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Eta Invariants and the Odd Index Theorem for Coverings

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1. Introduction. In this paper we develop the "odd" analog of the Atiyah-Singer Index Theorem for coverings [1, 50]. The odd index theorem for a compact manifold gives a topological formula for the integer index of a Toeplitz operator constructed from the compression of a unitary multiplier to the positive space of an elliptic self-adjoint psuedo-differential operator [8, 40]. Equivalently, this integer is a spectral flow invariant [6, 52]. When this construction is performed on an infinite covering the elliptic operator is lifted from the base, but there is a wider range of choices for the multiplier than exist on the compact base. The Toeplitz index or spectral flow invariant is calculated using a trace on an appropriate von Neumann algebra associated to the multiplier. If the multiplier is lifted from the base, then this index is an integer and we have the obvious direct generalization of the usual index theorem for coverings. However, if the multiplier is not a lift, then real-valued indices occur. We give several applications of the odd coverings index theorem, including the realization of the relative eta-invariant [3] as a type II-index, as announced in [28] and first proved in detail in [31].

Let M be a compact, oriented manifold without boundary of dimension m, with fundamental group $\pi_1(M)$. Consider a Galois covering

$$\Gamma \to \widetilde{M} \xrightarrow{\pi} M \tag{1.1}$$

with a surjection $\pi_1(M) \to \Gamma$ such that composing with the *right* action of Γ on $\widetilde{M}$ yields the "deck" action of $\pi_1(M)$ on $\widetilde{M}$. A subset M_0 of $\widetilde{M}$ is a *fundamental domain* if

(1.2a) M_0 is compact and path connected with connected interior $\mathring{M}_0$.

(1.2b) The boundary $\partial M_0 \equiv M_0 - \mathring{M}_0$ has Lebesgue measure zero.

(1.2c) M_0 maps onto M, and $\mathring{M}_0$ maps one-to-one into M.

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We choose and fix a fundamental domain M_0. For each $\gamma \in \Gamma$, let M_γ denote the right translate $M_0 \cdot \gamma$.

Consider an elliptic, first-order differential operator,

$$D : C^\infty(M, E_0) \to C^\infty(M, E_1) \tag{1.3}$$

acting on the smooth sections of Hermitian vector bundles E_0 and E_1. Let $\mathcal{H}_0$ and $\mathcal{H}_1$ denote the corresponding Hilbert space completions of these smooth section spaces. Let D also denote the unique closure of D to a densely defined unbounded operator from $\mathcal{H}_0$ to $\mathcal{H}_1$, and D^* will denote its adjoint. When $E_0 = E_1$ then D is *(essentially) self-adjoint* if $D = D^*$. The results of this paper are all formulated for D as above. They can also be proven for D pseudo-differential with distributional kernel supported in an ϵ-tube around the diagonal in $M \times M$, where ϵ is less than the injectively radius of M, but the necessary techniques are clumsy and so for clarity will be avoided. (The necessary methods are developed in sections 3 and 5 of [31].)

The operator D determines an even K-homology class, $[D] \in K_0^a(M)$. There is a pairing between K-homology and ordinary K-theory,

$$\mathrm{Ind} : K^0(M) \times K_0^a(M) \to \mathbf{Z}. \tag{1.4}$$

Given a complex vector bundle, $\xi \to M$, we equip it with an Hermitian connection ∇_ξ, then the pairing is defined as the analytic index of the extended operator

$$D \otimes \nabla^\xi : C^\infty(M, E_0 \otimes \xi) \to C^\infty(M, E_1 \otimes \xi) \tag{1.5}$$

denoted by $\mathrm{Ind}^a(D \otimes \nabla^\xi)$. This, of course, is the motivation for, and simplest example of, the external product in Kasparov's KK-theory (cf. [36], [40]).

Let us return to the problem of index theorems for coverings. We set $\widetilde{E}_i = \pi^! E_i$, $i = 0, 1$ and then lift the operator D to a differential operator $\widetilde{D}$ acting on *compactly supported* smooth sections

$$\widetilde{D} : C_c^\infty(\widetilde{M}, \widetilde{E}_0) \to C_c^\infty(\widetilde{M}, \widetilde{E}_1), \tag{1.6}$$

with formal adjoint $\widetilde{D}^*$. A standard, basic fact is that $\widetilde{D}$ is uniquely closable on the corresponding Hilbert space closures of these section spaces (cf. [1]), and we let $\widetilde{D}, \widetilde{D}^*$ also denote the closures. The key observation of Atiyah and Singer was that the projection operators onto $\ker(\widetilde{D})$ and $\ker(\widetilde{D}^*)$ are represented by *smooth* kernels $\tilde{p}_0(y_0, y_1)$ and $\tilde{p}_1(y_0, y_1)$ respectively on $\widetilde{M} \times \widetilde{M}$. Moreover, these kernels are invariant under the deck action

$$\tilde{p}_i(y_0 \cdot \gamma, y_1 \cdot \gamma) = \tilde{p}_i(y_0, y_1); i = 0, 1; \gamma \in \Gamma \tag{1.7}$$

and *their restrictions to the diagonal satisfy*

$$\tilde{p}_i(y, y) = p_i(\pi(y), \pi(y)) \tag{1.8}$$

where p_i is the smooth kernel on $M \times M$ representing the projection onto $\ker(D)$ and $\ker(D^*)$ as $i = 0$ or 1. Thus, if Tr denotes the fiberwise trace on $\mathrm{End}(\widetilde{E}_i)$ and we set

$$\mathrm{Ind}_\Gamma(\widetilde{D}) = \int_{M_0} \mathrm{Tr}(\tilde{p}_0(y, y)) - \mathrm{Tr}(\tilde{p}_1(y, y)) dy \tag{1.9}$$

then we have obtained part of the index theorem for coverings:

THEOREM (ATIYAH [1], SINGER [50]). *With respect to the Γ-module structure on $\widetilde{\mathcal{H}}_0$, $\widetilde{\mathcal{H}}_1$ the operator $\widetilde{D}$ is Breuer Fredholm, and its Γ-index is given by $Ind_\Gamma(\widetilde{D})$. Moreover,*

$$Ind_\Gamma(\widetilde{D}) = Ind^a(D) \tag{1.10}$$

$$= (-1)^m \int_M \psi^{-1}(ch(\sigma_D)) \cup Td(M) \tag{1.11}$$

where ψ is the Thom isomorphism. □

The justification for the claim that $\mathrm{Ind}_\Gamma(\widetilde{D})$ is the Breuer index requires some delicate analysis. A recent proof of this via the methods that are the basis for this work is given in (Chapter 13, [47]). Of course, the second line (1.11) is the usual Atiyah-Singer Index Theorem.

The primary application of the index theorem for coverings was to construct non-trivial, locally finite subspaces of $\widetilde{\mathcal{H}}_0$ or $\widetilde{\mathcal{H}}_1$ on which Γ represents. For Γ a compact lattice and D a generalized Dirac operator, Atiyah and Schmid [4] realized the discrete series representations of Γ in this way on spaces of harmonic spinors. A far-reaching generalization of the coverings index theorem (again starting from the observations (1.7) and (1.8)) to pairings with cyclic cocycles over the group algebra of Γ has been given by Connes and Moscovici, which has deep topological applications to the Novikov Conjecture [22, 25, 26].

We conclude this introduction to the "even" index theorem for coverings with a formulation of it via Kasparov's bivariant KK-theory. The principal symbol of the lifted operator $\widetilde{D}$ is elliptic and Γ-invariant, so determines a class denoted

$$[D]_\Gamma \in KK(C(M)),\ C_r^*(\Gamma)) \tag{1.12}$$

where $C_r^*(\Gamma)$ is the reduced C^*-algebra of Γ (cf. [7, 32]). The map to a point, $M \to \mathrm{pt}$, induces a class $i^* \in KK(\mathbf{C}, C(M))$ and the exterior product with i^* defines a map, μ, for which there is a commutative diagram.

$$\begin{array}{ccc} KK(C(M), C_r^*(\Gamma)) & \leftarrow & KK(C(M), \mathbf{C}) \\ \downarrow \mu & \swarrow \mu_\Gamma & \downarrow \mu \\ KK(\mathbf{C}, C_r^*(\Gamma)) & \leftarrow & KK(\mathbf{C}, \mathbf{C}) \\ \downarrow \mathrm{tr}_\Gamma & & \cong\downarrow \\ \mathbf{R} & \supset & \mathbf{Z} \end{array} \tag{1.13}$$

The composition $\mu_\Gamma([D])$ is the C^*-index of the Γ-equivariant operator $\widetilde{D}$ and lies in the domain of the natural trace, tr_Γ, on $C_r^*(\Gamma)$. The coverings index theorem implies that

$$\mathrm{Ind}_\Gamma(\widetilde{D}) = \mathrm{tr}_\Gamma(\mu_\Gamma([D])) \tag{1.14}$$

and commutativity of (1.13) implies that this is an integer. Let us now see how this formulation of the coverings index theorem motivates the "odd" version of the theorem.

Let $E = E_0 = E_1$ and D be self-adjoint. Then the analytic index of D vanishes. The fundamental observation of Baum-Douglas [8,9] and Kasaparov [40] is that in spite of this, D still carries index information in the form of a pairing corresponding to (1.4) where now $[D] \in K_1^a(M)$:

$$\text{Ind} : K^1(M) \boxtimes K_1^a(M) \to \mathbf{Z}. \tag{1.15}$$

The Toeplitz index interpretation of (1.15) associates to $[D]$ the projections $P^{\pm}$ onto the positive, respectively non-negative, spectral subspaces of D. Then for $u \in K^1(M)$ represented by a unitary $\varphi : M \to U(N) \subset GL(N.\mathbf{C})$ we form the multiplier operator on $\mathcal{H}^N$, again denoted by φ. The operator

$$T_D(\varphi) = P^+ \circ \varphi \circ P^+ - P^- \tag{1.16}$$

is Fredholm on $\mathcal{H}^N$ and (1.15) assigns to u and $[D]$ the integer $\text{Ind}^a(T_D(\varphi))$. This is the approach developed by Baum and Douglas (cf. §20, [8] and [9].) The spectral flow interpretation of (1.15) observes that the involutions

$$\mathcal{E}_0 = P^+ - P^- \text{ and } \mathcal{E}_1 = \varphi \circ (P^+ - P^-) \circ \varphi^*$$

differ by a compact operator (as P^+ is pseudo-differential with symbol of order 0 on M that commutes with the multiplier, φ). Thus, there is an integer invariant, the essential codimension, $EC(\mathcal{E}_1, \mathcal{E}_0)$, which measures the spectral flow (cf. [6]) of a family of operators $\mathcal{E}_t$ interpolating between $\mathcal{E}_0$ and $\mathcal{E}_1$. Wojciechowski [52] proves that

$$EC(\mathcal{E}_1, \mathcal{E}_0) = \text{Ind}^a(T_D(\varphi)) \tag{1.17}$$

so that $EC(\mathcal{E}_1, \mathcal{E}_0)$ also evaluates the pairing (1.15).

With the odd index theorem for coverings, the pairing (1.15) is the crucial aspect to be understood. Let us consider first the simplest case of how to generalize this pairing. Just as the elliptic operator D from $\mathcal{H}_0$ to $\mathcal{H}_1$ determined an even class $[D]_\Gamma$ in (1.12), a self-adjoint operator determines an *odd* class

$$[D]_\Gamma \in KK^1(C(M),\ C_r^*(\Gamma)). \tag{1.18}$$

A unitary $u \in K^1(M)$ determines a bivariant class $[u] \in KK^1(\mathbf{C}, C(M))$, and we can replace the index map $\mu \equiv i^* \boxtimes \_\_\_$ with the pairing map $\mu_* \equiv [u] \boxtimes \_\_\_$. The latter changes the parity of the KK-group, so that upon composing we obtain a map

$$[u]_\Gamma : KK^1(C(M), \mathbf{C}) \to KK(\mathbf{C}, C_\Gamma^*)). \tag{1.19}$$

A straightforward calculation of the exterior product with $[u]$ then yields our first line of the following extension of the coverings index theorem to self-adjoint operators:

THEOREM 1. *For D self-adjoint geometric operator on M,*

$$\mathrm{tr}_\Gamma([u]_\Gamma([D])) = \mathrm{tr}_\Gamma(\mu_\Gamma([T_D(\varphi)])) \tag{1.20}$$
$$= \mathrm{Ind}^a(T_D(\varphi)). \tag{1.21}$$

The content of Theorem 1 is that it calculates the analytic index of the Toeplitz operator $T_{\widetilde{D}}(\varphi)$ on $\widetilde{M}$, formed by lifting φ to a multiplier on $\widetilde{M}$ then compressing to the positive eigenspace of $\widetilde{D}$, in terms of data on M. The spectrum of $\widetilde{D}$ need not be discrete (in fact, it could be all of $\mathbf{R}$), so that we obtain from $\mathrm{Ind}(T_D(\varphi)) \neq 0$ the non-triviality of a continuous spectral flow for $\widetilde{D}$ acting on L^2-sections of $\widetilde{E} \to \widetilde{M}$. This is elaborated on in §2 below. The proof of Theorem 1 is given in §5.

Our interest in the odd index theorem lies deeper, though. We will formulate in section 2 the pairing (1.15) directly on $\widetilde{M}$ in terms of the spectrum of $\widetilde{D}$ and unitary multipliers on $\widetilde{M}$. The classes $u \in K^1(M)$ correspond to the Γ-periodic multipliers obtained by lifting from M to $\widetilde{M}$. It is of great interest to study other classes of unitaries on $\widetilde{M}$ for which the generalized Toeplitz compressions have an "index", with the constraint that it is possible to construct a continuous dimension function that measures the real-valued index of this compression. We discuss in detail this extention for the Γ-almost periodic multipliers which arise from finite-dimensional unitary representations, as described in sections 4 and 5. The more general construction of section 5 corresponds to "compactifications" of Γ, and allows consideration of multipliers with uniform recurrence under the Γ-action (denoted Γ-uniform in section 5), and multipliers which are Γ-random, corresponding to parameter spaces X which are Borel measure spaces. These latter cases are only briefly discussed.

