Differentiability, Rigidity and Godbillon-Vey Classes for Anosov Flows

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Introduction

In this paper we begin a study of C^∞ classification of area-preserving Anosov diffeomorphisms of the two-torus and transitive Anosov flows on 3-manifolds. One of our main results is a new type of rigidity phenomena. It is well-known [HP2] that the weak-stable and -unstable foliations of such dynamical systems always have transverse differentiability class C^1. We show that if these foliations are C^2 (actually, much less will do—for example, they need only be $C^{1,1}$ at every periodic orbit) then they are C^∞. For an area-preserving Anosov toral automorphism, we deduce from this that if either stable or unstable foliation is C^2, then the automorphism is C^∞-conjugate to a linear automorphism (cf. Theorem 0.6).

Similarly, for an Anosov flow arising as the geodesic flow of a metric of negative curvature on a compact surface which is close to a metric of constant negative curvature in C^4 topology, we prove that if the weak unstable (geodesic-horocycle) foliation is C^2 (or $C^{1,1}$ at periodic orbits) then the metric itself has constant negative curvature (cf. Theorem 0.13).

We believe the small perturbation assumption can be removed.

Conjecture If the weak unstable foliation of the geodesic flow associated to a metric of negative curvature on a compact surface is C^2 then the metric must have constant negative curvature.

Another purpose of this paper is to introduce two new real-valued invariants for volume-preserving Anosov flows on 3-manifolds. These invariants are the

Godbillon-Vey numbers of the weak-stable and weak-unstable foliations. The existence of these invariants entails proving that there is a well-defined notion of the Godbillon-Vey class for codimension-one foliations with transverse differentiability class $C^{1+\alpha}$, for $\alpha > \frac{1}{2}$. We further show that the resulting Godbillon-Vey invariants for geodesic flows characterizes those flows arising from metrics of constant negative curvature. This follows from properties of the Mitsumatsu defect, Def(g), of a metric g with negative curvature on a closed surface.

In our study of the Godbillon-Vey invariants for Anosov flows, we obtain continuous families of $C^{1+\alpha}$ foliations, for any $\alpha < 1$, which are topologically conjugate but not C^1-conjugate, and which have continuously varying Godbillon-Vey invariants. We believe these provide the first examples of topologically conjugate geometric structures with differing real characteristic classes.

The remainder of this paper is organized as follows:

§ 1. Statement of Results

I. Smoothness of Foliations

§2. C^∞-Regularity of functions C^∞ along leaves

§3. Modulus of continuity for weak-unstable foliations

§4. The Anosov cocycle and local obstructions to smoothness

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§ 1. Statement of Results

1. Let M be a closed orientable Riemannian 3-manifold. Let $f_t: M \to M$ be a C^∞-flow on M with generating vector field $\xi = \frac{d}{dt} f_t|_{t=0}$. The flow $\{f_t\}$ is Anosov if there is a continuous Df_t — invariant splitting $TM = E^+ \oplus E^0 \oplus E^-$, with E^0 spanned by ξ at each point $p \in M$; and E^+, respectively E^- are spanned by unit vector fields h^+, respectively h^- for which there are constants $c_1, c_2, \gamma > 0$ such that for all $p \in M$

$$\|Df_t(h^+(p))\| \geq c_1 e^{\gamma t} \text{ for } t \geq 0$$
$$\|Df_t(h^-(p))\| \leq c_2 e^{-\gamma t} \text{ for } t \geq 0.$$

A C^k, 1-form τ on M satisfying

$$\tau(\xi) = 1;\ \tau(h^+) = 0;\ \tau(h^-) = 0 \tag{0.1}$$

is called a transverse invariant 1-form to $\{f_t\}$. For $k \geq 1$ it makes sense to write $d\tau$, and form the exterior product $\tau \wedge d\tau$. The 3-form $\tau \wedge d\tau$ is flow invariant. If it is not identically zero, then both its positive and negative part define absolutely continuous invariant measures for the flow $\{f_t\}$. By the theorem of Livshitz and Sinai [LS] any absolutely continuous invariant measure for an Anosov flow is given by a positive density. Thus, $\tau \wedge d\tau$ is either identically zero or does not vanish at all. In the first case, by Plante (Theorem 3.1, [P1]) there exists a compact smooth section for the flow which must be a 2-torus, and hence the flow is smoothly conjugate to the suspension of an Anosov diffeomorphism of the 2-torus. In the second case, τ is a contact form, and we will refer to this case as a contact Anosov

flow. Such a flow can be extended to a Hamiltonian flow on $M \times R$ with a homogeneous Hamiltonian function (cf. Appendix, [Ar])

For an Anosov flow $\{f_t\}$ with splitting $TM = E^+ \oplus E^0 \oplus E^-$, define the weak-stable distribution $E^{ws} = E^0 \oplus E^-$, and the weak-unstable distribution $E^{wu} = E^+ \oplus E^0$. Both are uniquely integrable 2-plane bundles on M, and the resulting foliations are denoted W^{ws} and W^{wu}, respectively.

The individual integral manifolds of the distributions E^{wu} and E^{ws} are C^∞-submanifolds of M, and depend continuously on ξ in the C^∞-topology [HP1]. However, the distributions E^{wu} and E^{ws} are only known to be of class C^1, again by Hirsch and Pugh [HP2]. Our first main purpose is to determine precisely the degree of differentiability of these distributions.

Let S be a closed orientable surface with a Riemannian metric g of negative curvature. Let M denote the unit tangent bundle of S, and $\pi:M \to S$ the projection along the circle fibers. The geodesic flow $\{f_t\}$ on M associated to the geodesic vector field ξ_g is contact Anosov. For further discussion, see [An], [Eb] or [GK]. The integral curves of the vector fields h^+ and h^- on M project under π to the horocycles on S.

2. A continuous function $f:(a,b) \to \mathbb{R}$ is in the <u>Zygmund class</u>, $\Lambda_*(a,b)$ on just Λ_*, if

$$\Lambda_*(f) = \sup_{a<x<b} \limsup_{h\to 0} \frac{|f(x+h) + f(x-h) - 2f(x)|}{|h|} \tag{0.2}$$

is finite. A function in Λ_* has modulus of continuity $\omega(S) = K \cdot S \cdot |\log S|$, $S > 0$, for appropriate $K > 0$ (cf. Theorem 3.4, [Z]). Thus, every $f \in \Lambda_*$ is α-Hölder continuous for all $\alpha < 1$, but need not be Lipshitz or of bounded variation. The definition of Zygmund classs extends to functions on $\mathbb{R}^n$ using essentially the same norm as in (0.2), and where the supremum is taken over any open set in $\mathbb{R}^n$. Denote by $C^{k+\alpha}$, respectively $C^{k,1}$, C^{k,Λ_*}, $C^{k,\omega}$ the class of functions f whose k-th derivative exists everywhere, and is respectively α-Hölder, Lipshitz, Zygmund or with modulus of continuity ω.

