Spectral theory of foliation geometric operators

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1 Coarse spectral geometry and foliation ergodic theory

Spectral geometry on a complete open manifold V of bounded geometry studies the relations between the geometric and topological aspects of V, and the properties of the spectrum of symmetric elliptic differential operators on V. The local aspects of this study are classical, and center on curvature invariants of V and the asymptotic distribution of the spectrum. *Coarse spectral geometry* concerns the global aspects of spectral geometry – investigating the spectral properties of differential operators which depend only on the quasi-isometry type of the space. The first result of this type was Brooks' theorem that the essential spectrum of the Laplacian on functions on a covering space V of a compact manifold contains 0 if and only if the covering group is amenable [5, 6, 2]. The Novikov-Šubin invariants of a covering are more recent examples of coarse spectral invariants [33, 32, 15, 20, **?**, **?**].

The leaves of foliations provide a natural class of complete open manifolds endowed with a Riemannian metric of bounded geometry. The quasi-isometry class of the leafwise Riemannian metrics depends only on the foliation, so their coarse spectral geometry is a natural property of the foliation. This paper explores the coarse spectral geometry of leaves, and presents some basic aspects of this study. Our main results relate the essential spectra of leafwise geometric operators $\{\mathcal{D}_L \mid L \subset V\}$ to the ergodic theory of the leaves. The basic observation is that a point λ is in the essential spectrum of leafwise geometric operator $\mathcal{D}_L$ if and only if there exists a sequence of compactly-supported, approximate λ-eigensections on the leaf L whose supports tend to infinity. This characterization of essential spectrum is combined with techniques from foliation ergodic theory to obtain sharper forms of results in the literature known for $\mathbf{R}^n$-actions.

A foliation geometric operator is actually a collection of operators, acting along each leaf of $\mathcal{F}$. These can be combined to yield an operator denoted by $\mathcal{D}_\mathcal{F}$ on the total space $C^\infty(V, \mathbf{E})$. Each transverse invariant measure μ for the foliation $\mathcal{F}$ yields a Hilbert space completion $L^2(V, \mathbf{E}, \mu)$ of $C^\infty(V, \mathbf{E})$, and $\mathcal{D}_\mathcal{F}$ has a densely defined closure $\mathcal{D}_\mu$ in $L^2(V, \mathbf{E}, \mu)$. The *spectral coincidence problem* [28, 29, 34, 40, 27] asks how the spectrum of $\mathcal{D}_\mu$ is related to the spectra of the the operators $\{\mathcal{D}_L \mid L \subset spt(\mu)\}$. One expects such a relationship, because given a leaf L in the support of μ, an eigensection for $\mathcal{D}_\mu$ restricts to a generalized eigensection for $\mathcal{D}_L$. For an amenable leaf, each generalized eigensection gives rise to a point in the essential spectrum. Foliation ergodic theory methods yield in a converse to this, relating the spectra on leaves to the spectra of $\mathcal{D}_\mu$.

The study of spectral geometry of foliation operators is closely related to results from the physics literature concerning the spectra of almost periodic (deterministic) and random operators. However, these works tend to emphasize operator theory methods and not foliation methods. One of the incidental goals of this paper is to exposit (in section 2 below) some basic foliation techniques useful for spectral questions.

Let $(V, \mathcal{F})$ be a compact foliated measure space. (See § 2 for this and other basic definitions.) The leaf through a point $x \in V$ will be denoted by L_x. The leaves of $\mathcal{F}$ admit a partial ordering where $L_x \leq L_y$ means that L_x is contained in the topological closure $\overline{L_y} \subset V$. The first result relates the spectra for the leafwise operators $\mathcal{D}_L$ with this partial ordering.

THEOREM 1.1 *Let $\mathcal{D}_\mathcal{F}$ be a leafwise geometric operator for a foliated measure space $(V, \mathcal{F})$. Suppose that $L_x \leq L_y$ and the holonomy group of L_x is trivial when restricted to the closure $\overline{L_y}$. Then $\sigma(\mathcal{D}_{L_x}) \subset \sigma_e(\mathcal{D}_{L_y})$.*

Recall that the ω-limit set $\omega(L)$ of a leaf L is the closure of the ends of L. A leaf L is *proper* if the induced topology on L from V agrees with the manifold topology on L. This is equivalent to L being open in the closure $\mathcal{C}\ell_V(L)$, so that $\omega(L) = \mathcal{C}\ell_V(L) - L$. A leaf is *non-proper* otherwise.

COROLLARY 1.2 *Let L be a non-proper leaf of $\mathcal{F}$ without holonomy. Then for every geometric operator $\mathcal{D}_\mathcal{F}$ along the leaves of $\mathcal{F}$, the leafwise spectrum is all essential: $\sigma(\mathcal{D}_L) = \sigma_e(\mathcal{D}_L)$.* □

A more general result (Theorem 4.1) than Theorem 1.1 is true, but its hypotheses is more technical and requires precise definitions from section 2. The exact statement is postponed till section 4. Theorem 1.1 follows from Theorem 4.1 and Proposition 2.9.

A subset $Z \subset V$ is called a *minimal set* for a foliated space $(V, \mathcal{F})$ if Z is closed, a union of leaves of $\mathcal{F}$, and every leaf of $\mathcal{F}$ in Z is dense in Z. An equivalent condition is to say that for every pair of leaves $L_x, L_y \subset Z$ we have $L_x \leq L_y$. The foliation $\mathcal{F}$ is said to be minimal if V is a minimal set. Theorem 4.1 and Proposition 2.10 yield:

THEOREM 1.3 *Suppose that $\mathcal{F}$ is defined by a locally-free, smooth action of a connected nilpotent Lie group $\mathcal{N}$, and Z a minimal set $\mathcal{F}$. For a leafwise geometric operator $\mathcal{D}_{\mathcal{F}}$ and any leaf $L \subset Z$, the leafwise spectrum $\sigma(\mathcal{D}_L) = \sigma_e(\mathcal{D}_L)$ is independent of the choice of $L \subset Z$.* □

These last results address only the spectrum of the leaves as closed subsets of the line. It seems likely that there are generalizations of these results to the spectral measures associated to the leaves. For example, suppose that $\mathcal{D}_L$ has an L^2-eigensection (corresponding to a Dirac mass for the spectral measure). An interesting problem is to understand how this Dirac mass propagates to the spectral measures for the $\mathcal{D}_{L_y}$ where $L \leq L_y$. The work of J. Álvarez-López and the author [1] give geometric condition for the propagation of the pure-point spectrum.

Now assume that $\mathcal{F}$ admits an invariant transverse sigma-finite measure μ. The next results relate the spectrum of a leafwise operator $\mathcal{D}_L$ with that of the closure $\mathcal{D}_\mu$ in $L^2(V, \mathbf{E}, \mu)$.

THEOREM 1.4 *For every $L \subset spt(\mu)$, we have $\sigma(\mathcal{D}_L) \subset \sigma_e(\mathcal{D}_\mu)$.*

There is a converse to Theorem 1.4, but requires that $\mathcal{F}$ be *hyperfinite* with respect to μ (cf. [41, 13] and section 2 below).

THEOREM 1.5 *Suppose that $\mathcal{F}$ is μ-hyperfinite. Then*

$$\sigma(\mathcal{D}_{\mathcal{F}}) \subset \bigcup_{L \subset spt(\mu)} \sigma(\mathcal{D}_L) \tag{1}$$

COROLLARY 1.6 *Suppose that $\mathcal{F}$ is μ-hyperfinite. Then*

$$\sigma(\mathcal{D}_{\mathcal{F}}) = \bigcup_{L \subset spt(\mu)} \sigma(\mathcal{D}_L) \tag{2}$$

The conclusion of Corollary 1.6 is called *spectral coincidence* in the literature [27, 28, 29, 34, 40]. The usual context is in the study of almost periodic or random operators on $\mathbf{R}^n$, but the framework here shows that spectral coincidence follows from general dynamical principles. For example, we can conclude:

COROLLARY 1.7 *Let $\mathcal{N}$ be a simply connected nilpotent Lie group, and assume there is a minimal, locally-free, measure-preserving action of $\mathcal{N}$ on a Borel measure space $(V, \tilde{\mu})$. Then for every leafwise geometric operator $\mathcal{D}_{\mathcal{F}}$, the leafwise spectrum $\sigma(\mathcal{D}_L)$ is independent of the orbit L, and $\sigma(\mathcal{D}_L) = \sigma(\mathcal{D}_\mu) = \sigma_e(\mathcal{D}_\mu)$.*

This paper is dedicated to the memory of Bruce Reinhart, whose encouragement led the author to pursue the study of index theory and spectral properties of foliations. The results of this paper were first presented at the conference on "Foliations" in his memory at College Park, Maryland, September 1989.

2 Topological and measurable dynamics of foliations

This section gives a brief introduction to the topological and measurable dynamics of foliations. Foliated measure spaces were introduced by Feldman and Moore [19], and are natural generalizations of foliated manifolds and measured laminations. They are the correct context for studying the measure theoretic questions of foliations (cf. [3, 4, 36, 19, 39, 30, 25]). Proofs of the main results we need are given, as often only the idea of the proof is available in the published literature.

A compact foliated space is a pair $(V, \mathcal{F})$ where V is a compact metric space, and the foliation data $\mathcal{F}$ is specified by giving a finite covering $\{U_\alpha \mid \alpha = 1, \ldots, k\}$ and homeomorphisms $\phi_\alpha : U_\alpha \cong \mathbf{D}^m \times X_\alpha$ where $\mathbf{D}^m = (-1, 1)^m$ is the open m-disc and X_α is a compact metric space. The level sets $P_\alpha(x) = \phi_\alpha^{-1}(\mathbf{D}^m \times \{x\})$ are called the *plaques* of $\mathcal{F}$. Note that the sets X_α are not assumed to have a local Euclidean structure. For example, X_α may be a Cantor set - a perfect set without interior. At the other extreme, for a foliation of a smooth manifold the model for X_α can be assumed to be a closed disc in $\mathbf{R}^n$ where n is the codimension of the foliation.

