow \tau \\
L^2(E_{\rho\alpha}) & \xrightarrow[\quad D\otimes_{\nabla_{\rho\alpha}} I \quad]{} & L^2(E_{\rho\alpha})
\end{array}$$

where $\tau$ is induced by the trivialization of $\mathbb{R}\times_{\rho\alpha}\mathbb{C}$. We apply the previous formalism to the operators $\tau^{-1}\tilde{D}_\rho\tau$ and $D\otimes I$ on $L^2(S^1)\otimes V_\rho$. We write $\mathrm{Eta}(D,\rho,\alpha,\tau)$ for $\mathrm{Eta}(\tau^{-1}\tilde{D}_\rho\tau, D\otimes I)$ and similarly for spectral flow. For $R_\delta$ a finite set of $\rho$ set $\delta = \sum_{\rho\in R_\delta} k_{0,-\rho,\rho}$, and let $\mathrm{Eta}(D,\delta,\alpha,\tau) = \sum_{\rho\in R_\delta} \mathrm{Eta}(D,\rho,\alpha,\tau)$.

## §5 The renormalized transverse cocycle

The cocycle $C_D^{\pitchfork}$ is defined on the (non-commutative) smooth convolution algebra $C^\infty(T^2 \times S^1)$ which plays the role of the the smooth functions on a transversal to $\mathcal{F}_T$. This algebra, along with the commutative algebra $C^\infty(T^2)$, is represented on $L^2(T^2)$ in such a way that $C^\infty(T^2)$ acts as multipliers on $C^\infty(T^2 x S^1)$. We will "extend" the transverse cocycle to $C^\infty(T^2)$ by a renormalization process. If an approximate unit, $\delta_n$, is chosen for $C^\infty(T^2 x S^1)$, then $C_D^\pitchfork$ is defined on elements of the form $\varphi\delta_n$. Dividing by an appropriate quantity and letting n go to infinity yields the renormalized cocycle. It is sometimes useful to think of this process as analogous to the "transfer" in topology. In that situation, a class on the base of a fiber bundle is extended to a class on the total space. Here, one views the transversal as the base of a fiber bundle and the leaves of the foliation as the fibers.

We first choose, as an approximate identity for $C^\infty(T^2 x S^1)$, the elements

$$\delta_N = \sum_{-N}^{N} k_{0,n,-n}\ , \text{ where } k_{0,n,-n} = e^{i(n\varphi - n\delta)}.$$

<u>Definition:</u> The renormalized transverse cocycle, defined on $C^\infty(T^2)$, is

$$TC_D^\pitchfork(\varphi_1,\varphi_2) = \lim_{N\to\infty} \frac{C_D^\pitchfork(\varphi_1\delta_N,\varphi_2\delta_N)}{Tr_{L^2(S^1)}(\delta_N)}.$$

Theorem 5.1: The formula $TC_D^{\pitchfork}$ defines a cyclic cocycle on $C^{\infty}(T^2)$.

Proof: We must show that the limit exists and that the cyclic cocycle conditions are satisfied. For the first one computes

$$\frac{1}{Tr_{L^2(S^1)}(\delta_N)} C_D^{\pitchfork}(e_{m,n}\delta_N, e_{l,k}\delta_N) = \frac{1}{2N+1}\sum_{-N\le i\le N}([\alpha(l+i)]-[\alpha(i-k)])$$

if $l+k = -(n+m)$ and equals 0 otherwise.

Lemma 5.2: $\lim_{N\to\infty} C_D^{\pitchfork}(e_{m,n}\delta_N, e_{-m,-n}\delta_N) = -(\alpha n+m)$.

Proof: First note that $\frac{-1}{2N+1}\sum_{-N\le i\le N}([\alpha(l+i)]-[\alpha(i-k)]) =$

$\frac{-1}{2N+1}\sum_{-N\le i\le N}(\alpha(l+k)-\{\alpha(i+l)\}+\{\alpha(i-k)\})$ . Now, both $\{\alpha(i+l)\}$ and $\{\alpha(i-k)\}$ are equidistributed sequences in $(0,1)$ so $\frac{1}{2N+1}\sum_{-N\le i\le N}\{\alpha(i+l)\}$ and $\frac{1}{2N+1}\sum_{-N\le i\le N}\{\alpha(i-k)\}$ approach the same limit, namely 1/2, as $N \longrightarrow \infty$. This yields the result.∎

Now, if $\varphi_1 = \sum \lambda_{m,n}e_{m,n}$, and $\varphi_2 = \sum \mu_{p,q}e_{p,q}$ are in $C^{\infty}(T^2)$, then $\lambda_{m,n}$ and $\mu_{p,q}$ are rapidly decreasing so that $-\sum \lambda_{m,n}\mu_{-m,-n}(\alpha(n+m)) < \infty$, hence $TC_D^{\pitchfork}$ exists.