A 2-plane bundle E on a 3-manifold is said to be of class C^{k,Λ_*} if it is spanned locally by vector fields which are of class C^{k,Λ_*} in local C^∞-coordinates.

<u>Theorem</u> <u>0.1</u>. *Let* $\{f_t\}$ *be a volume-preserving* C^∞*-Anosov flow on a Riemannian 3-manifold* M.

a) *The distributions* E^{ws} *and* E^{wu} *are always of differentiability class* C^{1,Λ_*}

b) *If* $\{f_t\}$ *has a* C^2*-transverse invariant 1-form* τ, *then* E^+ *and* E^- *are* C^{1,Λ_*} *1-dimensional distributions on M.*

This theorem is proved in Section 3.

Note that b) follows immediately from a), for $E^+ = E^{wu} \wedge \ker(\tau)$ and similarly for E^-. It is not possible to remove the hypothesis that $\{f_t\}$ is a contact flow in b), as one can make a simple time-change to obtain a flow where E^+ and E^-

are not even C^1 (e.g., see Theorem 4.11, [P1]).

Under the hypothesis of Theorem 0.1, it is possible to deduce from Theorems 3.5 and 3.8 of Hirsch-Pugh-Shub [HPS] that there exists an α, $0 < \alpha < 1$, such that E^{wu} and E^{ws} are $C^{1+\alpha}$. The Hölder exponent α depends upon the global Anosov exponent λ in the definition of the Anosov flow. The much sharper result of Theorem 0.1 is obtained from using a "local contraction principle", Proposition 3.3 below, versus the global contraction principle behind [HPS].

3. For an area-preserving C^3-Anosov diffeomorphism F of the 2-torus T^2, Anosov observed in ([An], Section 24, especially lemma 24.1) that if either the stable or unstable distributions of F are C^2, then at each periodic point p of F, of period n, there exists a differential relation of the third order which F^n must satisfy at p. Thus, there are countably many obstructions to F having either C^2-stable or unstable foliations. A considerable part of the present work is based upon our study and elaboration upon Anosov's observation. In particular, the next three results put his obstructions into a comprehensive framework.

For each $p \in M$, let $\Psi_p:(-\epsilon,\epsilon)^2 \to M$ be a transversal to the flow $\{f_t\}$ satisfying conditions (3.1) from Section 3, which imply that Ψ_p maps the x-axis in R^2 into the unstable leaf W_p^{wu} through p, the y-axis into the stable leaf W_p^{ws}, and

$$D\Psi_p(\partial/\partial x|_{(x,o)}) = h^+(\Psi_p(x,0)), \qquad |x| < \epsilon$$

$$D\Psi_p(\partial/\partial y|_{(o,y)}) = h^-(\Psi(o,y)), \qquad |y| < \epsilon$$

Let p be a periodic point for the flow $\{f_t\}$ with period t_0. Then f_{t_0} induces a Poincaré return map on the transversal T_p given by the image of Ψ_p. Let $F(x,y) =$

$\begin{bmatrix} \lambda x + \varphi(x,y) \\ \lambda^{-1}y + \Psi(x,y) \end{bmatrix} : (-\epsilon, \epsilon)^2 \to R^2$ denote this return map in coordinates. The partial

derivatives of $\varphi(x,y)$ and $\psi(x,y)$ are denoted by subscripts x and y. Let us define

$$(0.3) \qquad A_u^{\Psi}(p,t_0) = \frac{4\lambda_2-\lambda_3}{2(\lambda-1)} \psi_{xy}\psi_{yy} = \varphi_{xy}\psi_{xy} + \lambda\psi_{xyy}$$

and similarly, define $A_u^{\Psi}(p,t_0)$ by reversing the roles of φ and ψ and interchanging x and y in (0.3).

<u>Theorem</u> <u>0.2</u> *Let* $\{f_t\}$ *be a volume-preserving* C^3 *Anosov flow on the closed 3-manifold* M *and let* $p \in M$ *be a periodic point for* $\{f_t\}$ *of period* t_0.

(i) *If the first derivative of the weak unstable distribution* E^{wu} *(respectively, the weak stable distribution* E^{ws}*) has near* p *modulus of continuity* ω, *where* $\omega(s) = 0(s\,|\log s|)$ *or satisfies a Lipshitz condition at p, then* $A_u^{\Psi}(p,t_0) = 0$ *(respectively,* $A_s^{\Psi}(p,t_0) = 0$*)*

(ii) *If in addition* $\{f_t\}$ *is* C^4, *and if* E^{wu} *(respectively* E^{ws}*) has a measurable second derivative almost everywhere, then* $A_u^{\psi}(p,t_0) = 0$. *(Respectively,* $A_s^{\psi}(p,t_0) = 0$.*)*

Statement (i) is completely local and is proved in Section 4. Statement (ii)

uses the construction of cocycles of A_u^{Ψ} and A_s^{Ψ} (see below) in Section 4. It is proved in Section 5.

Since every Anosov flow has infinitely many periodic points, Theorem 0.2 shows there are a countable number of relations on the 3-jet of the local return maps of the flow which must hold whenever the weak-stable or -unstable distributions are $C^{1,1}$ or even if their first derivatives have modulus of continuity $O(s \mid \log s\mid)$.

In Section 4 we also show that the numbers $A_u^{\Psi}(p,t)$ (and in the corresponding stable case $A_s^{\Psi}(p,t)$) of (0.3) can be defined for any point $p \in M$ and time t, and the values satisfy a <u>cocycle law</u>:

$$A_u^{\Psi}(p,t) + A_u^{\Psi}(f_t p,s) = A_u^{\Psi}(p,t+s) \tag{0.4}$$

$$A_s^{\Psi}(p,t) + A_s^{\Psi}(f_t p,s) = A_s^{\Psi}(p,t+s).$$

Let $H^1(\{f_t\};R)$ denote the group of C^1-cocycles over the flow $\{f_t\}$, modulo the C^1-coboundaries.

<u>Theorem</u> <u>0.3</u>. *Let* $\{f\}$ *be a volume-preserving* C^3*-Anosov flow on the closed* 3-*manifold* M. *Then the cohomology classes* $A^+ = [A_u^{\Psi}]$ *and* $A^- = [A_s^{\Psi}]$ *in* $H^1(\{f_t\}; R)$ *are well-defined, independent of the choices of transversals* Ψ_p *and the ambient metric on* M. *Furthermore,* $\{f_t\}$ is C^∞ *and* $\{\tilde{f}_t\}$ *is another volume-preserving* C^∞*-Anosov flow on a* 3*-manifold* $\tilde{M}$, *and* $\theta: M \to \tilde{M}$ *is a* C^1*-diffeomorphism*

conjugating the flows, then $\theta * \tilde{A}^+ = A^+$ *and* $\theta * \tilde{A}^- = A^-$.