The sets U_α are not assumed to be open (recall that the transversal space X_α is closed) so we must impose a covering condition on their interiors, which requires formulating a notion of the interior. Let $\overline{V - U_\alpha}$ denote the closure of the complement of U_α, and define the *rim* of U_α to be the intersection $R(U_\alpha) = \overline{V - U_\alpha} \cap \overline{U_\alpha}$. The rim of U_α has a disjoint decomposition $R(U_\alpha) = R^\ell(U_\alpha) \cup R^t(U_\alpha)$ where the longitudinal part $R^\ell(U_\alpha)$ corresponds to the boundaries of the plaques, so is homeomorphic via an extension of ϕ_α^{-1} to $\mathbf{S}^{m-1} \times X_\alpha$. The transverse part $R^t(U_\alpha)$ corresponds to the "boundary" of the metric space X_α and is homeomorphic via ϕ_α^{-1} to $\mathbf{D}^m \times \partial X_\alpha$ where ∂X_α is a closed subset of X_α. The interior of X_α is the complement $X_\alpha^{int} = X_\alpha - \partial X_\alpha$, and define the interior $U_\alpha^{int} = \phi_\alpha^{-1}(\mathbf{D}^m \times X_\alpha^{int})$ which is open in V. We require that the interiors $\{U_\alpha^{int} \mid \alpha = 1, \ldots, k\}$ cover V.

The collection of plaques endow V with a second, much finer topology whose basic open sets are the open sets (in the relative topology) in the plaques $P_\alpha(x)$. A leaf of $\mathcal{F}$ is a maximal connected open set in this finer topology. That is, a leaf of $\mathcal{F}$ is an increasing union of plaques

$$L = \bigcup_{N=1}^{\infty} \left\{ \bigcup_{i=1}^{N} P_{\alpha_i}(x_i) \right\} \tag{3}$$

where each

$$P_{\alpha_N}(x_N) \cap \left\{ \bigcup_{i=1}^{N-1} P_{\alpha_i}(x_i) \right\} \neq \emptyset \tag{4}$$

and if $P_\alpha(x) \cap L \neq \emptyset$ then $P_\alpha \subset L$. We will call the collection of plaques $\{P_{\alpha_i}(x_i) \mid i \in \mathcal{N}^+\}$ a *tiling* of L if it also satisfies the condition that

$$P_{\alpha_N}(x_N) \not\subset \bigcup_{i=1}^{N-1} P_{\alpha_i}(x_i) \tag{5}$$

If L is compact, then a tiling must be finite. A tiling is intuitively described as a connected chain of open discs of uniform diameter which cover the leaf without redundancy, so that their increasing unions eventually engulf every compact subset $K \subset L$.

This formal definition includes the usual examples of foliations on smooth manifolds, and also allows for constructions that arise in dynamical systems and probablistic considerations. Section 6 includes some of the basic examples and constructions to illustrate the possibilities.

The foliation $\mathcal{F}$ is *leafwise smooth* if for each non-empty intersection $P_\alpha(x) \cap U_\beta$ and $x \in X_\alpha$, the transition function $\ell_{\alpha\beta}(x) = \phi_\beta \circ \phi_\alpha^{-1}: \mathbf{D}^m_{\alpha\beta} \times \{x\} \to \mathbf{D}^m \times \{y\}$ is uniformly C^∞. The uniformity means that there are estimates on the sup norm over $P_\alpha(x) \cap U_\beta$ of the derivatives of $\ell_{\alpha\beta}(x)$ which are independent of α, β and x. We assume that each leaf $L \subset V$ is given a Riemannian metric $\langle \cdot, \cdot \rangle_L$ of bounded geometry, so that the leafwise metrics restricted to a chart U_α vary continuously with the transverse parameter $x \in X_\alpha$. Moreover, by refining a given foliation covering, we can assume that each plaque is convex for the leafwise geodesic metric. This implies that the covering is *good*: for all choices $\alpha, \alpha_2, \ldots, \alpha_k$ and $x \in X_\alpha$ if the intersection $P_\alpha(x) \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_k}$ is non-empty, then it is convex and hence connected.

Let ∇^L denote the associated Riemannian connection on the leaf L, and let $\nabla^{\mathcal{F}}$ denote the collection of all the leafwise connections.

A foliation cover as above of a foliated space admits a subordinate partition-of-unity $\{\lambda_\alpha \mid \alpha = 1, \ldots, k\}$, constructed by adapting the standard method. The first step is to produce "model bump functions" on the standard spaces $\mathbf{D}^m \times X_\alpha$. Choose a smooth function $\xi: \mathbf{D}^m \to [0,1]$ which vanishes on the boundary of $\mathbf{D}^m$ and is non-zero on the interior of the disc. Choose a continuous function $\zeta_\alpha: X_\alpha \to [0,1]$ which vanishes on ∂X_α and is positive on the interior X_α^{int}. For $p \in U_\alpha$ set $\xi_\alpha(p) = \xi(\pi_1(\phi_\alpha(p))) \cdot \zeta_\alpha(\pi_2(\phi_\alpha(p)))$ where $\pi_1: \mathbf{D}^m \times X_\alpha \to \mathbf{D}^m$ is projection onto the first factor, and $\pi_2: \mathbf{D}^m \times X_\alpha \to X_\alpha$ is projection onto the second. The support of ξ_α is all of U_α and it vanishes along the rim of U_α, so we can extend it to a continuous function on V which vanishes outside of U_α. Note that ξ_α has uniform bounds on it's leafwise covariant derivatives $(\nabla^{\mathcal{F}})^s \xi_\alpha$ for all $s \geq 0$. Then define

$$\lambda_\alpha(p) = \frac{\xi_\alpha(p)}{\sum_{\beta=1}^k \xi_\beta(p)}$$

A set $W \subset V$ which consists of a union of leaves of $\mathcal{F}$ is said to be *saturated* with respect to $\mathcal{F}$. The *saturation* $S_{\mathcal{F}}Z$ of a set $Z \subset V$ consists of the union of all leaves which intersect Z. If $Z \subset U_\alpha$ we also define the local saturation

$$S_\alpha Z = \bigcup_{P_\alpha(x) \cap Z \neq \emptyset} P_\alpha(x)$$

A *local cross-section* is a Borel subset $Z \subset U_\alpha$ which intersects each plaque $P_\alpha(x)$ in at most one point. Define local cross-sections $T_\alpha = \phi_\alpha^{-1}(\{0\} \times X_\alpha)$. For each β with $U_\alpha \cap U_\beta \neq \emptyset$ define compact spaces $T_{\alpha\beta} = T_\alpha \cap S_\alpha(U_\alpha \cap U_\beta)$ and $X_{\alpha\beta} = \phi_\alpha(T_{\alpha\beta}) \subset X_\alpha$.

A *transversal* $T \subset V$ to $\mathcal{F}$ is a Borel subset which intersects each leaf in at most a countable set. A cross-section is thus a special case of a transversal, and one can show (via a Borel selection process) that every transversal is a countable union of local cross-sections.

A pair (α, β) is *admissible* if $U_\alpha \cap U_\beta \neq \emptyset$. For each admissible pair (α, β) there is a well-defined transition function $\gamma_{\alpha\beta}: X_{\alpha\beta} \to X_{\beta\alpha}$, which for $x \in X_{\alpha\beta}$ is given by

$$\gamma_{\alpha\beta}(x) = \phi_\beta\left(S_\beta(\phi_\alpha^{-1}(\mathcal{D}^m \times \{x\}) \cap U_\beta) \cap T_\beta\right) \in X_{\beta\alpha}$$

The continuity of the charts ϕ_α implies that each $\gamma_{\alpha\beta}$ is homeomorphism from $X_{\alpha\beta}$ onto $X_{\beta\alpha}$.

A *plaque-chain of length* k between a pair of points $x, y \in L$ on a leaf of $\mathcal{F}$ is a finite collection of plaques $\mathcal{P} = \{P_{\alpha_1}(y_1), \ldots, P_{\alpha_k}(y_k)\}$ with $x \in P_{\alpha_1}(y_1)$ and $y \in P_{\alpha_k}(y_k)$ so that $P_{\alpha_i}(y_i) \cap P_{\alpha_{i+1}}(y_{i+1}) \neq \emptyset$ for $1 \leq i < k$. A plaque-chain $\mathcal{P}$ is said to be a *shortcut* from x to y if there does not exists a plaque-chain of length less than k between x and y.

DEFINITION 2.1 *The* plaque length *between two points $x, y \in L$ is defined to be the length of a shortcut between x and y.*

Each plaque-chain $\mathcal{P} = \{P_{\alpha_1}(y_1), \ldots, P_{\alpha_k}(y_k)\}$ determines a local homeomorphism

$$\gamma_{\mathcal{P}} = \gamma_{\alpha_1\alpha_2} \circ \cdots \gamma_{\alpha_{k-1}\alpha_k}$$

from an open neighborhood $y_1 \in W^s_{\mathcal{P}} \subset X_{\alpha_1}$ to an open neighborhood $y_k \in W^r_{\mathcal{P}} \subset X_{\alpha_k}$ where $W^s_{\mathcal{P}} \subset X_{\alpha_1}$ is the maximal subset on which the composition is defined.

For $x \in X_\alpha$, a *closed plaque-chain* based at x is a plaque-chain starting and ending at $\phi_\alpha^{-1}(x)$, and defines a local homeomorphism from a neighborhood of x to itself. The collection of all such local homeomorphisms defines a pointed groupoid $\psi\mathbf{Hol}_{\mathcal{F}}(x) \subset \mathbf{Emb}(X_\alpha, x)$. The germs of these maps at x form a subgroup $\mathbf{Hol}_{\mathcal{F}}(x) \subset \mathbf{Germ}(X_\alpha, x)$ of the germs at x called the *holonomy group* of $\mathcal{F}$ at x. The isomorphism class of $\mathbf{Hol}_{\mathcal{F}}(x)$ is well-defined, independent of the choice of transversal and basepoint x (cf. Chapter IV, [7]).