Clearly, $TC_D^{\pitchfork}(\varphi_1,\varphi_2) = -TC_D^{\pitchfork}(\varphi_2,\varphi_1)$, so it remains to show that the cocycle condition holds. Since $C_D^{\pitchfork}$ is a cocycle one has

$$C_D^{\pitchfork}(\varphi_1\delta_N*\varphi_2\delta_N,\varphi_3\delta_N) - C_D^{\pitchfork}(\varphi_1\delta_N,\varphi_2\delta_N*\varphi_3\delta_N) + C_D^{\pitchfork}(\varphi_3\delta_N*\varphi_1\delta_N,\varphi_2\delta_N) = 0$$

so it will suffice to show that

$$\lim_{N\to\infty} C_D^{\pitchfork}(\varphi_1\delta_N*\varphi_2\delta_N,\varphi_3\delta_N) = \lim_{N\to\infty} C_D^{\pitchfork}(\varphi_1\varphi_2\delta_N,\varphi_3\delta_N).$$

To this end consider

$$\mathrm{Tr}(F[F,\varphi_1\delta_N*\varphi_2\delta_N][F,\varphi_3\delta_N]) - \mathrm{Tr}(F[F,\varphi_1\varphi_2\delta_N][F,\varphi_3\delta_N])$$

and observe that it will be sufficient to look at

$$[F,\varphi_1\delta_N*\varphi_2\delta_N] - [F,\varphi_1\varphi_2\delta_N] = [F,\varphi_1\delta_N*\varphi_2\delta_N-\varphi_1\varphi_2\delta_N].$$

But $\delta_N$ satisfies

$$\|\delta_N*(\varphi_2\delta_N)-\varphi_2\delta_N\| \longrightarrow 0$$

which yields the result.∎

§6 <u>The $\eta$-invariant and the renormalized transverse cocycle</u>

In this section we will establish a direct relation between $TC_D^{\pitchfork}$ and the $\eta$-invariant. Our starting point is the following formula. Let $\rho = e_{0,n} \in C^{\infty}(T^2)$, where $\rho$ is identified with a unitary representation $\rho:S^1 \longrightarrow S^1$, with $\deg(\rho) = n$.

<u>Theorem 6.1</u>: $C_D^{\pitchfork}(\rho\delta_N,\rho^{-1}\delta_N) = \mathrm{Sf}(D,\delta_N,\alpha,\tau) - \mathrm{Sf}(D,\rho\delta_N,\alpha,\tau).$

Proof: We have

$$C_D^{\pitchfork}(\rho\delta_N,\rho^{-1}\delta_N) =$$

$$C_D^{\pitchfork}(\sum\rho k_{0,i,-i},\sum\rho^{-1}k_{0,j,-j}) =$$

$$\sum_{i,j} C_D^{\pitchfork}(k_{0,i,-i+\rho},k_{0,j,-j-\rho}).$$

Next, note that a single term in the sum satisfies

$$C_D^{\pitchfork}(k_{0,i,-i+\rho},k_{0,j,-j-\rho}) = \begin{cases} C_D^{\pitchfork}(\rho k_{0,i+\rho,-i-\rho},\rho^{-1}k_{0,i,-i}) & \text{if } i=j+\rho \\ 0 & \text{otherwise} \end{cases}.$$

Thus, adding up we obtain,

$$C_D^{\pitchfork}(\rho\delta_N,\rho^{-1}\delta_N) = \sum_{|i|\leq N} C_D^{\pitchfork}(\rho k_{0,i+\rho,-i-\rho},\rho^{-1}k_{0,i,-i}).$$

Finally, we note that each term in this sum is equal to $[\alpha i] - [\alpha(i+\rho)]$ which is equal to $Sf(D,i,\alpha,\tau) - Sf(D,i+\rho,\alpha,\tau)$. The result follows immediately from this.∎

The final step is contained in the next result.

Theorem 6.2: $\lim_{N\to\infty} [Sf(D,\delta_N,\alpha,\tau) - Sf(D,\rho\delta_N,\alpha,\tau)] = Eta(D,\rho,\alpha,\tau)$

Proof: We have

(1) $$\xi(D\otimes_{\nabla_{i\alpha}} I) - \xi(D\otimes I) = \mathrm{Eta}(D,i,\alpha,\tau) - \mathrm{Sf}(D,i,\alpha,\tau)$$

and

(2) $$\xi(D\otimes_{\nabla_{(i+\rho)\alpha}} I) - \xi(D\otimes I) = \mathrm{Eta}(D,i+\rho,\alpha,\tau) - \mathrm{Sf}(D,i+\rho,\alpha,\tau).$$

Now, $\xi(D\otimes_{\nabla_{(i+\rho)\alpha}} I) - \xi(D\otimes_{\nabla_{i\alpha}} I) = \{\alpha(i+\rho)\} - \{\alpha i\}$, and $\mathrm{Eta}(D,i+\rho,\alpha,\tau) - \mathrm{Eta}(D,i,\alpha,\tau) = -\alpha\rho$, a quantity independent of i. Thus, subtracting (1) and (2), summing from $-N$ to $N$ and dividing by $2N+1$ we obtain that $\mathrm{Eta}(D,\rho,\alpha,\tau) = \lim_{N\to\infty} \frac{1}{2N+1} \left\{\sum_{|i|\le N} \mathrm{Sf}(D,i,\alpha,\tau) - \mathrm{Sf}(D,i+\rho,\alpha,\tau)\right\}$. ■

<u>Corollary</u>: $TC_D^{\pitchfork}(\rho,\rho^{-1}) = \mathrm{Eta}(D,\rho,\alpha,\tau)$.

