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HOMOTOPY CHARACTERISTIC CLASSES OF FOLIATIONS

BY

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1. Introduction

The main point of this paper is to use minimal model theory to define new, higher order cohomology invariants of concordance classes of foliations, and then to apply these to establish the existence of uncountable families of distinct foliations on a much wider class of manifolds than previously had been shown. In some sense, this work can be considered as a sequel to the earlier paper [H2] of the first author.

Let $\mathscr{F}$ be a smooth (i.e., C^∞) foliation on a manifold V without boundary. Let $f_{\mathscr{F}}$: $WO_q \to \Omega_{DR}(V)$ denote the characteristic homomorphism of $\mathscr{F}$ (e.g., see [B] or [KT]), where we define the differential graded commutative $\mathbf{R}$-algebra (dgca)

$$W_q = \Lambda(h_1, h_2, \dots, h_q) \otimes \mathbf{R}[c_1, c_2, \dots, c_q]/(\deg > 2q)$$

($\deg h_i = 2i - 1$, $\deg c_i = 2i$, $dh_i = c_i$), and WO_q denotes the subalgebra

$$WO_q = \Lambda(h_1, h_3, \dots, h_{q'}) \otimes \mathbf{R}[c_1, c_2, \dots, c_q]/(\deg > 2q)$$

with q' the largest odd integer $\leq q$. The construction of $f_{\mathscr{F}}$ depends upon the choice of a Bott connection and of a connection preserving some metric on the normal bundle Q to $\mathscr{F}$. The induced map in cohomology is easily shown to be independent of this choice, and to even depend only on the concordance class of $\mathscr{F}$. When the bundle Q admits a framing, or parallelization, denoted by s, then there is also defined a map of dgca's, $f_{\mathscr{F},s}$: $W_q \to \Omega_{DR}(V)$, which induces a map in cohomology $f^*_{\mathscr{F},s}$ depending only on the concordance class of $\mathscr{F}$ and the homotopy class $[s]$ of s. Note that the map $f^*_{\mathscr{F},s}$ will, in general, vary with the choice of $[s]$ (which was a key point in the paper [H3]).

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This paper is based on the following remark, which is a strong refinement of the above discussion: the *homotopy class* of $f_{\mathscr{F}}$ is a well-defined invariant of the concordance class of $\mathscr{F}$. More precisely, let $\rho_q\colon \mathscr{M}O_q \to WO_q$ denote a quasi-isomorphism of a minimal model for WO_q. Similarly, let $\rho_q\colon \mathscr{M}_q \to W_q$ denote the minimal model for W_q.

LEMMA 1. (a) *The homotopy class of $f_{\mathscr{F}} \circ \rho_q\colon \mathscr{M}O_q \to \Omega_{DR}(V)$ in the set of dgca-morphisms from $\mathscr{M}O_q$ to $\Omega_{DR}(V)$ is well defined, and depends only on the concordance class of $\mathscr{F}$ in the set of smooth foliations of codimension q.*

(b) *If $\mathscr{F}$ admits a framing, s, on the normal bundle Q, then the homotopy class of $f_{\mathscr{F},s} \circ \rho_q\colon \mathscr{M}_q \to \Omega_{DR}(V)$ is similarly well defined and depends only on the framed concordance class of $\mathscr{F}$ in the set of smooth foliations of codimension q with framed normal bundle.*

In this paper, we assume that V is a connected manifold, and all algebras, A, are connected; that is $H^0(A) \cong R$.

The proof of Lemma 1 is elementary, and exactly parallel to that of Theorem 2.11 of [H1]. (For completeness, we give a proof in §2 below.) A key point in the first author's Thesis [H4] was that one should regard the homotopy class $[f_{\mathscr{F}} \circ \rho_q]$ in $[\mathscr{M}O_q, \Omega_{DR}(V)]$ as a fundamental differential invariant of a foliation. The difficulty with this invariant is that its calculation may be extremely laborious, and so one considers approximations derived from it. For example, the morphism $f^*_{\mathscr{F}}$ is obtained by considering the induced map on cohomology. The dual homotopy invariants for $\mathscr{F}$ are obtained by considering the induced map between the indecomposable elements of $\mathscr{M}O_q$ and those of a minimal model $\mathscr{M}_V$ for $\Omega_{DR}(V)$. As these invariants play a key role in our examples, we elaborate upon this remark. By Theorem C of §2 below, there is a one-to-one correspondence between $[\mathscr{M}O_q, \mathscr{M}_V]$ and $[\mathscr{M}O_q, \Omega_{DR}(V)]$, with suitable hypothesis on V. Let

$$\mathscr{M}f_{\mathscr{F}}\colon \mathscr{M}O_q \to \mathscr{M}_V$$

denote a lift of $f_{\mathscr{F}} \circ \rho_q$. Let $A^+ = \operatorname{Ker}(\varepsilon\colon A \to \mathbf{R})$ denote the augmentation ideal. Let Q denote the functor which associates to a connected dgca A its space of indecomposables: $QA \equiv A/(A^+)^2$. Then the dual homotopy map is by definition

$$h^\sharp_{\mathscr{F}} = Q(\mathscr{M}f_{\mathscr{F}})\colon \pi^*(WO_q) \to \pi^*(V).$$

The justification for introducing the notation $\pi^*(WO_q) \equiv Q\mathscr{M}O_q$ and $\pi^*(V) \equiv Q\mathscr{M}_V$ follows from Theorem A of §2.

In this paper we will construct a third approximation to the invariant $[f_{\mathscr{F}} \circ \rho_q]$ by considering the induced maps on the Postnikov k-invariants of the model $\mathscr{M}O_q$. This third approximation has the advantage that it produces cohomology classes which can be nontrivial in degrees greater than $2q + q^2$, unlike the map $f^*_{\mathcal{F}}$, and takes advantage of the enormous number of invariants in the dual homotopy spaces $\pi^p(WO_q)$. As is usual for higher order homotopy operations, our tertiary invariants are only well defined for foliations satisfying an additional hypothesis on their primary and secondary characteristic classes, but these restrictions are stable under concordance, so that we obtain new cohomology concordance invariants for foliations.

The construction of the tertiary invariants is given in §§3 and 4, while §2 contains preliminary material. Then in §§5, 6, 7 and 8 we apply our invariants to the study of existence of foliations. For example, we show that the 9-torus, T^9, has uncountably many distinct concordance classes of foliations, all of whose primary, secondary and dual homotopy classes are trivial, but which are distinguished by their tertiary classes. More generally, we give simple conditions on the homotopy type of a manifold V which are sufficient to guarantee that if V has at least one codimension-q foliation, then V has many-parameter families of non-concordant foliations, detected only by the tertiary invariants. These examples greatly extend the examples constructed in §4 of [H2].

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2. Homotopy between dgca-morphisms

Let $(\mathcal{M}, d_{\mathcal{M}})$ denote a minimal algebra over $\mathbf{R}$ in the sense of Sullivan [S1], so that $\mathcal{M} = \Lambda X$ is a free graded commutative algebra on a space X of homogeneous elements. Let X^n denote the subspace of elements of degree n. We write $d = d_{\mathcal{M}}$ when there is no danger of confusion. Moreover, d satisfies the minimality condition

$$d\colon X^n \to \left(\mathcal{M}(n)^+\right)^2$$

where $\mathcal{M}(n) = \Lambda_{m<n} X^m$ is the subalgebra generated by elements of degree strictly less[3] than n. We will always assume that a dgc-algebra, (A, d_A), satisfies $H^0(A, d_A) \cong \mathbf{R}$.

Let $(\mathbf{R}(t, dt), d_{\mathbf{R}})$ denote the acyclic dgc-algebra, with t a polynomial variable of degree 0, dt an exterior variable of degree 1, and $d(t) = dt$.

[3]This convention disagrees with the usual one in Sullivan's theory (where $\mathcal{M}(n)$ denotes $\Lambda_{m\le n} X^m$) but will agree with the definition of the nth stage Y_n of the Postnikov tower according to Spanier, if we want $\mathcal{M}(n)$ to be the model of Y_n (see Theorem B below).

Let $(\mathscr{M}, d_{\mathscr{M}})$ and (A, d_A) be dgc-algebras, and f_0, f_1: $\mathscr{M} \to A$ dgca-morphisms. Then f_0 and f_1 are homotopic if there exists a dgca-morphism F: $\mathscr{M} \to A \otimes \mathbf{R}(t, dt)$ so that $i_0 \circ F = f_0$ and $i_1 \circ F = f_1$, where i_z: $\mathbf{R}(t, dt) \to \mathbf{R}$ is evaluation at $t = z$, and $A \otimes \mathbf{R}(t, dt)$ has the differential $d_A \otimes 1 \pm 1 \otimes d_{\mathbf{R}}$. Note that this tensor algebra also has an $\mathbf{R}(t, dt)$-linear "differential", $d_A \otimes 1$, which by abuse of notation will again be denoted by d_A.

For a minimal dgc-algebra $(\mathscr{M}, d_{\mathscr{M}})$, homotopy is an equivalence relation on the set of dgc-morphisms $\{\mathscr{M}, A\}$ from $\mathscr{M}$ into a dgc-algebra (A, d_A) (cf. Corollary 10.7, [GM]). Let $[\mathscr{M}, A]$ denote the set of homotopy classes.