Let $(V, \mathcal{F})$ be a foliated measure space with a fixed covering by foliation charts $\{(U_\alpha, \phi_\alpha : U_\alpha \to \mathbf{D}^m \times X_\alpha) \mid \alpha = 1, \ldots, k\}$. For each leaf $L \subset V$ and $y = \phi_\alpha^{-1}(x) \in L \cap T_\alpha$ there is a well-defined *holonomy homomorphism*

$$h_x \colon \pi_1(L, y) \to \mathbf{Hol}_{\mathcal{F}}(x),$$

defined by sending the leaf homotopy class in $\pi_1(L, y)$ determined by a closed plaque-chain $\mathcal{P}$ to the germ of $\gamma_{\mathcal{P}}$ at x.

The *holonomy covering* $\pi \colon \tilde{L}_h \to L$ is the covering associated to the homomorphism h_x. Each plaque $P_\alpha(x)$ lifts to a disjoint union of open subsets of $\tilde{L}_h$ and each connected component of the lift is homeomorphic to $P_\alpha(x)$. These connected components in $\tilde{L}_h$ are the plaques of $\tilde{L}_h$, so $\tilde{L}_h$ is covered by the union of its plaques. The defining property of the holonomy covering is that each closed plaque-chain $\tilde{\mathcal{P}}$ in $\tilde{L}_h$ projects under $\pi \colon \tilde{L} \to L$ to a closed plaque-chain $\mathcal{P}$ in L for which $\gamma_{\mathcal{P}}$ is the identity germ. That is, for each each closed plaque-chain $\tilde{\mathcal{P}}$ in $\tilde{L}_h$ there is an open neighborhood W_α of $x \in X_\alpha$ so that the restriction of $\gamma_{\mathcal{P}}$ to W_α is the identity.

A *transverse measure* μ for $\mathcal{F}$ is a locally-finite measure on transversals, whose *measure class* is invariant under the transverse holonomy transformations. The transverse measure class defined by μ is equivalently specified by giving finite Borel measures μ_α on each set X_α, so that each local holonomy map $\gamma_{\alpha\beta}$ pulls the *measure class* of $\mu_\beta | X_{\beta\alpha}$ back to that of $\mu_\alpha | X_{\alpha\beta}$. A local cross-section $Z \subset U_\alpha$ is said to have μ-measure zero if and only if $\mu_\alpha(\phi_\alpha(S_\alpha Z \cap T_\alpha)) = 0$.

We say that μ is an *invariant transverse measure* for $\mathcal{F}$ if the local holonomy preserves the measure; that is, $\gamma^*_{\alpha\beta}(\mu_\beta | X_{\beta\alpha}) = \mu_\alpha | X_{\alpha\beta}$ for all admissible (α, β). The measure of a local cross-section $Z \subset U_\alpha$ is defined as

$$\mu(Z) = \mu_\alpha(\phi_\alpha(S_\alpha Z \cap T_\alpha))$$

This is extended as a countably additive measure to all transversals $Z \subset V$: use a selection lemma to decompose

$$Z = \bigcup_{\alpha=1}^{k} \bigcup_{i=1}^{\infty} Z_{\alpha,i}$$

where each $Z_{\alpha,i} \subset U_\alpha$ is a local cross-section, then define

$$\mu(Z) = \sum_{\alpha=1}^{k} \sum_{i=1}^{\infty} \mu_\alpha(Z_{\alpha,i})$$

The holonomy invariance of the measures μ_α implies that $\mu(Z)$ is independent of the choices of the foliation covering of V and the particular decomposition of a set Z.

DEFINITION 2.2 *A* foliated measure space *is a triple* $(V, \mathcal{F}, \mu)$ *where* $(V, \mathcal{F})$ *is a foliated space and* μ *is an invariant transverse measure for* $\mathcal{F}$*.*

The *support* of a transverse measure μ is the smallest closed $\mathcal{F}$-saturated subset $spt(\mu) \subset V$ so that $\mu(Z) = 0$ for any transversal $Z \subset V \setminus spt(\mu)$. The triple $(spt(\mu), \mathcal{F}|spt(\mu), \mu)$ is a foliated measure space.

The ω-limit set $\omega(L)$ of a leaf L generalizes the usual notion of the asymptotic limit set for flows: choose a tiling $\{P_{\alpha_i}(x_i) \mid i \in \mathcal{N}^+\}$ of L, then

$$\omega(L) = \bigcap_{N=1}^{\infty} \left(\overline{\bigcup_{i=N}^{\infty} P_{\alpha_i}(x_i)} \right) \tag{6}$$

where the overline denotes closure in the topology on V. If L has only a finite number of ends, then $\omega(L)$ equals the union of the closures of the ends; otherwise, it may properly contain this union.

LEMMA 2.3 $\omega(L)$ *is a closed saturated set.*

Proof: $\omega(L)$ is the intersection of closed subsets of V hence is closed.

Choose a tiling $\{P_{\alpha_i}(x_i) \mid i \in \mathcal{N}^+\}$ of L. Let $z \in \omega(L)$, and choose a subsequence of plaques $\{P_{\alpha_{i_\ell}}(x_{i_\ell}) \mid \ell \in \mathcal{N}^+\}$ with z in their limit. Let $z' \in L_z$ be another point on the leaf through z. Choose a plaque-chain $\mathcal{P} = \{P_{\alpha_1}(y_1), \ldots, P_{\alpha_k}(y_k)\}$ from z to z', with local holonomy homeomorphism $\gamma_{\mathcal{P}}: W^s_{\mathcal{P}} \to W^r_{\mathcal{P}}$. It will suffice to show that $P_{\alpha_k}(y_k) \subset \omega(L)$. Choose $N > 0$ so that $x_{i_\ell} \in W^s_{\mathcal{P}}$ for all $\ell > N$. Continuity of $\gamma_{\mathcal{P}}$ implies that the sequence $\{\gamma_{\mathcal{P}}(x_{i_\ell}) \mid \ell > N\}$ contains y_k in its limit, and thus $P_{\alpha_k}(y_k) \subset \omega(L)$. □

Recall that $X \subset V$ is a *minimal set* for $\mathcal{F}$ if X is $\mathcal{F}$-saturated, closed and there is no proper $\mathcal{F}$-saturated closed subset $Y \subset X$.

COROLLARY 2.4 *A closed saturated non-empty subset* $X \subset V$ *contains a minimal set* $Z \subset X$*.*

Proof: The collection of closed saturated subsets of X is closed under intersections, hence by Zorn's Lemma contains a minimal element Z. For each leaf $L \subset Z$, $\omega(L) \subset Z$ is a closed saturated subset, hence must equal Z. □

The minimal set $Z \subset X$ need not be unique. For example, if $\mathcal{F}$ is a foliation with all leaves compact, then a minimal set for $\mathcal{F}$ consists of a single leaf, so that every closed saturated set with more than one leaf contains more than one minimal set. (There are also much more sophisticated examples of non-uniqueness.)

Associate to each leaf L the collection $\mathcal{Z}(L) = \{Z \subset \omega(L) \mid Z \text{ is minimal}\}$. These are the invariant sets for $\mathcal{F}$ onto which the leaf L "spirals" as we go to infinity. For codimension one foliations, Poincaré-Bendixson Theory [35, 8, 9, 14, 23] relates the geometry of L with the structure of the minimal sets in $\mathcal{Z}(L)$. It is easy to show that if *all* leaves of $\mathcal{F}$ are proper, then $\mathcal{Z}(L)$ consists of compact leaves [24]. When $\mathcal{F}$ is C^2 and codimension-one, for a proper leaf L, $\mathcal{Z}(L)$ again consists of compact leaves [9, 23]. Moreover, the leaves of $\omega(L)$ admit a partial-ordering with the minimal elements consisting of the leaves in $\mathcal{Z}(L)$.

We next formulate the notions of Fölner leaves and uniform amenability for foliated spaces (cf. section 1.7, [25]). Fix a leaf L of $\mathcal{F}$ and a tiling $\{P_{\alpha_i}(x_i) \mid i \in \mathcal{N}^+\}$ of L. For each subset $\mathcal{I} \subset \mathcal{N}^+$ define

$$F_{\mathcal{I}} = \left\{ \bigcup_{i \in \mathcal{I}} P_{\alpha_i}(x_i) \right\}$$

A plaque $P_{\alpha_i}(x_i)$ is said to be in the *interior* of $F_{\mathcal{I}}$ if $P_{\alpha_i}(x_i) \cap P_{\alpha_k}(x_k) = \emptyset$ for all $k \notin \mathcal{I}$. $P_{\alpha_i}(x_i)$ is on the *boundary* of F_N if $i \in \mathcal{I}$ and $P_{\alpha_i}(x_i) \cap P_{\alpha_k}(x_k) \neq \emptyset$ for some $k \notin \mathcal{I}$. Let $\partial F_{\mathcal{I}}$ denote the union of the boundary plaques for $F_{\mathcal{I}}$. Let $\#\mathcal{F}_{\mathcal{I}} = \#\mathcal{I}$ denote the number of plaques in $F_{\mathcal{I}}$ and $\#\partial F_{\mathcal{I}}$ denote the number of plaques on the boundary of $F_{\mathcal{I}}$.

DEFINITION 2.5 *A leaf L of a foliated space $(V, \mathcal{F})$ is Fölner if there exists an increasing sequence of finite subsets $\mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \mathcal{N}^+$ so that the associated nested sequence of compact subsets*

$$F_1 \subset F_2 \subset \cdots \subset F_N = F_{\mathcal{I}_N} \subset \cdots$$

exhausts L and satisfies

$$\lim_{N \to \infty} \frac{\#\partial F_N}{\#F_N} = 0 \tag{7}$$

The exhaustion $F_1 \subset F_2 \subset \cdots$ is called a *Fölner sequence* for L. The notion clearly generalizes the usual idea of a Fölner sequence for a Lie group G.