The basic theme is that associated to each unitary multiplier, φ, is a ***hull completion*** X_φ with continuous Γ-action. From this we obtain a C^*-algebra, A_φ, which is constructed from an associated foliated space, $(V, \mathcal{F}_\varphi)$. The algebra A_φ is always stably isomorphic to the cross-product $C(X_\varphi) \rtimes \Gamma$. If φ is Γ-***amenable***, then there is a trace, tr_{A_φ}, on A_φ which defines a continuous dimension function on $KK(\mathbf{C}, A_\varphi)$. The multiplier φ extends to Φ on the foliated space V, which induces a map in KK-theory

$$[\Phi] : KK^1(C(V), A_\varphi) \to KK(\mathbf{C}, A_\varphi) \tag{1.22}$$

that is the "hull-extension" of $[u]_\Gamma$. Composing $[\Phi]$ with tr_{A_φ} yields the generalized index map

$$\mathrm{tr}_{A_\varphi} \circ [\Phi] : KK^1(C(V), A_\varphi) \to \mathbf{R}. \tag{1.23}$$

The operator D lifts to a leafwise operator on $\mathcal{F}_\varphi$, and in anology to the construction of $[D]_\Gamma$, there is a class $[D_\mathcal{F}] \in KK^1(C(V), A_\varphi)$. The real number $\mathrm{tr}_{A_\varphi}([\Phi]([D_\mathcal{F}]))$ then represents the generalized von Neumann index of the

Toeplitz operator $T_{\widetilde{D}}(\varphi)$. (In the case where we allow Φ to be a "random multiplier", we forego the above KK-formalism and directly construct the random Toeplitz operators $T_\epsilon(\varphi_x)$ for $x \in X$ some measure space. The index class is a difference of projections in a von Neumann algebra, W_φ, with the index obtained by applying a trace on this algebra (cf. [17]).

In either the C^*-completion or W^*-completion case, the abstract index $\mathrm{tr}_{A_\varphi}([\Phi]([D_{\mathcal{F}}]))$ has a topological formula derived from the measured foliation index theorem of Connes [18, 19].

Our passing from the Γ-index for coverings to a foliation index is parallel to the use of the foliation index theorem by Connes and Moscovici in their announcement [24], in place of the coverings index theorem used by Atiyah and Schmid [4]. On a technical level, for X_φ compact there is an inclusion $C^*_r(\Gamma) \to A_\varphi$, and the trace tr_{A_φ}, on A_φ restricts to the usual trace on $C^*_r(\Gamma)$. The coverings index theorem applies to operators which are in the commutant of the representatiom of $C^*_r(\Gamma)$ on $\widetilde{\mathcal{H}}$ (cf. [50]). This allows for precisely the Γ-invariant Toeplitz operators. By extending the trace, and representing on a field of Hilbert spaces associated to the foliation, we enlarge the commutant to include our class of "amenable" multipliers and their compressions.

The abstract odd index theory developed in sections 2 through 5 below is applied in section 6 to obtain a relation between eta invariants and a continuous spectral flow on $\widetilde{M}$. Fix the data: D is a self-adjoint first-order elliptic operator on M, and $\alpha : \Gamma \to U(N)$ is an injective representation. We form the associated Hermitian flat $\mathbf{C}^N$-bundle, $E_\alpha \to M$, with Hermitian connection ∇^α. Assume also the existence of a trivialization $\Theta : E_\alpha \cong M \times \mathbf{C}^N$; then the product bundle $M \times \mathbf{C}^N$ has both a horizontal flat connection, ∇^0, and an Hermitian flat connection $\nabla^{\overline{\alpha}} = \Theta_*(\nabla^\alpha)$. Define symmetric extensions $D_0 = D \otimes \nabla^0$ and $D_1 = D \otimes \nabla^{\overline{\alpha}}$ acting on smooth sections of $E^N = E \oplus \ldots \oplus E$. The family

$$\{D_t = (1-t)D_0 + tD_1 \mid 0 \le t \le 1\}$$

will consist of elliptic self-adjoint operators to which we can associate a 1-parameter family of eta-onvariants, and following [3] we define the relative eta invariant

$$\langle [D], [\overline{\alpha}] \rangle_\eta \equiv \eta(D, \alpha, \Theta) \tag{1.24}$$

$$= \int_0^1 \dot{\eta}(D_t)dt. \tag{1.25}$$

The equality (1.24) is to indicate the heuristic viewpoint put forward by Atiyah, Patodi and Singer that $\eta(D, \alpha, \Theta)$ is a real-valued index obtained by pairing D with the "real" K-theory class of $[\overline{\alpha}] = (\alpha, \Theta) \in K^1(M) \otimes \mathbf{R}$ (cf. §7, [3]). In fact, in this geometric context it is possible to replace the right-side column of (1.13) with the above "eta-cup product" (1.24) with $[\overline{\alpha}]$ to obtain a map

$$\begin{array}{c} K^1(C(M), \mathbf{C}) \\ \downarrow [\overline{\alpha}]_\eta \\ \mathbf{R} \end{array} \tag{1.26}$$

We will show that there exists a corresponding φ and algebra A_φ as discussed previously so that the new diagram commutes:

THEOREM 2. *Let M be an odd-dimensional closed manifold and D a self-adjoint geometric operator on M. Then*

$$\mathrm{tr}_{A(\varphi)}([\Phi]([D_{\mathcal{F}}])) = \langle [D], [\overline{\alpha}] \rangle_{\eta}. \tag{1.27}$$

The content of Theorem 2 is to interpret the eta-invariant coupling of Atiyah, Patodi and Singer as a Breuer index in a naturally associated von Neumann context. The left-side of (1.27) was denoted by $\langle [D], [\overline{\alpha}] \rangle_{\Gamma}$ in the announcement [28]. We will give a new proof of (1.27) using methods distinct from those of [29, 31] where the first proofs were given.

The results of this paper grew out of the author's attempt to understand the main theorem of [28] in a context independent of cyclic cohomology. It is a pleasure to thank R. Douglas and J. Kaminker for many conversations which have had relevance to this work, and also M. Ramachandran for his helpful comments. The contents of this paper are an expanded version of the author's talk at the Joint Summer Research Conference of the NSF on "Geometric and Topological Invariants of Elliptic Operators, Bowdoin, July 1988." The author is grateful to the organizers for the opportunity to present this work.

§2. Toeplitz Operators for Coverings.

In this section we introduce the most general class of smooth Toeplitz operators on the covering $\widetilde{M}$ associated to a Γ-periodic elliptic first-order differential operator $\widetilde{D}$. The "index" of these operators will be defined via a boundary map in algebaric K-theory. Our development of the odd index theory on $\widetilde{M}$ exactly parallels J. Roe's discussion of even index theory on open manifolds in (§4, [44]). We postpone to later sections the discussion of two fundamental problems: first, to describe the possible closures of these Toeplitz algebras; and second, the related question of when there exists an appropriate trace with which to assign a real-valued dimension to the abstract indices. The first problem is the odd analog of the theme of Roe [45], with some simplification possible due to the periodicity of D. In section 3 below, we give one answer to the second problem using invariant means on Γ, which parallels (§6, [44]) in our context. However, a principal theme of this paper is that other solutions are possible by considering the "trace" to be define only on a subalgebra of the Toeplitz algebra generated by operators whose symbols come from some "geometric completion" of Γ.

The theory of (smooth) Toeplitz operators on the real line with symbols in the classes we consider here have been thoroughly studied, first by Curto, Muhly and Xia [27], Ji and Xia [39] and Ji and Kaminker [38]. A technical theme developed in Douglas, Hurder and Kaminker [30] and employed essentially in this section, is that Fourier transform on the line is replaced in higher dimensions by the use of the wave operator associated to $\widetilde{D}$. The use of this technique in index theory was pioneered by Roe in [44, 45,46].

We fix a covering $\widetilde{M} \to M$ with Galois group Γ and the lift $\widetilde{D}$ acting on $C_c^\infty(\widetilde{M}, \widetilde{E})$. A choice of Riemannian metric on TM lifts to a metric of bounded geometry on $\widetilde{M}$ (cf. §2, [44]). Introduce the Frechet-uniform algebra of smooth complex-valued functions,

$$\mathcal{A} \equiv C^{\infty}_{\text{unif}}(\mathcal{M})$$

which is characterized by the property that given $f \in \mathcal{A}$, for any contractable open set $U \subset M$ and single covering sheet $\widetilde{U} \subset \widetilde{M}$, the restriction $f \mid \widetilde{U}$ defines a function on U with uniform C^∞-estimates for the Frechet norm on M. Let $M(N, \mathcal{A})$ denote the algebra of $N \times N$ matrices with entries from $\mathcal{A}$.

Let P^+ denote the projection onto the positive eigenspaces of the closure $\widetilde{D}$ acting in $\widetilde{\mathcal{H}} = L^2(\widetilde{M}, \widetilde{E})$. As $\widetilde{D}$ need not have (any) discrete spectrum, the more precise definition of P^+ is to first introduce the characteristic function

$$h_0(x) = \chi_{(0,\infty)}(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases} \tag{2.1}$$

and use the functional calculus to define $P^+ = h_0(\widetilde{D})$. Then set $P^- = \mathrm{Id} - P^+$. We also need approximating functions

$$h_\epsilon(x) = \begin{cases} 1 & x \ge \epsilon \\ 0 & x \le \epsilon \end{cases} \tag{2.2}$$

defined by $h_\epsilon(x) = h_1(x/\epsilon)$ where we fix h_1 smooth with $h_1' \ge 0$ and $h_1'(0) = 0$. Set $P_\epsilon = h_\epsilon(\widetilde{D})$ for $\epsilon > 0$, and $P_0 = P^+$. For each $N \ge 1$, by abuse of notation we let P_ϵ also denote the diagonal extension to the direct sum

$$\widetilde{\mathcal{H}}^N = \widetilde{\mathcal{H}} \widehat{\oplus} \dots \widehat{\oplus} \widetilde{\mathcal{H}} \equiv L^2(\widetilde{M}, \widetilde{E} \otimes \mathbf{C}^N). \tag{2.3}$$

The last ingredient we need for the construction of the Toeplitz extension is the theory of uniform operators on $\widetilde{M}$ from Roe (§5, [44]). For $k \in \mathbf{Z}$, an operator

$$Q : C^\infty_c(\widetilde{M}, \widetilde{E} \otimes \mathbf{C}^N) \to C^\infty(\widetilde{M}, \widetilde{E} \otimes \mathbf{C}^N) \tag{2.4}$$

is *uniform of order* $\le k$ if for each $r \in \mathbf{R}$, it has a continuous extension to a quasi-local operator on the Sobolev closures,

$$Q_r : W^r(\widetilde{E} \otimes \mathbf{C}^N) \to W^{r-k}(\widetilde{E} \otimes \mathbf{C}^N). \tag{2.5}$$

By abuse of notation, we let $\mathcal{U}_k$ denote the collection of uniform operators of order $\le k$, which for $k \le 0$ form an algebra. We recall the characteristic property of $Q \in \mathcal{U}_k$.

PROPOSITION (5.4, [44]). *Let $Q \in \mathcal{U}_k$, then Q is represented by a distributional kernel $k_Q(x,y)$ on $\widetilde{M} \times \widetilde{M}$:*

$$Qu(x) = \int_{\widetilde{M}} k_Q(x,y) u(y) dy. \tag{2.6}$$

Moreover, k_Q is smooth off the diagonal in $\widetilde{M} \times \widetilde{M}$ and for $k < -m$ there is a function $\nu = \nu(r)$ tending to zero as $r \to \infty$ such that for $r > 0$,

$$\int_{\widetilde{M} - B(x,r)} \{k_Q(x,y)^2 + k_Q(y,x)^2\} dy \le \nu(r). \tag{2.7}$$ □

In the above, $B(x,r)$ is the metric ball of radius r in $\widetilde{M}$ about x. A key technical tool for index theory is Roe's formulation of kernel estimates due to Cheeger, Gromov and Taylor [54] for operators $f(\widetilde{D})$ defined by the functional calculus. Introduce the space $S^m(\mathbf{R})$ of *symbols* of order $\leq m$ on $\mathbf{R}$ defined as the subspace of $C^\infty(\mathbf{R})$ consisting of functions f with estimates

$$|f^{(\ell)}(x)| \leq C_\ell \cdot (1+|x|)^{m-\ell}. \tag{2.8}$$

The best constants, C_ℓ, for (2.8) define semi-norms on $S^m(\mathbf{R})$, so that this is a Frechet space.

THEOREM (5.5, [44]). *Let D be a geometric operator on M and $f \in S^m(\mathbf{R})$. Then the operator $f(\widetilde{D}) \in \mathcal{U}_m$.* □

We now construct the smooth Toeplitz extension of $\mathcal{A}$ by $\mathcal{U}_{-1}$. Fix $\epsilon > 0$ and let $\mathcal{T}^\infty_\epsilon$ be the algebra of finite sums and products of elements from $\mathcal{U}_{-1}$ and the space of operators

$$\{P_\epsilon \cdot \varphi \cdot P_\epsilon \mid \varphi \in \mathcal{A}\}.$$

There are also matrix versions of $\mathcal{T}^\infty_\epsilon$ for $\varphi \in M(N,\mathcal{A})$. For simplicity of exposition, we discuss the scalar case, and leave to the reader the elementary extensions to the matrix case.

LEMMA 2.1. *For $\varphi \in \mathcal{A}$, the commutator $[P_\epsilon, \varphi] = P_\epsilon\varphi - \varphi P_\epsilon \in \mathcal{U}_{-1}$.*

PROOF: P_ϵ is a pseudo-differential operator of order 0 on $\mathcal{M}$ whose principal symbol commutes with φ, so that $[P_\epsilon,\varphi]$ is a pseudo-differential operator of order -1 and induces maps $S^r(\widetilde{E}) \to S^{r-1}(\widetilde{E})$. We need to show that it is a uniform operator. Introduce the function $g_\epsilon(x) = h_\epsilon(x)/x$ and define $Q_\epsilon = g_\epsilon(\widetilde{D}) \in \mathcal{U}_{-1}$ by the above result of Roe. Calculate

$$\begin{aligned} P_\epsilon\varphi - \varphi P_\epsilon &= DQ_\epsilon\varphi - \varphi DQ_\epsilon \\ &= D[Q_\epsilon,\varphi] + [D,\varphi]Q_\epsilon. \end{aligned} \tag{2.9}$$

The assumption that φ is uniform is easily seen to imply $[Q_\epsilon,\varphi] \in \mathcal{U}_{-2}$ and $[D,\varphi] \in \mathcal{U}_{-1}$. By (Proposition 5.2 [44]) the products $D \cdot [Q_\epsilon,\varphi]$ and $[D,\varphi]Q_\epsilon$ lie in $\mathcal{U}_{-1}$. □

LEMMA 2.2. *The algebra $\mathcal{T}^\infty_\epsilon$ is independent of the choice of $\epsilon > 0$.*

PROOF: Let $\epsilon, \delta > 0$, then the difference $(h_\epsilon - h_\delta) \in S^{-\infty}$, so that

$$(P_\epsilon - P_\delta) \in \mathcal{U}_{-\infty} \subset \mathcal{U}_{-1}.$$ □

PROPOSITION 2.3. *There exists an exact sequence of uniform operators*

$$0 \to \mathcal{U}_{-1} \to \mathcal{T}^\infty \xrightarrow{\sigma} \mathcal{A} \to 0. \tag{2.10}$$

PROOF: Let $\mathcal{T}^\infty = \mathcal{T}^\infty_\epsilon$ for $\epsilon > 0$, and note that $h_\epsilon \in S^0$ so that $\mathcal{T}^\infty \subset \mathcal{U}_0$. The linear map

$$\begin{aligned} T_\epsilon &: \mathcal{A} \to \mathcal{T}^\infty \\ &: \varphi \to T_\epsilon(\varphi) = P_\epsilon\varphi P_\epsilon \end{aligned} \tag{2.11}$$

descends to an algebra map

$$\widehat{T} : \mathcal{A} \to \mathcal{T}^\infty / \mathcal{U}_{-1} \tag{2.12}$$

by Lemma 2.1. The map $\widehat{T}$ is clearly onto, and independent of ϵ as $\mathcal{U}_{-1}$ is stable under conjugation by elements of $\mathcal{A}$. So it suffices to show that $\widehat{T}$ is monic and set $\sigma = \widehat{T}^{-1}$. This is a corollary of the following, which we leave to the reader as an exercise. We say that an operator B on $\widetilde{\mathcal{H}}$ is *locally compact* if for all compactly supported functions $f_0, f_1 \in C_c(\widetilde{M})$ the composition $f_0 B f_1$ is a compact operator. Note that each element of $\mathcal{U}_{-1}$ is locally compact.