It follows that A^+ and A^- are C^1-invariants of Anosov flows as above. The proof of Theorem 0.3 is contained in Sections 4 and 5, except for the C^1-invariance which follows from results of [Ka].

A basic problem is to understand how much information about the flow $\{f_t\}$ is contained in the classes A^+ and A^-. Our next result identifies those flows with zero cohomology.

<u>Theorem</u> <u>0.4</u>. *Let* $\{f_t\}$ *be a volume-preserving transitive* C^∞*-Anosov flow on a compact 3-manifold* M. *Then* $A^+ = 0$ *if and only if the weak-unstable distribution* E^{wu} *is* C^∞, *and* $A^- = 0$ *if and only if* E^{ws} *is* C^∞.

This theorem is proved in several steps in Section 6.

Combining Theorem 0.4 with the results of Livshitz, discussed in Section 5, we deduce:

<u>Theorem</u> <u>0.5</u>. *Let* $\{f_t\}$ *be as in Theorem* (0.4). *If the obstruction* $A_u^\Psi(p,t_0)$ *of* (0.3) *vanishes for every periodic orbit of period* t_0, *then* E^{wu} *is* C^∞, *and similarly for* E^{ws}.

4. Let $F:T^2 \to T^2$ be an Anosov diffeomorphism. The suspension construction yields a flow $\{f_t^F\}$ on the 3-manifold $M = T^2 \times R/(x,r) \sim (F(x),r+1)$ to which the

results above apply. In Section 7 we prove a much stronger conclusion for this case.

Theorem 0.6. *Let* $F:T^2 \to T^2$ *be an area-preserving* C^∞*-Anosov diffeomorphism. Either both stable and unstable distributions* E^+ *and* E^- *are not* $C^{1,1}$ *at some (maybe different) periodic orbits of* F, *or these distributions are* C^∞ *and there is a* C^∞*-conjugacy between* F *and a linear Anosov automorphism of* T^2. *In particular, if* $A^+ = 0$ *or* $A^- = 0$, *then* F *is linear up to* C^∞*-conjugation.*

5. In part III of this paper, we define and study two new invariants for volume-preserving C^3-Anosov flows on closed 3-manifolds. As remarked above, this requires defining the Godbillon-Vey invariant for foliations of class $C^{1+\alpha}$, where $\alpha < 1$. Note that we say a foliation $\mathfrak{F}$ of M is of transverse class $C^{1+\alpha}$ if there is a covering of M by open foliation charts for which the local transverse transition functions are $C^{1+\alpha}$.

Theorem 0.7. *Let* $\mathfrak{F}$ *be a codimension-one foliation on a closed, orientable* 3-*manifold* M, *and assume that* $\mathfrak{F}$ *has transverse differentiability* $C^{1+\alpha}$ *for* $\alpha > \frac{1}{2}$. *Then there is a natural construction of the Godbillon-Vey class* $GV(\mathfrak{F}) \in H^3(M,R)$. *Furthermore,* $GV(\mathfrak{F})$ *depends only upon the* $C^{1+\beta}$ *conjugacy and* $C^{1+\beta}$ *concordance class of* $\mathfrak{F}$ *for any* $\beta > \frac{1}{2}$.

Recall that two $C^{1+\alpha}$-foliations $\mathfrak{F}_0$ and $\mathfrak{F}_1$ or M are $C^{1+\alpha}$-concordant if there is a codimension-one $C^{1+\alpha}$-foliation $\mathfrak{F}$ or $M \times I$ whose restrictions to $M \times \{0\}$ is $\mathfrak{F}_0$, and to $M \times \{1\}$ is $\mathfrak{F}_1$ (cf. Lawson [La]).

It follows from the work of Tsuboi [T] that the Godbillon-Vey class cannot be defined for all codimension-one C^1-foliations, and we suspect that his methods extend to show that for $\alpha \leq \frac{1}{2}$, $GV(\mathfrak{F})$ cannot be defined for all foliations $\mathfrak{F}$ of transverse class $C^{1+\alpha}$.

A surprising fact is that the invariance at $GV(\mathfrak{F})$ under diffeomorphism can be improved from the statement in Theorem 0.7.

<u>Theorem</u> <u>0.8</u>. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be codimension-one $C^{1+\alpha}$-foliations on closed orientable 3-manifolds M and M′, respectively, for $\alpha > \frac{1}{2}$. Assume there exists a diffeomorphism $\theta : M \to M'$ conjugating $\mathfrak{F}$ to $\mathfrak{F}'$, and either*

(0.5) $$\theta \text{ is } C^{1+\beta} \text{ where } \beta + \alpha > 1, \text{ or}$$

(0.6) $$\mathfrak{F} \text{ and } \mathfrak{F}' \text{ are } C^2, \text{ and } \theta \text{ is } C^1$$

Then $GV(\mathfrak{F}) = \theta * GV(\mathfrak{F}')$.

Theorems 0.7 and 0.8 are proved in Section 9.

If we add a topological hypothesis to the assumptions of Theorem 0.8, then it is possible to prove what appears to be an optimal result.

<u>Theorem</u> <u>0.9</u>. *Let M and M′ be the total spaces of S^1-fibrations over an orientable surface S. Let $\mathfrak{F}$ and $\mathfrak{F}'$ be $C^{1+\alpha}$ foliations on M and M′, respectively, which are transverse to the fibers of these fibrations and have $\alpha > \frac{1}{2}$. Also assume that*

$\mathfrak{F}$ is the weak-stable foliation of a transitive Anosov flow on M. *If there exists a homeomorphism* θ:M $\rightarrow$ M′ *conjugating* $\mathfrak{F}$ *to* $\mathfrak{F}'$, *with* θ *absolutely continuous transversally to* $\mathfrak{F}$, *then* $\theta^*GV(\mathfrak{F}') = GV(\mathfrak{F})$.

Theorem 0.9 is proved in Section 10.

It is not possible to remove the hypothesis of absolute continuity transversally, as examples show.

For codimension-one C^2-foliations, G. Rabi proved in an unpublished manuscript [R] that $GV(\mathfrak{F})$ is a C^1-invariant. The proof of Theorem 0.8 we give is adapted from a more general result valid in all codimensions, and is part of an analysis of secondary and Chern-Simons invariants for foliations of class $C^{1+\alpha}$, $\alpha < 1$, in [Hu 3]. However, the proof we give for Theorem 0.8 can be seen to be similar to that of Rabí.

We are grateful to E. Ghys for bringing Rabi's work to our attention.