<u>Proof</u>: We have $TC_D^{\pitchfork}(\rho,\rho^{-1}) = \lim_{N\to\infty} \dfrac{C_D^{\pitchfork}(\rho\delta_N,\rho^{-1}\delta_N)}{\mathrm{Tr}_{L^2(S^1)}(\delta_N)} =$

$\lim_{N\to\infty} (2N+1)^{-1}[\mathrm{Sf}(D,\delta_N,\alpha,\tau) - \mathrm{Sf}(D,\rho\delta_N,\alpha,\tau)] = \mathrm{Eta}(D,\rho,\alpha,\tau)$

## §7 <u>The longitudinal cocycle</u>

In this section we describe a second cocycle based on the ellipticity of the operator $\tilde{D}$ along the leaves. Its value on unitary elements of $C^\infty(T^2)$ gives the index of Toeplitz operators along the leaves of the foliation.

Let $W^*(T^2,\mathcal{F}_\alpha)$ denote the von Neumann algebra of the foliation $\mathcal{F}_\alpha$ defined using the trace $Tr_\mu$ determined by Haar measure on $S^1$; [6], [15]. There is a natural representation of $W^*(T^2,\mathcal{F}_\alpha)$ on $L^2(T^2 x\mathbb{R})$. If $\varphi \in C^\infty(T^2)$, then as a leafwise multiplication operator it defines an element $M_\varphi \in \mathcal{L}(L^2(T^2 x\mathbb{R}))$. Since the operator $\tilde{D}$ is self-adjoint and acts along the leaves one can take its leafwise positive projection $P \in \mathcal{L}(L^2(T^2 x\mathbb{R}))$. Let $F = 2P - I$. Then the element $F[F,M_{\varphi_1}][F,M_{\varphi_2}]$ will belong to $W^*(T^2,\mathcal{F}_\alpha)$ and have finite trace if $\varphi_i$ is in $C^\infty(T^2)$ for i=1,2.

The longitudinal cocycle is defined, for $\varphi_1,\varphi_2 \in C^\infty(T^2)$ by

$$C_D^L(\varphi_1,\varphi_2) = Tr_\mu(F[F,M_{\varphi_1}][F,M_{\varphi_2}]).$$

<u>Proposition 7.1</u>: Let $\varphi \in C^\infty(T^2)$ and let $T_\varphi$ denote the Toeplitz operator on $P(L^2(T^2))$ defined by $T_\varphi = PM_\varphi P$. If $\varphi$ is invertible, then $T_\varphi$ is Fredholm and

$$\mathrm{Index}(T_\varphi) = C_D^L(\varphi,\varphi^{-1})$$

<u>Proof</u>: [3] or [8].∎

Although, in form, the cocycles $C_D^L$ and $C_D^{\pitchfork}$ are similar, there are many differences. An essential point is that $C_D^L$ is defined on the commutative algebra $C^\infty(T^2)$, while $C_D^{\pitchfork}$ is defined on the non-commutative convolution algebra $C^\infty(T^2 x S^1)$.

There is a formula for $\mathrm{Index}(T_\varphi)$ in terms of $\varphi$.

Proposition 7.2: $\mathrm{Index}(T_\varphi) = \lim_{t\to\infty} \frac{1}{2t}\{\arg\varphi(-t) - \arg\varphi(t)\}$.

Proof: [3]. ∎

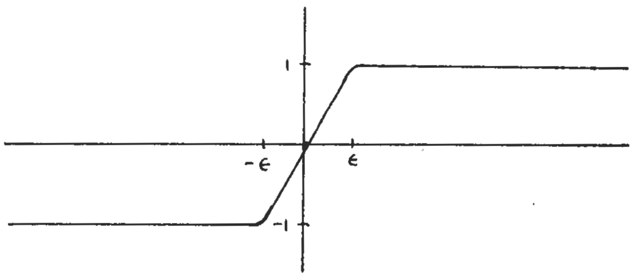
The map $\varphi: T^2 \longrightarrow S^1$ yields an element of $[T^2, S^1] \cong H^1(S^1, \mathbf{Z}) \cong \mathbf{Z}\oplus\mathbf{Z}$. Suppose that $\varphi$ corresponds to (m,n) under this identification. There is a topological formula for $\mathrm{Index}(T_\varphi)$, which is also meaningful for more general foliations.

Proposition 7.3: $\mathrm{Index}(T_\varphi) = \int_{S(T\mathcal{F}_\alpha)} \mathrm{Tch}(\varphi)\ \mathrm{Todd}(T\mathcal{F}_\alpha)\ \mathrm{ch}(\sigma_1(D))d\mu$.

Here, $Tch(\varphi)$ is the differential form $m d\theta + (\alpha n) d\varphi$, $ch(\sigma_1(D)) = \begin{cases} 1 & T^2 \times \{1\} \\ 0 & T^2 \times \{-1\} \end{cases}$, and $Todd(T\mathcal{F}_\alpha) = 1$. Thus, $Index(T_\varphi) = m + n\alpha$.

§8 The relation between the transverse and longitudinal cocycles

To relate the renormalized transverse cocycle to the longitudinal cocycle it is convenient to consider "approximate" cocycles. These are formed by using smooth approximations to the positive projection associated to $\tilde{D}$. Using the functional calculus on $L^2(T^2)$ we set $F_\epsilon = h_\epsilon(\tilde{D})$, where $h_\epsilon \in C^\infty(\mathbb{R})$ is a family of functions with graphs




satisfying $h'_\epsilon(x) > 0$ if $|x| < \epsilon$, $\lim_{\epsilon \to 0} h_\epsilon(x) = h_0(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ monotonically. Define $\kappa_\epsilon(\varphi_1, \varphi_2) = \frac{1}{4} F_\epsilon [F_\epsilon, \varphi_1][F_\epsilon, \varphi_2]$, where $\varphi_i \in C^\infty(T^2)$.

