
MR979832 (90g:58128)
Douglas, R. G. (1-SUNYS); **Hurder, S.** (1-ILCC); **Kaminker, J.** (1-INPI)
Eta invariants and von Neumann algebras.
Bull. Amer. Math. Soc. (N.S.) **21** (1989), *no. 1,* 83–87.
58G12 (19K56 46L80)

Let M be a compact, oriented, odd-dimensional Riemannian manifold. Let V be a complex vector bundle over M which is equipped both with a flat unitary connexion ∇ and with a trivialisation θ. The incompatibility between these two data is measured by a class $[V, \nabla, \theta]$ in $K^1(M; \mathbf{R})$. On the other hand, an operator D of Dirac type on M defines a K-homology class $[D] \in K_1(M)$; so one expects to be able to pair $[D]$ with $[V, \nabla, \theta]$ and obtain a real number. Such a pairing can be carried out either analytically (as a relative η-invariant) or topologically; the relative index theorem of M. F. Atiyah et al. [Math. Proc. Cambridge Philos. Soc. **79** (1976), no. 1, 71–99; MR **53** #1655c] asserts that these two definitions agree.

In the above-cited paper this result was proved via a study of boundary value problems, but it was suggested that an alternative proof via von Neumann algebras should be possible. It is this idea which has been taken up in the article under review. By comparing the holonomy of the flat connexion ∇ with the trivialisation θ, one obtains an "almost-periodic" unitary-valued function u on $\widetilde{M}$, the universal cover of M. The authors use u as a multiplier to define a "type II" Toeplitz operator on $\widetilde{M}$, which has a real-valued index; their key analytical result is that this real-valued index is equal to the relative η-invariant. They then relate this real-valued index to the odd case of A. Connes's foliation index theorem [in *Operator algebras and applications, Part I* (Kingston, ON, 1982), 521–628, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, RI, 1982; MR 84m:58140], where the foliation in question is that given by the flat connexion on the principal frame bundle of V.

The details of the authors' argument have been published for the simplest case [in *Index theory of elliptic operators, foliations, and operator algebras* (New Orleans, LA/Indianapolis, IN, 1986), 11–41, Contemp. Math., 70, Amer. Math. Soc., Providence, RI, 1988; MR 89j:58132], but the details of the general case are scheduled to appear elsewhere.

Reviewed by *John Roe*