For a dgc-algebra (A, d_A), define a linear map of degree -1 by setting:

$$\int_0^1 : A \otimes \mathbf{R}(t, dt) \to A,$$

$$\int_0^1 b \otimes t^i = 0,$$

$$\int_0^1 b \otimes t^i dt = (-1)^{|b|} \frac{b}{i+1}, \quad i \geq 0$$

where $|b|$ indicates the degree of a homogeneous element $b \in A$. A homotopy F: $\mathscr{M} \to A \otimes \mathbf{R}(t, dt)$ induces a cochain contraction operator

$$\hat{F} \equiv \int_0^1 \circ F: \mathscr{M} \to A.$$

The basic feature of $\hat{F}$ is then (cf. 10.3 of [GM]):

LEMMA 2. *$\hat{F}$ is a linear map of degree -1 such that*

$$d \circ \hat{F} + \hat{F} \circ d = f_1 - f_0.$$

The above constructions can also be done for dgc algebras over the rationals $\mathbf{Q}$, and with this algebraic notion of homotopy, there is a one-to-one correspondence between the homotopy theory of 1-connected dgc-algebras of finite type over $\mathbf{Q}$ and the rational homotopy types of 1-connected CW-complexes of finite type (cf. [L], [S1], and also Chapters X, XI of [GM]). We will need three results from this theory. Let Y denote a 1-connected topological space. A Postnikov tower for Y is a sequence of CW-complexes $\{Y_n | n > 1\}$ and maps

$$p_n: Y_n \to Y_{n-1}, \qquad n > 1, \qquad f_n: Y \to Y_n$$

such that $p_n \circ f_n = f_{n-1}$, f_n is an n-equivalence (i.e., it induces an isomorphism of homotopy groups, $(f_n)_\#$: $\pi_p(Y) \to \pi_p(Y_n)$ for $p < n$), $\pi_p(Y_n) = 0$ for

$p \geq n$, and $p_{n+1}: Y_{n+1} \to Y_n$ is a principal fibration of type (Π_n, n) for the abelian group $\Pi_n = \pi_n(Y)$. We explain this third condition in more detail. Let $K(\Pi_n, n+1)$ denote the Eilenberg-Maclane space for Π_n, then for each n there is a characteristic map $\theta_n: Y_n \to K(\Pi_n, n+1)$ so that Y_{n+1} is weak-homotopy equivalent to the homotopy fiber of θ_n. The characteristic element of the fibration is the class

$$k^{n+1} = \theta_n^*(i_n) \in H^{n+1}(Y_n, \Pi_n),$$

where $i_n \in H^{n+1}(K(\Pi_n, n+1); \Pi_n)$ is the canonical generator as a Π_n-module (cf. Chapter 8, [Sp]).

We return to a minimal algebra $\mathscr{M}$, and recall that X^n denotes a choice of a spanning space for the indecomposable elements of degree n. That is, X^n defines a section of the surjection

$$\mathscr{M}^n \to \mathscr{M}^+/(\mathscr{M}^+)^2 \cap \mathscr{M}^n \cong (QM)^n$$

and so can be identified with $(QM)^n$. The basic theorems of Sullivan's rational homotopy theory are then:

THEOREM A. *Let Y be a 1-connected CW-complex of finite type, and $\mathscr{M}$ a minimal model for a rational deRham model A_Y of Y. Then there is a* **natural** *isomorphism*

$$(Q\mathscr{M})^n \cong \mathrm{Hom}(\pi_n(Y), \mathbf{Q}).$$

Thus, each element of X^n naturally defines a group homomorphism on $\pi_n(Y)$.

THEOREM B. *For Y and $\mathscr{M}$ as above*:

(a) *$\mathscr{M}(n)$ is a minimal model for the n-th stage, Y_n, of the Postnikov tower of Y.*

(b) *The Hirsch extension corresponds to the localization over $\mathbf{Q}$ of the principal fibration*

$$K(\pi_n, n) \to Y_{n+1} \xrightarrow{p_{n+1}} Y_n.$$

Thus, $H^*(Y_n; \mathbf{Q}) \cong H^*(\mathscr{M}(n), d_{\mathscr{M}})$.

(c) *The $(n+1)$-st k-invariant $k^{n+1} \in H^{n+1}(Y_n; \pi_n)$ is identified with the map $d: X^n \to H^{n+1}(\mathscr{M}(n))$ via the isomorphisms*

$$\begin{aligned} H^{n+1}(Y_n; \pi_n(Y) \otimes \mathbf{Q}) &\cong \pi_n(Y) \otimes H^{n+1}(\mathscr{M}(n)) \\ &\cong \mathrm{Hom}\big(X^n, H^{n+1}(\mathscr{M}(n))\big). \end{aligned}$$

THEOREM C. *Let $\mathscr{M}$ be a minimal dgc-algebra, and let $\varphi: A \to B$ be a dgca-morphism which induces isomorphisms on cohomology. Then the induced* *map of mapping spaces,*

$$\varphi_*: \{\mathscr{M}, A\} \to \{\mathscr{M}, B\},$$

induces a bijection of homotopy classes,

$$\varphi_*: [\mathscr{M}, A] \xrightarrow{\cong} [\mathscr{M}, B].$$

For proofs of these results, see either [L] or Chapters X, XI of [GM]. By tensoring the above **Q**-vector spaces with **R**, we get corresponding statements for real algebras. Let us conclude this section with a proof of Lemma 1. For a manifold V, the natural inclusion

$$\Omega_{DR}(V) \bigotimes_{\mathbf{R}} \mathbf{R}(t, dt) \to \Omega_{DR}(V \times \mathbf{R})$$

is evidently an isomorphism on cohomology. It is a standard result of characteristic class theory (cf. [B]) that for concordant foliations $\mathscr{F}_0$ and $\mathscr{F}_1$ and any choice of Bott connections on their normal bundles, there is a Bott connection on the normal bundle to the concordance foliation $\mathscr{F}$ on $V \times \mathbf{R}$ extending the two. Thus, one obtains a homomorphism F': $WO_q \to \Omega_{DR}(V \times \mathbf{R})$ whose restrictions to the slices $V \times \{0\}$ and $V \times \{1\}$ satisfy

$$F_0' = f_{\mathscr{F}_0} \quad \text{and} \quad F_1' = f_{\mathscr{F}_1}.$$

By Theorem C, $f_{\mathscr{F}_0} \circ \rho_q$ and $f_{\mathscr{F}_1} \circ \rho_q$ determine the same class in $[\mathscr{M}O_q, \Omega_{DR}(V)]$, as was to be shown. For framed foliations, obviously the same method works again for the algebra $\mathscr{M}_q$.

3. Obstruction to the existence of a homotopy

In this section, we develop a characteristic class for the obstruction to extending a partial homotopy between two dgca-morphisms. Our main result gives a criterion for the vanishing of the indetermancy ideal for this obstruction, which will be used for the applications to foliation theory, developed in the next section.

Let $(\mathscr{M}, d_{\mathscr{M}})$ denote a minimal dgc-algebra over **R**, and f_0, f_1: $\mathscr{M} \to \mathscr{A}$ two dgca-morphisms. A partial n-homotopy from f_0 to f_1 is a dgca-morphism F_n: $\mathscr{M}(n) \to \mathscr{A} \otimes \mathbf{R}(t, dt)$ such that $i_z \circ F_n = f_z|\mathscr{M}(n)$ for $z = 0, 1$. From such, we construct a degree 0, linear map

$$\gamma^n = \gamma_{F_n}: X^n \to \mathscr{A}, \tag{3.1}$$

$$\gamma^n(x) = f_1(x) - f_0(x) - \hat{F}_n(dx).$$

An algebra $\mathscr{M}$ is r-connected if $\mathscr{M}(r+1) \cong \mathbf{R}$, and one then has

$$d\colon X^n \to \left(\mathscr{M}(n-r)^+\right)^2.$$

Thus, for $\mathscr{M}$ r-connected, the map γ^n can be defined using a partial $(n-r)$-homotopy from f_0 to f_1. The basic properties of γ^n are given by the next two results.

PROPOSITION 1. (a) *γ^n takes values in the cocycles of degree n.*
(b) *γ^n is functorial in $\mathscr{A}$.*
(c) *Given a partial n-homotopy F_n, it is both necessary and sufficient for the existence of a partial $(n+1)$-homotopy F_{n+1} extending F_n that the cohomology classes $[\gamma^n(x)] \in H^n(\mathscr{A})$ vanish for all $x \in X^n$.*

Proof. (a)

$$\begin{aligned} d\gamma^n(x) &= df_1(x) - df_0(x) - d\big(\hat{F}_n(dx)\big) \\ &= f_1(dx) - f_0(dx) - d\big(\hat{F}_n(dx)\big) \\ &= \hat{F}_n(d(dx)) = 0 \end{aligned}$$

by Lemma 2.
(b) Let $g\colon \mathscr{A} \to \mathscr{B}$ be a dgca-morphism. Then

$$\begin{aligned} g^*(\gamma^n(x)) &= g\circ f_1(x) - g\circ f_0(x) - g\big(\hat{F}_n(dx)\big) \\ &= g\circ f_1(x) - g\circ f_0(x) - (g\otimes 1\circ F_n)^\wedge(dx), \end{aligned}$$

where the last equality follows from the rule

$$g\left[\int_0^1 a\otimes t^m dt\right] = \int_0^1 g(a)\otimes t^m dt.$$

But this last expression above is clearly the obstruction associated to the partial n-homotopy $g\circ F_n$.
(c) If F_{n+1} extending F_n exists, then by Lemma 2, $\gamma^n(x)$ is exact, and more precisely $\gamma^n(x) = d_{\mathscr{A}}\hat{F}_{n+1}(x)$. The converse is proved in proposition 10.4 of [GM]. □

Let $(\gamma^n)^*\colon X^n \to H^n(\mathscr{A})$ be the linear map obtained by setting

$$(\gamma^n)^*(x) = [\gamma^n(x)],$$

its cohomology class. Via adjunction, $(\gamma^n)^*$ can also be viewed as an element of $H^n(\mathscr{A}; \mathrm{Hom}(X^n; \mathbf{R}))$ where it clearly corresponds to the desuspension of the Postnikov k^{n+1}-invariant for the lifting of the relative n-homotopy F_n. By part (c) above, $(\gamma^n)^*$ is the obstruction to extending F_n; but to show $(\gamma^n)^*$ obstructs the existence of any homotopy from f_0 to f_1, one must determine to what extent $(\gamma^n)^*$ depends upon the data F_n. This is the customary problem with applying Postnikov theory to obtain characteristic classes. For a related discussion in the context of foliations, see [Shul]. Our next result enables us to overcome this problem.

A minimal algebra $(\mathscr{M}, d_{\mathscr{M}})$ is said to be *N-spherical* if $d_{\mathscr{M}}$ vanishes on $\mathscr{M}(N+1)$. Equivalently, for some choice of the spaces $\{X^n | 1 \le n \le N\}$, each $x \in X^n$ is closed. The rational Hurewicz Theorem states that an r-connected minimal algebra is $2r$-spherical. Another important example is provided by the minimal model $\mathscr{M}O_q$ for WO_q, which is $2q$-spherical. The algebra $\mathscr{M}_q$ is $4q$-spherical.