It is necessary now to shift our point of view and notation a little. Recall that there is a representation of the smooth convolution operator of the foliation $\mathcal{F}_\alpha$, $C_c^\infty(T^2 x\mathbb{R})$, on $L^2(T^2)$. (The analogous represntation of $C^\infty(T^2 x S^1)$ on $L^2(T^2)$ was used in §3 in constructing $C_D^\ast$.) In general this representation may not extend to the entire foliation algebra, but in the present case it does. Using the Fourier transform we will represent the foliation algebra, $C^*(T^2,\mathcal{F}_\alpha)$, on $\ell^2(\mathbf{Z}^2)$, rather than $L^2(T^2)$. It is the subalgebra of $\mathcal{L}(\ell^2(\mathbf{Z}^2))$ generated by operators of the form $\delta_{(m,n)} f\iota_\alpha$, where $f \in C_0(\mathbb{R})$, $\delta_{(m,n)}$ denotes translation by $(m,n)$, and $\iota_\alpha(m,n) = m+n\alpha$, and $f\iota_\alpha(m,n) = f(m+n\alpha)$. This is easily checked to be the Fourier transform of the standard representation of $C^*(T^2,\mathcal{F}_\alpha)$ on $L^2(T^2)$, and hence is faithful. There is a trace on $C^*(T^2,\mathcal{F}_\alpha)$ which is given in this representation by

$$\tau(\delta_{(m,n)} f\iota_\alpha) = \begin{cases} \int_{\mathbb{R}} f(x)dx & \text{if } (m,n) = (0,0) \\ 0 & \text{if } (m,n) \neq (0,0) \end{cases}$$

<u>Proposition 8.1</u>: The operator $\kappa_\epsilon(\varphi_1,\varphi_2)$ is in $C_c^\infty(T^2 x\mathbb{R})$ and has finite trace.

<u>Proof</u>: Observe that $[F_\epsilon,\varphi]$ is of the form

$$\sum \lambda_{m,n}\delta_{(m,n)} f_{(m,n)}\iota_\alpha$$

where $\lambda_{(m,n)}$ is rapidly decreasing and $f_{(m,n)} \in C_c^\infty(\mathbb{R})$. Thus $[F_\epsilon,\varphi]$ belongs to

$C_c^\infty(T^2 \times \mathbb{R})$ and hence $\kappa_\epsilon(\varphi_1,\varphi_2)$ does as well. ∎

Let $\Delta$ denote $\frac{\partial^2}{\partial\varphi^2}$ acting on $L^2(T^2)$ and let $e^{-t\Delta}$ denote the associated heat kernel. In general, neither $\kappa_\epsilon(\varphi_1,\varphi_2)$ nor $e^{-t\Delta}$ will be trace class. However, we have the following result.

Proposition 8.2: The operator $\kappa_\epsilon(e^{-t\Delta}\varphi_1, e^{-t\Delta}\varphi_2)$ is trace class on $\ell^2(\mathbb{Z}^2)$.

Proof: We note that $[F_\epsilon, e^{-t\Delta}\varphi] = e^{-t\Delta}[F_\epsilon,\varphi] \in \mathcal{L}(\ell^2(\mathbb{Z}^2))$, since $\Delta$ commutes with $\tilde{D}$, so that it will be sufficient to check that the trace norm, $\|e^{-t\Delta}[F_\epsilon,\varphi]\|_1$, is finite for any $\varphi \in C^\infty(T^2)$. For this one observes that $\|e^{-t\Delta}[F_\epsilon,\varphi]\|_1 \le \mathrm{Tr}(e^{-t\Delta})\Sigma|\lambda_{n,m}|$ where $\varphi = \Sigma\lambda_{n,m}e_{n,m}$. Since $\varphi$ is smooth the Fourier coefficients $\lambda_{n,m}$ are rapidly decreasing and the result follows. ∎

A basic result which we will need is the following.

Theorem 8.3: $$\lim_{t\to 0} \frac{\mathrm{Tr}_{L^2(T^2)}(\kappa_\epsilon(e^{-t\Delta}\varphi_1, e^{-t\Delta}\varphi_2))}{\mathrm{Tr}_{L^2(S^1)}(e^{-2t\Delta})} = \mathrm{Tr}_\mu(\kappa_\epsilon(\varphi_1,\varphi_2)).$$

Proof: We start off with a lemma whose proof is in an appendix.

Lemma 8.4: Let $f \in C_c^\infty(\mathbb{R})$. Then one has

$$\lim_{t\to 0} \frac{\sum_{n,m} f(m+\alpha n)e^{-tn^2}}{\sum_n e^{-tn^2}} = \int_{\mathbb{R}} f(x)dx$$

It is a consequence of Lemma 8.4 that

$$\lim_{t\to 0} \frac{Tr_{L^2(T^2)}(e^{-t\Delta}\delta_{(m,n)}f\iota_\alpha)}{Tr_{L^2(S^1)}(e^{-t\Delta})} = \begin{cases} \int_{\mathbb{R}} f(x)dx & \text{if } (m,n) = (0,0) \\ 0 & \text{if } (m,n) \neq (0,0) \end{cases}.$$

Now, note that $\kappa_\epsilon(e^{-t\Delta}\varphi_1, e^{-t\Delta}\varphi_2) = F_\epsilon[F_\epsilon, e^{-t\Delta}\varphi_1][F_\epsilon, e^{-t\Delta}\varphi_2] =$