A foliation $\mathcal{F}$ defined by a locally-free action of a connected solvable Lie group on V has every leaf Fölner [6, 25]. This example, however, can be misleading, for it suggests that the typical Fölner sequence has a "uniformity" corresponding to the homogeneity of the Lie group. This need not be the case in more intricate, non-uniform examples [22, 23]. For results concerning the transference of spectra from a leaf L to another L', we require a formulation of amenability which is localized to neighborhoods of L. The simplest approach is to require that the holonomy group $\mathbf{Hol}_{\mathcal{F}}(x)$ of L_x be amenable. However, it is the holonomy groupoid $\psi\mathbf{Hol}_{\mathcal{F}}(x)$ which describes the orbits of the leaves near to L_x. Composition is not always defined for elements of $\psi\mathbf{Hol}_{\mathcal{F}}(x)$, so the usual Fölner condition must be reformulated in different terms. We give two extensions to pseudo-groups – one for invidual orbits, and the other for the entire action of $\psi\mathbf{Hol}_{\mathcal{F}}(x)$. The purpose of these definitions is to ensure that a gievn subset of a leaf has an arbitrarily close "covering" by a set in nearby leaves which again satisfies the Fölner condition.

Fix a leaf L_x and representative of the groupoid $\psi\mathbf{Hol}_{\mathcal{F}}(x) \subset \mathbf{Emb}(X_\alpha, x)$ defined by the closed plaque-chains based at x. For each integer $N > 0$, let $\psi\mathbf{Hol}_{\mathcal{F},N}(x)$ denote the sub-groupoid generated by the plaque-chains of length at most $2N$. There are at most a finite number of plaque-chains which are short-cuts of length at most N, so there exists a countable subset

$$\{h_0, h_1, \ldots, h_{i_1}, \ldots, h_{i_1+\cdots+i_N}, \ldots\} \subset \psi\mathbf{Hol}_{\mathcal{F}}(x)$$

so that each sub-collection

$$\Delta_N = \{h_0, h_1, \ldots, h_{i_1}, \ldots, h_{i_1+\cdots+i_N}\}$$

(where $h_0 = \mathrm{Id}\colon X_\alpha \to X_\alpha$) generates the sub-groupoid $\psi\mathbf{Hol}_{\mathcal{F},N}(x)$.

Let L_y be a leaf with $L_x \leq L_y$, where $y \in T_\alpha = \phi_\alpha^{-1}(\{0\} \times X_\alpha)$. The maps in the groupoid $\psi\mathbf{Hol}_{\mathcal{F},N}(x)$ have domains depending upon the element selected, so that the orbit of y under this groupoid is defined by a partial action as follows: for $y_\alpha = \pi_1 \circ \phi_\alpha(y) \in X_\alpha$, set

$$\psi\mathbf{Hol}_{\mathcal{F},N}(x) \cdot y_\alpha = \{h(y_\alpha) \mid h \in \psi\mathbf{Hol}_{\mathcal{F},N}(x) \ \&\ y_\alpha \in \mathrm{Domain}(h)\} \subset X_\alpha$$

Let

$$\psi\mathbf{Hol}_{\mathcal{F}}(x) \cdot y_\alpha = \bigcup_{N>0} \psi\mathbf{Hol}_{\mathcal{F},N}(x) \cdot y_\alpha$$

For example, the assumption $L_x \leq L_y$ implies that $x_\alpha = \phi_\alpha(x) \in \mathrm{Closure}_{X_\alpha}(\psi\mathbf{Hol}_{\mathcal{F}}(x) \cdot y_\alpha)$.

Finally, given any subset $\mathcal{W} = \{w_1, \ldots, w_\ell\} \subset X_\alpha$ define the Δ_N-*penumbra*

$$P_N\mathcal{W} = \{z \in X_\alpha \mid \text{ either } z = h(w_i) \text{ or } w_i = h(z) \text{ for some } h \in \Delta_N \ \& \ w_i \in \mathcal{W}\}$$

Note that $h_0 = \mathrm{Id}$ implies that $\mathcal{W} \subset P_N\mathcal{W}$ always. The Δ_N-*penumbra* consists of the orbits of the groupoid which can be obtained from an element of the given set $\mathcal{W}$ by application of an element of the generating set Δ_N.

DEFINITION 2.6 *For $L_x < L_y$, we say $\psi\mathbf{Hol}_\mathcal{F}(x)$ is* uniformly Fölner *on L_y if, for every $N > 0$, $\epsilon > 0$ and open neighborhood $x \in U \subset X_\alpha$, there exists a finite subset $\mathcal{W}_{U,N,\epsilon} \subset U \cap \psi\mathbf{Hol}_\mathcal{F}(x) \cdot y_\alpha$ such that*

$$\#\{P_N\mathcal{W}_{U,N,\epsilon}\} \leq (1+\epsilon) \cdot \#\mathcal{W}_{U,N,\epsilon} \tag{8}$$

It follows by standard techniques that this notion is depends only on L_x and L_y and not on the choices of basepoints and covering of V by foliation charts. The union of the plaques containing the points of $\mathcal{W}_{U,N,\epsilon}$ define a compact subset of L_y intersecting the open neighborhoof U of x which has the Fölner condition.

DEFINITION 2.7 *$\psi\mathbf{Hol}_\mathcal{F}(x)$ is* uniformly Fölner *if or every $N > 0$, $\epsilon > 0$ and open neighborhood $x \in U \subset X_\alpha$, there exists a finite subset $\mathcal{W}_{U,N,\epsilon} \subset U$ such that*

$$\#\{P_N\mathcal{W}_{U,N,\epsilon}\} \leq (1+\epsilon) \cdot \#\mathcal{W}_{U,N,\epsilon} \tag{9}$$

The following proposition has an almost obvious proof:

PROPOSITION 2.8 *Suppose that L_x has trivial holonomy group $\mathbf{Hol}_\mathcal{F}(x)$. Then $\psi\mathbf{Hol}_\mathcal{F}(x)$ is uniformly Fölner.*

Proof: Suppose that $x_\alpha \in U \subset X_\alpha$, $N > 0$ and $\epsilon > 0$ are given. By hypothesis, the germ of each map $h \in \psi\mathbf{Hol}_\mathcal{F}(x)$ is the identity at x_α, so for the finite collection Δ_N there exists an open set $x_\alpha \in V_N \subset U$ on which each $h \in \Delta_N$ restricts to the identity. Thus, for any subset $\mathcal{W} \subset V_N$ we have $P_N\mathcal{W} = \mathcal{W}$ so that it satisfies the Fölner condition (9). □

The same idea also establishes:

PROPOSITION 2.9 *Suppose that $L_x \leq L_y$ and the holonomy group of L_x is trivial when restricted to the closure $\overline{L_y}$. Then $\psi\mathbf{Hol}_\mathcal{F}(x)$ is uniformly Fölner on L_y.* □

A nilpotent Lie group admits an exhaustion by open neighborhoods of the identity e which are Fölner with respect to a fixed compact neighborhood of e (i.e., the group satisfies the usual Fölner condition) and the orbits of the group inherit this condition. Standard foliation techniques then yields the following class of examples:

PROPOSITION 2.10 *Suppose that $\mathcal{F}$ is defined by the locally-free action of a connected nilpotent Lie group $\mathcal{N}$ on V. Then $\psi\mathbf{Hol}_\mathcal{F}(x)$ is uniformly Fölner for all $x \in V$.*

Hyperfinitesness for a foliated measure space $(V, \mathcal{F}, \mu)$ (cf. section 4, [19]; section 5, [31]) generalizes the Fölner condition from individual orbits to a uniform condition over all orbits. A *Borel equivalence relation* on X is a Borel subset $\mathcal{R} \subset X \times X$ such that $x \sim_{\mathcal{R}} y$ whenever $(x, y) \in \mathcal{R}$ defines an equivalence relation. A *subequivalence relation* is Borel subset $\mathcal{S} \subset \mathcal{R}$ which also defines an equivalence relation.

For the disjoint union of the transversal spaces to $\mathcal{F}$,

$$X = \bigcup_{\alpha=1}^{k} X_\alpha$$

introduce the equivalence relation $\mathcal{R}(\mathcal{F})$ defined by setting $x \sim_{\mathcal{F}} y$ if and only if the plaques $P_\alpha(x)$ and $P_\beta(y)$ are on the same leaf of $\mathcal{F}$. The $\mathcal{R}(\mathcal{F})$-saturation of a subset $Z \subset X$ is

$$\mathcal{R}(\mathcal{F}) \cdot Z = \bigcup_{x \in Z} \{y \sim \mathcal{F}x\}$$

If $Z \subset X_\alpha$ is μ_α-measurable and $\mu_\alpha(Z) = 0$, then for every admissible pair α, β the image $\gamma_{\alpha\beta}(Z \cap T_{\alpha\beta}) \subset X_\beta$ also has measure zero. Hence, the saturation $\mathcal{R}(\mathcal{F}) \cdot Z$ will have μ-measure zero. This implies that $\mathcal{R}(\mathcal{F})$ is a *measured equivalence relation* in the terminology of [18, 19].

An equivalence relation $\mathcal{S}$ on V is *finite* if the number of elements in each equivalence class is finite. The intuitive definition of a finite equivalence relation is that it partitions the leaves of $\mathcal{F}$ into bounded Borel subsets, and two points x, y are $\mathcal{S}$-equivalent if and only if they are on the same leaf *and* they belong to the same bounded subset. For example, the *nearest neighbor partition* of the leaves defines a finite subequivalence relation: Endow the leaves of $\mathcal{F}$ with the Riemannian distance function, then for each plaque $P_\alpha(x)$ we obtain a Borel subset consisting of the points which are "closer" to the transversal $T_\alpha(x)$ than to any other transversal $T_\beta(y)$. Continuity of the Riemannian metric from leaf to leaf implies that the resulting partition of the leaves is Borel.