6. Let $\{f_t\}$ be a volume-preserving C^3 Anosov flow on a closed 3-manifold M. The distributions E^{wu} and E^{ws} are uniquely integrable, and we set $\mathfrak{F}^u(f_t) = W^{wu}$, and $\mathfrak{F}^s(f_t) = W^{ws}$, the corresponding C^{1,Λ_*}-codimension-one foliations of M. From Theorem 0.8, both foliations have well-defined Godbillon-Vey classes, and for [M] the fundamental class of M, define the secondary characteristic numbers of $\{f_t\}$ by

$$\begin{cases} gv^u(f_t) = \langle GV(\mathfrak{F}^u(f_t)), [M]\rangle \\ gv^s(f_t) = \langle GV(\mathfrak{F}^s(f_t)), [M]\rangle \end{cases} \tag{0.7}$$

If M is not orientable, then lift $\{f_t\}$ to the orientable double cover of M, and divide the numbers in (0.7) by 2. (Also note: the foliations $\mathfrak{F}^u(f_t)$ and $\mathfrak{F}^s(f_t)$ are generically not of class C^1 plus bounded variation, by Theorem 0.2 (ii), so the extension of the Godbillon-Vey class given by Duminy-Sergiescu in [DS] does not suffice to define (0.7).) We next collect into one theorem our results about these invariants.

<u>Theorem</u> <u>0.10</u>. *Let* $\{f_t\}$ *be a volume-preserving* C^3 *Anosov flow on a closed 3-manifold* M.

a) $gv^u(f_t)$ *and* $gv^s(f_t)$ *depend continuously on* $\{f_t\}$ *in the* C^3 *topology on flows.*

b) *There are* C^3 *1-parameter families of such flows on which* $gv^u(f_t)$ *and* $gv^s(f_t)$ *vary continuously and non-trivially.*

c) *Let* $\{f_t\}$ *and* $\{\tilde{f}_t\}$ *be two such flows on 3-manifolds* M *and* $\tilde{M}$, *respectively. Let* $\theta: M \to \tilde{M}$ *be a* $C^{1+\beta}$*-diffeomorphism conjugating* $\mathfrak{F}^u(f_t)$ *to* $\tilde{\mathfrak{F}}^u(\tilde{f}_t)$, *with* $\beta > 0$. *Then* $gv^u(f_t) = \pm\, gv^u(\tilde{f}_t)$, *with* $+$ *if* θ *is orientation preserving, and* $-$ *otherwise. Similarly for* $gv^s(f_t)$.

d) *Let* $\{f_t(g)\}$ *be the geodesic flow for a metric* g *of negative curvature on a a surface* S. *Then*

$$gv^u(f_t(g)) = gv^s(f_t(g)),$$

and we set $gv(g) \overset{\text{def}}{=} gv^u(f_t(g))$.

e) *Define the* <u>*Mitsumatsu*</u> <u>*Defect*</u> *of a metric* g *of negative curvature on a surface* S *of Euler characteristic* $\chi(S)$ *to be*

$$\mathrm{Def}(g) = (2\pi)^2\, \chi(S) - gv(g) \tag{0.8}$$

Then $\mathrm{Def}(g) \geq 0$, *with equality if and only if* g *has constant curvature.*

f) gv(g) *is an invariant of the length function* $\ell_g : \pi_1(S) \to R^+$ *which assigns to each* $[\gamma] \in \pi_1(S)$ *the length of the unique closed geodesic representing* $[\gamma]$: *if* g *and* $\tilde{g}$ *are metrics of negative curvature on* S *and* $\ell_g = \ell_{\tilde{g}}$ *then* $gv(g) = gv(\tilde{g})$.

The conclusion of Theorem 0.10 e) was first proved by Mitsumatsu in [Mi] for C^2-foliations. In Section 11 we show how his proof can be adapted to the critical case of $C^{1+\alpha}$-foliations. Mitsumatsu also derived an integral expression for Def(g), again under the hypothesis that $\mathfrak{F}^u(f_t(g))$ is C^2. This integral expression also holds for the extended invariant gv(g), which we now briefly describe. Let $w = \partial/\partial\theta$ be the unit vector field on M tangent to the fibers of the projection $\pi : M \to S$ along the circle fibers. Let $k(g) : S \to R$ be the Gaussian curvature function for g. Let $H = H^+ : M \to R$ denote the positive C^1-solution on M of the global Ricatti equation

$$\xi_g H + H^2 + k(g) \circ \pi = 0. \tag{0.9}$$

In Section 11 we prove:

<u>Theorem</u> <u>0.11</u>. *For* g *of negative curvature,*

$$\text{(0.10)} \qquad \mathrm{Def}(g) = 3\int_M \left(\frac{\partial H}{\partial \theta}\right)^2 \mathrm{dvol}.$$

It is well-known that $\int_M H\cdot\mathrm{vol}$ is the metric entropy of the flow $\{f_t(g)\}$ with respect to the smooth (Liouville) invariant measure (cf. [Pe]). Formula (0.10) suggests that Def(g) should be viewed as a kind of mean variation of the distribution of the closed orbits of $\{f_t(g)\}$, based on an analogy with the results of [Ka].

7. Now we can state two of the main results of this paper:

<u>Theorem</u> <u>0.12</u>. *Let* S *be a compact surface with negative Euler characteristic, and let* M *denote the unit tangent bundle to* M. *Then there exists a family of codimension-one,* C^{1,Λ_*}*-foliations* $\{\mathfrak{F}_s | 0 \leq s \leq 1\}$ *on* M *such that* $gv(\mathfrak{F}_s)$ *varies non-trivally and continuously with respect to* S. *Furthermore, all of the foliations* $\mathfrak{F}_s$ are topologically conjugate.

<u>Proof</u>: Choose a C^∞-path of metric $\{g_s | 0 \leq s \leq 1\}$ of negative curvature from g_0, a metric of constant negative curvature, to a metric g_1 of non-constant negative curvature.

Then set $\mathfrak{F}_s = \mathfrak{F}^u(f_t(g_s))$, and the conclusions follow. #

<u>Theorem</u> <u>0.13</u>. *Let* g *be a metric with negative curvature on a closed surface* S. *Assume that the foliation* $\mathfrak{F}^u(f_t(g))$ *is* C^2, *and that at every point* $p \in M$ *the solution* H *of* (0.9) *satisfies*

$$(0.11) \qquad H(p)^2 + 2(\frac{\partial H}{\partial \theta})(p)^2 - H(p)\frac{\partial^2 H}{\partial \theta^2}(p) > 0.$$

Then the metric g *has constant negative curvature. In particular, if* g *is* C^4*-close to a metric of constant negative curvature, then the above-mentioned conditions are satisfied so that* g *has constant negative curvature.*

This theorem is proved in Section 11. It is deduced from the results of Section 8, which assert that under the assumption of the theorem the foliation $\mathfrak{F}^u(f_t(g))$ is C^∞ conjugate to the geodesic-horocycle foliation of a metric of constant negative curvature, and from Theorems 0.10 and 0.11.

8. We conclude this section by formulating five open questions.

<u>Problem</u> <u>0.14</u>. *Does there exist a volume-preserving* C^3*-Anosov flow* $\{f_t\}$ *on a closed 3-manifold for which* $gv^u(f_t) \neq gv^s(f_t)$?