$e^{-2t\Delta}F_\epsilon[F_\epsilon,\varphi_1][F_\epsilon,\varphi_2] + e^{-t\Delta}[e^{-t\Delta}, F_\epsilon[F_\epsilon,\varphi_1]][F_\epsilon,\varphi_2]$. It is necessary to show that

Lemma 8.5: Let $\varphi_1, \varphi_2 \in C^\infty(T^2)$. Then one has

$$\lim_{t\to 0} \frac{Tr_{L^2(T^2)}(e^{-t\Delta}[e^{-t\Delta}, F_\epsilon[F_\epsilon,\varphi_1]][F_\epsilon,\varphi_2])}{Tr_{L^2(S^1)}(e^{-2t\Delta})} = 0.$$

This shall be done in an appendix. Granted this, we then have

$$\lim_{t\to 0} \frac{Tr_{L^2(T^2)}(\kappa_\epsilon(e^{-t\Delta}\varphi_1, e^{-t\Delta}\varphi_2))}{Tr_{L^2(S^1)}(e^{-2t\Delta})} = \lim_{t\to 0} \frac{Tr_{L^2(T^2)}(e^{-2t\Delta}\kappa_\epsilon(\varphi_1,\varphi_2))}{Tr_{L^2(S^1)}(e^{-2t\Delta})}.$$ If one now

expresses $\kappa_\epsilon(\varphi_1,\varphi_2)$ as $\Sigma\, \lambda_{m,n}\delta_{(m,n)}f_{(m,n)}\iota_\alpha$ then we obtain

$$\lim_{t\to 0} \frac{\mathrm{Tr}_{L^2(T^2)}(e^{-2t\Delta}\kappa_\epsilon(\varphi_1,\varphi_2))}{\mathrm{Tr}_{L^2(S^1)}(e^{-2t\Delta})} = \lim_{t\to 0}\left\{\Sigma\lambda_{m,n}\frac{\mathrm{Tr}_{L^2(T^2)}(e^{-2t\Delta}\delta_{(m,n)}f_{(m,n)}\iota_\alpha)}{\mathrm{Tr}_{L^2(S^1)}(e^{-2t\Delta})}\right\} =$$

$$\int_{\mathbb{R}} f_{(0,0)}(x)dx = \mathrm{Tr}_\mu(\kappa_\epsilon(\varphi_1,\varphi_2)).\blacksquare$$

In §3 we defined the renormalized transverse cocycle via the approximate unit $\delta_N$. We now choose a different approximate unit, $e^{-t\Delta}$, and perform an analogous construction. It will be shown in the end that the resulting cocycles are the same. Thus, we set

$$TC^{\pitchfork}_{D,\epsilon}(\varphi_1,\varphi_2) = \lim_{t\to 0} \frac{\mathrm{Tr}_{L^2(T^2)}(\kappa_\epsilon(e^{-t\Delta}\varphi_1, e^{-t\Delta}\varphi_2))}{\mathrm{Tr}_{L^2(S^1)}(e^{-2t\Delta})},$$

and set $C^L_{D,\epsilon}(\varphi_1,\varphi_2) = \mathrm{Tr}_\mu(\kappa_\epsilon(\varphi_1,\varphi_2))$. Then a restatement of what we have proved is the following.

<u>Theorem 8.6</u>: If $\varphi_i \in C^\infty(T^2)$ then

$$C^L_{D,\epsilon}(\varphi_1,\varphi_2) = TC^{\pitchfork}_{D,\epsilon}(\varphi_1,\varphi_2).\blacksquare$$

The final step is to show that, for $\varphi_1$, $\varphi_2 \in C^\infty(T^2)$ one has $TC^{\pitchfork}_D(\varphi_1,\varphi_2) =$

$C_D^L(\varphi_1,\varphi_2)$. We accomplish this in several steps.

Recall that $TC_D^{\pitchfork}(\varphi_1,\varphi_2) = \lim\limits_{N\to\infty} \dfrac{C_D^{\pitchfork}(\varphi_1\delta_N,\varphi_2\delta_N)}{Tr_{L^2(S^1)}(\delta_N)}$ where $\delta_N$ is an approximate identity for $C^\infty(T^2xS^1)$. Let

$$\Psi(\epsilon,N) = \frac{Tr_{L^2(T^2)}(\kappa_\epsilon(\varphi_1\delta_N,\varphi_2\delta_N))}{Tr_{L^2(S^1)}(\delta_N)}.$$

The essential point is the following interchange of limits formula.

<u>Theorem 8.7</u>: $\lim\limits_{N\to\infty}\lim\limits_{\epsilon\to 0}\Psi(\epsilon,N) = \lim\limits_{\epsilon\to 0}\lim\limits_{N\to\infty}\Psi(\epsilon,N)$