Let $\mathscr{Q}$ be a subalgebra generated by closed homogeneous indecomposable elements in a minimal algebra $\mathscr{M}$. Then define a subspace of $\mathscr{M}$,

$$\mathscr{Q}' = \{u \in \mathscr{M} | d_{\mathscr{M}} u \in \mathscr{Q}\}.$$

For example, if $\mathscr{M}$ is r-connected, then $\mathscr{M}(3r) \subseteq \mathscr{M}(2r)'$.

THEOREM 1. *Let $\mathscr{Q}$ and $\mathscr{Q}'$ be as above. Let f_0, f_1: $\mathscr{M} \to \mathscr{A}$ be given dgca-morphisms.*

(a) *Suppose that a partial homotopy F: $\mathscr{Q} \to \mathscr{A} \otimes \mathbf{R}(t, dt)$ from $f_0 | \mathscr{Q}$ to $f_1 | \mathscr{Q}$ is given. Then there is a well-defined obstruction morphism γ^*: $\mathscr{Q}' \to H^*(\mathscr{A})$. For $u \in \mathscr{Q}'$ of degree N, the value of $\gamma^*(u)$ in the quotient space*

$$H^N(\mathscr{A}) / (\mathrm{Image}\{f_1^* : \mathscr{Q}(N) \to H^*(\mathscr{A})\} \wedge H^*(\mathscr{A})) \tag{3.2}$$

is independent of the choice of F. Moreover, for $u \in \mathscr{Q}'$ with $du = 0$, then $\gamma^*(u) = f_1^*(u) - f_0^*(u)$.

Proof. The last conclusion is evident from the definition of $\gamma^*(u)$. We need a standard device from simplicial DeRahm theory for the proof of the first statement. Let $\mathscr{N}$ denote a minimal dgc-algebra, and F, F': $\mathscr{N} \to A \otimes \mathbf{R}(t, dt)$ denote two homotopies from f_0 to f_1. Then there exists a dgca-morphism

$$J: \mathscr{N} \to \mathscr{A} \otimes \mathbf{R}(r, dr) \otimes \mathbf{R}(s, ds)$$

such that

$$J|_{(r,s)=(t,0)} = F \quad \text{and} \quad J|_{(r,s)=(0,t)} = F'.$$

The map J is constructed via obstruction theory of minimal models, or can sometimes be written down explicitly. Then let F'' denote the restriction of J to the opposite face, $r + s = 1$:

$$F'' = J|_{\substack{r+s=1\\ t=s}} : \mathscr{N} \to \mathscr{A} \otimes \mathbf{R}(t, dt).$$

Define an operator, integration on the 2-simplex Δ_2, by

$$\int_{\Delta_2} : \mathscr{A} \otimes \mathbf{R}(r, dr) \otimes \mathbf{R}(s, ds) \to \mathscr{A}$$

$$\int_{\Delta_2} a \cdot r^l \cdot s^m dr ds = \frac{(-1)^{|a|}}{(m + l + 2)(l + 1)} \cdot a$$

and let it be zero for other monomials. Then let $\hat{J} = \int_{\Delta_2} \circ J \colon \mathscr{N} \to \mathscr{A}$, a linear map of degree -2.

LEMMA 3 (*Simplicial Stokes' Theorem*). *Let $x \in \mathscr{N}$ be closed. Then*

$$d_{\mathscr{A}} \hat{J}(x) = \hat{F}(x) - \hat{F}'(x) + \hat{F}''(x).$$

Proof. See §10 of [B]. □

We apply Lemma 3 to the maps F and F' to obtain for $u \in \mathscr{Q}'$, $\gamma'(u) - \gamma(u) = \hat{F}''(du) - d_{\mathscr{A}}\hat{J}(du)$, where F'' is a partial homotopy restricted to $\mathscr{N} = \mathscr{Q}$ from f_1 to f_1. The theorem now follows from the next lemma by taking $\mathscr{N} = \mathscr{Q}$ and $F = F''$:

LEMMA 4. *Let $\mathscr{N}$ be a minimal dgc-algebra with trivial differential, and F: $\mathscr{N} \to \mathscr{A} \otimes \mathbf{R}(t, dt)$ a homotopy from $f\colon \mathscr{N} \to \mathscr{A}$ to itself. Then for $x \in \mathscr{N}$ of degree n and decomposable,*

$$\hat{F}(x) \in \operatorname{Image}\Big\{ f^* \colon \bigoplus_{i<n} H^i(\mathscr{N}) \to H^*(\mathscr{A}) \Big\} \cdot H^*(\mathscr{A}). \tag{3.3}$$

Proof. Write $x = \sum_I a_I x_I$, where each x_I is a product of closed indecomposable elements of $\mathscr{N}$. It suffices to establish (3.3) for each $\hat{F}(x_I)$. Let $x_I = x_1 \wedge \cdots \wedge x_s$ with x_k of degree n_k and indecomposable. Define $u_k, v_k \in \mathscr{A} \otimes \mathbf{R}[t]$ by the rule

$$F(x_k) = f(x_k) + u_k + (-1)^{n_k} v_k \, dt.$$

It follows from $dF(x_k) = 0$ that

$$\frac{d}{dt}u_k = d_{\mathscr{A}}v_k, \qquad u_k(0) = u_k(1) = 0, \tag{3.4}$$

where $u(z)$ represents evaluation at $t = z$. Define an $(n_k - 1)$-form $w_k \in \mathscr{A}$ by setting $w_k = \int_0^1 v_k\, dt$ and observe that $d_{\mathscr{A}}w_k = 0$ by (3.4). Define for each x_k an element of $\mathscr{A} \otimes \mathbf{R}(r, dr) \otimes \mathbf{R}(s, ds)$ by setting

$$J(x_k) = f(x_k) + d\left(s \cdot \int_0^r v_k\, dt\right),$$

and observe that

$$\begin{aligned} J(x_k)(1, s) &= f(x_k) + ds \wedge w_k, \\ J(x_k)(t, 1) &= f(x_k) + u_k + v_k\, dt, \\ J(x_k)(0, s) &= J(x_k)(t, 0) = 0. \end{aligned} \tag{3.5}$$

Then let $J(x_I) = J(x_1) \wedge \cdots \wedge J(x_s)$, and observe that integration over r and s yields

$$\hat{J}(x_I) = \int_0^1 \int_0^1 J(x_I) \in \mathscr{A}$$

with degree $n - 2$. By Stokes' Theorem, we have

$$\begin{aligned} d_{\mathscr{A}}\hat{J}(x_I) &= \int_0^1 J(x_I)(t, 1) - \int_0^1 J(x_I)(1, s) \\ &= \hat{H}(x_I) - \sum_{k=1}^{s} \pm f(x_1) \cdots \wedge w_k \wedge \cdots f(x_s) \end{aligned}$$

by (3.5). This last expression lies in the ideal of (3.3) as each w_k is closed, so the proof of Lemma 4 is complete. □

We draw an important Corollary of Theorem 1:

COROLLARY 1. *Let $\mathscr{Q}$ be a subalgebra generated by closed homogeneous indecomposable elements of a minimal algebra $\mathscr{M}$, and let $\mathscr{Q}'$ be as above. If F: $\mathscr{Q} \to \mathscr{A} \otimes \mathbf{R}(t, dt)$ is a partial homotopy between two dgca-morphisms f_0, f_1: $\mathscr{M} \to \mathscr{A}$ such that the obstruction class $[\gamma]$ with values in the quotient in (3.2) is non-zero, then there does not exist a homotopy defined on $\mathscr{M}$ between f_0 and f_1.*

Proof. If a homotopy $\tilde{F}: \mathscr{M} \to \mathscr{A} \otimes \mathbf{R}(t, dt)$ exists between f_0 and f_1, then by restricting $\tilde{F}$ to $\mathscr{G}$, we obtain a characteristic map $\tilde{\gamma}^*$ which is identically zero by Proposition 1(c). However, the quotient classes $[\gamma]$ and $[\tilde{\gamma}]$ agree by Theorem 1, implying $[\gamma] = 0$ which contradicts our hypothesis. □

4. Tertiary classes for foliations

For the remainder of this paper, V will denote a connected smooth manifold without boundary and having the homotopy type of a finite CW complex. Let $\mathrm{Fol}_q(V)$ denote the set of concordance classes of smooth (i.e., C^∞) codimension q foliations on V, and $\mathrm{Fol}_q^+(V)$ the subset of foliations with orientable normal bundle. In this section we introduce the subsets of $\mathrm{Fol}_q(V)$ on which the tertiary homotopy invariants are defined. Our basic tools are the Postnikov obstruction classes of §3, and the theory of Haefliger-Gromov-Phillips-Thurston relating concordance classes of foliations on V to sets of homotopy classes of maps into classifying spaces.

Let $B\Gamma_q$ denote the Haefliger classifying space for smooth codimension-q foliations [Hae]. A foliation $\mathscr{F}$ on V determines a well-defined homotopy class of maps $c_{\mathscr{F}}: V \to B\Gamma_q$, such that if two foliations are concordant, then their classifying maps are homotopic. There is a natural map $\nu: B\Gamma_q \to BO_q$ such that

$$\nu \circ c_{\mathscr{F}}: V \to BO_q$$

classifies the normal bundle to $\mathscr{F}$. One says that ν classifies the universal normal bundle on $B\Gamma_q$. Given a map $\tau: V \to BO_q$, let $\{V, B\Gamma_q\}_\tau$ denote the set of continuous maps $c: V \to B\Gamma_q$ for which $\nu \circ c$ is homotopic to τ, and topologize the set with the usual function space topology. Similarly, let $\mathrm{Fol}_q(V)_\tau$ denote the concordance classes of foliations whose normal bundles are classified by a map homotopic to τ. Then the above discussion implies there is a well-defined map $\mathrm{Fol}_q(V)_\tau \to [V, B\Gamma_q]_\tau$.