One can create larger partitions of the leaves by assembling the nearest neighbor partitions into "blocks". However, the selection process by which these basic leaf blocks are grouped together is required to yield a Borel subset $\mathcal{S} \subset X \times X$. The hyperfinite property asserts that this can be done with arbitrarily large nested subequivalence relations:

DEFINITION 2.11 *The equivalence relation* $\mathcal{R}(\mathcal{F})$ *is* μ-hyperfinite *if there exists an increasing sequence of finite subequivalence relations*

$$*X \subset \mathcal{R}_1 \subset \mathcal{R}_2 \subset \cdots \subset \mathcal{R}$$

where $*X$ *denotes the trivial equivalence relation* ($x \sim_* y$ *if and only if* $x = y$) *and the union* $\cup \mathcal{R}_N$ *defines an equivalence relation* $\sim_\infty$ *which agrees* μ*-a.e. with* $\sim_{\mathcal{F}}$ *(i.e., for* μ*-a.e.* $x \in X$ *we have* $x \sim_\infty y \iff x \sim_{\mathcal{F}} y$*). We then say that* $(V, \mathcal{F}, \mu)$ *is a hyperfinite foliated measure space, and* $\{\mathcal{R}_N(\mathcal{F})\}$ *is an exhaustion of* $\mathcal{R}(\mathcal{F})$*.*

Connes-Feldman-Weiss [13] proved that an equivalence relation is μ-hyperfinite if and only if the quivalence relation is *amenable* in the sense of Zimmer [41]. Amenability of $\mathcal{R}(\mathcal{F})$ can often be established by a variety of methods (cf. [41]; Chapter [42]) so this is a very useful criterion for the existence of a hyperfinite partition.

The equivalence relation $\mathcal{R}(\mathcal{F})$ has an orbit metric $d_{\mathcal{F}}: \mathcal{R}(\mathcal{F}) \to \mathcal{N}^+$ (cf. section 1.1, [25]) where the distance $d_{\mathcal{F}}(x, y)$ between points $x \in X_\alpha$ and $y \in X_\beta$ is 1 less than the length of a shortcut between the plaques $P_\alpha(x)$ and $P_\beta(y)$.

Let $\mathcal{S}$ be a finite subequivalence relation for $\mathcal{R}(\mathcal{F})$. A point $x \in X$ is said to be in the *interior* of $\mathcal{S}$ if all points $y \sim_{\mathcal{F}} x$ with $d_{\mathcal{F}}(x,y) = 1$ satisfy $x \sim_{\mathcal{S}} y$. Otherwise, y is on the boundary of $\mathcal{S}$. The set of interior points for $\mathcal{S}$ will be denoted by $X^i(\mathcal{S})$ and the boundary points by $X^{\partial}(\mathcal{S})$.

The key property that we need about a hyperfinite foliated measure space is given by:

PROPOSITION 2.12 *Let $(V, \mathcal{F}, \mu)$ be a hyperfinite foliated measure space, with exhaustion $\{\mathcal{R}_N(\mathcal{F})\}$. Then the μ-measure of the set $X^{\partial}(\mathcal{R}_N(\mathcal{F}))$ tends to zero as $N \to \infty$.*

Proof: The condition $\mathcal{R}_N(\mathcal{F}) \subset \mathcal{R}_{N+1}(\mathcal{F})$ implies that $X^{\partial}(\mathcal{R}_{N+1}(\mathcal{F})) \subset X^{\partial}(\mathcal{R}_N(\mathcal{F}))$. Clearly, $X^{\partial}(\mathcal{R}(\mathcal{F}))$ has μ-measure 0, so by the dominated convergence theorem the sequence $\{\mu(X^{\partial}(\mathcal{R}_N(\mathcal{F})))\}$ tends to 0. □

Adopt the notation $\sim_N$ for the equivalence relation determined by $\{\mathcal{R}_N(\mathcal{F})\}$.

COROLLARY 2.13 *Let $(V, \mathcal{F}, \mu)$ be a hyperfinite foliated measure space, with exhaustion $\{\mathcal{R}_N(\mathcal{F})\}$. There exists Borel subsets $\{X_N\}$ of X so that*

1. *X_N intersects each equivalence class of $\mathcal{R}_N(\mathcal{F})$ exactly once.*
   That is, for $x, y \in X_N$ if $x \sim_N y$ then $x = y$
2. *The μ-measure of the complement $X \setminus \mathcal{R}_N(\mathcal{F})X_N$ tends to 0 as $N \to \infty$.*
3. *For μ-almost every $x \in X_\alpha$ the sequence of sets*

$$F_N(x) = \bigcup_{y \sim_N x} P_{\alpha(y)}(y)$$

   is a Folner sequence for the leaf L through $P_\alpha(x)$.

Proof: A finite Borel equivalence relation admits a Borel cross-section [18]. For each N let X_N be such a cross-section for $\mathcal{R}_N(\mathcal{F})$, then conclusions 1 and 2 follow directly from Proposition 2.12. The proof of 3 follows from a standard counting argument in the theory of measure preserving group actions (cf. [17]): suppose there exists a subset $X^b \subset X_\alpha$ of positive μ_α-measure such that for each $x \in X_\alpha$ the leaf L_x through $P_\alpha(x)$ is not Fölner. Then we can assume there exists a constant $\epsilon > 0$ so that for each $x \in X^b$ and $N > 0$ we have the estimate

$$\mu((X^{\partial}(\mathcal{R}_N(\mathcal{F}) \cap X^b) \geq \epsilon \cdot \mu((X^i(\mathcal{R}_N(\mathcal{F}) \cap X^b)$$

Since $X^{\partial}(\mathcal{R}_i(\mathcal{F}) \cup X^i(\mathcal{R}_i(\mathcal{F})$ tends to a set of full μ-measure, for all i sufficiently large

$$\mu((X^i(\mathcal{R}_i(\mathcal{F}) \cap X^b) \geq \mu(X^b)/2 > 0.$$

and hence $\mu((X^{\partial}(\mathcal{R}_N(\mathcal{F}) \cap X^b)$ is bounded from below by $\epsilon \cdot \mu(X^b)/2 > 0$, a contradiction. □

(2.13.3) implies in particular that μ-almost every leaf of $\mathcal{F}$ is Fölner (cf. Théorème 4, [10]).

We say that a transverse measure μ is *continuous*, or *without atoms*, if for every $x \in X$ and $\epsilon > 0$ there exists an open set about x in X whose μ-measure is less than ϵ. Every invariant transverse measure μ can be decomposed $\mu = \mu_c + \mu_a$ into an invariant continuous part μ_c plus an atomic part μ_a where the atomic measure is a direct integral of measures supported on a countable set of compact leaves of $\mathcal{F}$. Thus, if $\mathcal{F}$ has no compact leaves then every invariant transverse measure is continuous (cf. section 1, [26]).

We conclude our discussion of the dynamics of foliated spaces with a structure theorem for neighborhoods of leaves, which is a direct generalization of the fundamental structure theorem for a neighborhood of a compact leaf with finite holonomy (cf. Theorem 3, §4 [7]). Given a set $Z \subset V$ and $\epsilon > 0$ let $\mathcal{N}(Z, \epsilon)$ be the open neighborhood of K consisting of points which lie within ϵ of Z.

PROPOSITION 2.14 *Let L be a leaf in a foliated space $(V, \mathcal{F})$. Given a compact subset $K \subset \tilde{L}_h$ and $\epsilon > 0$, there exists $x \in X_\alpha$ for some α and $\delta > 0$ so that for the open metric ball $B(x, \delta) \subset X_\alpha$ there is a foliated immersion $\Pi: K \times B(x, \delta) \to V$ with $\Pi | K \times \{x\}$ the restriction to K of the covering map $\pi: \tilde{L}_h \to L$, and*

$$\Pi(K \times B(x, \delta)) \subset \mathcal{N}(\pi(K), \epsilon)$$

Proof: There exists $N > 0$ so that K is covered by the union of a finite plaque-chain $\{\tilde{P}_1, \ldots, \tilde{P}_N\}$ in $\tilde{L}_h$. Let $\tilde{P}_i$ project to the plaque $P_{\alpha_i}(x_i)$ on L. For each $1 < \ell \leq N$ the plaque-chain $\tilde{\mathcal{P}}_\ell = \{\tilde{P}_1, \ldots, \tilde{P}_\ell\}$ projects to a plaque-chain $\mathcal{P}_\ell = \{P_{\alpha_1}(x_1), \ldots, P_{\alpha_\ell}(x_\ell)\}$ on L. The holonomy map $\gamma_{\mathcal{P}_\ell}$ for $\mathcal{P}_\ell$ is a homeomorphism from an open neighborhood $x_1 \in W_\ell^s \subset X_{\alpha_1}$ to an open neighborhood $x_\ell \in W_\ell^r \subset X_{\alpha_\ell}$. Choose $\delta_1 > 0$ sufficiently small so that

$$B(x_1, \delta_1) \subset W_1^s \cap \cdots \cap W_N^s \text{ and } \phi_{\alpha_\ell}^{-1}\left(\mathbf{D}^m \times \gamma_\ell(B(x_1, \delta_1))\right) \subset \mathcal{N}(\pi(K), \epsilon) \text{ for all } \quad 1 \leq \ell \leq N$$

That is, the ball $B(x_1, \delta_1)$ is in the domain of $\gamma_{\mathcal{P}_\ell}$ for all ℓ, and foliated product centered on the plaque $P_{\alpha_\ell}(x_\ell)$ is contained in the ϵ-tube about $\pi(K)$.

The definition of the map $\Pi: K \times B(x_1, \delta_1) \to V$ requires a small technical nuance, for though the above choices show how to define this map on individual boxes $\tilde{P}_\ell \times B(x_1, \delta_1)$, a point $p \in K$ may be contained in several plaques. We must use a standard trick from differential topology to get a well-defined map.