The study of the relation between the length function ℓ_g of a metric g and the geometry of g has an extensive tradition (cf. [BK], [GK], [Ka]). The second author has conjectured that the function ℓ_g uniquely characterizes the metric g. Detailed discussions are given in the problem survey [BK] and the references therein. Based on Theorem 0.10f), we ask:

<u>Problem</u> <u>0.15</u>. *Does there exist a formula for* $gv(f_t(g))$ *in terms of the length function* ℓ_g? *Can* $gv(f_t(g))$ *be derived from knowing the* ζ*-function for* ℓ_g?

The class of Zygmund functions most commonly arises in the study of

regularity of kernels for singular operators (cf. [St], [Kr]). This suggests the following speculative question, to which a positive solution would fit perfectly with the program of A. Connes in [Co].

Problem 0.16. *In the case where* $\{f_t\}$ *is a geodesic flow, can the conclusion of Theorem* 0.1 *be deduced from an analytic principle? For example, does there exist a natural singular operator associated to a metric of variable negative curvature (possibly associated to the induced representation of the fundamental group on the circle at infinity), whose regularity properties can be used to deduce that* E^{wu} *is* C^{1,Λ_*}*?*

Next, we pose a question whose solution would remove the hypothesis (0.11) from Theorem 0.13, and settle affirmatively the Conjecture of the Introduction:

Problem 0.17. *Can the invariant* $gv^u(f_t)$ *be calculated from the cohomology class* A^+*? For a geodesic flow* $\{f_t(g)\}$*, does* $A^+ = 0$ *imply that* $gv(\mathfrak{F}^u(f_t(g)) = (2\pi)^2\ \chi(S)$*?*

Finally, we would like to understand what information about the flow is contained in the cohomology classes A^+ and A^- in general, i.e. when those classes do not vanish. We restrict our discussion to the case of geodesic flows. In that case A^- is determined by A^+. For, let $I{:}M \to M$ be the "flip" transformation which sends tangent vector v with footpoint $p \in S$ to the vector $-v$ with the same footpoint. Then $I^*A^+ = A^-$.

By Livshitz Theorem [L1], [L2] the class A^+ is determined by the number $A_u^{\Psi}(p,t_0)$ for periodic points (cf. 0.3). That number actually depends only on a periodic

orbit, and not on the choice of a particular point on it. In our case periodic orbits are closed geodesics and each such orbit is uniquely determined by a non-zero free homotopy class of closed curves on the surface S. The set Γ of all such classes is independent of the metric. Thus, the class A^+ determines a real-valued function on Γ and we will say that two metrics have the same classes A^+ if corresponding functions on Γ coincide. Obviously, any diffeomorphism of S isotopic to the identity, applied to a metric, does not change the class A^+.

Problem 0.18. *Characterize the set of metrics of negative curvature on* S *with the same class* A^+. *More specifically, is this set, factorized by the action of the group of diffeomorphism of* S *isotopic to the identity, always finite-dimensional?*

I. Smoothness of Foliations

§2. C^∞-regularity of functions C^∞ along leaves.

Rigidity for Anosov dynamical systems often turns on proving rigidity first for the solutions of equations along the stable and unstable manifolds. Hyperbolicity along these manifolds accounts for the "algebraic" local rigidity encountered. To then prove global rigidity, one must have results implying that local behavior along stable and unstable manifolds suffices to determine global behavior. (cf. [An], [Ka], [Gk], [L1], [L2], [LMM].)

In this section, we prove a result, first proven by (Lemma 2.3, [LMM]), characterizing the C^∞-functions on R^n by their local restrictions to complementary $C^{0,\infty}$-foliations of R^n, with a transverse regularity hypothesis. Our proof is based on an idea of C. Toll, and is presented both for the sake of completeness (Theorem 2.1 will be applied in a crucial way in §6, for the case n=2), and because of the simplicity of the approach. Recently, yet another proof using Taylor approximations has been given by J.-L. Journé [J].

The basic data is the following: $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are C^0 transversal foliations of R^n with dimensions of leaves k and (n-k), respectively. They also satisfy:

(2.1) For i = 1,2, each leaf L_p^i of $\mathfrak{F}_i$ through $p \in R^n$ is a C^∞-submanifold of R^n, and the family of submanifolds $p \mapsto L_p^i$ is a continuous function of p, with C^∞-

submanifold topology on the L_p^i.

(2.2) For each $p \in R^n$, there are coordinates $\Phi:(-\epsilon,\epsilon)^n \to R^n$, such that for $(\bar{x},\bar{y}) = (x_1,\ldots,x_k, y_1,\ldots,y_{n-k})$, $\Phi(0,0) = p$, $\Phi(\bar{x},\bar{y}_0)$ for $\bar{x} \in (-\epsilon,\epsilon)^k$ is into the leaf L_p^1 through $\Phi(0,\bar{y}_0)$, and for $d\bar{y} = dy_1\wedge\ldots\wedge dy_{n-k}$, $\omega_1^\Phi = \Lambda^{n-k}(\Phi^{-1})^*(d\bar{y})$ defines a continuous $(n-k)$-form on image Φ. Here, $\Lambda^{n-k}(\Phi^{-1})^*$ denotes the push forward map on $(n-k)$-measures. Similarly, $\omega_2^\Phi = \Lambda_k(\Phi^{-1})^*(d\bar{x})$ is a continuous k-form on image Φ transverse to $\mathfrak{F}_2$.

(2.3) For $i = 1,2$; for each integer $\ell > 0$, consider an ℓ-triple $\{v_1,\ldots,v_\ell\}$ of vector fields on R^n tangent to $\mathfrak{F}_i$ as in (2.1). Then $D(v_1) \circ\cdots\circ D(v_\ell)\omega_i$ is a continous form on the image of Φ, for each chart Φ as in (2.2).

By the results of Hirsch-Pugh-Shub [HPS] and Section 1, the complementary stable and unstable foliations for an Anosov diffeomorphism, restricted to a coordinate patch, provide examples of foliations satisfying (2.1)–(2.3). For an Anosov flow one has to take a smooth transversal to the flow and the foliations are the intersections of weak-stable and weak-unstable manifolds with the transversal.

<u>Theorem 2.1</u>. *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be transverse foliations on R^n satisfying* (2.1)–(2.3) *alone. Let $f:R^n \to R$ be a continous function satisfying*

(2.4) *For $i = 1,2$; for each $p \in R^n$, the restriction of f to the leaf L_p^i of $\mathfrak{F}_i$ through p is C^∞, and the leafwise C^∞-jet of f depends continuously on p.*

Then f is C^∞ on R^n.

Proof: The conclusion on f is local, so we can assume that f has compact support in a neighborhood of the origin which is the image of a coordinate map, as given in (2.2) for $p = (0,0)$. Next, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are transverse, so without loss of generality, we can assume that, for $\epsilon > 0$ given, after a C^∞-coordinate change on R^n then in coordinates $(\bar{x},\bar{y}) = (x_1,\ldots,x_k, y_1,\ldots,y_{n-k})$ on R^n.