Deep theorems of Gromov-Phillips (for V open) and Thurston (for V compact) provide a converse to the above discussion (cf. Chapter 4 of [La 1]). We recall the precise result. Let $T: V \to BO_m$ classify the tangent bundle to V, and suppose that T factors:

$$V \xrightarrow{\tau \times l} BO_q \times BO_{m-q} \longrightarrow BO_m$$

so that τ determines a rank q-sub-bundle of TV. The main theorem is that *if* there is a lifting $c: V \to B\Gamma_q$ so that $\nu \circ c \simeq \tau$, then there is a foliation $\mathscr{F}_c$ of codimension q on V for which $c_{\mathscr{F}} \simeq c$. Moreover, homotopic lifts yield concordant foliations. Thus, for τ determining a sub-bundle of TV, there is a bijection

$$\mathrm{Fol}_q(V)_\tau \mapsto [V, B\Gamma_q]_\tau. \tag{4.1}$$

From (4.1) it follows that the task of (abstractly) constructing foliations on a manifold V can be broken into two steps:

(a) Produce a splitting (τ, l): $V \to BO_q \times BO_{m-q}$

(b) Characterize the lifts c: $V \to B\Gamma_q$ of τ.

The first step is a relatively easy problem about the differential topology of V. The second step is traditionally solved via Postnikov tower constructions (see the next section).

There is a special case of (4.1) worth noting. Suppose that TV contains a rank-q trivial bundle $\varepsilon^q \subset TV$; then we can take τ: $V \to BO_q$ to be the constant map onto a basepoint $* \in BO_q$. In this case we use the notation for (4.1)

$$\mathrm{Fol}_q(V)_* \mapsto [V, B\Gamma_q]_*. \tag{4.1$'$}$$

Let $\mathscr{Q}$ be a subalgebra of $\mathscr{M}O_q$ generated by closed indecomposable homogeneous elements. For $c \in [V, B\Gamma_q]$, let f_c: $WO_q \to \Omega(V)$ denote the characteristic homomorphism for the foliation $\mathscr{F}_c$. (This works even when $\tau = \nu \circ c$ is not a sub-bundle of TV (cf. Chapter 3, [La 1]).) Then define the subset

$$[V, B\Gamma_q|\mathscr{Q}]_\tau = \{c \in [V, B\Gamma_q]_\tau \quad \text{such that} \quad (f_c \circ \rho_q)^*|\mathscr{Q} \equiv 0\}. \tag{4.2}$$

and also let

$$[V, B\Gamma_q|WO_q]_\tau \equiv \{c \in [V, B\Gamma_q]_\tau \quad \text{such that} \quad f_c^* \equiv 0\}.$$

THEOREM 2. *Let $\mathscr{Q}$ be as above, and let $u \in \mathscr{Q}'$. That is, u is an indecomposable element of $\mathscr{M}O_q$ with $du \in \mathscr{Q}$. Then there is a well-defined tertiary characteristic class*

$$\gamma^*(u)\colon [V, B\Gamma_q|\mathscr{Q}] \to H^*(V; \mathbf{R})$$

which is functorial in V. If u is homogeneous of degree p, then $\gamma^*(u)$ takes values in $H^p(V; \mathbf{R})$. Moreover, γ^* induces a universal map

$$\gamma\colon [V, B\Gamma_q|\mathscr{Q}] \to H^*(V; \mathrm{Hom}(\mathscr{Q}'; \mathbf{R})).$$

Next, let τ define a sub-bundle of TV. For $c \in [V, B\Gamma_q|\mathscr{Q}]_\tau$, let $\gamma_c^*(u)$ denote the corresponding tertiary class in $H^*(V; \mathbf{R})$ and

$$\gamma_c^*\colon \mathscr{Q}' \to H^*(V; \mathbf{R})$$

the tertiary characteristic homomorphism for c and $\mathscr{Q}'$.

COROLLARY 2. *Let $\mathscr{Q}$ be as above and $u \in \mathscr{Q}'$. Suppose that $c_0, c_1 \in [V, B\Gamma_q|\mathscr{Q}]$ satisfy $\gamma_{c_0}^*(u) \neq \gamma_{c_1}^*(u)$. Then $\mathscr{F}_{c_0}$ and $\mathscr{F}_{c_1}$ are not concordant.*

Proof of Theorem 2. Given $c\colon V \to B\Gamma_q$ with characteristic map $f_c \circ \rho_q\colon \mathscr{M}O_q \to \Omega(V)$, we define $\gamma^*(u)[c]$ to be the Postnikov obstruction class $\gamma_c^*(u)$ obtained from a homotopy, F, between $f_c \circ \rho_q|\mathscr{Q}$ and the zero map. The homotopy F exists because $\mathscr{Q}$ is itself a minimal dgc-algebra (with trivial differential) and $(f_c \circ \rho_q)^*|\mathscr{Q}$ is identically zero, so that a choice for each indecomposable generator $z_i \in \mathscr{Q}$ of a form $w_i \in \Omega(V)$ with $dw_i = f_c \circ \rho_q(z_i)$ determines a homotopy F from $f_0 \circ \rho_q$ to 0.

By Theorem 1 and our definition (4.2), the value of $\gamma_c^*(u)$ does not depend on the choice of F, for u homogeneous. Extend γ_c^* by linearity to all of $\mathscr{Q}'$ to obtain the tertiary characteristic map γ_c. □

We next discuss how the Postnikov invariants yield tertiary classes for framed codimension-q foliations. It is slightly more convenient to work with $B\Gamma_q^+$, the classifying space of foliations with orientable normal bundles. Let $B\overline{\Gamma}_q$ denote the principal SO_q-fibration over $B\Gamma_q^+$ associated to the universal map $\nu\colon B\Gamma_q^+ \to BSO_q$. It is well known that $B\overline{\Gamma}_q$ is weak homotopy equivalent to the homotopy fiber of ν (cf. [H3]). For $c_{\mathscr{F}} \in [V, B\Gamma_q^+]_*$ classifying a foliation $\mathscr{F}$, then a lift $\bar{c}_{\mathscr{F}}\colon V \to B\overline{\Gamma}_q$ classifies $\mathscr{F}$ with a choice of framing of the normal bundle Q.

There is a Puppe sequence of fibrations

$$SO_q \to B\overline{\Gamma}_q \to B\Gamma_q^+ \to BSO_q \tag{4.3}$$

inducing an exact sequence of sets

$$[V, SO_q] \to [V, B\overline{\Gamma}_q] \to [V, B\Gamma_q^+]_* \to * \tag{4.4}$$

Note that $[V, SO_q]$ has a natural group structure, and acts on the next term in the sequence. The exactness of (4.4) at $[V, B\overline{\Gamma}_q]$ is the statement that the orbits of this group action are precisely the inverse images of elements of $[V, B\Gamma_q^+]_*$. This is the homotopic formulation of the principle that $[V, B\Gamma_q^+]_*$ counts concordance classes of foliations with trivial normal bundles, while $[V, B\overline{\Gamma}_q]$ counts the homotopy classes of normally framed foliations, which are parametrized by the homotopy gauge group $[V, SO_q]$. Thus, invariants for framed foliations may separate into distinct classes maps $\bar{c}: V \to B\bar{\Gamma}_q$, which coalesce in $[V, B\Gamma_q^*]_*$, but the coalescing is precisely determined by the group $[V, SO_q]$ and (4.4). These points are just elaborations on the meaning of (4.4), but it is useful to make these "well known" facts precise, as they are applied in §§7 and 8 below.

Recall that $\bar{c}: V \to B\bar{\Gamma}_q$ determines a dgca-morphism $f_{\bar{c}}: W_q \to \Omega(V)$, and we can apply the Postnikov invariants to the morphism

$$f_{\bar{c}} \circ \rho_q: \mathscr{M}_q \to \Omega(V).$$

The same method of proof as used for Theorem 2 now yields:

THEOREM 3. *Let $\mathscr{Q} \subset \mathscr{M}_q$ be a subalgebra generated by closed indecomposable homogeneous elements, and let $u \in \mathscr{Q}'$. Then there is a well-defined tertiary characteristic map for framed foliations*

$$\gamma^*(u): [V, B\Gamma_q|\mathscr{Q}] \to H^*(V; \mathbf{R})$$

which induces a universal map

$$\gamma: [V, B\Gamma_q|\mathscr{Q}] \to H^*(V; \mathrm{Hom}(\mathscr{Q}'; \mathbf{R})). \qquad \square$$

The algebra W_q is $2q$-connected, so we can take $\mathscr{Q} = \mathscr{M}_q(4q + 1)$ and then $\mathscr{M}_q(6q + 1) \subset \mathscr{Q}'$. Recall that $\pi^p(W_q)$ denotes the space of indecomposables for $\mathscr{M}_q$ in degree p. With the obvious notation for maps vanishing on $H^*(W_q)$ and $H^p(W_q)$, we obtain:

COROLLARY 3. (a) *For all $p \leq 6q$, there is a well-defined tertiary map*

$$\gamma^p: [V, B\bar{\Gamma}_q|H^*(W_q)] \to H^p(V; \pi^p(W_q))$$

(b) *For $p = 4q + 1$, there is a well-defined tertiary map*

$$\gamma^{4q+1}: [V, B\bar{\Gamma}_q|H^{2q+1}(W_q)] \to H^{4q+1}(V; \pi^{4q+1}(W_q)).$$

Corollaries 2 and 3 just indicate some of the possible choices for the subalgebra $\mathscr{Q}$. Other choices will also be used in §7 below for which u has degree p greater than the range of degrees for which $H^*(W_q)$ is non-trivial.

5. Applications of tertiary classes-universal constructions

In this section, we give some abstract constructions of elements in $[V, B\Gamma_q]_\tau$. For a manifold V possessing a codimension-q foliation $\mathscr{F}$ with normal bundle τ, sufficient conditions on V are described which imply the existence of a map (to be defined below)

$$\hat{c}_\sigma \colon \pi_p(B\overline{\Gamma}_q) \to [V, B\Gamma_q]_\tau.$$

A key point is that the evaluation of the tertiary classes on the image of $\hat{c}_\sigma$ can be reduced to a calculation of dual homotopy invariants, yielding the two main results, Theorems 4 and 5, of this section.

First, consider the case when TV admits a rank-q trivial sub-bundle, so that the constant map $*\colon V \to B\Gamma_q$ induces a codimension-q foliation $\mathcal{F}_*$ on V. Let $\Pi^p(V) = [V, S^p]$ be the p-th cohomotopy group.