LEMMA 2.4. *For $\varphi \in \mathcal{A}$ non-zero, the smooth Toeplitz operator $T_\epsilon(\varphi)$ is not locally compact.* □

We call (2.10) the *smooth Toeplitz extension* on $\widetilde{M}$ determined by $\widetilde{D}$. Our notation corresponds to both the smoothness of the symbol algebra $\mathcal{A}$ and the property that the quasi-projector P_ϵ is a uniform operator.

There are several variations on the construction of Toeplitz extensions. First, the algebra $\mathcal{A}$ can be replaced by an algebra of uniformly continuous (matrix-valued) functions on $\widetilde{M}$. However, our Lemma 2.1 fails for φ only continuous, which neccesitates replacing $\mathcal{U}_{-1}$ with a larger algebra. For $\mathcal{M}$ compact, it is customary to replace $\mathcal{U}_{-1}$ with its $*$-closure the Banach algebra of compact operators. For $\widetilde{M}$ identified with the leaf of a foliation, by restricting the symbols to continuous functions which extend continuously to the ambient foliated (compact) manifold, then $\mathcal{U}_{-1}$ can be replaced with the C^*-algebra of the foliation. This is the approach taken in previous works on Toeplitz extensions on open manifolds (cf. [27,30,38,39].) One goal of our approach is to develop the Toeplitz index for the full $\mathcal{A}$, then to study its "completion" as a process associated with defining a numerical index of the Toeplitz operators.

The second alternative Toeplitz extension is formed by replacing P_ϵ with the projection P_0 in $\widetilde{\mathcal{H}}$. Define $\mathcal{T}_0^\infty$ to be the algebra generated by finite sums and products of elements of $\mathcal{U}_{-1}$ and the space of bounded operators

$$\{T_0(\varphi) = P_0 \varphi P_0 \mid \varphi \in \mathcal{A}\}.$$

Introduce the class S_0^{-1} of symbols with a jump discontinuity at the origin. More precisely, for a function f on $\mathbf{R}$ and constant $z \in \mathbf{C}$ set:

$$f_z(x) = \begin{cases} f(x) + z e^{-x^2}, & x > 0 \\ \\ f(x), & x \le 0 \end{cases} \quad , \text{and}$$

$$S_0^{-1} = \{f \mid f_z \in S^{-1} \text{ for some } z \in \mathbf{C}\}. \tag{2.13}$$

Let $\widehat{\mathcal{U}}_{-1}$ be the algebra generated by finite sums and products of elements of $\mathcal{U}_{-1}$ and the space $\{f(\widetilde{D}) \mid f \in S_0^{-1}\}$. Each operator $f(\widetilde{D})$ for $f \in S_0^{-1}$ is locally compact, and as $P_0 = P_\epsilon + f_\epsilon(\widetilde{D})$ where $f_\epsilon = h_0 - h_\epsilon \in S_0^{-1}$, the method of Proof of Proposition 2.4 also yields:

Proposition 2.5. *There is an exact sequence*

$$0 \to \widehat{\mathcal{U}}_{-1} \to \mathcal{T}_0^{\infty} \to \mathcal{A} \to 0 \tag{2.14}$$ □

The algebras $\mathcal{U}_{-1}$ and $\widehat{\mathcal{U}}_{-1}$ are generally distinct, but this depends upon the spectral measure of $\widetilde{D}$ at 0. For example, $\widetilde{M} = \mathbf{R}$ and $\widetilde{D} = -i\frac{d}{dx}$ then the spectral measure is simply $d\xi$, Lebesgue measure on the line, and the Theorem 2.5 of Ji and Xia shows that $K_1(\mathcal{U}_{-1})$ and $K_1(\widehat{\mathcal{U}}_{-1})$ differ (cf. Proposition 3.1, [39]). The operators $f_z(\widetilde{D})$ define bounded maps from $W^r(\widetilde{E} \otimes \mathbf{C}^N)$ to itself by the spectral theorem, but the difficulty is that they need not be uniform operators. The function f_z is not absolutely continuous, hence its Fourier transform is not integrable so that the estimator constructed by Roe (Formula 5.6, [44]) will not tend to zero at infinity. It is possible to impose hypotheses on $\widetilde{D}$ which give $f_z(\widetilde{D})$ is uniform - for example, if the spectral measure of $\widetilde{D}$ is smooth at 0 with vanishing first derivative. Such hypotheses tend to be restrictive on the geometry of $\widetilde{M}$ and operator $\widetilde{D}$ (cf. Chapter 7, [37]).

We next consider the "index" invariants associated to the smooth Toeplitz extension. The algebras $\mathcal{A}$ and $\mathcal{U}_{-1}$ are not Banach closed so that their K-theory need not be periodic. Nonetheless, we investigate only the group $K_1(\mathcal{A})$, as the higher groups $K_n(\mathcal{A})$ require more sophisticated methods of cyclic cohomology (cf. [21,23]). The connecting map in the six-term exact sequence of algebraic K-theory,

$$\partial : K_1(\mathcal{A}) \to K_0(\mathcal{U}_{-1}) \tag{2.15}$$

is called the abstract index map. Following the model of (§4, [44]), we will make this map completely explicit.

Recall that each class in $K_1(\mathcal{A})$ is represented by some $\varphi \in M(N, \mathcal{A})$ such that the pointwise evaluations of φ are invertible and lie in a compact neighborhood of the identity in $GL(N, \mathbf{C})$. For scalar φ, this is the condition that there exists a constant $c > 0$ for which $c > |\varphi| > 1/c$, and the class of φ is equivalent to its unitary part $\varphi/|\varphi|$.

We must add a unit to $\mathcal{T}^{\infty}$, so let $\widetilde{\mathcal{T}}^{\infty}$ denote the algebra spanned by $\mathcal{T}^{\infty}$ and the identity operator, I, on $\widetilde{\mathcal{H}}^N$. Define an operator via the spectral theorem,

$$P_\epsilon^- = h_\epsilon(-\widetilde{D}).$$

Then $I = P_\epsilon + P_\epsilon^- + F_\epsilon$, where $F_\epsilon \in \mathcal{U}_{-\infty} \subset \mathcal{U}_{-1}$. We augment the algebra $\mathcal{U}_{-1}$ by adding P_ϵ^{-1} to obtain $\widetilde{\mathcal{U}}_{-1}$, then there is an exact sequence

$$0 \to \widetilde{\mathcal{U}}_{-1} \to \widetilde{\mathcal{T}}^{\infty} \to \mathcal{A} \to 0. \tag{2.16}$$

We will first construct the index map

$$\widetilde{\partial} : K_1(\mathcal{A}) \to K_0(\widetilde{\mathcal{U}}_{-1}).$$

Fix $\varphi \in GL(N, \mathcal{A})$ representing $[\varphi] \in K_1(\mathcal{A})$. Define operators on $\widetilde{\mathcal{H}}^N$:

$$\begin{aligned} P &= \widetilde{T}_\epsilon(\varphi) = P_\epsilon \varphi P_\epsilon - P_\epsilon^- \\ Q &= \widetilde{T}_\epsilon(\varphi^{-1}) = P_\epsilon \varphi^{-1} P_\epsilon - P_\epsilon^{-1} \\ S &= I - PQ \\ T &= I - QP \end{aligned}$$

We then observe the identities

$$(2.17)\qquad \begin{cases} P_\epsilon P_\epsilon^- &= 0 = P_\epsilon^- P_\epsilon \\ QS &= TQ \\ SP &= PT \end{cases}$$

LEMMA 2.6. $S, T \in \mathcal{U}_{-1}$.

PROOF:

$$(2.18)\qquad \begin{cases} I - PQ &= I - (P_\epsilon \varphi P_\epsilon - P_\epsilon^-)(P_\epsilon \varphi^{-1} P_\epsilon - P_\epsilon^-) \\ &= (I - (P_\epsilon^-)^2) - P_\epsilon \varphi P_\epsilon^2 \varphi^{-1} P_\epsilon \end{cases}$$

which lies in $\mathcal{U}_{-1}$ as the principal symbols of φ and P_ϵ commute, so that both expressions in the last line of (2.18) are congruent to P_ϵ modulo $\mathcal{U}_{-1}$. A similar calculation applies for $T = I - QP$. □

For each integer $\ell > 0$ define

$$(2.19)\qquad \begin{aligned} A_\ell &= (I + S + \ldots + S^{2\ell-1})P - I \\ &= P(I + T + \ldots + T^{2\ell-1}) - I. \end{aligned}$$

PROPOSITION 2.7. *For each integer $\ell > 0$, the class $\widetilde{\partial}[\varphi] \in K_0(\widetilde{\mathcal{U}}_{-1})$ is represented by the difference of projections in $M(2, \widetilde{\mathcal{U}}_{-1})$*

$$(2.20)\qquad \begin{bmatrix} (I + A_\ell)Q & (I + A_\ell)T^\ell \\ T^\ell Q & T^{2\ell} \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

PROOF: For $\ell = 1$ the connecting homomorphism $\widetilde{\partial}$ of algebraic K-theory is explicitly evaluated by Roe (§4. [44]), so we indicate only the necessary changes. Let $R \in GL(2, \widetilde{\mathcal{T}}^\infty)$ be the matrix

$$(2.21)\qquad R = \begin{bmatrix} I + A_\ell & -S^\ell \\ T^\ell & Q \end{bmatrix}$$

with inverse

$$R^{-1} = \begin{bmatrix} Q & T^\ell \\ -S^\ell & I + A_\ell \end{bmatrix}.$$

Note that $R = \text{Ind}$ modulo $\mathcal{U}_{-1}$, then evaluating $\{R\pi R^{-1} - \pi\}$ yields (2.20). □

Note that the difference in (2.20) can be rewritten as

$$(2.22)\qquad \widetilde{\partial}[\varphi] \sim \begin{bmatrix} -S^\ell & * \\ * & T^\ell \end{bmatrix} \in M(2, \mathcal{U}_{-1})$$

using Lemma 2.6, the identities (2.17) and the fact that $P_\epsilon^- \in \mathcal{U}_0$ is a multiplier on $\mathcal{U}_{-1}$. We can therefore identify $\widetilde{\partial}[\varphi]$ with a class $\partial[\varphi] \in K_0(\mathcal{U}_{-1})$ in the kernel of the augmentation-induced map

$$(2.23)\qquad K_0(\mathcal{U}_{-1}) \to K_0(\widetilde{\mathcal{U}}_{-1}) \to K_0(\widetilde{\mathcal{U}}_{-1}/\mathcal{U}_{-1}).$$

A standard diagram chase shows that $\partial[\varphi]$ represents the boundary map of (2.10) applied to $[\varphi]$.

The formula (2.22) for $\partial[\varphi]$ is familiar as the standard expression to which applying a trace yields the index of the "elliptic" operator P (cf. Proposition 6, Appendix 2, [20].) However, the above derivation of (2.22) is purely algebraic without making assumptions on the spectrum of $\widetilde{D}$. We next impose the hypothesis on $\widetilde{D}$ that P_ϵ is a projection, so that $\widetilde{D}$ has a gap in the spectrum between 0 and ϵ. (We could also proceed with the weaker hypothesis that $P_0 \in \mathcal{U}_{-1}$ and $\lim_{\epsilon \to 0} P_\epsilon = P_0$ in the uniform topology; this case is left to the reader.) The formula (2.22) can then be simplified to resemble the more familiar expression involving Hankel operators that is associated with the index of Toeplitz operators.

Let $U(N, \mathcal{A}) \subset GL(N, \mathcal{A})$ be the subgroup of unitary matrices.

Let $\varphi \in U(N, \mathcal{A})$ be unitary-valued with the inverse denoted by φ^*. Set

$$\begin{aligned} T^+(\varphi) &= P_\epsilon \varphi P_\epsilon \\ T^-(\varphi) &= (I - P_\epsilon)\varphi(I - P_\epsilon) \\ H^+(\varphi) &= (I - P_\epsilon)\varphi P_\epsilon \\ H^-(\varphi) &= P_\epsilon \varphi(1 - P_\epsilon). \end{aligned}$$

The identities $\varphi\varphi^* = I = \varphi^*\varphi$ yield corresponding identities for the Toeplitz operators $T^\pm(\varphi)$ and the Hanekl operators $H^\pm(\varphi)$. We need only the following

$$T^+(\varphi)T^+(\varphi^*) + H^-(\varphi)H^+(\varphi^*) = P_\epsilon. \tag{2.24}$$

PROPOSITION 2.8. *Let $\widetilde{D}$ have no spectrum in the interval $(0, \epsilon)$, and assume $[\varphi] \in K_1(\mathcal{A})$ is represented by a unitary-valued $\varphi \in GL(N, \mathcal{A})$. Then*

$$\partial[\varphi] \sim \begin{bmatrix} -(H^-(\varphi)H^+(\varphi^*))^\ell & * \\ * & (H^-(\varphi^*)H^+(\varphi))^\ell \end{bmatrix}. \tag{2.25}$$

PROOF: In the proof of Proposition 2.7, we replace the operator P_ϵ^- with $(I - P_\epsilon)$ in the definition of P and Q. The remainder of the proof proceeds as before, as we need only the property

$$(I - P_\epsilon) \cdot P_\epsilon = P_\epsilon - P_\epsilon^2 = 0 \tag{2.26}$$

corresponding to the first identity (2.17). Then formula (2.18) simplifies so that from (2.24) we obtain

$$S = H^-(\varphi)H^+(\varphi^*).$$

The corresponding formula for T then yields (2.25). □

For P_ϵ a projection, the Hankel operator $H^+(\varphi)$ represents the spectral flow from the range of P_ϵ to the range of $(1 - P_\epsilon)$ induced by multiplication by φ. This spectral flow is heuristically the kernel of $P_\epsilon \varphi P_\epsilon - (I - P_\epsilon)$. Similarly, $H^-(\varphi)$ represents the spectral flow from range P_ϵ to range $(1 - P_\epsilon)$ induced by φ^*, so represents the kernel of the adjoint $P_\epsilon \varphi^* P_\epsilon - (I - P_\epsilon)$. Our next task is to assign "dimensions" to these spectral flows.

§3. Amenable Covers.