(2.5) The leaf at $\mathfrak{F}_1$ through (0,0) is $\bar{y} = 0$; and the leaf at $\mathfrak{F}_2$ through (0,0) is $\bar{x} = 0$.

(2.6) The leaf at $\mathfrak{F}_1$ through $(0,\bar{y})$ is given by a graph

$$L^1_{0,\bar{y}} = \{(\bar{x}, \bar{\psi}(\bar{x},\bar{y}))|\bar{x} \in R^k\}.$$

where $\bar{\psi}:R^n \to R^{n-k}$ satisfies

(2.7) $\bar{\psi}(\bar{x},\bar{0}) = \bar{0}$; $\|\bar{\psi}(\bar{x},\bar{y}) - \bar{y}\|_{R^{n-k}} \leq \epsilon$; for $i = 1,\ldots,k$, $\|\frac{\partial}{\partial x_i}\bar{\psi}|_{(\bar{x},\bar{y})}\|_{R^{n-k}} \leq \epsilon$, all $\bar{x} \in R^k$, $\bar{y} \in R^{n-k}$.

(2.8) The leaf at $\mathfrak{F}_2$ through $(\bar{x}, \bar{o})$ is given by a graph $L^2_{\bar{x},\bar{o}} = \{(\bar{\varphi}(\bar{x},\bar{y}), \bar{y})|\bar{y} \in R^{n-k}\}$ where $\bar{\varphi}:R^n \to R^k$ satisfies

(2.9) $\bar{\varphi}(\bar{0},\bar{y}) = \bar{0}$; $\|\bar{\varphi}(\bar{x},\bar{y}) - \bar{x}\|_{R^k} \leq \epsilon$; for $j = 1,\ldots,n-k$, $\|\frac{\partial}{\partial y_j}\bar{\varphi}|_{(\bar{x},\bar{y})}\| \leq \epsilon$, all $\bar{x} \in R^k$, $\bar{y} \in R^{n-k}$.

The hypothesis (2.1) implies that $\bar{\psi}(\bar{x}, \bar{y}_0)$ is a C^∞ function of $\bar{x}$ for fixed $\bar{y}_0$, and the C^∞-jet on R^k of $\bar{\psi}$ depends continuously on $\bar{y}_0$. A similar conclusion holds

for $\bar{\varphi}(\bar{x}_0,\bar{y})$. The hypothesis (2.2) further guarantees that the coordinates on R^n can be chosen so that for fixed $\bar{x}_0$, $\bar{y} \to \bar{\psi}(\bar{x}_0, \bar{y})$ induces a continuous map on (n-k)-measures, and similarly for $\bar{x} \to \varphi(\bar{x},\bar{y}_0)$.

After these lengthy preliminaries, the proof is rather immediate. Let $(\bar{\xi}, \bar{\eta}) = (\xi,\ldots,\xi_k, \eta_1,\ldots,\eta_{n-k})$ denote coordinates on $R^n \cong R^{n^*}$, and let $\hat{f}$ denote the Fourier transform:

$$\hat{f}(\bar{\xi},\bar{\eta}) = (2\pi)^{-\frac{n}{2}} \int_{R^n} \exp\{i(\bar{\xi} \cdot \bar{x} + \bar{\eta} \cdot \bar{y})\} f(\bar{x},\bar{y})d\bar{x}d\bar{y}. \tag{2.10}$$

<u>Lemma 2.2</u>. *For each integer* $m > 0$, *there exists constants* $C(m)$, $T(m) > 0$ *such that*

$$| \hat{f}(t\bar{\varepsilon} + t\bar{\eta}) | < C(m)t^{-m} \text{ for } t > T(m) \tag{2.11}$$

for all unit vectors $(\bar{\xi},\bar{\eta}) \in R^n$.

Theorem 2.1 follows immediately from Lemma 2.2, for f has compact support in R^n and (2.11) implies that $\hat{f} \in H^s(R^n)$, the s-Sobolev space, for all $s \geq 0$. By the Sobolev Lemma, f is C^∞ on R^n.

<u>Proof of Lemma 2.2</u>:

Either $|\bar{\eta}| \leq 2|\bar{\xi}|$, or $|\bar{\xi}| < 2|\bar{\eta}|$, possibly both. We will assume the first holds and establish (2.11). The second case follows by the same proof, interchanging the

roles of $\mathfrak{F}_1$ and $\mathfrak{F}_2$.

Consider $F(t) = \hat{f}(t\bar{\xi},t\bar{\eta})$. Introduce a change of coordinate $\bar{y} = \bar{\psi}(\bar{u},\bar{v})$, $\bar{x} = \bar{u}$ for $\bar{u} \in R^k$, $\bar{v} \in R^{n-k}$, into (2.10) and separate out the variable $\bar{v}$ to obtain

$$(2.12) \qquad \hat{F}(t) = (2\pi)^{-\frac{n}{2}} \int_{R^{n-k}} d\bar{v} \int_{R^k} \exp\{it\,(\bar{\xi}\cdot\bar{u} + \bar{\eta}\cdot\bar{\psi}(\bar{u},\bar{v}))\}\cdot f(\bar{u},\bar{\psi}(\bar{u},\bar{v}))\cdot |\bar{\psi}|_{\bar{v}}(\bar{u},\bar{v})d\bar{u}$$

where $|\bar{\psi}|_{\bar{v}}\, d\bar{v}$ is the image of $d\bar{y}$ under the map $\bar{y} \to \bar{\psi}(\bar{x},\bar{y})$. By assumption, $|\bar{\psi}|_{\bar{v}}$ is continuous in $\bar{v}$ and C^∞- in $\bar{u}$.

Let A be an invertible $k \times k$ matrix whose first row is $\bar{\xi}$, and whose subsequent rows form an orthonormal basis for the complement to $\bar{\xi}$. Let B be a $k \times (n-k)$ matrix whose first row is $\bar{\eta}$, and has 0 for all other entries. Introduce a new variable $\bar{z} = \bar{z}(\bar{v},\bar{u}):R^k \to R^k$ by

$$(2.13) \qquad \bar{z} = \bar{u} + A^{-1}\cdot B\cdot\bar{\psi}(\bar{u},\bar{v}).$$

The choice of A and B ensures that

$$\bar{\xi}\cdot\bar{z} = \bar{\xi}\cdot\bar{u} + \bar{\eta}\cdot\bar{\psi}(\bar{u},\bar{v}).$$