<u>Proof</u>: One must show that $\lim\limits_{\epsilon\to 0}\Psi(\epsilon,N)$ exists uniformly in N, for then standard interchange of limit theorems yield the result. To justify this it is sufficient to check that $|\Psi(\epsilon,N)-\Psi(\epsilon',N)| \longrightarrow 0$ as $\epsilon$ and $\epsilon'$ go to zero, independently of N. For this note that $\kappa_\epsilon(\delta_N\varphi_1,\delta_N\varphi_2) - \kappa_{\epsilon'}(\delta_N\varphi_1,\delta_N\varphi_2) =$ $F_{\epsilon'}[F_{\epsilon'},\delta_N\varphi_1][F_\epsilon-F_{\epsilon'},\delta_N\varphi_2] + F_{\epsilon'}[F_\epsilon-F_{\epsilon'},\delta_N\varphi_1][F_\epsilon,\delta_N\varphi_2] +$ $(F_\epsilon-F_{\epsilon'})[F_\epsilon,\delta_N\varphi_1][F_\epsilon,\delta_N\varphi_2]$. One must show that the trace of each term, divided by $Tr(\delta_N)$, goes to 0 independently of N. We will carry out the computation for the first term, the others being similar. It will be sufficient to do this

with $\varphi_1$ and $\varphi_2$ equal to $\delta_{(i,j)}$ and $\delta_{(l,k)}$, respectively. Then

$|Tr(F_{\epsilon'}[F_{\epsilon'},\delta_N\varphi_1][F_\epsilon - F_{\epsilon'},\delta_N\varphi_2])| \le const.Tr(|\delta_N[F_\epsilon - F_{\epsilon'},\varphi_2]|)$. Now,

$Tr(|\delta_N[F_\epsilon - F_{\epsilon'},\varphi_2]|) = Tr([F_\epsilon - F_{\epsilon'},\bar{\varphi}_2]\delta_N[F_\epsilon - F_{\epsilon'},\varphi_2]) = \sum_{(l,k)} \chi_{[-N,N]}(n+k)C_{m,n;l,k}$

where $|C_{m,n;l,k}| \le 2$ and $C_{m,n;l,k} = 0$ unless $|m+l+\alpha(n+k)| < \epsilon$ and $|l+\alpha k| < \epsilon$. Using the fact that the numbers $\{\alpha k\}$ are equidistributed in $[0,1]$ one can show that the number of such pairs $(l,k)$ in a box $B_{L,\epsilon}$ (as in Appendix 1) is less than $4\epsilon L$ for L large and $\epsilon$ small. Dividing by $Tr_{L^2(S^1)}(\delta_N)$, (restricted to the box), we get an upper bound of $\frac{1}{2L+1}(4\epsilon L)$ and letting L go to infinity yields the result.∎

Next observe that, for each fixed $\epsilon$, one has

$$\lim_{N\to\infty} \frac{Tr_{L^2(T^2)}(\kappa_\epsilon(\varphi_1\delta_N,\varphi_2\delta_N))}{Tr_{L^2(S^1)}(\delta_N)} = \lim_{t\to 0} \frac{Tr_{L^2(T^2)}(\kappa_\epsilon(e^{-t\Delta}\varphi_1, e^{-t\Delta}\varphi_2))}{Tr_{L^2(S^1)}(e^{-2t\Delta})}$$

One must now look at the longitudinal cocycle. Recall that there are two representations of $C_c^\infty(T^2 x\mathbb{R})$-- the one we have been using, $\rho: C_c^\infty(T^2 x\mathbb{R}) \longrightarrow \mathcal{L}(\ell^2(\mathbf{Z}^2))$, and $\tilde{\rho}: C_c^\infty(T^2 x\mathbb{R}) \longrightarrow \mathcal{L}(L^2(T^2 x\mathbb{R}))$. Both are faithful and $\tilde{\rho}\rho^{-1}(\kappa_\epsilon(\varphi_1,\varphi_2)) = \tilde{\kappa_\epsilon}(\varphi_1,\varphi_2) = \tilde{F}_\epsilon[\tilde{F}_\epsilon,\varphi_1][\tilde{F}_\epsilon,\varphi_2]$ where $\tilde{F}_\epsilon = h_\epsilon(\tilde{D})$ is obtained via functional calculus on $L^2(T^2 \times \mathbb{R})$ and $\varphi_i$ acts as a leafwise multiplier. Now, $\mathrm{Tr}_\mu$ is normal and each $\tilde{\kappa}_\epsilon(\varphi_1,\varphi_2)$ has finite trace as does the limit. Moreover, $\tilde{\kappa}_\epsilon(\varphi_1,\varphi_2) \longrightarrow \kappa_0(\varphi_1,\varphi_2)$ weakly and all are dominated by $|\tilde{\kappa}_0(\varphi_1,\varphi_2)|$ so we obtain

<u>Theorem 8.9</u>: $\lim_{\epsilon\to 0} \mathrm{Tr}_\mu(\tilde{\kappa}_\epsilon(\varphi_1,\varphi_2)) = C_D^L(\varphi_1,\varphi_2)$. ∎

Combining these results we finally obtain

<u>Theorem 8.10</u>: $TC_D^{\pitchfork}(\varphi_1,\varphi_2) = C_D^L(\varphi_1,\varphi_2)$

<u>Theorem 8.11</u>: Let $\varphi \in K_1(C^\infty(T^2)) \cong \mathbb{Z}\oplus\mathbb{Z}$. If $\varphi$ corresponds to $(m,n)$ under this isomorphism, then

$$\int_{S(T\mathcal{F}_\alpha)} \mathrm{Tch}(\varphi)\ \mathrm{Todd}(T\mathcal{F}_\alpha)\ \mathrm{ch}(\sigma_1(D))d\mu = \langle C_D^L,\varphi\rangle = \langle TC_D^{\pitchfork},\varphi\rangle = \mathrm{Eta}(D,\rho,\alpha).$$

<u>Remark</u>: This provides a direct proof, of a different nature than previous ones, of the Atiyah-Patodi-Singer Theorem in this special case.