Let $\{\lambda_\alpha\}$ be the partition-of-unity on V constructed previously. Fix $p \in K$ and $y \in B(x, \delta_1)$, then let

$$\mathcal{I}(p) = \{i \mid\ p \in \tilde{P}_i\} \subset \{1, \ldots, N\}$$

The collection of points $\{\phi_{\alpha_i}^{-1}(\pi(p), y) \mid i \in \mathcal{I}(p)\}$ lie in the geodesically convex intersection

$$P_{\mathcal{I}(p)}(y) = \bigcap_{i \in \mathcal{I}(p)} P_{\alpha_i}(\gamma_i(y))$$

Define $\Pi(p, y)$ to be the unique point in $P_{\mathcal{I}(p)}(x)$ which is the center of mass for the collection $\{\phi_{\alpha_i}^{-1}(\pi(p), x) \mid i \in \mathcal{I}(p)\}$ with respect to the weights $\{\lambda_{\alpha_i}(\pi(p)) \mid i \in \mathcal{I}(p)\}$. When $y = x_1$ all of the points $\{\phi_{\alpha_i}^{-1}(\pi(p), x_1) \mid i \in \mathcal{I}(p)\}$ are equal, so that $\Pi | K \times \{x_1\} = \pi$ is an immersion. Moreover, for all $y \in B(x_1, \delta_1)$ the restriction $\Pi | K \times \{y\}$ is smooth along K, and depends continuously on y in the C^1-topology on immersions. Hence, there exists a $0 < \delta < \delta_1$ for which the restriction $\Pi | K \times B(x_1, \delta) \to V$ is an immersion. By construction, this map preserves the foliated product structure and has image contained in the tube $\mathcal{N}(\pi(K), \epsilon)$. □

3 Geometric operators on foliated measure spaces

In this section, we introduce the class of geometric operators on foliated spaces, and recall some of their basic spectral properties.

Let $(V, \mathcal{F})$ be a foliated space with the ancillary data as determined in section 2. A Hermitian vector bundle $\mathbf{E} \to V$ is a *foliated Hermitian bundle* if for each foliation chart U_α, there is a trivialization $\Phi_\alpha : \mathbf{E}|U_\alpha \cong \mathbf{C}^N \times \mathbf{D}^m \times X_\alpha$, such that for each admissible pair α, β and $x \in X_{\alpha\beta}$, the transition function

$$\Phi_\beta \circ \Phi_\alpha^{-1}(x) : \mathbf{C}^N \times \mathbf{D}^m_{\alpha\beta} \times \{x\} \longrightarrow \mathbf{C}^N \times \mathbf{D}^m_{\beta\alpha} \times \{\gamma_{\alpha\beta}(x)\}$$

is an Hermitian isomorphism of $\mathbf{C}^N$ which depends C^∞ on $\mathbf{D}^m_{\alpha\beta}$. Moreover, we require that $\Phi_\beta \circ \Phi_\alpha^{-1}(x)$ vary continuously on the parameter x in the C^∞ topology on bundles maps. Thus, the restrictions $\mathbf{E}|L \to L$ to the leaves of $\mathcal{F}$ are smooth Hermitian bundles which depend continuously on the transverse parameter.

Let $\nabla^{\mathbf{E}_L}$ denote the leafwise Hermitian connection for $\mathbf{E}_L$

$\mathbf{E} \to V$ is a *foliated flat Hermitian bundle* if the transition functions $\Phi_\beta \circ \Phi_\alpha^{-1}(x)$ are constant on $\mathbf{D}^m_{\alpha\beta}$ for all x. Thus, the restrictions $\mathbf{E}|L \to L$ to the leaves of $\mathcal{F}$ are flat Hermitian bundles whose flat structures depends continuously on the transverse parameter.

For each leaf $L \subset V$, let $\mathcal{S}_L \to L$ denote the Clifford bundle of spinors associated to the Clifford algebra bundle $\mathcal{C}(TL)$, and let $\not{\mathcal{D}}_L : C^\infty_c(\mathcal{S}_L) \to C^\infty_c(\mathcal{S}_L)$ be the corresponding Dirac operator. The union of the leafwise Spinor bundles $\mathcal{S} \to V$ forms a foliated Hermitian bundle over V.

Fix an Hermitian vector bundle $\mathbf{E}^0 \to V$. For each leaf L, form the leafwise generalized Dirac operator $\mathcal{D}_L = \not{\mathcal{D}}_L \otimes \nabla^{\mathbf{E}^0_L}$ defined on the compactly supported sections $C^\infty_c(\mathbf{E}_L)$ of the bundle $\mathbf{E} = \mathcal{S} \otimes \mathbf{E}^0$ restricted to L. That is, $\mathcal{D}_L$ is the Dirac operator associated to the leafwise Clifford bundles $\mathcal{S}_L \otimes E^0_L \to L$ (cf. Definition 2.4, [38]): At a point $x \in L$, choose an orthogonal framing $\{e_1, \ldots, e_m\}$ of T_xL and extend these to local synchronous vector fields $\{\tilde{e}_1, \ldots, \tilde{e}_m\}$ about x (cf. (1.29) of [38].) The Clifford algebra $\mathcal{C}(T_xL)$ is spanned by the monomials $\{e_I = e_{i_1} \otimes \cdots \otimes e_{i_p} \mid I = (1_1 < \cdots < i_p)\}$. Define $|I| = p$ for $I = (1_1 < \cdots < i_p)$. Choose also a unitary framing $\{f_1, \ldots, f_q\}$ of $\mathbf{E}^0_x$ and extend to local $\nabla^{\mathbf{E}^0_L}$-synchronous sections $\{\tilde{f}_1, \ldots, \tilde{f}_q\}$. Then for a general local section

$$s \quad = \quad \sum_{\substack{|I| \leq m \\ 1 \leq j \leq q}} s_{I,j} \cdot \tilde{e}_I \otimes \tilde{f}_j \qquad \text{set}$$

$$\mathcal{D}_L(s)(x) \quad = \quad \sum_{\substack{|I| \leq m \\ 1 \leq \alpha \leq m \\ 1 \leq j \leq q}} \nabla^L_{e_\alpha}(s_{\alpha,j})|_x \cdot e_\alpha \otimes \tilde{e}_I(x) \otimes \tilde{f}_j(x) \tag{10}$$

DEFINITION 3.1 *A* foliation geometric operator $\mathcal{D}_\mathcal{F}$ *for* $(V, \mathcal{F})$ *is a collection of leafwise Dirac operators* $\{\mathcal{D}_L \mid L \subset V\}$ *associated to a leafwise Riemannian metric for* $\mathcal{F}$ *and a foliated Hermitian vector bundle* $\mathbf{E}^0$.

PROPOSITION 3.2 (Chernoff [12, 37]) *Let $\mathcal{D}_L: C_c^\infty(\mathbf{E}_L) \to C_c^\infty(\mathbf{E}_L)$ be a leafwise generalized Dirac operator. Then $\mathcal{D}_L$ is essentially self-adjoint.* □

Let $\mathcal{D}_\mathcal{F}: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ be a foliation geometric operator. For each bounded Borel function $\phi: \mathbf{R} \to \mathbf{R}$, we can apply the spectral theorem leafwise for each $x \in V$ to the essentially self-adjoint operator

$$\mathcal{D}_{L_x}: C_c^\infty(\mathbf{E}_{L_x}) \to C_c^\infty(\mathbf{E}_{L_x})$$

to obtain a family of bounded operators $\phi(\mathcal{D}_\mathcal{F}) = \{\phi(\mathcal{D}_{L_x}) | \ x \in V\}$. The finite-propagation method [11, 37] can be used to show that this family depends Borel on the transverse parameter.

PROPOSITION 3.3 *Let $\mathcal{D}_L: C_c^\infty(\mathbf{E}_L) \to C_c^\infty(\mathbf{E}_L)$ be a leafwise generalized Dirac operator. Then $\lambda \in \sigma_e(\mathcal{D}_L)$ if and only if there exists a sequence $\{\phi_1, \phi_2, \ldots\} \subset C_c^\infty(\mathbf{E}_L)$ of unit-norm sections whose supports are disjoint, tend to infinity in L, and $||(D_L - \lambda)\phi_i||_2 \leq 1/i$ for all $i \geq 1$.*

Proof: Clearly, if such a sequence of test functions exists, then $(D_L - \lambda)$ does not admit a bounded inverse so that $\lambda \in \sigma(\mathcal{D}_L)$. If λ is not in the essential spectrum, then it must be an isolated point of finite-multiplicity pure-point spectrum. Hence there is a finite-rank projection Π_λ so that $\mathcal{D}_L - \lambda + \Pi_\lambda$ is invertible. The condition on the supports of the $\{\phi_i\}$ imply that for $i \gg 0$, the sections ϕ_i are almost orthogonal to the range of Π_λ, hence

$$\lim_{i \to \infty} ||(D_L - \lambda + \Pi_\lambda)\phi_i||_2 = 0$$

contradicting invertibility.

We use three properties of geometric operators to establish the theorem in the converse direction. A leafwise operator $A_L: L^2(\mathbf{E}_L) \to L^2(\mathbf{E}_L)$ is *smoothing* if for each $\psi \in L^2(\mathbf{E}_L)$ the image $\chi_\mathbf{B}(\mathcal{D}_L)\psi \in C^\infty(\mathbf{E}_L)$. It is *locally-compact* if for each compactly-supported bounded Borel function $g: L \to \mathbf{R}$ the compositions $A_L \circ M(g)$ and $M(g) \circ A_L$ are compact operators. (M(g) denotes the mulptiplication operator by g.)