In this section we assume that the Galois group Γ of the covering $\widetilde{M} \to M$ is amenable. Following (§6, [44]) we call $\widetilde{M}$ an *amenable covering*. This hypothesis yields a "trace", tr_Γ, defined on the algebra $\mathcal{U}_{-k}$ for $k > m$ so that we define the Γ-Toeplitz index

$$\begin{aligned} \mathrm{Ind}_\Gamma : K_1(\mathcal{A}) &\to \mathbf{R} \\ [\varphi] &\to \mathrm{tr}_\Gamma(\partial[\varphi]). \end{aligned} \tag{3.1}$$

The dimension function defined by tr_Γ is a renormalized dimension $\widetilde{\mathcal{H}}^N$, so that the spectral flow induced by $[\varphi]$ is assigned a renormalized dimension by (3.1) via a process analogous to the procedure for measuring the index of almost periodic operators on $\mathbf{R}^m$ by dividing the index on domains by the volume of the domain. The Γ-trace is the natural generalization of this idea (cf. §6, [44]).

There are three common classes of amenable discrete groups, each more general than the previous, but as the generality increases the known results about their Γ-index theory decreases:

(QA) Quasi-abelian. There is a finite group G and an exact sequence

$$0 \to \mathbf{Z}^m \to \Gamma \to G \to 1.$$

(QN) Quasi-nilpotent. There is a finite group G, nilpotent group N and exact sequence

$$1 \to N \to \Gamma \to G \to 1.$$

(QS) Quasi-solvable. There is a finite group G, solvable group S and exact sequence

$$1 \to S \to \Gamma \to G \to 1.$$

A theme of this paper is that while the full algebra $\mathcal{A}$ of symbols may be very hard to explicitly describe, there are important subalgebras of symbol classes corresponding to the representation theory of the group Γ, and for symbols from these subalgebras the index map (3.1) can be reduced to a more familiar (topological) form. For the class (QA), there are the quasi-periodic functions in $\widetilde{M}$ with additional symmetry group G. For Γ quasi-nilpotent, the Kirillov theory parametrizing the unitary representations provides a large class of "Γ-quasi-periodic" symbols. Similarly, finite unitary representations of Γ-quasi-solvable lead to distinguished symbols in $\mathcal{A}$, but for this case both the representation theory and the Toeplitz index theory are less-well understood.

Recall that by Fölner's theorem, Γ is amenable if and only if Γ admits a Fölner sequence: there is an ascending sequence of finite subsets,

$$\{1\} = \Gamma_0 \subset \Gamma_1 \subset \ldots \subset \Gamma; \quad \bigcup_{n=0}^{\infty} \Gamma_n = \Gamma \tag{3.2}$$

so that for each $\gamma \in \Gamma$ and $\epsilon > 0$, there is $N(\gamma, \epsilon)$ such that $n > N(\gamma, \epsilon)$ implies

$$\mathrm{Card}(\Gamma_n \; \Delta \; \Gamma_n \cdot \gamma) \leq \epsilon \cdot \mathrm{Card}(\Gamma_n) \tag{3.3}$$

where $(D\Delta B) = (A\backslash B) \cup (B\backslash A)$. For example, when $\Gamma = \mathbf{Z}^m$ we take for Γ_n the cube $\{-n, \ldots, n\}^m$. For the classes (QN) and (QS), an inductive procedure starting with the case $\mathbf{Z}^\ell$ yields explicit choices for the Γ_n. (cf. Feldman [33].) We will assume that a choice of $\{\Gamma_n\}$ for Γ has been made.

Define an averaging sequence for $\widetilde{M}$ via the Fölner sequence for Γ by setting

$$M_n = M_0 \cdot \Gamma_n = \bigcup_{\gamma \in \Gamma_n} M_\gamma. \tag{3.4}$$

Each closed set M_n is a union of translates of the fundamental domain whose overlaps have measure zero. Thus $\mathrm{vol}(M_n) = \mathrm{Card}(\Gamma_n) \cdot \mathrm{vol}(M_0)$. For a closed set $A \subset \widetilde{M}$, introduce its r-boundary for $r > 0$:

$$\partial_r(A) = \{y \in \widetilde{M} \mid \mathrm{dist}(y, A) < r, \mathrm{dist}(y, \widetilde{M} - A) < r\}. \tag{3.5}$$

An averaging sequence is *Fölner* if for all $r > 0$,

$$\lim_{n \to \infty} \frac{\mathrm{vol}(\partial_r(M_n))}{\mathrm{vol}(M_n)} = 0. \tag{3.6}$$

PROPOSITION 3.1. *$\{\Gamma_n\}$ is a Fölner sequence for Γ if and only if $\{M_n\}$ is a Fölner averaging sequence for $\widetilde{M}$.*

PROOF: This is a direct consequence of ideas of Plante [42]. Details of the proof are given in Proposition 6.6 of Roe [44]. □

A key idea of Roe's work [44,45] is that a Fölner averaging sequence of $\widetilde{M}$ defines a trace, denoted tr_Γ, on the algebras $\mathcal{U}_{-\ell}$ for $\ell > m$. By a *trace* we mean a linear functional $\tau : \mathcal{U}_{-\ell} \to \mathbf{C}$ such that $\tau(AB) = \tau(BA)$ for all $A, B \in \mathcal{U}_{-\ell}$. A *normal trace* will mean that for suitable topology on $\mathcal{U}_{-\ell}$, if there is a monotone increasing sequence $\{A_1, A_2, \ldots\}$ with limit A_∞ and $\lim \tau(A_n) = a_\infty$, then $\tau(A_\infty)$ exists and equals a_∞. The traces obtained from Fölner averaging sequences are generally not normal on all of $\mathcal{U}_{-\ell}$.

To define the linear functional, tr_Γ, first note that each operator $Q \in \mathcal{U}_{-\ell}$ for $\ell > m$ is represented by a continuous kernel, k_Q, on $\widetilde{M} \times \widetilde{M}$, so that its restriction to the diagonal is well-defined. (cf. Proposition 5.8, [47]). For each n, set

$$m_n(Q) = \mathrm{vol}(M_n)^{-1} \cdot \int_{M_n} \mathrm{Tr}(k_Q(y, y))dy. \tag{3.7}$$

The integral in (3.7) depends only on the C^0-topology on kernels, so we can use the weak-$*$ compactness of the bounded linear functionals on $C^0(\widetilde{M})$ to select a convergent net, then set

$$\mathrm{tr}_\Gamma(Q) = \lim{}^* \{m_n(Q)\}. \tag{3.8}$$

THEOREM (THEOREM 6.7, [44]). *The linear functional tr_Γ defines a trace of $\mathcal{U}_{-\ell}$ for $\ell > m$.* □

We remark that a key technical idea behind the proof of this theorem is the *uniform* decay of the kernels in $\mathcal{U}_{-\ell}$, including the L^2-estimate (2.7).

Note that our notation tr_Γ is imprecise, as the weak-$*$ limit process can (and will for $\widetilde{M}$ not compact) produce an uncountable number of inequivalent means, with the axiom of choice used to select one of these. Each such choice yields a distinct "continuous dimension" function on projections in $K_0(\mathcal{U}_{-\ell})$. However, it is often the case that $\mathcal{U}_{-\ell}$ admits preferred subalgebras on which all choices for tr_Γ agree. For example, the kernels on $\mathbf{R}^m$ whose diagonal restriction is almost-periodic have this property. For more general groups Γ we can define Γ-almost-periodic functions on $\widetilde{M}$ with similar properties. Thus, we settle on the notation tr_Γ in recognition that for most practical examples, tr_Γ is determined by Γ.

The tracial property of tr_Γ implies that it defines a linear map $\mathrm{tr}_\Gamma : K_0(\mathcal{U}_{-\ell}) \to \mathbf{R}$ for $\ell > m$. Combining this remark with Roe's Theorem we obtain

Proposition 3.2. *For $\epsilon > 0$, $\ell > m$ and unitary $\varphi \in U(N, \mathcal{A})$, there is a well-defined index, independent of ϵ and ℓ,*

$$\mathrm{Ind}_\Gamma(T_\epsilon(\varphi)) = \mathrm{tr}_\Gamma(I - QP)^\ell - \mathrm{tr}_\Gamma(I - PQ)^\ell. \tag{3.9}$$ □

The Toeplitz operator $T_0(\varphi) = P_0\varphi P_0$ is the strong operator limit of the operators $T_\epsilon(\varphi) = P_\epsilon \varphi P_\epsilon$, and it is natural to ask whether the value of (3.9) is equal to that for $\epsilon = 0$, corresponding to the classical case. However, if tr_Γ is not normal then we cannot show this for all symbols in $\mathcal{A}$. The difficulty is the same as that encountered in Roe [45]. In section 5 we discuss additional regularity hypotheses on the symbol φ which will ensure that $\epsilon = 0$ case yields the same Toeplitz index.

§4. $K_1(\mathcal{A})$ and Representation Theory.

In this section we consider the symbols $\varphi \in GL(N, \mathcal{A})$ arising from the finite-dimensional unitary representations of Γ. We introduce the concept of a *homotopy trivial* representation, then parametrize the resulting symbols by the *gauge group* of maps from M to $U(N)$.

Let $\alpha : \Gamma \to U(N)$ be a representation. A smooth function $\varphi : \widetilde{M} \to U(N)$ is *α-related* if

$$\varphi(x \cdot \gamma^{-1}) = \alpha(\gamma) \cdot \varphi(x); \; x \in \widetilde{M}, \; \gamma \in \Gamma. \tag{4.1}$$

Let $\mathcal{R}_\alpha$ denote the set of α-related maps. Γ acts via isometries on $\widetilde{M}$, so that $\mathcal{R}_\alpha \subset GL(N, \mathcal{A})$. Let us obtain an alternative characterization.

The representation α determines a flat $U(N)$-bundle over M,

$$P_\alpha = \widetilde{M} \times U(N)/(x, g) \sim (x\dot{\gamma}^{-1}, \alpha(\gamma) \cdot g). \tag{4.2}$$

Denote by $C^\infty(M, P_\alpha)$ the set of smooth sections of $\pi : P_\alpha \to M$.

Proposition 4.1. $\mathcal{R}_\alpha \cong C^\infty(M, P_\alpha)$.

Proof: A section $s : M \to P_\alpha$ is equivalent to a Γ-equivariant map

$$\begin{aligned} \tilde{s} : & \widetilde{M} \to \widetilde{M} \times U(N) \\ & \tilde{x} \to (\tilde{x}, \varphi(\tilde{x})) \end{aligned}$$

where

$$\begin{aligned}\tilde{s}(\tilde{x}\gamma^{-1}) &= (\tilde{x}\cdot\gamma^{-1}, \varphi(\tilde{x}\cdot\gamma^{-1})) \\ &= (\tilde{x}\cdot\gamma^{-1}, \alpha(\gamma)\cdot\varphi(\tilde{x}))\end{aligned}$$

yields that φ is α-related. Conversely, given $\varphi \in \mathcal{R}_\alpha$ we define a smooth section s by specifying $\tilde{s}(\tilde{x}) = (\tilde{x}, \varphi(\tilde{x}))$ then observe that (4.1) implies that $\tilde{s}$ is Γ-equivariant. □

We will say that α is *homotopically trivial* if the bundle $P_\alpha \to M$ admits a continuous section. As a continuous map always admits smooth approximations, this is equivalent to the set $\mathcal{R}_\alpha$ being non-empty. Moreover, we can use Proposition 4.1 to completely characterize the set $\mathcal{R}_\alpha$ when it is non-empty.

Given $s_0 \in C^\infty(M, P_\alpha)$ and a smooth function $g : M \to U(N)$, we use the right $U(N)$-action on P_α to obtain $s_0 \cdot g \in C^\infty(M, P_\alpha)$, where

$$\widetilde{s_0 \cdot g}(\tilde{x}) = (\tilde{x}, s_0(\tilde{x}) \cdot g(\pi(\tilde{x}))). \tag{4.3}$$

Conversely, given two sections $s_0, s_1 \in C^\infty(M, P_\alpha)$ for each $x \in M$ there is a unique $g(x) \in U(N)$ such that $s_0(x) \cdot g(x) = s_1(x)$. The resulting function $g : M \to U(N)$ is smooth and satisfies $s_1 = s_0 \cdot g$. Define the gauge group

$$\mathcal{G}(N) = C^\infty(M, U(N)) \tag{4.4}$$

then we have shown

PROPOSITION 4.2. *For α homotopically trivial, choose $s_0 \in C^\infty(M, P_\alpha)$. Then there is topological equivalence*

$$\mathcal{R}_\alpha \cong s_0 \cdot \mathcal{G}(N) \equiv \{s_0 \cdot g \mid g \in \mathcal{G}(N)\}. \tag{4.5}$$ □

COROLLARY 4.3. *Each $\varphi \in \mathcal{R}_\alpha$ determines a homomorphism*

$$\varphi^\# \cdot K^1(M) \to K_1(\mathcal{A}). \tag{4.6}$$

PROOF: Identify $K^1(M) \cong K_1(C^\infty(M))$, then for $g : M \to U(N)$ let $\varphi^\#([g])$ be the class of $\varphi \cdot \tilde{g} \in U(N, \mathcal{A})$. This is well-defined since the inclusion $\pi^* : C^\infty(M) \to \mathcal{A}$ is continuous and the product $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is continuous. □

For Γ amenable we can compose the map (4.6) with the Γ-index map, (3.9):

$$\mathrm{Ind}_\Gamma \circ \varphi^\# \cdot K^1(M) \to \mathbf{R}. \tag{4.7}$$

This index map generalizes the periodic symbol index map $\mathrm{tr}_\Gamma \circ [u]_\Gamma$ of (1.19) and (1.20). For α the trivial representation and $\varphi \equiv e$ the constant map onto the identity in $U(N)$, we have for $u = [\varphi]$

$$\mathrm{Ind}_\Gamma \circ e^\#(u) = \mathrm{tr}_\Gamma \circ [u]_\Gamma([1]). \tag{4.8}$$

Unlike the index formula for (1.21), the topological formula for (4.7) derived in section 5 will involve the characteristic classes of the flat bundle foliation on P_α. Equivalently, we must incorporate the Cheeger-Chern-Simons classes of P_α with section s_0 into the index formula.

Let us conclude this section with a discussion of the example $\Gamma = \mathbf{Z}^m$ and $\widetilde{M} = \mathbf{R}^m$, so that $M = \mathbf{T}^m$. A representation $\alpha : \Gamma \to U(N)$ is equivalent to specifying:

(4.9) a compact toral subgroup $\mathbf{T}^\ell \subset U(N)$.

(4.10) a choice of m points $\{\tilde{x}_1, \ldots, \tilde{x}_m\}$ in the cover $\mathbf{R}^\ell \to \mathbf{T}^\ell$ whose images $\{x_1, \ldots, x_m\}$ in $\mathbf{T}^\ell$ generate a dense subgroup.