## §9 Secondary invariants for elliptic operators

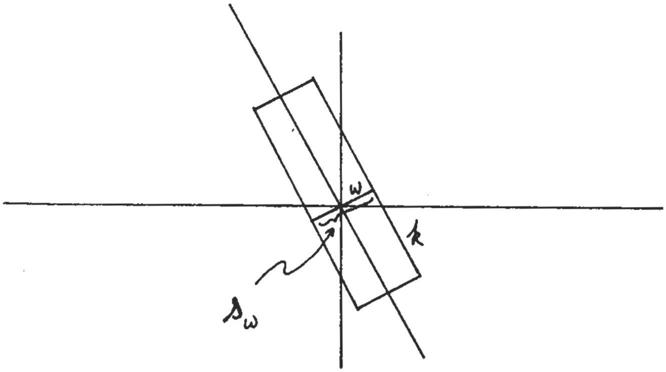
To indicate some of the possible applications of our results, we formulate them in a more general, and functorial manner. We thank Alain Connes for suggesting this point of view. Rather than restricting to $S^1$ immediately, we will work with $U_n$ at first. Let $\overline{BU}_n$ denote the fiber of the map $BU_n^{\delta} \longrightarrow BU_n$ where $U_n^{\delta}$ is the unitary group with the discrete topology. Then elements in $K_1(\overline{BU}_n)$ are represented by triples $(M,D,f)$, $M$ a closed $\mathrm{Spin}^c$ manifold, $D$ a self-adjoint elliptic operator on $M$ and $f:M \longrightarrow \overline{BU}_n$ a map. Recall that $f$ determines a homomorphism $\alpha:\pi_1(M) \longrightarrow U_n$ and a trivialization $\theta:\tilde{M}\times_{\alpha}U_n \longrightarrow M\times U_n = V$, where $\tilde{M}$ is the universal cover of $M$. Note that $V$ has a foliation, $\mathcal{F}_{\alpha}$, by images of $\tilde{M}\times\{g\}$, $g \in U_n$. This is precisely the data needed to obtain an operator longitudinally elliptic along the leaves of the foliation $\mathcal{F}_{\alpha}$ and, hence, an element $[\tilde{D}] \in KK^1(C(V),C^*(V,\mathcal{F}_{\alpha}))$. Note that $C^*(V,\mathcal{F}_{\alpha})$ is Morita equivalent to $C(U_n)\rtimes\pi_1(M)/\ker\alpha$ so there is a map from $KK^1(C(V),C^*(V,\mathcal{F}_{\alpha}))$ to $KK^1(C(V),C(U_n)\rtimes U_n^{\delta})$. Pairing an element $[u] \in KK^1(\mathbb{C},C(V))$ with the image of $[\tilde{D}]$ one obtains $[u]\otimes_{C(V)}[\tilde{D}] \in K_0(C(U_n)\rtimes U_n^{\delta})$. Now we look for cyclic classes in $HC^{ev}(C^{\infty}(U_n)\rtimes U_n^{\delta})$. One possibility is the trace, $[Tr]$, obtained from Haar measure on $U_n$. If we specialize again to $U_1 = S^1$, then the results of this paper show that

$$<[Tr],[\rho]\otimes_{C(T^2)}[\tilde{D}]> = Eta(D,\rho,\alpha).$$

This suggests that by using cyclic classes other than [Tr] one will obtain a useful family of higher order invariants of elliptic operators.

Appendix 1: Proof of Lemma 8.4

Proof of lemma: Let $B_{k,w}$ be a box of width 2w and length 2k situated as in the picture




Let $\delta_w$ be the segment on $y-\alpha x = 0$ cut off by $B_{k,w}$. Let $u(B_{k,w},t)$ $= \sum_{m,n\in B_{k,w}} f(m+\alpha n)e^{-tn^2}$ and $d(B_{k,w},t) = \sum_{n\in pr_2(B_{k,w})} e^{-tn^2}$ Then, setting $\zeta(k,t) = \dfrac{u(B_{k,w},t)}{d(B_{k,w},t)}$ we have that $\lim_{k\to\infty} \zeta(k,t)$ exists uniformly in t and $\lim_{t\to 0} \zeta(k,t)$ exists for each k, so the iterated limit exists and can be taken in k and t simultaneously. We claim that $\lim_{(k,t)} \zeta(k,t) = \int_{\delta_w} f(x)dx$. To see this, let $X_k$ denote the points on $\delta_w$ obtained by intersecting lines through lattice points in $B_{k,w}$ parallel to the sides corresponding to k. Then $\cup X_k$ is equidistributed and $\lim_{k\to\infty} \frac{1}{|X_k|}\left\{ \sum_{m,n\in B_{k,w}} f(m+\alpha n)\right\} = \int_{\delta_w} f(x)dx$.

For each k, there is a $t_k$ such that $0 < t < t_k$ implies that

$$\frac{1}{|X_k|} - \epsilon_k < \frac{e^{-tn^2}}{d(B_{k,w},t)} < \frac{1}{|X_k|} + \epsilon_k$$

Choosing $\epsilon_k$ to converge to zero sufficiently rapidly and summing appropriately, the result is obtained.