Proposition 2. *Let V have the homotopy type of a finite CW complex X. For p odd, the cohomology Hurewicz homomorphism*

$$\Pi^p(V) \to H^p(V; \mathbf{Z})$$

is onto a subgroup of finite index. For p even and X of dimension less than $2p$, the same conclusion holds.

Proof. Let $a \in H^p(V; \mathbf{Z})$. There is a classifying map $g_a\colon V \to K(\mathbf{Z}, p)$ such that $g_a^*(i_p) = a$, where i_p is the canonical general of $H^p(K(\mathbf{Z}, p), \mathbf{Z})$. The inclusion of the p-cell, $S^p \to K(\mathbf{Z}, p)$, induces a rational homotopy equivalence for p odd, and is a $(2p-1)$-equivalence for p even. We can assume that g_a is the trivial map on the $(p-1)$-skeleton of X. The map g_a on the p-skeleton X^p lifts to a map $\tilde{g}_a\colon X^p \to S^p$. Since X has only finitely many cells, and $\pi_r(S^p)$ is a finite torsion group for $r > p$, p odd, (and $2p-1 > r > p$ for p even) a sufficiently high power of $\tilde{g}_a\colon X^p \to S^p$ lifts to a map $\tilde{g}_a\colon X \to S^p$ (cf. the proof of Lemma 1.2 of [H2].) Thus, each element $a \in H^p(V; \mathbf{Z})$ has a power in the image of the Hurewicz map, and the proposition follows. □

We say a map $\sigma\colon V \to S^p$ is *non-torsion* if the class $\sigma^*(i_p) \in H^p(V; \mathbf{Z})$ is not torsion.

For each $\sigma \in \Pi^p(V)$, define

$$\hat{*}_\sigma \colon \pi_p(B\overline{\Gamma}_q) \to [V, B\Gamma_q^+]_*$$

by setting, for $\alpha \in \pi_p(B\Gamma_q)$,

$$\hat{*}_\sigma(\alpha) \colon V \xrightarrow{\sigma} S^p \xrightarrow{\alpha} B\overline{\Gamma}_q \xrightarrow{i} B\Gamma_q^+.$$

This extends to a pairing

$$\hat{*}: \Pi^p(V) \times \pi_p(B\overline{\Gamma}_q) \to [V, B\Gamma_q^+]_*.$$

For p odd and $H^p(V; \mathbf{Q}) \neq 0$, one knows that non-torsion σ exist by Proposition 2, and this becomes a very effective method of constructing elements of $[V, B\Gamma_q^+]_*$.

For the case when τ is not homotopic to a constant, we must first assume the existence of a map $c \in [V, B\Gamma_q]_\tau$.

We say that V has a p-splitting, σ, if there exists σ: $V \to V \vee S^p$ satisfying:

(5.1a) $\sigma^*(i_p) \in H^p(V; \mathbf{Q})$ is non-trivial, where $i_p \in H^p(V \vee S^p; \mathbf{Q})$ is the non-zero integral class arising from the S^p-summand.

(5.1b) The composition

$$p_1 \circ \sigma: V \to V \vee S^p \to V \vee pt = V$$

is homotopic to the identity.

Given c and a p-splitting, σ, as above, there is a well-defined map

$$\hat{c}_\sigma: \pi_p(B\overline{\Gamma}_w) \to [V, B\Gamma_q^+]_\tau,$$

defined for $\alpha \in \pi_p(B\overline{\Gamma}_q)$ to be the composition

$$\hat{c}_\sigma(\alpha): V \xrightarrow{\sigma} V \vee S^p \xrightarrow{c \vee \alpha} B\Gamma_q^+ \vee B\overline{\Gamma}_q \xrightarrow{\mathrm{id} \vee i} B\Gamma_q^+. \tag{5.2}$$

It is only necessary to check that $\nu \circ \hat{c}_\sigma(\alpha) \simeq \tau$, and this follows from (5.1b).

Note that if $\mathcal{G}$ is a subalgebra generated by closed elements of degree less than p, and $c \in [V, B\Gamma_q^+ | \mathcal{G}]_\tau$, then the image of $\hat{c}_\sigma$ will also lie in this subset.

Remark 1. The customary abstract approach to calculating $[V, B\Gamma_q^+]_\tau$ is to replace the map ν: $B\Gamma_q^+ \to BSO_q$ with a Postnikov tower approximation (cf. Chapter 8, [Sp]):

$$B\Gamma_q^+ \to \cdots Y_n \to Y_{n-1} \to \cdots Y_1 \to BSO_q$$

such that $\varprojlim Y_n$ is weak homotopy equivalent to $B\Gamma_q^+$. Then there is a filtration with terms $[V, Y_n]_{\tau_n}$ increasing to $[V, B\Gamma_q^+]_\tau$, which is the $E^1_{0,n}$-term of a spectral sequence converging more generally to $H^*(\{V, B\Gamma_q^+\}_\tau)$. The differentials arise from the boundary maps in the fibrations

$$H^p\big(V;\pi_p\big(B\overline{\Gamma}_q\big)\big) \cong \big[V, K\big(\pi_p(B\Gamma_q), p\big)\big] \to \big[V, Y_{p+1}\big] \to \big[V, Y_p\big].$$

Thus, one obtains natural candidates for elements in $[V, B\Gamma_q^+]_\tau$ from the fiber groups $H^p(V;\pi_p(B\overline{\Gamma}_q))$. In general, it is difficult to determine which classes in these fiber groups live to E^∞ in the spectral sequence. The idea of introducing p-splittings is that we identify a family of classes in the fiber groups which are known to persist to E^∞ by the direct construction (5.2). □

We next study the relation between the classes in the image of (5.2) for a given c, the tertiary classes for c and the dual homotopy invariants $h^\#\colon \pi^*(WO_q) \to \pi^*(B\Gamma_q)$. For $u \in \mathscr{M}O_q$, let $[u] \in \pi^p(WO_q)$ denote its equivalence class.

THEOREM 4. *Let $\mathscr{Q}$ be as before, with all generators of degree less than p, and let $u \in \mathscr{Q}'$. Let $c \in [V, B\Gamma_q|\mathscr{Q}]_\tau$ and suppose that $\alpha, \beta \in \pi_p(B\Gamma_q)$ are given with*

$$h^\#([u])(i_\#\alpha) \neq h^\#([u])(i_\#\beta),$$

where $i\colon B\Gamma_q \to B\Gamma_q$. Then for all p-splittings σ of V,

$$\gamma^*_{\hat{c}_\sigma(\alpha)}(u) \neq \gamma^*_{\hat{c}_\sigma(\beta)}(u) \quad \text{in} \quad H^p(V;\mathbf{R}).$$

In particular, $\hat{c}_\sigma(\alpha)$ and $\hat{c}_\sigma(\beta)$ are not homotopic in $[V, B\Gamma_q]_\tau$.

Proof. By the functoriality of the tertiary invariants, it will suffice to show that $\gamma^*_{c\vee\alpha}(u)$ and $\gamma^*_{c\vee\beta}(u) \in H^p(V \vee S^p;\mathbf{R})$ have distinct values when projected onto the $H^p(S^p;\mathbf{R})$ factor. Again by functoriality, this is equivalent to showing that $\gamma^*_\alpha(u)$ and $\gamma^*_\beta(u) \in H^p(S^p;\mathbf{R})$ have distinct values. To calculate these γ-invariants, we can assume the partial homotopy is the constant map, for the minimal model $\mathscr{M}_{S^p}(p)$ is $(p-1)$-connected. Thus, the only term arising in the definition of $\gamma^*_\alpha(u)$ is $[\mathscr{M}\alpha(u)] \in H^p(S^p;\mathbf{R})$. But this is exactly $h^\#_\alpha(u)$ via the identification $\pi^p(S^p) \cong H^p(S^p;\mathbf{R})$. Similarly for β, and so by hypothesis, $\gamma^*_\alpha(u) = h^\#_\alpha(u) \neq h^\#_\beta(u) = \gamma^*_\beta(u)$. □

Actually, Theorem 4 can be cast in a stronger form, which we give as a corollary to the above proof. Let $\mathscr{Q} \subset \mathscr{M}O_q$ be as before, and let $\{u_1,\ldots,u_s\} \subset \mathscr{Q}'$ be indecomposable elements of degree p. Let $\bar{u} = (u_1,\ldots,u_s)$, and let $h^\#(\bar{u})\colon \pi_p(B\overline{\Gamma}_q) \to \mathbf{R}^s$ denote the evaluation of the corresponding dual homotopy invariants.

COROLLARY 4. *Let $\{u_1, \ldots, u_s\}$ be as above. Suppose that there exists a p-splitting σ of V, a class $c \in [V, B\Gamma_q | \mathscr{Q}]_\tau$ and a subset*

$$W(p,q) \subset \pi_p\big(B\overline{\Gamma}_q\big)$$

for which $h^\sharp(\bar{u}) | W(p,q)$ is a bijection onto $\mathbf{R}^s$. Then the composition

$$W(p,q) \xrightarrow{\hat{c}_\sigma} \big[V, B\Gamma_q | \mathscr{Q}\big]_\tau \xrightarrow{\gamma^p(\bar{u})} H^p(V; \mathbf{R}^s)$$

is a bijection. In particular, $\hat{c}_\sigma$ is injective on $W(p,q)$ and $\gamma^p(\bar{u})$ is surjective.

We formulate a version of Corollary 4 for maps into the classifying space $B\overline{\Gamma}_q$.

THEOREM 5. *Let $\mathscr{Q} \subset \mathscr{M}_q$ be a subalgebra generated by closed indecomposable homogeneous elements of degree less than p. Suppose that $H^p(V; \mathbf{Q})$ is not trivial, and for p even that in addition, V has the homotopy type of a CW complex of dimension less than $2p$. Let $\{u_1, \ldots, u_s\} \subset \mathscr{Q}'$ be indecomposable elements such that there exists a set $W(p,q) \subset \pi_p(B\overline{\Gamma}_q)$ for which $h^\sharp(\bar{u})$: $W(p,q) \to \mathbf{R}^s$ is a bijection. Then, for each $\sigma \in \Pi^p(V)$ with non-trivial image in $H^p(V; \mathbf{Q})$, there is an inclusion*

$$\hat{*}_\sigma : W(p,q) \to \big[V, B\overline{\Gamma}_q | \mathscr{Q}\big] \subset \big[V, B\overline{\Gamma}_q\big].$$

Moreover, the evaluation of $\gamma^p(\bar{u})$ on the image of $\hat{*}_\sigma$ in $[V, B\overline{\Gamma}_q | \mathscr{Q}]$ is faithful.