From this characterization, it is clear that each representation α can be continuously deformed to the trivial representation. Thus, the bundle P_α is always topologically a product, so α is homotopically trivial. Given the data (4.9) and (4.10), there is a canonical (homotopy class of a) section s_0 of P_α, determining $\varphi_0 \in \mathcal{R}_\alpha$. Let α_t be the representation determined by the data (4.9) and the new points $\{t \cdot x_1, \ldots, t \cdot x_m\}$ in (4.10), so that $\alpha = \alpha_1$ and α_0 is trivial. The family of representations for $t \in I \equiv [0,1]$ defines a (U,N)-bundle $\pi : \widetilde{P}_\alpha \to M \times I$. Its restriction to $M \times \{0\}$ is explicitly a product $M \times U(N) \times \{0\}$, with section $(x,0) \to (x,e,0)$. Extend the flat partial connection on $\widetilde{P}_\alpha$ defined by the representations α_t to a full connection on $M \times I$, so that parallel transparent in $\widetilde{P}_\alpha$ along the I-coordinate is defined. Then we define $s_0(x)$ to be the transport from $t = 0$ to $t = 1$ of $(x,e,0) \in \pi^{-1}(x,0)$ to $s_0(x) \in \pi^{-1}(x,1)$. The resulting section s_0 is smooth, and depends up to homotopy only on the choice of points $\{x_0, \ldots, x_m\}$ in (4.10). When applied to the simplest case $\Gamma = \mathbf{Z}$ and $N = 1$, this procedure defines a product structure on $P_\alpha \cong \mathbf{T}^2$ for which the flat foliation is by lines of slope $= x_1$ (cf. examples of [29]).

The integral Chern character

$$\text{(4.11)} \qquad \mathrm{ch}^* : K^1(\mathbf{T}^m) \to \bigoplus_{i \text{ odd}} H^i(\mathbf{T}^m; \mathbf{Z})$$

is injective onto a subgroup of finite index. Thus, for $N > m/2$ and for each odd degree cohomology class z on $\mathbf{T}^m$ in the image of ch^*, there corresponds a map $g_z \in \mathcal{G}(N)$. The lifted map, $\varphi_0 \cdot \tilde{g}_x \in \mathcal{R}_\alpha$, then has the properties that it is α-related, and its restriction to each fundamental domain M_γ carries the cohomological structure of z.

The manifold $\mathbf{T}^m$ is $\mathrm{Spin}_{\mathbf{C}}$, so that the Dirac operator D acting on the spinors over $\mathbf{T}^m$ is a geneator of the K-homology group $K_1^a(\mathbf{T}^m)$. This means that for index theory on $\mathbf{T}^m$, the operators $D_\xi \equiv D \otimes \nabla^\xi$, for $\xi \to \mathbf{T}^m$ an Hermitian vector bundle, form a complete set of representatives. Let $\widetilde{D}_\xi$ denote the lift to $\mathbf{R}^m$, and form the corresponding Toeplitz index map

$$\text{(4.12)} \qquad \mathrm{Ind}_{\Gamma,\xi} : K_1(\mathcal{A}) \to \mathbf{R}.$$

This map depends only on the K-theory class of ξ, so composing with a morphism $\varphi^\#$ we obtain

PROPOSITION 4.4. *For $\Gamma = \mathbf{Z}^m$ and α specified by (4.9) and (4.10), there is a well-defined non-degenerate pairing*

$$\text{(4.13)} \qquad \mathrm{Ind}_\Gamma \circ \varphi_0^\# : K^0(\mathbf{T}^m) \times K^1(\mathbf{T}^m) \to \mathbf{R}. \qquad \square$$

The non-degeneracy of (4.13) follows from the topological formula for index. Note that when α is trivial, (4.13) is the K-theory Poincaré Duality pairing of the $\text{Spin}_{\mathbf{C}}$-oriented manifolds $\mathbf{T}^m$. (cf. [9]). Thus, (4.13) represents Poincaré Duality in "α-twisted K-theory."

The spectrum of the lifted operators D_ξ is of band type; that is, a countable union of closed intervals (cf. Chapter 7, [37]). When the origin lies in a gap in the spectrum, we can apply the analysis at the end of section 3 along with non-degeneracy of (4.13) to conclude there are Toeplitz operators $T_0(\varphi)$ on $\mathbf{R}^m$ with non-trivial renormalized spectral flows.

§5. Hull Closures and the Foliation Toeplitz Index Theorem.

In this section we discuss the most general class of unitary multipliers for which we can define a continuous dimension function on their corresponding Toeplitz index class. We begin with the hull-completion construction - this is analogous to the technique used in the study of almost periodic operators to construct the Bohr space on which the operator admits a continuous extension. We then consider a generalization of the hull-completion in which the Γ-orbits need not be dense. For Γ amenable, or for an appropriate hypothesis on the Γ-action on the hull otherwise, there exists a trace $\mathrm{tr}_{\mathfrak{m}}$ on the index classes arising from the symbols defined over the hull. These Toeplitz operators are part of the theory of foliation Toeplitz operators, which we relate to the Toeplitz extension defined in section 2. The measured foliation index theorem yields a topological formula for the Toeplitz indices with respect to $\mathrm{tr}_{\mathfrak{m}}$. To illustrate our final results, we consider two special cases. The first case of periodic operators yields Theorem 1 of section 1. The second case, for α-related multipliers, introduces Cheeger-Simons characters into the Toeplitz index.

The C^1-hull-closure, X_φ, of a symbol $\varphi \in U(N,\mathcal{A})$ is a topological space with a continuous left Γ-action. We first define the set of translates

$$\Gamma \cdot \varphi = \{(\gamma \cdot \varphi)(y) = \varphi(y \cdot \gamma) \mid \gamma \in \Gamma\}. \tag{5.1}$$

The uniform C^1-norm on $U(N.\mathcal{A})$ is given by

$$||\varphi||_1 = \sup_{y \in \widetilde{M}} ||\varphi(y)|| + \sup_{y \in \widetilde{M}} ||\nabla \varphi \mid_y ||. \tag{5.2}$$

This induces a topology on $\Gamma \cdot \varphi$, and we let $X_\varphi = \overline{\Gamma \cdot \varphi}$ denote the sequential closure of $\Gamma \cdot \varphi$.

Lemma 5.1. *There is a canonical continuous Γ-action on X_φ.*

Proof: Each $\gamma \in \Gamma$ acts uniformly continuously on $\mathcal{A}$ in the uniform C^1-norm. Given $x = \{\gamma_n \cdot \varphi\} \in X_\varphi$, the sequence $\{\gamma \cdot \gamma_n \cdot \varphi\}$ will again be convergent. We set

$$\gamma \cdot \{\gamma_n \cdot \varphi\} = \{\gamma \cdot \gamma_n \cdot \varphi\}.$$

This action is unformly continuous on the dense set $\Gamma \cdot \varphi$, so its continuous extension to X_φ is continuous. □

For each $\gamma \cdot \varphi \in \Gamma \cdot \varphi$, let $\widehat{\gamma\varphi} \in X_\varphi$ denote the stationary sequence it determines. The diagonal action of Γ on $\widetilde{M} \times X_\varphi$ is defined by $\gamma \cdot (y, x) = (y \cdot \gamma^{-1}, \gamma \cdot x)$.

LEMMA 5.2. *There is a continuous map* $\widetilde{\Phi} : \widetilde{M} \times X_\varphi \to U(N)$ *such that*

a) $\widetilde{\Phi}(\gamma(y,x)) = \widetilde{\Phi}(y,x)$, *so descends to a continuous map*

$$\Phi : V \equiv \Gamma\backslash(\widetilde{M} \times X_\varphi) \to U(N).$$

b) *For each* $x \in X_\varphi$, *the restriction*

$$\varphi_x \equiv \widetilde{\Phi}_x : \widetilde{M} \times \{x\} \to U(N)$$

is uniformly C^1.

c) *For each* $\gamma \cdot \varphi \in \Gamma \cdot \varphi$, *the restriction*

$$\widetilde{\Phi} : \widetilde{M} \times \{\widehat{\gamma \cdot \varphi}\} \to U(N)$$

is equal to $\gamma \cdot \varphi$.

PROOF: Given $y \in \widetilde{M}$ and $x = \{\gamma_n \cdot \varphi\}$, set

$$\widetilde{\Phi}(y,x) = \lim_{n\to\infty} \varphi(y \cdot \gamma_n). \tag{5.3}$$

This converges C^1-uniformly, so that for x fixed, $\widetilde{\Phi}(y,x)$ is uniformly C^1 in y. Moreover, the uniform C^1-norm of φ_x depends continuously on x.

The stationary sequence $\gamma \cdot \varphi$ is defined by taking $\gamma_n = \gamma$ for all n, so c) follows from (5.3) directly.

Finally, to prove b) we calculate

$$\begin{aligned}
\widetilde{\Phi}(\gamma \cdot (y,x)) &= \widetilde{\Phi}(y \cdot \gamma^{-1}, \gamma x) \\
&= \lim_{n\to\infty} \varphi((y \cdot \gamma^{-1}) \cdot (\gamma \cdot \gamma_n)) \\
&= \lim_{n\to\infty} \varphi(y \cdot \gamma_n) \\
&= \widetilde{\Phi}(y,x).
\end{aligned}$$

□

We use the C^1-norm to define X_φ due to b) above. To obtain that the commutator $[P_\epsilon, \varphi] \in \mathcal{U}_{-1}$ in section 2 we needed φ to be uniformly C^1. Thus, b) guarantees a similar result for each multiplier φ_x, $x \in X_\varphi$. The definition of X_φ is obviously modeled on the hull-completion technique used in the theory of almost periodic operators cf. [10,51]. However, in that case, no commutator estimates are needed, so that the uniform C^0-topology for closing $\Gamma \cdot \varphi$ suffices. It is easy to construct multipliers φ on $\mathbf{R}^m$ for which the C^0 and C^1-closures differ, so this is an essential point.

In analogy with the almost periodic case on $\mathbf{R}^n$, we make the following definitions for $\varphi \in U(N, \mathcal{A})$:

φ is *γ-uniform* if X_φ admits a Borel probability measure, $\mathfrak{m}$, which is Γ-invariant.

φ is Γ-*normal* if φ is Γ-uniform and X_φ is compact.

φ is Γ-*almost periodic* if φ is Γ-normal, and $\mathfrak{m}$ is the *unique* Γ-invariant Borel probability measure on X_φ.

For example, if φ is $\mathbf{Z}^m$-normal then a standard lemma of Fourier series theory implies that φ is Γ-almost periodic, as X_φ is a compact abelian group with $\mathbf{Z}^m$ as a dense subgroup. If Γ is not abelian, then there are examples which are normal but not almost periodic.

For Γ amenable, every continuous action on a compact space admits an invariant probability measure $\mathfrak{m}$. Thus, if X_φ is compact then φ is Γ-normal.

In Lemma 5.2 we introduced the quotient space $V = \Gamma\backslash(\widetilde{M} \times X_\varphi)$. Note that V has a foliation, $\mathcal{F}_\varphi$, whose leaves are the images of the level sets $\{\widetilde{M} \times \{x\} \mid x \in X_\varphi\}$. In general, V is only a topological manifold, but the leaves have a natural C^∞-manifold structure. The multiplier φ on $\widetilde{M}$ induces a global function $\Phi : V \to U(N)$, whose restrictions to leaves in C^1, uniformly in V.

Let us consider a special case of this construction, for $\varphi \in \mathcal{R}_\alpha$ where $\alpha : \Gamma \to U(N)$. First, observe

$$\Gamma \cdot \varphi = \{\alpha(\gamma) \cdot \varphi \mid \gamma \in \Gamma\}.$$

Multiplication by $U(N)$ on $U(N, \mathcal{A})$ is C^1-uniformly continuous, so we can identify

$$X_\varphi = \overline{\Gamma \cdot \varphi} = \mathrm{Hull}(\alpha) \cdot \varphi \tag{5.4}$$

where $\mathrm{Hull}(\alpha)$ denotes the closure of the subgroup $\alpha(\Gamma) \subset U(N)$. Clearly, Γ acts on X_φ via multiplication on $\mathrm{Hull}(\alpha)$ on the left with dense orbits. By the compactness of $U(N)$, hence of $\mathrm{Hull}(\alpha)$, and uniqueness of Haar measure on compact Lie groups we conclude that φ is Γ-almost periodic. If α is faithful, then the Γ-action on X_φ will be fixed-point free. Finally, identify $X_\varphi = \mathrm{Hull}(\alpha) \subset U(N)$, then $\widetilde{\Phi}$ is given by

$$\widetilde{\Phi} : \widetilde{M} \times \mathrm{Hull}(\alpha) \to U(N)$$
$$\widetilde{\Phi}(y, g) = g \cdot \varphi(y).$$

The construction of the foliated space V above has an extension that yields the most general class of symbols for which a trace function will be defined. Let X denote a topological space and $\mathfrak{m}$ a Borel probability measure. Denote by $\mathrm{Homeo}(X, \mathfrak{m})$ the subsets of the homeomorphisms of X that preserve the measure $\mathfrak{m}$. Given a group homomorphism

$$\alpha : \Gamma \to \mathrm{Homeo}(X, \mathfrak{m}),$$

we associate a topological manifold

$$V = \widetilde{M} \times X/(y, x) \sim (y \cdot \gamma^{-1}, \alpha(\gamma) \cdot x)$$

with foliation $\mathcal{F}_\alpha$ whose leaves L_x are the image of level sets $\{\widetilde{M} \times \{x\} \mid x \in X\}$. Again, the leaves of $\mathcal{F}_\alpha$ have a natural smooth manifold structure. Each leaf of $\mathcal{F}_\alpha$ is a covering of M via the fibration map $\pi : V \to M = \Gamma\backslash\widetilde{M}$.

There are several variations on the above definition. We can restrict attention to X a compact smooth manifold with a volume form ω of total mass 1, then let $\alpha : \Gamma \to \mathrm{Diffeo}(X, \omega)$ represents into ω-preserving diffeomorphisms. The quotient V will be a smooth compact foliated manifold. At the other extreme, we can take X to be a Borel measure space with probability measure $\mathfrak{m}$, and let $\mathrm{Aut}(X, \mathfrak{m})$ be the group of Borel isomorphisms of X that preserve $\mathfrak{m}$. Then V is a foliated measure space in the broadest sense of [41].

Define $\mathcal{C}(V/\mathcal{F}_\alpha)$ to be the set of uniformly continuous maps $\varphi : V \to \mathbf{C}$ whose restrictions to leaves of $\mathcal{F}_\alpha$ are C^1 uniformly in V. Given φ, for each $x \in X$ we thus obtain a uniformly C^1-map

$$\varphi_x : \widetilde{M} = \widetilde{M} \times \{x\} \to \Gamma \backslash (\widetilde{M} \times X) \to \mathbf{C}. \tag{5.5}$$

(In the case X is a Borel measure space, we consider the functions $\mathcal{M}(V/\mathcal{F}_\alpha)$ which are Borel on V and whose restrictions to leaves are uniformly C^1, with Borel dependence on the parameter x.)