The norm of the matrix $A^{-1}\cdot B$ satisfies

$$(2.14) \qquad \|A^{-1}\cdot B\| \le \frac{\|\bar{\eta}\|}{\|\bar{\xi}\|} \le 2,$$

so for $\epsilon > 0$ appropriately small, the differential of $\bar{u} \to \bar{z}(\bar{v},\bar{u})$ is invertible and uniform in $\bar{v}$. Furthermore, $\bar{z}_{\bar{v}}(\bar{u})$ is injective in $\bar{u}$, so we can define a differentiable

inverse, $\bar{u} = \alpha(\bar{v},\bar{z})$, whose $\bar{z}$ — C^∞-jet is uniform in $\bar{v}$. Now substitute this change-of-variable into (2.12) to obtain

$$F(t) = (2\pi)^{-\frac{n}{2}} \int_{R^{n-k}} d\bar{v} \int_{R^k} \exp(it\, \bar{\xi} o \bar{z})\, F(\bar{z},\bar{v}) d\bar{z} \tag{2.15}$$

where

$$F(\bar{z},\bar{v}) = f(\alpha(\bar{v},\bar{z}), \bar{\psi}(\alpha(\bar{v},\bar{z}),\bar{v})) \cdot |\bar{\psi}|_{\bar{v}}(\alpha(\bar{v},\bar{z}),\bar{v}) \cdot |\alpha|_{\bar{z}}(\bar{v},\bar{z}) \tag{2.16}$$

has compact support in $(\bar{z},\bar{v})$, is C^∞ in $\bar{z}$ and the C^∞-$\bar{z}$-jet of F is continuous in $\bar{v}$. Thus, the Fourier transform

$$\hat{F}_{\bar{v}}(t) = (2\pi)^{-\frac{k}{2}} \int_{R^k} \exp(it\, \overline{\xi} \cdot \bar{z})\, F(\bar{z},\bar{v}) d\bar{z} \tag{2.17}$$

has super-polynomial decay in t, uniformly in $\bar{v}$. As (2.15) is obtained from (2.17) by integrating a compactly supported function over $\bar{v}$, the formula (2.11) follows. #

<u>Remark</u> <u>2.3</u>. In the above proof, it was not necessary to assume that all quantities considered are continuous in the transverse variables. In order to derive the estimate (2.11), a weaker hypothesis at appropriate local L^p-integrability on the various quantities would suffice. We leave it to the reader to make these changes as needed. In our applications, $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are transversally C^1, so these considerations are immaterial.

§3. Modulus of Continuity for Weak-unstable Foliations

In this section we prove Theorem 0.1. We begin with a somewhat weaker result, namely we will show that the weak stable and unstable bundles for a volume-preserving C^3-Anosov flow $\{f_t\}$ on a 3-dimensional manifold M are $C^{1,\omega}$, where $\omega(s) = K \cdot s \cdot |\log s|$ for an appropriate constant K. Our proof of this is close in spirit to the classical approach to proving that these bundles are C^1 via the Arzela-Ascoli Theorem (cf. [Pu]). However, the argument given below is much more delicate than the classical case, and more precise than the global hyperbolic contraction methods of Hirsch-Pugh-Shub [HPS]. We will use the facts that the weak-stable and -unstable bundles are C^1, and that their integral submanifolds are as smooth as the Anosov flow with C^1-dependence on the base point of these submanifolds in the C^3-topology. The proofs of these facts are in [HPS].

Let $\{f_t\}$ be a fixed volume-preserving C^3-Anosov flow on M. Denote by ξ the vector field on M generating $\{f_t\}$. Endow TM with a Riemannian metric whose volume form is invariant under the flow and for which ξ is a unit vector field.

For each point $p \in M$, consider a smooth transversal $T_p \subset M$ to ξ of a fixed size which depends upon the point p in a C^∞ way. Let W^u_p and W^s_p denote the connected components containing p of the intersection of the weak-unstable and weak-stable manifolds at p with T_p. Let E^u_p and E^s_p be the unit tangents to W^u_p and W^s_p at p; so $\{E^u_p | p \in M\} = E^u$ and $\{E^s_p | p \in M\} = E^s$ define C^1 unit vector fields on M. (Caution: E^u and E^+ need not agree, and E^s and E^- need not agree.) Introduce a coordinate system on each T_p so that W^u_p and W^s_p correspond to the x-axis and y-axis, and so that the unique invariant smooth transverse measure for $\{f_t\}$ becomes the standard area form in R^2 in these coordinates. We can also assume that the

standard coordinate fields $\partial/\partial x$ and $\partial/\partial y$ along W^u_p and W^s_p are unit vector fields. An example of such a coordinate system is the exponentiation of a coordinate system in the subspace of the tangent space T_pM spanned by E^u_p and E^s_p.

We can also insist that the coordinate systems on T_p depend upon p in a C^1-way, when considered as C^∞-maps of a subset of R^2 into M. This is possible since we know that the weak-stable and weak-unstable manifolds and their C^∞-jet bundles depend C^1 on the base point. For future reference, we formalize the properties of these coordinates:

For some $\epsilon > 0$, there is a C^1-map

$$\Psi : M \times (-\epsilon,\epsilon)^2 \to M$$

such that $\Psi_p(x,y) = \Psi(p,x,y)$ is the above coordinate system on T_p, and we have:

(3.1.i) $\Psi_p:(-\epsilon,\epsilon)^2 \to M$ is a C^∞ embedding, with $\Psi_p(x,0) \in W^u_p$, $\Psi_p(0,y) \in W^s_p$ for $|x| < \epsilon$, $|y| < \epsilon$.

(3.1.ii) The curves $\Psi_p(x,0)$, $|x| < \epsilon$ and $\Psi_p(0,y)$, $|y| < \epsilon$ depend C^1 on p in the C^∞-topology of curves in M.

(3.1.iii) For $d\upsilon = i(\xi) \cdot d(vol)$, where d(vol) is the Riemannian volume form on M, $\Psi_p{*}(d\upsilon) = dx\wedge dy$.

For $\delta > 0$, consider the set $V(\delta)$ of all C^1-vector fields v on M satisfying:

(3.2.i) At each point $p \in M$, $v(p)$ is tangent to T_p

(3.2.ii) $D(\Psi_p^{-1})(v(p) = (1,\tilde{v}_p(0)) \in T_{(0,0)}R^2$, with $|\tilde{v}_p(0)| \leq \delta$

(3.2.iii) v is δ-C^1-close to the line field E^u.

The set $V(\delta)$ is clearly convex, and $V(0)$ is the unit vector field spanning E^u.