6. Examples in codimension one

In this and the next two sections, we give applications of the tertiary classes to the construction of non-concordant foliations. The case of codimension-one is most elementary, for all of the γ-invariants are secondary classes corresponding to multiples of the Godbillon-Vey class.

THEOREM 6. *Let V have zero Euler characteristic, so that there exists a non-vanishing vector field on V. Then for each class $\alpha \in H^3(V; \mathbf{Q})$ and real number $\lambda \in \mathbf{R}$, there exists a foliation $\mathscr{F}$ on V such that*

$$f_{\mathscr{F}}^*(h_1 c_1) = \lambda \cdot \alpha \in H^3(V; \mathbf{R}).$$

Proof. There exists an integral class $\beta \in H^3(V; \mathbf{Z})$ and an integer m such that $\alpha = (1/m) \cdot \beta$ in $H^3(V; \mathbf{Q})$. Let g_β: $V \to K(\mathbf{Z}; 3)$ classify β. Then as in §5, a multiple $\tilde{g}_\beta$: $V \to K(\mathbf{Z}; 3)$ factors through the inclusion $S^3 \to K(\mathbf{Z}; 3)$.

Let $\alpha = p^{-1}(\tilde{g}_\beta^*(i_3))$ in $H^3(V;\mathbf{Q})$, then by results of Thurston one can choose $c\colon S^3 \to B\Gamma_1^+$ for which $f_c^*(h_1c_1) = p \cdot \lambda \cdot [i_3]$, where $i_3 \in H^3(S^3;\mathbf{Z})$ is a generator. Then $f_{\mathscr{F}} = c \circ \tilde{g}_\beta$ determines a codimension-one foliation as desired. □

The above examples are the best possible using that ρ_3 surjects onto $\mathbf{R}$. To obtain more, for example that $f_c^*(h_1c_1) \in H^3(V;\mathbf{R})$ can be arbitrary, one must show that the map $B\Gamma_1^+ \to K(\mathbf{R};3)$ classifying the Godbillon-Vey class admits a cross-section. For a further discussion, see [M].

7. Examples in higher codimension

For codimension two and above, the algebra $\mathscr{M}O_q$ has great complexity (cf. [HK], [Sh]), and there are many classes for which the tertiary construction can be applied. Moreover, there are large families of indecomposable classes in $\mathscr{M}O_q$ whose corresponding dual homotopy invariants define surjections of $\pi_*(B\overline{\Gamma}_q)$ onto real vector spaces. We first recall the data needed, then describe examples.

Fact 1 (cf. [H2]). Let $z_1 = h_1c_1^2$, $z_2 = h_1c_2$ denote the Vey basis for $H_5(WO_2)$. For the vector $\bar{z} = (z_1, z_2)$, let $\Delta(\bar{z})\colon H^5(B\Gamma_2;\mathbf{Z}) \to \mathbf{R}^2$ denote evaluation on 5-cycles. Then composition with the Hurewicz homomorphism $\mathscr{H}$ yields a surjection from $\pi_5(B\overline{\Gamma}_2)$ onto $\mathbf{R}^2$:

$$\pi_5(B\overline{\Gamma}_2) \xrightarrow{i_\sharp} \pi_5(B\Gamma_2) \xrightarrow{\mathscr{H}} H_5(B\Gamma_2;\mathbf{Z}) \xrightarrow{\Delta(\bar{z})} \mathbf{R}^2 \to 0.$$

Moreover, there is a set $W(5,2) \subset \pi_5(B\overline{\Gamma}_2)$ on which $\Delta(\bar{z}) \circ \mathscr{H} \circ i_\sharp$ induces a bijection onto $\mathbf{R}^2$.

The above result admits a generalization to all codimensions $q > 2$.

Fact 2 (cf. [H1], [H2]). There is a set

$$V_q = \{z_1, \ldots, z_{r(q)}\} \subset H^{2q+1}(WO_q)$$

of linearly independent classes such that:

(a) $r(q) \geq q$.

(b) The corresponding vector-valued functional $\Delta(\bar{z})$ on $H_{2q+1}(B\Gamma_q;\mathbf{Z})$ when restricted to the image of

$$\pi_{2q+1}(B\overline{\Gamma}_q) \xrightarrow{i_\sharp} \pi_{2q+1}(B\Gamma_q) \xrightarrow{\mathscr{H}} H_{2q+1}(B\Gamma_q;\mathbf{Z})$$

is onto $\mathbf{R}^{r(q)}$. Moreover, there is a subset

$$W(2q+1,q) \subset \pi_{2q+1}(B\overline{\Gamma}_q)$$

on which $\Delta(\bar{z})\circ \mathscr{H} \circ i_{\#}$ is a bijection onto $\mathbf{R}^{r(q)}$. □

The classes $\{z_1, \dots, z_{r(1)}\}$ from Facts 1 and 2 have the important additional property that each has the form $h_1 c_J$ where degree $c_J = 2q$. Thus, the products $z_i z_j$ in WO_q are identically zero. Choose indecomposable cocycles $\{x_i\} \subset \mathscr{M}O_q$ such that $\rho_q(x_i) = z_i$. Then for each $1 \le i < j \le r(q)$, there exists an indecomposable homogeneous class $u_{ij} \in \mathscr{M}O_q$ of degree $4q+1$ such that

$$du_{ij} = -x_i \wedge x_j \operatorname{mod}\langle p_J \rangle,$$

where $\langle p_J \rangle$ denotes the ideal in $\mathscr{M}O_q$ generated by the even Chern classes. Moreover, the terms in du_{ij} containing a factor of the p_J have as other factors secondary classes from $H^*(WO_q)$ (cf. [HK]).

Fact 3 (cf. [H1]). For $q \ge 2$, set

$$s(q) = \frac{1}{2}\left(r(q)^2 - r(q)\right), \quad \bar{u} = \left(u_{12}, u_{13}, \dots, u_{r(q)-1, r(q)}\right).$$

Then evaluation of the dual homotopy classes

$$h^{\#}(\bar{u}) \colon \pi_{4q+1}(B\overline{\Gamma}_q) \to \mathbf{R}^{s(q)}$$

yields a surjection. Let $W(4q+1,q)$ denote a subset on which $h^{\#}(\bar{u})$ restricts to a bijection onto.

Before giving the more abstract systematic examples, let us consider the case for T^9 from the introduction.

THEOREM 7. *For each trivial rank-2 bundle $\varepsilon^2 \subset T(T^9)$, there is a family of codimension-2 foliations $\{\mathscr{F}_\lambda | \lambda \in \mathbf{R}\}$, all of whose normal bundles are homotopic to this embedded subbundle, and for which*:

(a) *The secondary maps*

$$f_\lambda^* \colon H^*(WO_2) \to H^*(T^9; \mathbf{R})$$

are identically zero.

(b) *The dual homotopy classes*

$$f_\lambda^\sharp\colon \pi^*(WO_2) \to \pi^*(T^9)$$

are identically zero.

(c) *The indecomposable* $u = u_{12}$ *defines a tertiary invariant*

$$\gamma_{\mathcal{F}_\lambda}(u) \in H^9(T^9;\mathbf{R}) \cong \mathbf{R}$$

which has the value λ *on* $\mathcal{F}_\lambda$, *and hence the* $\mathcal{F}_\lambda$ *are pairwise non-concordant.*

Proof. Let $\alpha_\lambda\colon S^9 \to B\overline{\Gamma}_2$ satisfy $h^\sharp([u_{12}])(\alpha_\lambda) = \lambda \in \pi^9(S^9) \cong \mathbf{R}$, which exists by Fact 3. Let $\sigma\colon T^9 \to S^9$ denote the map obtained by collapsing the 8-skeleton of T^9 to a point. Then clearly each $c_\lambda = \alpha_\lambda \circ \sigma \in [T^9, B\Gamma_2 | H^*(WO_2)]$ and we evaluate $\gamma^*_{\mathcal{F}_\lambda}(u)$ using functoriality and the Hurewicz identification of $\gamma^*_{\alpha_\lambda}(u)$ and $h^\sharp_{\alpha_\lambda}([u])$ to obtain (c). □

In Theorem 7, we can replace T^9 by any manifold V satisfying the following condition: there exists a codimension 2 foliation $\mathcal{F}$ on V such that $f^*_{\mathcal{F}} = 0$ and $h^\sharp_{\mathcal{F}} = 0$, and V admits a 9-splitting. For example, given any compact orientable 7-manifold M_0, and any other manifold W, then $V = S^2 \times M_0 \times W$ satisfies the conditions above. We leave it to the reader to construct further examples in codimension two, and next consider the situation for codimension $q > 2$.