We fix the differential operator D on $C^\infty(M, E)$, then use the covering property of the leaves of $\mathcal{F}_\alpha$ to lift it to a leafwise operator $D_{\mathcal{F}}$ for $\mathcal{F}_\alpha$ acting on sections (over the holonomy covers of leaves) of $\widetilde{E} = \pi^* E$. (cf. §3, [31]). These operators are *leafwise* essentially self-adjoint, so that we can leafwise apply the spectral theorem to obtain a family of operators,

$$\begin{cases} P_\epsilon & = \{P_{\epsilon,x} : \widetilde{\mathcal{H}}_x^N \to \widetilde{\mathcal{H}}_x^N \mid x \in X\} \\ \widetilde{\mathcal{H}}_x^N & = L^2(\widetilde{M}_x, \widetilde{E} \mid_{\widetilde{M}_x} \otimes \mathbf{C}^N). \end{cases} \tag{5.6}$$

where $\widetilde{M}_x \to M$ is the covering of M that is also the holonomy cover of the leaf, L_x, through x.

For a symbol $\varphi \in U(N, \mathcal{C}(V/\mathcal{F}_\alpha))$ we construct leafwise Toeplitz operators

$$T_\epsilon(\varphi) = \{T_{\epsilon,x}(\varphi) = P_{\epsilon,x} \cdot \varphi \cdot P_{\epsilon,x} \mid x \in X\}. \tag{5.7}$$

The proof of Lemma 2.1 required only that φ be uniformly C^1, so that it applies leaf-by-leaf in the above context. Introduce the algebra $\mathcal{U}_{-1}(V/\mathcal{F}_\alpha)$ of leafwise operators on $\widetilde{\mathcal{H}}$, so that the representing kernels $k_{Q,x} \in \mathcal{U}_{-1}(\widetilde{M}_x)$ and depend uniformly continuously on x. We then obtain the foliated version of (2.10):

PROPOSITION 5.3. *There exists an exact sequence of uniform foliation operators*

$$0 \to \mathcal{U}_{-1}(V/\mathcal{F}_\alpha) \to \mathcal{T}(V/\mathcal{F}_\alpha) \to \mathcal{C}(V/\mathcal{F}_\alpha) \to 0. \tag{5.8}$$ □

By its construction, the sequence (5.8) maps naturally to the Toeplitz C^*-extension associated to $D_{\mathcal{F}}$. For each $x \in X$, there is a representation of the algebras $\mathcal{U}_{-1}(V/\mathcal{F}_\alpha)$ and $\mathcal{T}(V/\mathcal{F}_\alpha)$ on $\widetilde{\mathcal{H}}_x$ which defines a C^*-semi-norm. Let $C^*(V/\mathcal{F}_\alpha)$ and $\overline{\mathcal{T}}(V/\mathcal{F}_\alpha)$ denote their respective closures with respect to the supremum over X of these semi-norms.

PROPOSITION 5.4. *There is a commuting diagram of exact sequences*

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{U}_{-1}(V/\mathcal{F}_\alpha) & \to & \mathcal{T}(V/\mathcal{F}_\alpha) & \to & \mathcal{C}(V/\mathcal{F}_\alpha) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C^*(V/\mathcal{F}_\alpha) & \to & \overline{\mathcal{T}}(V/\mathcal{F}_\alpha) & \xrightarrow{\sigma} & C^0(V) & \to & 0 \end{array} \tag{5.9}$$

PROOF: Exactness of the bottom row is proved in [30]. □

A Γ-invariant probability measure, $\mathfrak{m}$, on X induces a transverse invariant measure for $\mathcal{F}_\alpha$ (cf. [42]). This in turn, defines a densely defined trace $\mathrm{tr}_{\mathfrak{m}}$ on $C^*(V/\mathcal{F}_\alpha)$ (cf. Chapter 4, [41]), which is given (locally) by restricting a kernel to the leaf diagonals and integrating, then integrating transversally with $\mathfrak{m}$. In particular, the trace is defined on the image of $\mathcal{U}_{-\ell}(V/\mathcal{F}_\alpha)$ for $\ell > m$ which consists of operators represented by continuous kernels.

The K-theory connecting map for the bottom row of (5.9),

$$\partial_{\mathcal{F}} : K_1(C^0(V)) \to K_0(C^*(V/\mathcal{F}_\alpha)) \tag{5.10}$$

has range in the domain of $\mathrm{tr}_{\mathfrak{m}}$, so we obtain an odd analytic index map

$$\mathrm{Ind}_{\mathfrak{m}} = \mathrm{tr}_{\mathfrak{m}} \circ \partial_{\mathcal{F}} : K_1(C^0(V)) \to \mathbf{R}. \tag{5.11}$$

The Toeplitz operator analogue of Connes' measured foliation index theorem [18] yields a topological formula for this index. Let $C_{\mathfrak{m}}$ denote the Ruelle-Sullivan homology class on V associated to $\mathfrak{m}$. The following is proved in [30]:

THEOREM. *For $\varphi : V \to U(N)$,*

$$Ind_{\mathfrak{m}}[\varphi] = (-1)^m \langle \psi^{-1} ch^*(\sigma_D) \cup ch^*(\varphi) \cup Td(\mathcal{F}), C_{\mathfrak{m}} \rangle \tag{5.12}$$

where ψ is the Thom isomorphism for the leafwise unit cotangent bundle $S^1\mathcal{F} \to V$, $ch(\sigma_D)$ is the Chern character for the positive eigenbundle over $S^1\mathcal{F}$ determined by the fiberwise involution σ_D which is the principal symbol of $D_{\mathcal{F}}$, and $Td(\mathcal{F})$ is the leafwise Todd class. □

REMARK 5.5: The index formula (5.12) also holds for symbols $\varphi \in \mathcal{M}(V/\mathcal{F}_\alpha)$ which are essentially bounded on V. This uses the full force of the measured foliation index theorem, as it applies to foliated measure spaces with uniformly bounded leaf geometry. The current $C_{\mathfrak{m}}$ is transverse to $\mathcal{F}$, so all of the differential data in (5.12) is integrated leafwise. Thus, $\mathrm{ch}^*(\varphi)$ need only be a leafwise form for the right-hand-side of (5.12) to be defined. □

We let $\mathcal{C}(\widetilde{M})$ denote the closure of $\mathcal{A}$ in the uniform C^1-topology, and introduce $\mathcal{T}$, the algebra generated by $\mathcal{U}_{-1}$ and operators $P_\epsilon \varphi P_\epsilon$ for $\varphi \in \mathcal{C}(\widetilde{M})$. We again have an exact sequence

$$0 \to \mathcal{U}_{-1} \to \mathcal{T} \to \mathcal{C}(\widetilde{M}) \to 0, \tag{5.13}$$

containing (2.10) as a sub-exact-sequence. We want to define a map from the top row of (5.9) to (5.13). For each $x \in X$ the composition (5.5) defines a restriction map

$$r_x : \mathcal{C}(V/\mathcal{F}_\alpha) \to \mathcal{C}(\widetilde{M}). \tag{5.14}$$

However, to obtain maps between the first two terms of these exact sequences, we need an additional geometric hypothesis.

The action $\alpha : \Gamma \times X \to X$ is *essentially free at* $x_0 \in X$ if for each $e \neq \gamma \in \Gamma$, x_0 is *not* an interior point of the fixed-point set

$$X_\gamma = \{x \mid \alpha(\gamma)x = x\}.$$

We say α is *essentially free* if X_γ has no interior for all $\gamma \in \Gamma$ not the identity.

LEMMA 5.6. *Let X_φ be the hull-completion of φ. If $\hat{\varphi} \in X_\varphi$ is not a fixed-point for all non-identity $\gamma \in \Gamma$, then $\Gamma \times X_\varphi \to X_\varphi$ is essentially free.*

PROOF: Let $x \in U \subset X_\varphi$ and suppose that $U \subset X_\gamma$ is open. Then $x \in \overline{\Gamma \cdot \hat{\varphi}}$ implies some $\delta \cdot \hat{\varphi} \in U$. Thus $\gamma \cdot \delta\hat{\varphi} = \delta\hat{\varphi}$, so $\hat{\varphi}$ is a fixed-point for $\delta^{-1}\gamma\delta \in \Gamma$ which must be the identity, and so $\gamma = e$. □

The following is a standard property of the holonomy cover (cf. Connes [19], Haefliger [35]):

LEMMA 5.7. *If α is locally free at x, then the holonomy cover $\widetilde{M}_x$ of the leaf L_x of $\mathcal{F}_\alpha$ through x is the full Galois cover $\widetilde{M} \to M$. If α is essentially free, then all holonomy covers are equal to $\widetilde{M}$.* □

COROLLARY 5.8. *If α is locally free at $x \in X$, then there is a restriction map of exact sequences:*

$$
(5.15)\qquad
\begin{array}{ccccccccc}
0 & \to & \mathcal{U}_{-1}(V/\mathcal{F}_\alpha) & \to & \mathcal{T}(V/\mathcal{F}_\alpha) & \to & \mathcal{C}(V/\mathcal{F}_\alpha) & \to & 0 \\
 & & \downarrow r_x & & \downarrow r_x & & \downarrow r_x & & \\
0 & \to & \mathcal{U}_{-1} & \to & \mathcal{T} & \to & \mathcal{C}(\widetilde{M}) & \to & 0
\end{array}
$$

PROOF: Each operator on the top row restricts to an operator on $\widetilde{\mathcal{H}}_x$ which is canonically identified to $\widetilde{\mathcal{H}} = L^2(\widetilde{M}, \widetilde{E})$. The restrictions clearly lie in the appropriate algebras. □

For x such that $\widetilde{M}_x$ is not equal to $\widetilde{M}$, it is not possible to define (in general) the diagram (5.15). The difficulty is that the uniform operators do not behave well when lifted to a covering, for both their domains can drastically change and the decay condition away from the diagonal can fail.

We can now define the generalized Toeplitz index of special multipliers. We say that $\varphi \in U(N, \mathcal{A})$ is *amenable* if φ is Γ-uniform, and a choice of invariant measure $\mathfrak{m}$ on X_φ is called a *mean* fo φ. We say that φ is *uniquely amenable* if φ is Γ-almost periodic. Our previous discussion has shown:

LEMMA 5.9. *If φ is α-related for $\alpha : \Gamma \to U(N)$, then φ is uniquely amenable.* □

Continuing the discussion of the introduction, we set for a choice of mean $\mathfrak{m}$,

$$(5.16)\qquad A_\varphi = C^*(V/\mathcal{F}_\varphi); \mathrm{tr}_A = \mathrm{tr}_{\mathfrak{m}}.$$

For an amenable multiplier φ, the generalized index of the Toeplitz operator $T_\epsilon(\varphi) \in \mathcal{T}$ on $\widetilde{M}$ is

$$(5.17)\qquad \mathrm{Ind}_{\mathfrak{m}}([\varphi]) = \mathrm{tr}_{\mathfrak{m}} \partial_{\mathcal{F}}[\Phi]$$

where Φ is the continuous extension of φ to V.

In general, there is no closed procedure for describing the index (5.17) directly in terms of the kernel $\partial[\varphi] \in \mathcal{U}_{-1}$. Let us recapitulate its construction: The set of Γ-translates of $\partial[\varphi]$ form a subset of the image

$$r_x : \mathcal{U}_{-1}(V/\mathcal{F}_\varphi) \to \mathcal{U}_{-1}.$$

The next step is to define the mean in terms of these translates. Unfortunately, for a Γ-uniform multiplier φ it is possible that the orbit of $\hat{\varphi} \in X_\varphi$ is not even contained in the support of $\mathfrak{m}$. As X_φ consists of the limit sequences of translates of φ, $\Gamma \cdot \hat{\varphi}$ disjoint from support $(\mathfrak{m})$ is equivalent to $\mathfrak{m}$ being defined only on the ∞-points in Γ, or equivalently is determined by the behavior of Γ "at infinity". As such, the abstract mean $\mathfrak{m}$ is then impossible to calculate from the restrictions of $\partial[\varphi]$ to compact sets in $\mathcal{M}$.

Suppose that Γ is amenable, and φ is uniquely amenable and Γ acts freely on $\hat{\varphi}$, then we can derive an explicit recipe for $\mathrm{Ind}_{\mathfrak{m}}(T_\epsilon(\varphi))$. Moreover, this formula calculates the index for $\epsilon = 0$. By Lemma 5.6, Γ acting on X_φ is essentially free, so that for $x = \hat{\varphi} \in X_\varphi$ we have the restriction

$$\mathrm{tr}_{\mathfrak{m}} \circ r_x : \mathcal{U}_{-\ell}(V/\mathcal{F}_\varphi) \to \mathcal{U}_\ell \to \mathbf{C}. \tag{5.18}$$

Uniqueness of measure $\mathfrak{m}$ implies the uniqueness of the trace on $\mathcal{U}_{-\ell}(V/\mathcal{F}_\varphi)$, so that $\mathrm{tr}_{\mathfrak{m}} \circ r_x$ must agree (up to a scale factor) with the traces on $U_{-\ell}$ constructed in §3, restricted to this subalgebra. Moreover, by uniqueness the lim*-construction reduces to an ordinary limit on these kernels. Thus we have:

THEOREM 5.10. *Let Γ be amenable and φ a uniquely amenable multiplier on which Γ acts freely. Then for any Fölner sequence $\{\Gamma_n\} \subset \Gamma$,*

$$Ind_{\mathfrak{m}}([\varphi]) = vol(M)^{-1} \lim_{n\to\infty} \frac{1}{\#\Gamma_n} \int_{M_n} Tr(\partial[\varphi]) \cdot dy. \tag{5.19}$$

Moreover, there is a topological formula

$$= vol(M)^{-1} \lim_{n\to\infty} \frac{1}{\#\Gamma_n} \int_{M_n} ch^*(\varphi) \cup \psi^{-1} \cdot ch(\sigma_D) \cup Td(\widetilde{M}) \tag{5.20}$$

where $ch^*(\varphi)$ is the pull-back to $\widetilde{M}$ of the universal Chern form on $U(N)$. □

The identification of (5.20) with the right-hand-side of (5.12) follows from the usual description of the transverse invariant measure for $\mathcal{F}_\varphi$ obtained from $\mathfrak{m}$ as an asymptotic cycle [42], which is the domain of integration M_n viewed as a subset of a typical leaf in V.

Theorem 5.10 is the Toeplitz index analogue of Roe's Index Theorem (8.2, [44]) for open manifolds with mean derived from Γ. Deriving this formula from the foliation index theorem has the additional advantage that we can embed the problem into a von Neumann context. As proven in [30], for $\epsilon = 0$ the Toeplitz operator $T_0(\varphi)$ has an index class in $W^*(V/\mathcal{F}_\varphi)$ the von Neumann algebra of $(\mathcal{F}_\varphi, \mathfrak{m})$, and its $\mathfrak{m}$-index equals the expressions (5.19) and (5.20). In the special case that φ is α-related, the extended multiplier $\widetilde{\Phi}$ is related to φ by multiplication by $h \in \mathrm{Hull}(\alpha)$. Thus, the corresponding leafwise Toeplitz operators are similarly related, so their index forms (2.22) are independent of $x \in X_\varphi$. Thus, we conclude the limits $T_\epsilon(\varphi)$ as $\epsilon \to 0$ are leafwise uniform, so that for Γ amenable formula (5.20) remains true for $T_0(\varphi)$.