Now, we let the box become wider. Let $\eta(w,t) = \lim_{k\to\infty} \varsigma(w,k,t)$. Then $\lim_{t\to 0} \eta(w,t)$ exists, uniformly in w and $\lim_{w\to\infty} \eta(w,t)$ exists for each t. Thus $\int_{\mathbb{R}} f(x)dx$

$$= \lim_{w\to\infty} \lim_{t\to 0} \eta(w,t) = \lim_{t\to 0} \lim_{w\to\infty} \eta(w,t) = \lim_{t\to 0} \frac{\sum_{n,m} f(m+\alpha n)e^{-tn^2}}{\sum_n e^{-tn^2}}.$$

Appendix 2: Proof of Lemma 8.5.

First note that $F_\epsilon[F_\epsilon,\varphi_1] = \Sigma\mu_{i,j}\delta_{i,j}(f_{i,j}\iota_\alpha)$ and $[F_\epsilon,\varphi_2] = \Sigma\lambda_{m,n}\delta_{m,n}(g_{m,n}\iota_\alpha)$. Shift the first $e^{-t\Delta}$ to the right inside the trace in the numerator and denote the resulting operator by X. Then $Tr_{L^2(T^2)}(X)$ $= \sum_{l,k} \langle X1_{l,k}, 1_{l,k}\rangle$. Expanding this out yields $\sum_{m,n;l,k} C(l,k;m,n)e^{-tk^2}(e^{-tk^2} - e^{-t(k+n)^2})$ where m,n and l,k run through the lattice points in the plane. One can check that it is enough to fix m,n and show that

$$\lim_{t\to 0}\left\{ \sum_{l,k} C(l,k;m,n)e^{-tk^2}(e^{-tk^2} - e^{-t(k+n)^2})\frac{1}{\Sigma e^{-tk^2}} \right\} = 0.$$ Let $\mathcal{E}_t(k,n) =$

$(e^{-tk^2}-e^{-t(k+n)^2})$ and let $\mathcal{D}_t(l,k;m,n)$ be the other factor in each term. Then the sum is $\sum_{l,k} \mathcal{D}_t(l,k;m,n)\mathcal{E}_t(k,n)$, and we have the following properties: (i) for each k,n $\lim_{t\to 0} \mathcal{E}_t(k,n) = 0$, (ii) $\sum_{l,k} \mathcal{D}_t(l,k;m,n) < \infty$, (iii) $\sum_{l,k} \mathcal{D}_t(l,k;m,n)\mathcal{E}_t(k,n) < \infty$. Moreover, $|\mathcal{E}_t(k,n)|$ is uniformly bounded. From this, an elementary argument yields the desired result.

## REFERENCES


1. M. F. Atiyah, Elliptic Operators and Compact Groups, Lecture Notes in Math. 401, Springer Verlag, 1974

2. M. F. Atiyah, V. K. Patodi, and I. M. Singer, "*Spectral asymmetry and Riemannian geometry I, II, III*", Math. Proc. Camb. Phil. Soc. 77(1975), 43-69, 78(1975), 405-432, 79(1976), 71-99

3. L. A. Coburn, R. G. Douglas, D. G. Schaeffer, I. M. Singer, "*C*-algebras of operators on a half-space-II: Index theory*", Publ. Math. I.H.E.S. 40(1971) 69-79

4. A. Connes, "*Non-commutative differential geometry-I: the Chern character in K-homology*", Publ. Math. I.H.E.S. 62 (1986), 41-93

5. ________ , "*Non-commutative differential geometry-II: De Rham homology and non-commutative algebra*", Publ. Math. I.H.E.S. 62 (1986), 94-144.

6. ________ , "*Sur la theorie non commutative de l'integration*", Lecture Notes in Math. 725, Springer-Verlag, Berlin-New York, (1979), 19-143.

7. ________ , "*A survey of foliations and operator algebras*", Proc. Symposia Pure Math. 38 (1982), Part I, American Math. Society, Providence, 521-628.

8. A. Connes and G. Skandalis, "*The longitudinal index theorem for foliations*", Publ. RIMS, Kyoto Univ. 20 (1984), 1139-1183

9. R. G. Douglas, "*On the C*-algebras of a one-parameter semi-group of isometries*", Acta Math., 128(1972), 143-151

10. ____________ ,"*Another look at real-valued index theory*", Proceedings of Special Year on Operator Theory, Bloomington, 1985-86

11. R. G. Douglas, S. Hurder, and J. Kaminker,"*The longitudinal cyclic cocycle and the index of Toeplitz operators*", in preparation

12. ________________________________________ ,"*The $\eta$-invariant and the transverse Chern character*", in preparation

13. P. Gilkey, "*The eta invariant and secondary characteristic classes of locally flat bundles*", Algebraic and differential topology, Teubner-Texte zur Math., Teubner, Leipzig, 1984, 49-87

14. C. Lazarov, "*Spectral invariants of foliations*", Michigan J. Math. 33(1986), 231-243

15 C. Moore and C. Schochet, Global Analysis on Foliated Spaces, Springer-Verlag, to appear.

16. I. M. Singer, "*Recent applications of index theory for elliptic operators*", Proc. Symp. Pure Math. 23(1973), American Math. Society, Providence, 11-31

17. E. Witten, "*Global gravitational anomalies*", Comm. Math. Phys. 100(1985), 197-229

Addresses of authors:

Ronald G. Douglas
Department of Mathematics
SUNY Stony Brook
Stony Brook, NY 11794

Steven Hurder
Department of Mathematics
University of Illinois-Chicago
Chicago, IL 60680

Jerome Kaminker
Department of Mathematical Sciences
IUPUI
Indianapolis, IN 46223