THEOREM 8. *Suppose that either:*

(a) *There exists a codimension-q foliation* $\mathcal{F}$ *on* V *for which*

$$f^*_{\mathcal{F}}\colon H^*(WO_q) \to H^*(V;\mathbf{R})$$

is zero, and there exists a $(4q+1)$*-splitting* σ *of* V

(b) TV *contains a rank-q trivial subbundle and there exists a non-torsion map* $\sigma\colon V \to S^{4q+1}$ (*that is,* $H^{4q+1}(V;\mathbf{Q}) \neq 0$). *Then for each real vector* $\bar{\lambda} = (\lambda_{ij}) \in \mathbf{R}^{s(q)}$, *there exists a codimension-q foliation* $\mathcal{F}_{\sigma,\bar{\lambda}}$ *such that*

$$\gamma_{\mathcal{F}_{\sigma,\bar{\lambda}}}(u_{ij}) = \lambda_{ij} \cdot \sigma^*(i_{4q+1}) \in H^{4q+1}(V;\mathbf{R}).$$

In particular, σ *determines an inclusion of sets*

$$\hat{c}_\sigma\colon W(4q+1, q) = \mathbf{R}^{s(q)} \to \left[V, B\Gamma_q\right]_\tau = \mathrm{Fol}_q(V)_\tau$$

where $\tau = \nu \circ c_{\mathcal{F}}$ *in case a) or* $\tau = *$ *in case b).*

Proof. This follows from combining Fact 3 with Corollary 4 of §5. □

Note that in Theorem 8 case b), if we also assume that the map σ: $V \to S^{4q+1}$ induces the trivial map on rational homotopy, then the dual homotopy invariants of all the foliations $\mathscr{F}_{\sigma,\bar{\lambda}}$ will be zero. If we require that each $\alpha_{\bar{\lambda}}$: $S^{4q+1} \to B\overline{\Gamma}_q$ induces the zero map on secondary classes, then all secondary classes of $\mathscr{F}_{\sigma,\bar{\lambda}}$ also vanish. Again, it is easy to produce explicit examples where this holds; the simplest examples are for $V = T^p$ with $p \geq 4q + 1$.

For the classifying space $B\overline{\Gamma}_q$ of framed foliations, the homotopy permanence principle (Corollary 6.10 of [H1]; see also the treatment of the cohomology permanence principle in [H3]) yields many more secondary classes from $H^*(W_q)$ which detect spherical cycles in $H_*(B\overline{\Gamma}_q; \mathbf{Z})$. A rather complete list of the realizable variable classes is described in Remarks 2.4 and 2.9 of [H2].

We consider here just the case $q = 3$, and leave the general construction for $q \geq 3$ to the reader. Consider the following elements in a Vey basis for W_3:

$$\begin{aligned} z_1 &= h_1 c_1^3 && \text{of degree } 7 \\ z_2 &= h_1 c_1 c_2 && \text{of degree } 7 \\ z_3 &= h_1 h_2 c_1^3 && \text{of degree } 10 \\ z_4 &= h_1 h_2 c_1 c_2 && \text{of degree } 10. \end{aligned}$$

In W_3, all of these cocycles have pairwise trivial products, so that for lifts $\{x_1, x_2, x_3, x_4\}$ to $\mathscr{M}_3$, there are indecomposable classes $\{u_{ij} | 1 \leq i < j \leq 4\}$ with $du_{ij} = -x_i \wedge x_j$. Moreover, by Theorem 2.7b) of [H2], for $\bar{v}_1 = (u_{12})$, $\bar{v}_2 = (u_{13}, u_{14}, u_{23}, u_{24})$, $\bar{v}_3 = (u_{34})$ there are surjections

$$\begin{aligned} h^\#[\bar{v}_1]&: \pi_{13}(B\Gamma_3) \to \mathbf{R} \\ h^\#[\bar{v}_2]&: \pi_{16}(B\Gamma_3) \to \mathbf{R}^4 \\ h^\#[\bar{v}_3]&: \pi_{19}(B\Gamma_3) \to \mathbf{R} \end{aligned}$$

Consider the case of $\bar{v}_3 = u_{34}$.

THEOREM 9. *Let TV contain a trivial rank-3 subbundle, and suppose that $H^{19}(V; \mathbf{Q}) \neq 0$. Then for each non-torsion σ: $V \to S^{19}$ and real $\lambda \in \mathbf{R}$, there is a codimension-3 foliation $\mathscr{F}_{\sigma,\lambda}$ on V for which*

$$\gamma_{\mathscr{F}_{\sigma,\lambda}}(u_{34}) = \lambda \cdot \sigma^*(i_{19}) \in H^{19}(V; \mathbf{R}).$$

Thus, $\hat{*}_\sigma\colon \pi_{19}(B\overline{\Gamma}_3) \to [V, B\overline{\Gamma}_3]$ *has uncountable image. Moreover,* $\mathrm{Fol}_3(V)_*$ *contains an uncountable set of distinct concordance classes of foliations.*

Proof. Theorem 5 and the given fact about the range of $h^\sharp([u_{34}])$ on $\pi_{19}(B\overline{\Gamma}_3)$ yields the first part of the claim. For the conclusion about $\mathrm{Fol}_3(V)_*$, note that $[V, B\Gamma_3^+]_*$ is the quotient of $[V, B\overline{\Gamma}_3]$ by the orbits of the group $[V, SO_3]$. As this latter group is countable, this second claim follows. Note that if $H^3(V;\mathbf{Q}) = 0$, then $[V, SO_3]$ is a finite group, so that any measurable structure on the set of maps in $[V, B\overline{\Gamma}_3]$ constructed above will be preserved into the quotient. □

Here again, if we assume that σ induces 0 on dual homotopy, then neither secondary classes nor dual homotopy classes are sufficient to detect the elements of $\mathrm{Fol}_3(V)_*$ produced in the proof of Theorem 9.

We conclude this section with the generalization of Theorem 6 to higher codimension.

THEOREM 10. *Suppose that V admits a codimension-q foliation $\mathscr{F}$, classified by $c\colon V \to B\Gamma_q$ with $\tau = \nu \circ c$. Let σ be a $(2q+1)$-splitting of V, and let $p_2\colon V \vee S^{2q+1} \to S^{2q+1}$ denote the projection. Then*

$$\hat{c}_\sigma\colon \mathbf{R}^{r(1)} \cong W(2q+1, q) \to \left[V, B\Gamma_q\right]_\tau$$

is injective. For each $\alpha \in W(2q+1, q)$, let $\mathscr{F}_\alpha$ denote the foliation of V corresponding to $\hat{c}_\sigma(\alpha)$. Then the secondary classes of $\mathscr{F}_\alpha$ are given by

$$f^*_{\mathscr{F}_\alpha}(\bar{z}) = f^*_{\mathscr{F}}(\bar{z}) + (p_2 \circ \sigma)^*\left(h^\sharp_\alpha[\bar{z}]\right)$$

where we identify

$$h^\sharp_\alpha[\bar{z}] \in \mathrm{Hom}\left(\pi_{2q+1}(S^{2q+1}), \mathbf{R}^{r(q)}\right) \cong H^{2q+1}(S^{2q+1}; \mathbf{R}^{r(q)}),$$

and $W(2q+1, q)$ and $\bar{z}$ are defined as in Fact 2.

THEOREM 11. *Suppose that there exists a rank-q trivial subbundle $\varepsilon^q \subset TV$, $q > 1$. Let $\bar{s} = \{s_1, \dots, s_a\} \subset H^{2q+1}(V;\mathbf{Q})$ be a linearly independent set such that each cup product $s_i \cup s_j = 0$, $1 \le i, j \le a$. Then for each choice of vector $\overline{\lambda}(i) \in \mathbf{R}^{r(q)}$ where $1 \le i \le a$, there is a codimension-q foliation $\mathscr{F}_{\bar{s},\overline{\lambda}}$ on V with normal bundle homotopic to $\varepsilon^q \subset TV$, and whose characteristic map satisfies*

$$f^*_{\mathscr{F}_{\bar{s},\overline{\lambda}}}(z_j) = \sum_{i=1} \lambda_j(i) \cdot s_i.$$

Proof. The hypothesis about the set $\bar{s}$ implies that there exists a map into a bouquet, $V \to \bigvee_{i=1}^{a} K(\mathbf{Q}, 2q+1)$, such that the ith canonical class in degree $(2q+1)$ pulls back to s_i. Then the method of proof of Proposition 2 yields a map $g\colon V \to \bigvee_{i=1}^{a} S^{2q+1}$ such that the ith-canonical integral generator of $H^{2q+1}(S^{2q+1}; \mathbf{Q})$ pulls back to a non-zèro integral multiple $n_i \cdot s_i$. For each i, by Fact 2 we can choose a map $\alpha_i\colon S^{2q+1} \to B\overline{\Gamma}_q$ for which

$$f_{\alpha_i}^*(\bar{z}) = \bar{\lambda}(i) \in H^{2q+1}(S^{2q+1}; \mathbf{R}^{r(q)}).$$

Then the classifying map $c_{\bar{s},\bar{\lambda}}$ for $\mathscr{F}_{\bar{s},\bar{\lambda}}$ is defined to be the composition

$$V \xrightarrow{g} \bigvee_{i=1}^{a} S^{2q+1} \xrightarrow{\vee \alpha_i} B\overline{\Gamma}_q \xrightarrow{i} B\Gamma_q. \quad \square$$

Theorems 10 and 11 do not utilize the tertiary invariants, but rather illustrate a second principle of this paper, that knowing the existence of spherical cycles in $B\overline{\Gamma}_q$ detected by secondary classes (or tertiary classes) provides an access towards the calculation of the sets $[V, B\Gamma_q]_\tau$, without requiring the calculation of the spectral sequence of Remark 1 in Section 5.

8. Rigid tertiary classes

Two codimension-q foliations $\mathscr{F}_0$ and $\mathscr{F}_1$ on V are homotopic if there is a smooth 1-parameter family of codimension-q foliations, $\{\mathscr{F}_t | 0 \le t \le 1\}$ between them. The *rigid* secondary classes are precisely those which are a priori invariant under homotopy, and are characterized as the image of the restriction map

$$\mathscr{R}_q = \text{image}\ \mu^*\colon H^*(W_{q+1}) \to H^*(W_q)$$

in the case of framed foliations, where $\mu\colon W_{q+1} \to W_q$ is the natural restriction map. In the papers [H1], [H2], examples were given of foliations with non-trivial rigid classes. We discuss next rigid tertiary classes, and use them to exhibit much larger families of foliated manifolds in high codimensions with non-homotopic foliations. Moreover, these foliations can sometimes be chosen to have homotopic tangential distributions, giving further examples which "solve" Problem 3 of [La 2].

The rigid tertiary classes for framed foliations are constructed as follows. Let $\{z_1, \ldots, z_r\} \subset W_q$ be a cocycle basis for $\mathscr{R}_q$. For each $1 \le i,\ j \le r$, choose $u_{ij} \in \mathscr{M}_q$ such that $du_{ij} = x_i \wedge x_j$, where $\rho_q(x_i) = z_i$ with x_i closed. Let $\mathscr{D}_{ij}$ denote the subalgebra of $\mathscr{M}_q$ generated by the elements x_i and x_j.