Let us describe the main application we have for formula (5.12) and Theorem 5.10. Fix a closed subgroup $H \subset U(N)$ and a faithful representation $\alpha : \Gamma \to H$ with dense image. Form the principal bundles

$$\widehat{P}_\alpha = \widetilde{M} \times H/(y,h) \sim (y \cdot \gamma^{-1}, \alpha(\gamma)h)$$
$$P_\alpha = \widetilde{M} \times U(N)/(y,g) \sim (y \cdot \gamma^{-1}, \alpha(\gamma)g).$$

There is an inclusion of bundles

$$\begin{tikzcd} H \arrow[r, phantom, "\subseteq"] \arrow[d] & U(N) \arrow[d] \\ \widehat{P}_\alpha \arrow[r] \arrow[dr] & P_\alpha \arrow[d] \\ & M \end{tikzcd}$$

A section $s : M \to P_\alpha$ is equivalent to an α-related multiplier $\varphi : \widetilde{M} \to U(N)$ by Proposition 4.2, and the above construction yields all of the multipliers in $\mathcal{R}_\alpha$ for $H = \mathrm{Hull}(\alpha)$. The symbol φ is Γ-almost periodic, with $\mathfrak{m}$ the left-invariant Haar measure on H. The generalized index, $\mathrm{Ind}_{\mathfrak{m}}([\varphi])$ is defined by (5.11) with Φ given by $\widetilde{\Phi}(y \cdot \gamma^{-1}, \alpha(\gamma) \cdot h) = h \cdot \varphi(y)$ for $(y,h) \in \widetilde{M} \times H$. The topological formula (5.12) simplifies to (5.20) if Γ is amenable.

The remarkable property of the generalized Toeplitz index of $\varphi \in \mathcal{R}_\alpha$ is that the topological formula for $\mathrm{Ind}_{\mathfrak{m}}([\varphi])$ has an altenative description in terms of the Cheeger-Simons class of the flat bundle $P_\alpha \to M$ with section s, [16]. This is an odd dimensional cohomology class which we denote by

$$\mathrm{Tch}(\varphi) \in H^{\mathrm{odd}}(M; \mathbf{R}),$$

as both α and s are determined by φ. The identification $H = \mathrm{Hull}(\alpha) = X_\varphi$ identifies $\widehat{P}_\alpha = V_\varphi$. The fibration map $\pi : \widehat{P}_\alpha \to M$ lifts $\mathrm{Tch}(\varphi)$ to a class $\pi^*(\mathrm{Tch}(\varphi))$ on $\widehat{P}_\alpha$.

Theorem (Theorem 4.8, [31]). *Let $\varphi \in \mathcal{R}_\alpha$. Then*

$$\mathrm{Ind}_{\mathfrak{m}}([\varphi]) = (-1)^m \langle \pi^*(\mathrm{Tch}(\varphi) \cup \mathrm{Td}(M)) \cup \psi^{-1}\mathrm{ch}(\sigma_D), C_{\mathfrak{m}} \rangle \tag{5.21}$$ □

For Γ amenable, the content of (5.21) is to equate the leafwise differential forms representing $\pi^*(\mathrm{Tch}(\varphi))$ with the form $\mathrm{ch}^*(\varphi)$ appearing in (5.20).

We conclude this section by deriving the proof of Theorem 1 of the introduction. Introduce the topological space

$$X = \times_{\delta \in \Gamma} \{0,1\}_\delta$$

with typical point $x = \{x_\delta \mid \delta \in \Gamma\}$. The δ-coordinate of x is x_δ. The basic open sets are the cylinders: for $\Delta = \{\delta_1, \ldots, \delta_p\} \subset \Gamma$ and $n_1, \ldots, n_p \in \{0,1\}$

$$X_\Delta = \{x \in X \mid x_{\delta_i} = n_i\}.$$

Γ acts on X via translation:

$$\gamma \cdot \{x_\delta\} = \{x_{\gamma \cdot \delta}\}$$

which is continuous. Give each factor $\{0,1\}_\delta$ the $(1/2,1/2)$-measure and X the resulting product measure, $\mathfrak{m}$. Observe, for example, that $\mathfrak{m}(X_\Delta) = 2^{-p}$. The measure $\mathfrak{m}$ is Γ-invariant. It is an easy exercise to show the Γ-action is essentially-free, although it is never free. (The constant sequences provide two fixed points, and if $\Gamma = \mathbf{Z}^m$ then the periodic orbits are dense.)

Form the topological space $V = \Gamma \backslash (\widetilde{M} \times X)$ as above, where all of the holonomy covers $\widetilde{M}_x = \widetilde{M}$ for $\mathcal{F}$. Observe that $\varphi : M \to U(N)$ lifts to $\varphi : V \to U(N)$ which is smooth on leaves. We can thus define the Toeplitz foliation index for $T_\epsilon(\varphi)$, and (5.12) gives a topological formula for $\mathrm{Ind}_{\mathfrak{m}}([\varphi])$. The key point is that both the lifts $D_{\mathcal{F}}$ and φ are independent of x so that φ, $T_\epsilon(\varphi)$ and $\partial[\varphi]$ are independent of x under the identification $\widetilde{M}_x \cong \widetilde{M}$. Consequently, the restriction to the diagonal of the pointwise trace of (2.22) descends to a function on M. The $\mathfrak{m}$-trace used in $\mathrm{tr}_{\mathfrak{m}} \partial[\varphi]$ is thus calculated by restricting $\mathrm{Tr}(\partial[\varphi])$ to a fundamental domain $M_0 \subset \widetilde{M} \cong \widetilde{M}_x$ in a typical leaf as the fiber X has $\mathfrak{m}$-mass 1. Hence

$$\mathrm{Ind}_{\mathfrak{m}}[\varphi] = \mathrm{vol}(M)^{-1} \int_{M_0} \mathrm{Tr}(\partial[\varphi]) dy. \tag{5.22}$$

The right side of (5.22) is precisely the Γ-trace of the difference of projections $\partial[\varphi]$. By (§7, Kasparov [40]), this is equal to the left-hand-side of (1.20).

The topological formula (5.12) for $\mathrm{Ind}_{\mathfrak{m}}[\varphi]$ also reduces to an integral over M as all of the symbol data is lifted from M. The topological index (5.12) thus agrees with the topological index for $T_D(u)$, defined in the introduction. Combining the foliation index theorem with the odd index theorem for compact manifolds yields equality (1.21).

§6. The Relative Eta Invariant for Coverings.

A principal theme of this paper is that the odd index theorem for coverings applies to more than just periodic multipliers on the covering. The α-related multipliers provide the next largest natural class, and for these the analytic index is calculated via a trace on a foliation C^*-algebra associated to the hull closure $\mathrm{Hull}(\alpha)$ of the unitary representation α. It was observed in section 4 that there is a close relationship between α-multipliers and sections of associated flat bundles. The purpose of this section is to prove that correspondingly, the Breuer index $\mathrm{Ind}_{\mathfrak{m}}(\varphi)$ for $\varphi \in \mathcal{R}_\alpha$ is equal to an analytic invariant defined directly on M in terms of D and the flat bundle data (P_α, s) determined by φ. This is the content of Theorem 2 of the Introduction for which we give a direct analytic proof below.

Theorem 2 was first proven in the special case Γ is abelian in [29] and in full generality in [31]. Both papers use the method of renormalization applied in cyclic cohomology, but their essence was to reduce the equality between the two sides of (1.27) to an equality between two integer-valued index problems, and then observe that the symbol data in both Fredholm contexts were homotopic. In contrast, we use here the definition of the eta-invariant in terms of an integral involving the heat kernel to directly relate the relative eta invariant to a spectral flow on $\widetilde{M}$ involving P_ϵ and φ.

More precisely, it is possible to define a "type II eta invariant" of $\widetilde{D}$ directly on $\widetilde{M}$. For the signature operator this was done by Cheeger and Gromov [14], and for more general geometric (Dirac) operators independently by G. Peric [53] and M. Ramachandran [43]. This type II eta invariant is the same for $\widetilde{D}$ and its unitary conjugate $\varphi^* \widetilde{D}\varphi$, as φ extends to a unitary Φ on the full C^*-algebra A_φ associated to φ. Thus, a path between $\widetilde{D}$ on $\varphi^*\widetilde{D}\varphi$ yields a family of eta invariants for which the integral of its distributional derivative is zero. This distributional derivative is the difference of two terms, the first is smooth and locally calculable from $\widetilde{D}$ and $\varphi^*\widetilde{D}\varphi$. Its integral is equal to the relative eta-invariant on M. The second term is non-local, and integrates to the spectral flow between $\widetilde{D}$ and $\varphi^*\widetilde{D}\varphi$. In the type II context, this spectral flow is again equal to the Breuer index of $T_\epsilon(\varphi)$, proving Theorem 2. The rest of this section is devoted to filling in the details of this sketch of proof.

Fix the geometric operator D, a representation $\alpha : \pi_1(M) \longrightarrow U(N)$ with image isomorphic to a quotient Γ of $\pi_1(M)$, and section s of the associated flat principal $U(N)$-bundle P_α. The vector bundle E_α of the Introduction is associated to P_α via the natural action of $U(N)$ on $\mathbf{C}^N$. This bundle lifts to product $\widetilde{M} \times \mathbf{C}^N$ over the Γ-covering $\widetilde{M} \longrightarrow M$, and s lifts to an α-related multiplier $\varphi : \widetilde{M} \longrightarrow U(N)$. Note that the section s induces a trivialization

$$\begin{aligned} M \times U(N) &\cong P_\alpha \\ (x, g) &\longrightarrow s(x) \cdot g \end{aligned}$$

which in turn induces Θ of the Introduction.

We have two extension of D to $C^\infty(M, E \otimes \mathbf{C}^N)$: the product extension $D_0 = D \otimes \nabla^0$, and that defined via the flat connection $\nabla^{\overline{\alpha}}$ from E_α, $D_1 = D \otimes \nabla^{\overline{\alpha}}$. The operator D_0 lifts to $\widetilde{D}_0 = \widetilde{D} \otimes \nabla^0$ on $C^\infty(\widetilde{M}, \widetilde{E} \otimes \mathbf{C}^N)$.

LEMMA 6.1. *The operator D_1 lifts to $\widetilde{D}_1 = \varphi^* \widetilde{D}_0 \varphi$ on $C^\infty(\widetilde{M}, \widetilde{E} \otimes \mathbf{C}^N)$.*

PROOF: The map induced by Θ on sections is implemented by φ when lifted to $\widetilde{M}$. Given $f \in C^\infty(M, E \otimes \mathbf{C}^N)$, the lift $\tilde{f}$ is Γ-periodic, so the product $\varphi \cdot \tilde{f}$ is α-related so descends to a section $\varphi(f) \in C^\infty(M, E_\alpha)$. Clearly, the inverse is multiplication by the adjoint φ^*. By definition, the extension $D \otimes \nabla^\alpha$ on $C^\infty(M, E_\alpha)$ lifts to the product extension $\widetilde{D} \otimes \nabla^0$ on $\widetilde{M}$, and D_1 is the push-forward of $D \otimes \nabla^\alpha$ via Θ, so that lifting to $\widetilde{M}$ we have

$$\widetilde{D}_1 = \varphi^*(\widetilde{D} \otimes \nabla^0)\varphi = \varphi^* \widetilde{D}_0 \varphi. \tag{6.1}$$ □

For $0 \leq t \leq 1$ we define

$$\begin{cases} D_t &= (1-t)D_0 + t \cdot D_1 = D_0 + t \cdot G \\ \widetilde{D}_t &= (1-t)\widetilde{D}_0 + t \cdot \widetilde{D}_1 = \widetilde{D}_0 + t \cdot \widetilde{G} \end{cases} \tag{6.2}$$

where $\widetilde{G} = \varphi^*[D, \varphi]$ is a matrix-valued Γ-periodic smooth function on $\widetilde{M}$, which descends to G on M. The first-order elliptic operators D_t and $\widetilde{D}_t$ are of geometric type as D is assumed to be a geometric operator (cf. §2, [44]). In particular, they are essentially self-adjoint. As M is compact, each D_t has isolated pure-point spectrum, while the spectrum of each $\widetilde{D}_t$ can be a mixture of continuous and pure-point. In particular, 0 need not be isolated in $\mathrm{Spec}(\widetilde{D}_t)$. A geometric operator, A, has the Bismut-Freed local cancellation property ([11], cf. also [43]) that the local trace of $Ae^{-\xi A^2}$ has an asymptotic expansion holomorphic in ξ, beginning with $\xi^{1/2}$. Define the eta function

$$\eta(D_t, s) = \frac{1}{2} \cdot \Gamma((s+1)/2)^{-1} \int_0^\infty \xi^{(s-1)/2} \mathrm{tr}_M\{D_t e^{-\xi D_t^2}\} d\xi. \tag{6.3}$$

As 0 is isolated in $\mathrm{Spec}(D_t)$, this defines a holomorphic function for $s > m$. However, the Bismut-Freed cancellation property implies that $\eta(D_t, s)$ is in fact holomorphic for $s > -1/2$, a considerable sharpening of the usual results. In particular, we define

$$\begin{aligned} \eta(D_t) &= \eta(D_t, 0) \\ &= \frac{1}{2\sqrt{\pi}} \int_0^\infty \xi^{-1/2} \mathrm{tr}_M\{D_t e^{-\xi D_t^2}\} d\xi \end{aligned} \tag{6.4}$$

the latter being an improper, conditionally convergent integral at $\xi = 0$. Let us also introduce the truncated integral

$$\eta^a(D_t) = \frac{1}{2\sqrt{\pi}} \int_0^a \xi^{-1/2} \mathrm{tr}_M\{D_t e^{-\xi D_t^2}\} d\xi. \tag{6.5}$$

We would also like to define $\eta_\Gamma(\widetilde{D}_t, s)$ by replacing the trace on M with the Γ-trace on $\widetilde{\mathcal{H}}^N$ in (6.3). However, this requires that 0 be isolated in $\mathrm{Spec}(\widetilde{D}_t)$, a condition that patently fails in spectral flow applications. We adopt an alternative strategy, and introduce for $a > 0$

$$\eta_\Gamma^a(\widetilde{D}_t, s) = \frac{1}{2} \cdot \Gamma((s+1)/2)^{-1} \cdot \int_0^a \xi^{(s-1)/2} \mathrm{tr}_\Gamma\{\widetilde{D}_t e^{-\xi \widetilde{D}_t^2}\} d\xi. \tag{6.6}$$

By the Bismut-Freed local cancellation, this is holomorphic for $s > -1/2$. Moreover, for $s = 0$ and D the signature operator, Cheeger and Gromov [14] proved that the limit exists as $a \to \infty$. Ramachandran extended this to include all geometric operators, using a clever trick based on defining a tempered distribution associated to D and tr_Γ ([43]). So we define for D geometric:

$$\eta_\Gamma(\widetilde{D}_t) = \lim_{a \to \infty} \eta_\Gamma^a(\widetilde{D}_t, 0). \tag{6.7}$$

We next make a simple observation that is the basis for our proof of Theorem 2.