LEMMA 3.4 *Let $\mathcal{D}_L: C_c^\infty(\mathbf{E}_L) \to C_c^\infty(\mathbf{E}_L)$ be a leafwise generalized Dirac operator. Then for each bounded Borel set $\mathbf{B} \subset \mathbf{R}$ the spectral projection $\chi_\mathbf{B}(\mathcal{D}_L)$ is a locally-compact, smoothing operator.*

Proof: By the spectral theorem, the operators $\mathcal{D}_L^\ell \circ \chi_\mathbf{B}(\mathcal{D}_L)$ are bounded for all $\ell > 0$, so by the Sobolev Lemma for open complete manifolds (cf. Chapter 3, [38]) the range of $\chi_\mathbf{B}(\mathcal{D}_L)$ is contained in $C^\infty(\mathbf{E}_L)$. Local compactness follows from Rellich's Theorem in the usual way. □

LEMMA 3.5 *Let $\mathcal{D}_L: C_c^\infty(\mathbf{E}_L) \to C_c^\infty(\mathbf{E}_L)$ be a leafwise generalized Dirac operator. Then there exists a constant $C_1 > 0$ (independent of the choice of L) so that for any smooth function $f: L \to \mathbf{R}$ with compact support, and section $\psi \in C_c^\infty(\mathbf{E}_L)$ we have*

$$||\mathcal{D}_L(f \cdot \psi) - f \cdot D_L(\psi)||_2 \leq C_1 \cdot \sup_{x \in L} ||\nabla_x f|| \cdot ||\psi||_2 \tag{11}$$

Proof: The definition (10) for generalized Dirac operators yields a pointwise commutator formula

$$\mathcal{D}_L(f \cdot \psi)|_x = f(x) \cdot (\mathcal{D}_L \psi)_x + \nabla_x \cdot A_x(\psi)$$

where A_x is a 0-order linear operator on sections associated to the symbol of $\mathcal{D}_L$; (11) follows. □

Let $B(x, R) = \{y \in L \mid \mathrm{dist}_L(x, y) < R\}$ be the open ball in L of radius R centered at x.

LEMMA 3.6 *Let $\mathcal{D}_L: C_c^\infty(\mathbf{E}_L) \to C_c^\infty(\mathbf{E}_L)$ be a leafwise generalized Dirac operator. Then for each $x \in L$, $R > 0$ and $\lambda \in \mathbf{R}$, the restriction of $\mathcal{D}_L - \lambda$ to the space*

$$C^\infty(B(x, R), \mathbf{E}_L; \partial) = \{\psi \in C_c^\infty(\mathbf{E}_L) \ \textit{so that spt}(\psi) \subset B(x, R)$$

is Fredholm. □

Our assumption $\lambda \in \sigma_e(\mathcal{D}_L)$ implies that for all $\epsilon > 0$ the range of the spectral projection $\chi_{(\lambda-\epsilon,\lambda+\epsilon)}(\mathcal{D}_L)$ is infinite dimensional. We use this conclusion to inductively construct sections $\{\phi_i \mid i = 1, 2, \ldots\} \subset C_c^\infty(\mathbf{E}_L)$ and choose a sequence of radii $S_1 < S_2 < \cdots$ so that

1. $||\phi_i||_2 = 1$
2. There exists $x_* \in L$ so that $B(x*, S_i) \subset spt(\phi) = K_i \subset B(x_*, S_{i+1})$ for all $i \geq 1$
3. $||(\mathcal{D}_L - \lambda)\phi_i||_2 < 1/i$

Choose a basepoint $x_* \in L$. Then choose ψ_1 in the range of $\chi_{(\lambda-1/10,\lambda+1/10)}(\mathcal{D}_L)$. There exists $R_1 > 0$ so that the restriction $\psi_1|B(x_*, R_1)$ to the ball of radius R_1 centered at x_* has norm $||\psi_1|B(x_*, R_1)||_2 > 99/100$. Fix a constant $C_1 > 1$ as in Lemma 3.5, set $S_1 = R_1 + 20C_1$ then choose a function $0 \leq g_1 \leq 1$ which is identically 1 on $B(x_*, R_1)$, identically 0 on the complement of $B(x_*, S_1)$ and has gradient bounded by $(10C_1)^{-1}$. Set

$$\phi_1 = ||g_1 \cdot \psi_1||_2^{-1} \cdot g_1 \cdot \psi_1$$

and $K_1 = B(x_1, S_1)$. Then by Lemma 3.5 we have

$$\begin{aligned} ||(\mathcal{D}_L - \lambda)\phi_1||_2 &\leq ||g_1 \cdot \psi_1||_2^{-1} \cdot \{||g_1 \cdot (\mathcal{D}_L - \lambda)\psi_1||_2 + ||(\mathcal{D}_L - \lambda)\phi_1 - g_1 \cdot (D_L - \lambda)\psi_1||_2\} \\ &\leq 100/99 \cdot \left\{1/10 + C_1 \cdot (10C_1)^{-1} \cdot ||\psi_1||\right\} \\ &\leq 1. \end{aligned}$$

Assume we have constructed unit-norm sections $\{\phi_1, \phi_2, \ldots, \phi_n\} \subset C_c^\infty(\mathbf{E}_L)$ with the support of ϕ_i contained in the compact set K_i with $B(x_*, S_{i-1}) \subset K_i \subset B(x_*, S_i)$ for $1 < i \leq n$, and

$$||(D_L - \lambda)\phi_i||_2 \leq 1/i \quad \text{for} \quad 1 \leq i \leq n.$$

Let χ_n be the characteristic function for $B(x_*, S_n)$. Set $\epsilon(n) = (20(n+1))^{-1}$. The range of the projection $\chi_{(\lambda-\epsilon(n),\lambda+\epsilon(n))}(\mathcal{D}_L)$ is infinite dimensional, while the composition

$$M(\chi_n) \circ \chi_{(\lambda-\epsilon(n),\lambda+\epsilon(n))}(\mathcal{D}_L)$$

is compact, so there exists a unit-norm ψ_{n+1} in the range of $\chi_{(\lambda-\epsilon(n),\lambda+\epsilon(n))}(\mathcal{D}_L)$ so that

$$||\psi_{n+1} \mid B(x_*, S_n + 20nC_1)||_2^2 < \epsilon(n)$$

Choose a compact set K'_{n+1} disjoint from $B(x_*, S_n + 20nC_1)$ such that

$$||\psi_{n+1} \mid K_{n+1}||_2^2 > 1 - 2\epsilon(n)$$

Choose a compactly-supported smooth function $0 \leq g_{n+1} \leq 1$ which is identically 1 on K'_{n+1}, identically 0 on $B(x_*, S_n)$ and has gradient bounded by $(10nC_1)^{-1}$. Set $\phi_{n+1} = g_{n+1} \cdot \psi_{n+1}$, with $K_{n+1} = spt(\phi_{n+1}$. Calculation as before yields

$$||(\mathcal{D}_L - \lambda)\phi_{n+1}||_2 \leq (1 - \epsilon(n)) \cdot \left\{(10(n+1)C_1)^{-1} + C_1 \cdot (10(n+1))^{-1}\right\} < 1/(n+1)$$

Finally, choose $S_{n+1} > S_n$ so that $K_{n+1} \subset B(x_*, S_{n+1})$. □

A *transverse measure* μ for $\mathcal{F}$ is a Radon measure on the Borel subsets of the transversals to $\mathcal{F}$, which takes finite value on compact subsets [36]. A transverse measure is *quasi-invariant* if, for every transversal Z with μ-measure zero, all holonomy transports of Z also have μ-measure zero. A transverse measure is *invariant* if the μ-measure of a transverse set Z does not change under holonomy transport. A transverse measure for $\mathcal{F}$ is said to be *non-atomic* if it has no atoms. That is, it assigns measure zero to each countable transverse set Z. Conversely, if μ is supported on a countable collection of compact leaves, then we say μ is *atomic*.

Given an invariant transverse measure μ for $\mathcal{F}$ let $\tilde{\mu}$ be its Haar extension to a locally-finite Borel measure on V. Let $L^2(V, \mathbf{E}, \mu)$ denote the completion of $C^\infty(V, \mathbf{E})$ with respect to the Hilbert space inner product defined by $\tilde{\mu}$. The resulting Hilbert space depends substantially on the geometric properties of the measure μ. For example, if μ is the transverse Dirac measure associated to a compact leaf L, then $L^2(V, \mathbf{E}, \mu) \cong L^2(L, \mathbf{E}|L)$ is the Hilbert space completion of the smooth sections over L with respect to the smooth leafwise Riemannian measure. While if $\tilde{\mu}$ is equivalent to the smooth Lebesgue measure on V, then $L^2(V, \mathbf{E}, \mu)$ is isomorphic to the Hilbert space completion of the smooth sections over V with the usual Riemannian inner product.

A foliation geometric operator $\mathcal{D}_\mathcal{F}$ acts on the smooth sections $C^\infty(\mathbf{E})$ of a foliated Hermitian bundle. Lemma 2.1 of Chernoff [12] implies that $\mathcal{D}_\mathcal{F}$ and each of its powers is essentially self-adjoint:

PROPOSITION 3.7 *Let $\mathcal{D}_\mathcal{F}\colon C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ be a foliation geometric operator. Then $\mathcal{D}_\mathcal{F}$ has a unique densely-defined extension to a closed operator $\mathcal{D}_\mu$ on $L^2(V, \mathbf{E}, \mu)$.* □

Hence, for any bounded Borel function $\phi\colon \mathbf{R} \to \mathbf{R}$ we can define the bounded operator $\phi(\mathcal{D}_\mu)$ on $L^2(V, \mathbf{E}, \mu)$ using the spectral theorem.

4 Spectrum and recurrence

In this section we prove the results of section 1 relating the spectrum of a leafwise geometric operator to dynamics.

THEOREM 4.1 *Let $\mathcal{D}_{\mathcal{F}}$ be a leafwise geometric operator for a foliated measure space $(V, \mathcal{F})$. Fix L_x and let $L_x \leq L_y$ with $\psi\mathbf{Hol}_{\mathcal{F}}(x)$ uniformly Fölner on L_y. Then $\sigma(\mathcal{D}_{L_x}) \subset \sigma_e(\mathcal{D}_{L_y})$.*

Proof: Let $\lambda \in \sigma(\mathcal{D}_{L_x})$, then by Proposition 3.3, there exists a sequence of unit-norm sections $\{\phi_i \mid i = 1, 2, \ldots\} \subset C_c^\infty(\mathbf{E}_{L_x})$ with $||(\mathcal{D}_{L_x} - \lambda)\phi_i|| < 1/i$. The strategy is to lift these test functions to a sequence of compact regions in L_y so they are approximate eigensections for $\mathcal{D}_{L_y}$.

Let $K_i = spt(\phi_i)$ which we can assume is a covered by a plaque-chain

$$\mathcal{P}_i = \{P_{\alpha(i,1)}, \ldots, P_{\alpha(i,N(i))}\}$$

In particular, every plaque in $\mathcal{P}_i$ can be connected to $P_{\alpha(i,1)}$ by a chain of length at most N_i.