PROPOSITION 3. *The tertiary class $\gamma(u_{ij})$ is rigid on the set $[V, B\Gamma_q | \mathscr{D}_{ij}]$.*

Proof. It suffices to follow the usual proof that the rigid secondary classes are homotopy invariant. A homotopy $\{\mathscr{F}_t\}$ yields a codimension-$(q+1)$

foliation $\hat{\mathscr{F}}$ on $V \times \mathbf{R}$ for which $i_t^* \hat{\mathscr{F}} = \mathscr{F}_t$, where $i_t \colon V \to V \times [0,1]$ is the inclusion at time t. The characteristic maps $f_{\mathscr{F}_t} \circ \rho_q \circ \hat{\mu}$, for $0 \le t \le 1$, are determined up to homotopy as the restrictions $i_t^* \circ f_{\hat{\mathscr{F}}} \circ \rho_{q+1} \colon \mathscr{M}_{q+1} \to \Omega(V)$. Here, $\hat{\mu} \colon \mathscr{M}_{q+1} \to \mathscr{M}_q$ denotes a lift of the map μ. Choose an indecomposable $\bar{u}_{ij} \in \mathscr{M}_{q+1}$ such that $\hat{\mu}(\bar{u}_{ij}) = u_{ij}$. Then the tertiary class $\gamma_{\hat{\mathscr{F}}}(\bar{u}_{ij})$ is well defined in $H^*(V \times \mathbf{R}; \mathbf{R})$ and its restrictions to $t = 0$ and $t = 1$ yield $\gamma_{\mathscr{F}_0}(u_{ij})$ and $\gamma_{\mathscr{F}_1}(u_{ij})$, respectively, which must then agree. □

Fact 4 (cf. §3 [H2]). Let $q = 2k \ge 4$, and set

$$R_3 = \{h_2 c_2^3, h_6 c_6\}$$
$$R_k = \{h_2 h_I c_2^k \mid I \subseteq (4, 6, \ldots, 2k+2)\}, \quad k \text{ odd}$$
$$R_k = \{h_{2k} c_{2k}, h_2 c_2^k, h_2 h_I c_2^k \mid I \subset (4, 6, \ldots, 2k)\}, \quad k \text{ even}$$

Then there exists a bouquet of spheres X_k and a map $g_k \colon X_k \to B\overline{\Gamma}_{2k}$ for which the composition

$$H_*(X_k; \mathbf{Z}) \xrightarrow{(g_k)_*} H_*(B\overline{\Gamma}_{2k}; \mathbf{Z}) \xrightarrow{\Delta(R_k)} \mathbf{R}^{r_k}$$

is bijectively onto a lattice in $\mathbf{R}^{r_k}$, where r_k denotes the cardinality of the set R_k. In other words, all of the rigid classes in $\mathbf{R}_k$ can be independently evaluated on spherical cycles. Thus, by Theorem 4.4 of [H1], for each $1 \le i < j \le r_k$, there is a map $g_{ij} \colon S^{n_{ij}} \to B\overline{\Gamma}_{2k}$ for which $h^\#(u_{ij})(g_{\mu\nu})$ is non-zero precisely when $i = \mu$ and $j = \nu$, where the u_{ij} are chosen as in Fact 3 for $\{z_i\}$ a basis of the set R_k.

We give an example of how to apply Fact 4 and Proposition 3 for $q = 6$, and leave to the reader the more general cases.

THEOREM 12. *Let TV contain a rank-*6 *trivial subbundle, and suppose that either*

(a) $H^{15}(V; \mathbf{Q}) \neq 0$, *or*

(b) $H^{37}(V; \mathbf{Q}) \neq 0$.

*Then V has an infinite set of codimension-*6 *foliations, all with distinct rigid classes in case a), and distinct rigid tertiary class in case b). If $H^i(V; \mathbf{Q}) = 0$ for $i = 3, 5, 7$ and* 11, *then* $\mathrm{Fol}_6(V)_*$ *contains an infinite set of foliations which are not homotopic as foliations. Moreover, if the set of lifts l of the tangent map T*

$$\begin{tikzcd} & BO_{m-6} \arrow[d] \\ V^m \arrow[ur, dashed, "l"] \arrow[r, "T"] & BO_m \end{tikzcd}$$

is finite, then there exists an infinite set of pairwise non-homotopic foliations with homotopic tangential distributions.

Proof. Case (a) is similar to Example 1, so we consider (b). Choose σ: $V \to S^{37}$ which is non-torsion. Let $u_{12} \in \mathscr{M}_6$ of degree 37 satisfy $du_{12} = x_1x_2$, where $\rho_6(x_1) = h_2c_2^3$ and $\rho_6(x_2) = h_6c_6$. Then apply Theorem 5 and Fact 4 to obtain a set of foliations $\{\mathscr{F}_{\sigma,n} | n \in \mathbf{Z}\}$ for which

$$\gamma_{\mathscr{F}_{\sigma,n}}(u_{12}) = c \cdot n \cdot \sigma^*(i_{37}) \in H^{37}(V;\mathbf{R}),$$

where c is a non-zero constant. This yields an infinite set of distinct elements in the image of

$$[V, B\overline{\Gamma}_6] \to [V, B\overline{\Gamma}_7].$$

Then consider the commutative diagram

$$\begin{array}{ccc} [V, SO_6] & \longrightarrow & [V, SO_7] \\ \downarrow & & \downarrow \\ [V, B\overline{\Gamma}_6] & \longrightarrow & [V, B\overline{\Gamma}_7] \\ \downarrow & & \downarrow \\ [V, B\Gamma_6]_* & \longrightarrow & [V, B\Gamma_7]_* \end{array}$$

With our last hypothesis, both groups at top are finite, so that the infinite set in the image of the middle line descends to an infinite set in the image from $B\Gamma_6 \to B\Gamma_7$. If the set of tangential lifts is finite, then an infinite subset of these foliations must have homotopic tangential distributions. □

Note that $V = S^6 \times S^{13} \times M$ satisfies (b) for any compact orientable 18-manifold M, and this example will have trivial secondary classes.

9. Universal Postnikov invariants

Let us close this paper with a remark on the implications of the γ-invariants for the topology of $B\Gamma_q$. On page 121 of [H2], a set $\overline{V}_q^s \subset H^*(W_q)$ was constructed, and a corresponding bouquet of spheres indexed by the elements of $\overline{V}_q^s$:

$$Y_q = \bigvee_{n=2q+1}^{2q+q^2} \left(\bigvee_{\substack{z \in \overline{V}_q^s \\ \dim z = n}} (S^n)_z \right)$$

In more detail we have

$$\begin{aligned}
\overline{V}_2^s &= \{h_1c_1^2, h_1c_2\},\\
Y_2 &= S^5 \vee S^5,\\
\overline{V}_3^s &= \{h_1c_1^3, h_1c_1c_2, h_2c_2, h_1h_2c_1^3, h_1h_2c_1c_2\},\\
Y_3 &= S^7 \vee S^7 \vee S^7 \vee S^{10} \vee S^{10}, \text{ etc.}
\end{aligned}$$

For each $z \in \overline{V}_q^s$ and real $\lambda \in \mathbf{R}$, there exists a continuous map

$$\psi_{z,\lambda}\colon S_z^n \to B\overline{\Gamma}_q$$

for which the characteristic map

$$f_{z,\lambda}^*\colon H^n(W_q) \to H^n(S^n;\mathbf{R}) \cong \mathbf{R}$$

satisfies $f_{z,\lambda}^*(z) = \lambda$, and $f_{z,\lambda}^*(w) = 0$ for $z \neq w \in \overline{V}_q^s$. Thus, for all families of reals $\overline{\lambda} = \{\lambda_z\} \subset \mathbf{R}$, there is a continuous map $\psi_{\overline{\lambda}}\colon Y_q \to B\overline{\Gamma}_q$ such that the characteristic map

$$f_{\overline{\lambda}}^*\colon H^*(W_q) \to H^*(Y_q;\mathbf{R})$$

maps $\overline{V}_q^s$ onto $\overline{\lambda}$.

THEOREM 13. *For each $\overline{\lambda}$ as above with all $\lambda_z \neq 0$, there exists a homomorphism*

$$\gamma_{\overline{\lambda}}\colon \mathcal{M}_{Y_q} \to \mathcal{M}_{B\overline{\Gamma}_q}$$

such that $\mathcal{M}\psi_{\overline{\lambda}} \circ \gamma_{\overline{\lambda}} = \mathrm{id}$ on $\mathcal{M}_{Y_q}$.

Proof. Let $f_0\colon W_q \to \mathcal{M}_{B\overline{\Gamma}_q}$ denote the map of minimal models covering the universal map $\Delta\colon W_q \to \Omega(B\overline{\Gamma}_q)$ where $\Omega(B\overline{\Gamma}_q)$ denotes the real simplicial deRham algebra of the singular simplicial complex of $B\overline{\Gamma}_q$. Then $\mathcal{M}\psi_{\overline{\lambda}} \circ f_0 = f_{\overline{\lambda}}$. Let $\sigma\colon \mathcal{M}_{Y_q} \to \mathcal{M}_q$ denote the map induced by the inclusion of $\overline{V}_q^s$ into W_q. The composition $\alpha_{\overline{\lambda}} = f_{\overline{\lambda}} \circ \rho_q \circ \sigma$ is an automorphism of $\mathcal{M}_{Y_q}$, so set $\gamma_{\overline{\lambda}} = f_0 \circ \rho_q \circ \sigma \circ \alpha_{\overline{\lambda}}^{-1}$. Then $\mathcal{M}\psi_{\overline{\lambda}} = \mathrm{id}$. □

COROLLARY 5. *Let u be an indecomposable element of degree N in $\mathcal{M}_q$ such that $0 \neq [du] \in H^{N+1}(\mathcal{M}_{Y_q}(N))$. Then the corresponding Postnikov invariant,*

$$\gamma[du] \in H^{N+1}\big((B\overline{\Gamma}_q)_N;\mathbf{R}\big)$$

is non-zero, where $(B\overline{\Gamma}_q)_N$ denotes the N-th stage of a Postnikov tower for $B\overline{\Gamma}_q$.

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