LEMMA 6.2.

$$\eta_\Gamma(\widetilde{D}_0) = \eta_\Gamma(\widetilde{D}_1). \tag{6.8}$$

PROOF: The operators $\widetilde{D}_0$ and $\widetilde{D}_1$ are unitarily conjugate by Lemma 6.1, but the unitary φ is not Γ-periodic so does not a priori satisfy $\mathrm{tr}_\Gamma(\varphi A) = \mathrm{tr}_\Gamma(A\varphi)$ for a Γ-trace-class operator A. However, for the foliated space $(V, \mathcal{F}_\varphi)$ obtained from the hull construction of section 5, the leafwise operators $D_{\mathcal{F},0}$ and $D_{\mathcal{F},1}$ are independent of $x \in X_\varphi$. Thus, the Γ-trace in (6.4) can be replaced with the foliation trace, $\mathrm{tr}_{\mathfrak{m}}$, and $\widetilde{D}_t$ with the corresponding leafwise operator $D_{\mathcal{F},t}$ lifted from D_t. The unitary φ extends to a unitary Φ defined on all of V, and acts as an outer automorphism of the algebra $\mathcal{U}_{-\infty}(\mathcal{F}_\varphi)$, so we have $\mathrm{tr}_{\mathfrak{m}}(\Phi \circ A) = \mathrm{tr}_{\mathfrak{m}}(A \circ \Phi)$ for $A \in \mathcal{U}_{-\infty}(\mathcal{F}_\varphi)$. Taking $A = \Phi^* \circ \widetilde{D}_{\mathcal{F},t} e^{-\xi \widetilde{D}^2_{\mathcal{F},t}}$ and integrating yields (6.8). □

It is standard fact that if 0 is not an eigenvalue of D_t for $t = t_0$, then the function $tmapsto\eta(D_t)$ is differentiable at t_0, and the derivative $\dot{\eta}(D_t)$ is given by a local formula derived from the total symbol of D_t. In particular, if the spectrum of all D_t is discrete, then the derivative extends to a smooth function for all $0 \leq t \leq 1$ and we can set

$$\eta(D, \alpha, \Theta) = \int_0^1 \dot{\eta}(D_t)dt. \tag{6.9}$$

The local formula for $\dot{\eta}(D_t)$ can be derived directly from the definition (6.3) as in Gilkey (page 83, [34]).

Our task is to make sense of $\frac{d}{dt}\{\eta_\Gamma(\widetilde{D}_t)\}$, then use (6.8) and the fundamental theorem of calculus to derive the identity which implies (1.27). The difficulty is that zero need not be isolated in $\mathrm{Spec}(D_t)$ for a continuum of values of t, so the "derivative" will be distribution-valued in an essential way. The key observation which overcomes the obstacle is that $\eta^a_\Gamma(\widetilde{D}_t, s)$ is holomorphic for $s > -1/2$ and all $a > 0$. We prove that it has a derivative, given by a local term coinciding with $\dot{\eta}(D_t)$, and a second boundary-value term that will be equated to the spectral flow interpretation of $\mathrm{Ind}_{\mathfrak{m}}(\varphi)$. Moreover,

$$0 = \lim_{a \to \infty} \int_0^1 \frac{d}{dt}\{\eta^a_\Gamma(\widetilde{D}_t, 0)\}dt \tag{6.10}$$

by (6.8) so that we obtain (1.27). It remains to justify these claims.

PROPOSITION 6.3. *For all $a > 0$, $\eta^a_\Gamma(\widetilde{D}_t)$ is differentiable as a function of t, and we have*

$$\int_0^1 \frac{d}{dt}\{\eta^a_\Gamma(\widetilde{D}_t)\}dt = \eta(D, \alpha, \Theta) - SF_\Gamma(\widetilde{D}, \varphi, a) \tag{6.11}$$

$$SF_\Gamma(\widetilde{D}, \varphi, a) = \sqrt{\frac{a}{\pi}} \cdot \int_0^1 \mathrm{tr}_\Gamma(\widetilde{G} e^{-a\widetilde{D}_t^2})dt. \tag{6.12}$$

PROOF: For $s > m+2$, the integrand in (6.6) is uniformly differentiable in t for $0 < \xi \leq a$, so that the derivative of $\eta_\Gamma^a(\widetilde{D}_t, s)$ exists and is equal to the integral of the derivative inside. Then compute as in (page 83, [34]) to obtain

$$\frac{d}{dt}\eta_\Gamma^a(\widetilde{D}_t, s) = \tag{6.13}$$
$$\frac{1}{2}\cdot\Gamma((s+1)/2)^{-1}\int_0^a \xi^{(s-1)/2}\left(1+2\xi\frac{d}{d\xi}\right)\mathrm{tr}_\Gamma(\widetilde{G}e^{-\xi\widetilde{D}_t^2})d\xi.$$

Integrate by parts to obtain

$$\left\{\begin{array}{l} -\frac{s}{2}\cdot\Gamma((s+1/2))^{-1}\int_0^a \xi^{(s+1)/2}\mathrm{tr}_\Gamma(\widetilde{G}e^{-\xi\widetilde{D}_t^2})d\xi \\ +\Gamma((s+1)/2)^{-1}\cdot\xi^{(s+1)/2}\mathrm{tr}_\Gamma(\widetilde{G}e^{-\xi\widetilde{D}_t^2})|_{\xi=0}^a \end{array}\right. \tag{6.14}$$

The heat equation asymptotics for the compact case, M, carry over to $\widetilde{M}$ as well (cf. Chapter 13, [47]), so that the first term in (6.14) is a meromorphic function of s with no pole at $s=0$. Its value at $s=0$ is determined by a local expression derived from the complete symbol of $\widetilde{G}e^{-\xi\widetilde{D}_t^2}$, which is equal to the local expression for $Ge^{-\xi D_t^2}$ on M. Thus, the Γ-trace of it reduces to an integral over M which equals $\dot{\eta}(D_t)$.

The second term in (6.14) for $s > m-1$ reduces to

$$\Gamma((s+1)/2)^{-1}\cdot a^{(s+1)/2}\cdot\mathrm{tr}_\Gamma(\widetilde{G}e^{-a\widetilde{D}_t^2}) \tag{6.15}$$

as the limit in the evaluation for $\xi \to 0$ vanishes. The term (6.15) is holomorphic for $s > -1/2$, so its value at $s=0$ is defined. Integrate (6.15) for $s=0$ to obtain (6.12), proving (6.11). □

The next step is to identify (6.12) with the Breuer index $\mathrm{Ind}_{\mathfrak{m}}(\varphi)$. Consider the function

$$\sigma^a(\lambda) = \frac{1}{\sqrt{\pi}}\int_0^a \xi^{-1/2}\lambda e^{-\xi\lambda^2}d\xi \tag{6.16}$$

which satisfies

$$\lim_{\lambda\to\pm\infty}\sigma^a(\lambda) = \pm 1 \tag{6.17}$$

$$\frac{d}{d\lambda}\sigma^a(\lambda) = 2\cdot\sqrt{\frac{a}{\pi}}\cdot e^{-a\lambda^2} > 0 \tag{6.18}$$

and hence

$$\left\{\begin{array}{ll} |\sigma^a(\lambda)-1| < c(\epsilon)e^{-(a\lambda^2)/2} & \text{for } \lambda > \epsilon \\ |\sigma^a(\lambda)+1| < c(\epsilon)e^{-(a\lambda^2)/2} & \text{for } \lambda < -\epsilon \end{array}\right. \tag{6.19}$$

For each $E > 0$ and $0 \leq t \leq 1$ introduce the spectral projection on $\widetilde{\mathcal{H}}^N$,

$$\chi(E,t) = \chi_{[-E,E]}(\widetilde{D}_t) \tag{6.20}$$

associated to $\widetilde{D}_t$ and the interval $[-E, E]$.

LEMMA 6.4.

$$SF_\Gamma(\widetilde{D},\varphi,a) = \lim_{E\to\infty} -\frac{1}{2}\int_0^1 \mathrm{tr}_\Gamma\left\{\left(\frac{d}{dt}\sigma^a(\widetilde{D}_t)\right)\circ\chi(E,t)\right\}dt. \tag{6.21}$$

PROOF: $\chi(E,t)$ is Γ-trace class for all E and commutes with D_t, so that we can repeat the integration-by-parts method of Proposition 6.3 to reduce the right-hand-side of (6.21) to the expression:

$$\lim_{E\to\infty}\sqrt{\frac{a}{\pi}}\cdot \mathrm{tr}_\Gamma\{\widetilde{G}\cdot e^{-a\widetilde{D}_t^2}\circ\chi(E,t)\}. \tag{6.22}$$

The proof is completed by noting that $\widetilde{G}$ is a bounded operator, and so for some constant

$$\mathrm{tr}_\Gamma\{\widetilde{G}\circ e^{-a\widetilde{D}_t^2}\circ(I-\chi(E,t))\}\le c\cdot||G||\cdot e^{-aE^2}. \tag{6.23}$$ □

The final step uses standard spectral properties of geometric (first-order) operators:

PROPOSITION 6.5.

$$\begin{aligned}&\mathrm{Ind}_m(\varphi)=\\ &\lim_{a,E\to\infty} -\frac{1}{2}\int_0^1 \mathrm{tr}_\Gamma\left\{\left(\frac{d}{dt}\sigma^a(\widetilde{D}_t)\right)\circ\chi(E,t)\right\}dt.\end{aligned} \tag{6.24}$$

PROOF: For an α-related multiplier φ, we noted that the Hull$(\alpha)\cong\overline{\alpha(\Gamma)}\subset U(N)$, and the restrictions of Φ to leaves is given by the translates $g\cdot\varphi$ of φ, for $g\in U(N)$. Thus, the leafwise operators appearing in (2.22) are leaf-independent after taking the pointwise trace. We can therefore let $\epsilon\to 0$ in the index construction of section 2, use normality of Tr_m and leaf independence to identify

$$\mathrm{Ind}_m(\varphi)=\mathrm{tr}_\Gamma\{\Pi(\varphi)\}-\mathrm{tr}_\Gamma\{\Pi(\varphi^*)\} \tag{6.25}$$

where $\Pi(\varphi)$ is projection onto the kernel of $P_0^+\circ\varphi\circ P_0^+ - P_0^-$, and $\Pi(\varphi^*)$ the corresponding projection for the adjoint.

The geometric operator D satisfies an elementary commutation rule with functions (viewed as diagonal multipliers) $[D,f]=\nabla f$, where ∇ is a first-order differential operator constructed from the Spin$^{\mathbf{C}}$-structure and covariant differentiation on $E\to M$. In particular, for φ unitary, this implied that $\widetilde{G}=\varphi^*[D,\varphi]$ is a uniformly bounded operator on $\widetilde{\mathcal{H}}^N$. Let $c_2=||\widetilde{G}||$. This gives the standard estimates.

LEMMA 6.6. *There are operator inequalities*

$$\widetilde{D} - c_2 \leq \varphi^* \widetilde{D} \varphi \leq \widetilde{D} - c_2 \tag{6.26}$$

and hence for $E > c_2$,

$$\chi_{[-E+c_2, E-c_2]}(\widetilde{D}) \leq \chi_{[-E,E]}(\varphi^* \widetilde{D} \varphi) \leq \chi_{[-E-c_2, E+c_2]}(\widetilde{D}). \tag{6.27}$$

□

As the range of $\Pi(\varphi)$ is the closure of the subspace range $(P_0^+) \cap$ range $(\varphi^* P_0^-)$, and $\Pi(\varphi^*)$ the closure of range $(P_0^+) \cap$ range (φP_0^-), we deduce

COROLLARY 6.7. *For* $E > c_2$,

$$\begin{cases} \Pi(\varphi) \circ \chi(E,0) & = \Pi(\varphi) \\ \Pi(\varphi^*) \circ \chi(E,0) & = \Pi(\varphi^*). \end{cases} \tag{6.28}$$

□

Combine (6.28) with the equality (6.25) and $P_0^+ + P_0^- = \mathrm{Id}$ to obtain

COROLLARY 6.8. *For* $E > c_2$

$$\begin{aligned} \mathrm{Ind}_{\mathrm{m}}(\varphi) = & -\frac{1}{2}[\mathrm{tr}_\Gamma\{(\varphi^* P_0^+ \varphi - P_0^+) \circ \chi(E,0)\} \\ & - \mathrm{tr}_\Gamma\{(\varphi^* P_0^- \varphi - P_0^-) \circ \chi(E,0\}]. \end{aligned} \tag{6.29}$$

□

To conclude the proof of Proposition 6.5, we must show that the right-hand-sides of (6.24) and (6.29) are equal. First note that by the estimate (6.23) and operation inequalties (6.27), we can replace $\chi(E,t)$ in (6.24) by $\chi(E,0)$. Then integrate the right-side of (6.24) to obtain

$$\lim_{a,E \to \infty} -\frac{1}{2}\mathrm{tr}_\Gamma\{(\sigma^a(D_1) - \sigma^a(D_0)) \circ \chi(E,0)\}. \tag{6.30}$$

Then combine the exponential estimate (6.19) with (6.26) and the Sobolev estimate for an open manifold (cf. Chapter 5, [47]) to deduce that (6.30) converges uniformly to

$$-\frac{1}{2} \cdot \lim_{E \to \infty} \mathrm{tr}_\Gamma\left\{ \left(\varphi^* \lim_{a \to 0} \sigma^a(\widetilde{D}) \varphi - \lim_{a \to 0} \sigma^a(\widetilde{D}) \right) \circ \chi(E,0) \right\} \tag{6.31}$$

which is equal to the right-hand-side of (6.29). □

REMARK 6.9. The above analytic proof of Theorem 2 also applies for Γ finite, even trivial. The multiplier φ is then a unitary over a finite cover of M, and we identify the spectral flow for a family of eta-invariants on M with a Toeplitz index. This equality is asserted in a preliminary form in (§7, [3]), and Booss-Wojciechowski gave a topological proof in [12]. As is evident from the results of this section, the analytic proof extends to the Γ-almost periodic case. □

Let us conclude this paper with the remark that our study of Toeplitz indices for *normal* multipliers φ on $\widetilde{M}$ could equally well be carried out for non-self-adjoint D coupled to *normal vector bundles* over $\widetilde{M}$. In this case, the approach of this paper yields index problems similar to those which arise for $C_r^*(\Gamma)$-bundles over M in Kasparov-Mischenko approach to the Novikov Conjecture (cf. [32]). On the other hand, the even case generalizes the almost periodic index theory on $\mathbf{R}^m$, [51], so deserves further investigation.

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