The leafwise operators $\mathcal{D}_L$ vary uniformly continuously on the compact space V, so for $\epsilon_i > 0$ sufficiently small, the variation of $\mathcal{D}_{L_z}$ restricted to $L_z \cap \mathcal{N}(K_i, \epsilon)$ becomes uniformly small independent of K_i. To be more exact, for $z \in T_{\alpha_i} = \phi_{\alpha_i}^{-1}(\{0\} \times X_{\alpha_i})$ set $z_{\alpha_i} = \phi_{\alpha_i}(y) \in X_{\alpha_i}$. Let $K(z_{\alpha_i}, N) \subset L_z$ be the union of the plaques of $\mathcal{F}$ which can be joined to $P_{\alpha_i}(z_{\alpha_i})$ by a plaque-chain of length at most N. For N fixed and z_{α_i} sufficiently close to x_{α_i}, projection along transversals defines a covering map from $K(z_{\alpha_i}, N)$ to a region of L_x. In particular, for $z_{\alpha_i} \in X_{\alpha_i}$ close to x_{α_i} the set $K(z_{\alpha_i}, N_i + 1)$ projects to a region in L_x containing K_i. We can thus lift $\mathcal{D}_{L_x}$ to an operator $(\widetilde{\mathcal{D}_{L_x}})_{z_{\alpha_i}}$ on the region $K(z_{\alpha_i}, N_i + 1)$ and compare coefficients with the operator $\mathcal{D}_{L_z}$. We choose $\epsilon_i > 0$ sufficiently small so that for $z_{\alpha_i} \in B(x_{\alpha_i}, \epsilon_i) \subset X_{\alpha_i}$

$$\sup_{w \in K(z_{\alpha_i}, N_i+1)} ||D_{L_z} - (\widetilde{\mathcal{D}_{L_x}})_{z_{\alpha_i}}||_z < 1/i$$

Following the notation of Definition 2.6, choose a finite subset $\mathcal{W}_i \subset B(x_{\alpha_i}, \epsilon_i) \cap \psi\mathbf{Hol}_{\mathcal{F}}(x) \cdot y_{\alpha_i}$ such that

$$\#\{P_{N_i}\mathcal{W}_i\} \leq (1 + 1/i) \cdot \#\mathcal{W}_i \tag{12}$$

Define $K(\mathcal{W}_i, N) \subset L_y$ to be the union of all plaques which can be joined to one of the plaques $\{P_{\alpha_i}(w) \mid w \in \mathcal{W}_i\}$ by a plaque-chain of length at most N.

LEMMA 4.2 *There exists $C_2 > 0$ so that for all i, there is a smooth function $g_i: L_y \to [0, 1]$ so that*

1. $g_i(z) = 1$ *for all* $z \in K(\mathcal{W}_i, N_i)$
2. $g_i(z) = 0$ *for all* $z \in L_y - K(\mathcal{W}_i, N_i + 1)$
3. $|\nabla_{\mathcal{F}} g_i| \leq C_2$

Proof: Recall that $\{\lambda_\alpha \mid \alpha = 1, \ldots, k\}$ is a partition-of-unity for the foliation covering of V. Let

$$C_2 = k \cdot \sup_{z \in V} |\nabla_{\mathcal{F}} \lambda_\alpha|_z$$

be the supremum of the gradients along leaves for the partition functions. Conditions (4.2.1) and (4.2.2) define g_i everywhere except on L_y for $z \in K(\mathcal{W}_i, N_i + 1) - K(\mathcal{W}_i, N_i)$, where we set

$$g_i(z) = \sum_{\beta \in \mathcal{B}_i(z)} \lambda_\beta(z)$$

with $\mathcal{B}_i(z) = \{\beta \mid P_\beta(z_\beta) \subset K(\mathcal{W}_i, N_i + 1)\}$. □

Let ψ_i be the lift to $K(\mathcal{W}_i, N_i + 1)$ of the test section ϕ_i, and set $\xi_i = g_i \cdot \psi_i \in C_c^\infty(\mathbf{E}_{L_y})$. Calculate the pointwise norms, for $z \in L_y$ using Lemma 3.5

$$||(\mathcal{D}_{L_y} - \lambda)\xi_i||_z \leq C_1 \cdot |\nabla_{\mathcal{F}} g_i|_z \cdot ||\psi_i||_z + g_i(z) \cdot ||(D_{L_y} - (\widetilde{\mathcal{D}_{L_x}})_{z_{\alpha_i}} \psi_i||_z + g_i(z) \cdot ||((\widetilde{\mathcal{D}_{L_x} - \lambda})\phi_i)_{z_{\alpha_i}}||_z \quad (13)$$

Integrate this estimate over $K(\mathcal{W}_i, N_i + 1)$ and use (4.2.1), (4.2.3) and (12) to obtain

$$\begin{aligned} ||(\mathcal{D}_{L_y} - \lambda)\xi_i||_2 &\leq C_1 C_2 \cdot ||\psi_i|(K(\mathcal{W}_i, N_i + 1) - K(\mathcal{W}_i, N_i))||_2 \\ &\quad + 1/i \cdot ||\psi_i|K(\mathcal{W}_i, N_i + 1)||_2 + 1/i \cdot \#\mathcal{W}_i && (14) \\ &\leq C_1 C_2 \cdot \#\mathcal{W}_i/i + 1/i \cdot \#\mathcal{W}_i + 1/i \cdot \#\mathcal{W}_i && (15) \\ &\leq (C_1 C_2 + 2)/i \cdot \#\mathcal{W}_i && (16) \end{aligned}$$

On the other hand, $K(\mathcal{W}_i, N_i)$ covers the region $K_i \subset L_x$ with multiplicity at least $\#\mathcal{W}_i$ so we have the estimate

$$||\xi_i||_2 \geq ||\psi_i|K(\mathcal{W}_i, N_i)||_2 \geq \#\mathcal{W}_i \cdot ||\phi_i||_2 = \#\mathcal{W}_i$$

Define $\Phi_i = \dfrac{\xi_i}{||\xi_i||_2}$ and we have shown that

$$||(\mathcal{D}_{L_y} - \lambda)\Phi_i||_2 \leq (C_1 C_2 + 2)/i$$

which tends to 0 as $i \to \infty$. Thus, $\lambda \in \sigma(\mathcal{D}_{L_y})$.

To establish that λ is a point of infinite multiplicity, note that for each i, the test function Φ_i constructed has compact support in the set $K(\mathcal{W}_i, N_i + 1) \subset L_y$. When making the choice of the next set $\mathcal{W}_{i+1}$ we can specify that the open neighborhood $B(x_{\alpha_{i+1}}, \epsilon_{i+1})$ be chosen sufficiently small so that its $N_i + 1$ saturation is disjoint from the supports of the previously constructed test functions $\{\Phi_1, \ldots \Phi_i\}$. This yields an infinite set $\{\Phi_1, \ldots \Phi_i, \ldots\}$ of test functions with disjoint supports, hence $\lambda \in \sigma_e(\mathcal{D}_{L_y})$ by Proposition 3.3. □

5 Spectral coincidence for completely amenable foliations

Throughout this section we assume $\mathcal{F}$ admits a holonomy-invariant transverse Radon measure μ. We prove the two spectral coincidence results of the introduction, relating the leafwise spectra $\sigma(\mathcal{D}_L)$ for a geometric operator $\mathcal{D}_{\mathcal{F}}$ to the spectrum of its closure $\mathcal{D}_\mu$ in $L^2(V, \mathbf{E}, \mu)$.

THEOREM 5.1 *Let $x \in spt(\mu)$ with $\psi\mathbf{Hol}_{\mathcal{F}}(x)$* uniformly Fölner*. Then $\sigma(\mathcal{D}_{L_x}) \subset \sigma_e(\mathcal{D}_\mu)$.*

Proof: We use the elementary observation

LEMMA 5.2 *Let μ_α be a Radon measure on X_α. Then $z \in spt(\mu_\alpha)$ if and only if, for every open set $z \in U \subset X_\alpha$ we have $\mu_\alpha(U) > 0$.* □

Let $x \in spt(\mu) \cap T_\alpha$ for some foliation chart (U_α, ϕ_α) with corresponding point $x_\alpha \in X_\alpha$. Then for every $\delta > 0$ we have $\mu_\alpha(B(x_\alpha, \delta)) > 0$.

Let $\lambda \in \sigma(L_x)$. For each $i > 0$ choose a unit-norm $\phi_i \in C_c^\infty(\mathbf{E}_L)$ for which $||(\mathcal{D}_L - \lambda)\phi_i|| < 1/i$. The leaves L_y of $\mathcal{F}|spt(\mu)$ with trivial holonomy group $\mathbf{Hol}_{\mathcal{F}}(y)|spt(\mu) = \{e\}$ form a dense G^δ subset of $spt(\mu)$ [16], so by the same method of proof as for Theorem 4.1, we deduce that for all $\epsilon > 0$, there exists a leaf $L_\epsilon \subset spt(\mu)$ without holonomy, and test section $\Phi_\epsilon \in C_c^\infty(\mathbf{E}_{L_\epsilon})$ with $||(\mathcal{D}_{L_\epsilon} - \lambda)\Phi_\epsilon|| < \epsilon$.

Every saturated neighborhood in $\mathcal{F}|spt(\mu)$ of the support K_ϵ of Φ_ϵ has positive measuse. So we can extend the section

There is a converse to Theorem 1.4, but requires that $\mathcal{F}$ be *hyperfinite* with respect to μ (cf. [41, 13] and section 2 below).

THEOREM 5.3 *Suppose that $\mathcal{F}$ is μ-hyperfinite. Then*

$$\sigma(\mathcal{D}_{\mathcal{F}}) \subset \bigcup_{L \subset spt(\mu)} \sigma(\mathcal{D}_L) \tag{17}$$

6 Examples

foliated manifold, invariant measure and pass to support of measure, group actions, constructions using probability spaces

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