For each $t \geq 0$, define an operation $\mathfrak{F}_t$ on $V(\delta)$: For $v \in V(\delta)$, $(\mathfrak{F}_t v)(p)$ is equal to the projection of $Df_t(v(f_{-t}p))$ to T_t along the flow, and then pointwise rescaled so that (3.2.ii) holds. It follows from the usual C^1-contraction arguments that for $\delta_0 > 0$ small enough, the image of $\mathfrak{F}_t$ lies in $V(\delta)$. In fact, for any $v \in V(\delta)$, $\mathfrak{F}_t v$ converges C^1 to the vector field E^u. We will prove the following:

<u>Theorem 3.1</u>. *Let $\mathfrak{F}_t$ be defined as above, for $t \geq 0$. Then there exists ϵ_0, δ, $K > 0$ such that for every $\epsilon < \epsilon_0$, there exists $T(\epsilon)$ so that for any $t > T(\epsilon)$ and $v \in V(\delta)$, if $q \in W_p^s$ and $\epsilon \leq d = \mathrm{dist}(p,q) \leq \epsilon_0$, then in the coordinate system Ψ_p,*

(3.3) $|D_s(\mathfrak{F}_t v)(q) - D_s(\mathfrak{F}_t v)(p)| \leq Kd \cdot |\log d|$, *where D_s denotes differentiation along W_p^s by $\partial/\partial y$.*

<u>Remark 3.2</u>. Since condition (3.3) is closed in the C^1-topology, and $E^u = \lim_{t\to\infty} \mathfrak{F}_t v$ for v non-vanishing, an obvious corollary of Theorem 3.1 is that the derivative $D_s E^u$ has module of continuity $\omega(s) = K \cdot s \cdot |\log s|$ in the stable direction. As $D_s E^u$ is C^3 along the weak-unstable manifolds, we conclude that $D_s E^u$ is $C^{1+\omega}$ on M.

<u>Proof of Theorem 3.1</u>: It is enough to consider only integer values of t. Let $\mathfrak{F} =$

$\mathfrak{F}_1$ and $f = f_1$ to simplify notation.

For each $\epsilon > 0$, choose $\delta(\epsilon) > 0$ so that (3.3) is satisfied for $K = 1$, $t = 0$, all $v \in V(\delta(\epsilon))$ and all $p \in M$, $q \in W^s_p$ with $\epsilon \geq \mathrm{dist}(p,q) \geq \epsilon^2$. Such $\delta(\epsilon)$ exists by virtue of the uniform continuity of D_sE^u.

For $q \in W^s_p$, let $q' \in W_{fp}$ be the image of q under the Poincaré map from T_p to T_{fp}. Let $d = \mathrm{dist}(p,q)$ and $d' = \mathrm{dist}(fp,q')$. We will show:

<u>Proposition</u> <u>3.3</u>. *There exists* $K \geq 1$ *and* $\epsilon_1 > 0$ *with* $\Psi(p,x,y)$ *defined for* $|x| < \epsilon_1$, $|y| < \epsilon_1$, *so that if* $v \in V(\delta(\epsilon_1))$, $d < \epsilon_1$ *and*

$$\left|D_s v(q) - D_s v(p)\right| \leq K \cdot d \cdot \left|\log d\right|, \tag{3.4}$$

then

$$\left|D_s \mathfrak{F} v(q') - D_s \mathfrak{F} v(fp)\right| \leq K \cdot d' \cdot \left|\log d'\right|. \tag{3.5}$$

Proposition 3.3 implies Theorem 3.1 via a simple inductive argument. First, choose $\epsilon_0 > 0$ with $\epsilon_0 < \epsilon_1$, and ϵ_0^{-1} greater than the expansion coefficient λ_p of f at every point $p \in M$. Set $\delta_0 = \delta(\epsilon_1)$. By the Anosov condition, for any $p \in M$ and $q \in W^s_p$ with $d = \mathrm{dist}(p,q) < \epsilon_0$, there exists $t = T(p,q)$ such that the Poincaré map from T_p to $T_{f^{-t}p}$ sends q to a point $\tilde{q} \in T_{f^{-t}p}$, and $\epsilon_0 \geq \mathrm{dist}(f_p^{-1}, \tilde{q}) \geq \epsilon_0^2$. Again using the Anosov condition, for fixed $\epsilon > 0$ there is an integer $T(\epsilon)$ such that for any $p \in M$, $q \in W^s_p$ with $\mathrm{dist}(p,q) > \epsilon$, then $T(p,q) \leq T(\epsilon)$. Let $v \in V(\delta_0)$, then (3.3) holds for $\tilde{p},\tilde{q}$ as above, and K as given by Proposition 3.3. Applying Proposition 3.3 inductively to the iterates $\mathfrak{F}^n v$ of v, and the iterates $f^n\tilde{p}$ and $F^n\tilde{q}$, we arrive at the

conclusion of formula (3.3) when $n = t = T(p,q)$.

To establish formula (3.3) when $t > T(p,q)$, starting with $v \in V(\delta_0)$, observe that $w = \mathfrak{F}_{t-T(p,q)}\, v \in V(\delta_0)$, and we apply the previous inductive argument to the vector field w.

<u>Proof of Proposition 3.3</u>. Let F be the Poincaré map from T_p to T_{fp}. We write F in our coordinates $\Psi_p(x,y)$ and $\Psi_{fp}(x',z)$:

$$F(x,y) = (\lambda_p x + \varphi_p(x,y),\ \lambda_p^{-1} y + \psi_p(x,y)) \tag{3.6}$$

where the first jets of the functions φ_p and ψ_p vanish at (0,0), and we have $|\lambda_p| > \lambda_0 > 1$ for some λ_0, independent of p. Differentiating gives us

$$Df_{(x,y)}(\xi,\eta) = (\lambda_p\xi + (\varphi_p)_x(x,y)\xi + (\varphi_p)_y(x,y)\eta, \lambda_p^{-1}\eta + (\psi_p)_x(x,y)\xi + (\psi_p)_y(x,y)\eta). \tag{3.7}$$

For the rest of §3, we will suppress the dependence on p in the expansion (3.7). Furthermore, we will only consider what happens along the y-axis. We then have $F(0,y) = (0,z(y))$, where

$$z(y) = \lambda^{-1}y + \psi(o,y) = \lambda^{-1} + \frac{d}{2}y^2 + o(y^2). \tag{3.8}$$

with $d = \psi_{yy}(0,0)$. Here, and throughout when we write o( ) or 0( ), we are giving a result uniform in p. From (3.8) we derive

$$y(z) = \lambda z - \frac{\lambda^3 d}{2} \cdot z^2 + o(z^2). \tag{3.9}$$

Let us now write down the local expressions for the partial derivatives of the Poincaré map F:

$$
\begin{aligned}
&\varphi_x(0,y) = ay + \alpha y^2 + o(y^2);\ a = \varphi_{xy}(0,0);\ \alpha = \tfrac{1}{2}\,\varphi_{xyy}(0,0)\\
&\varphi_y(0,y) = by + \beta y^2 + o(y^2);\ b = \varphi_{yy}(0,0);\ \beta = \tfrac{1}{2}\,\varphi_{yyy}(0,0)\\
(3.10)\qquad &\psi_x(0,y) + cy + \gamma y^2 + o(y^2);\ c = \psi_{xy}(0,0);\ \gamma = \tfrac{1}{2}\,\psi_{xyy}(0,0)\\
&\psi_y(0,y) = dy + \delta y^2 + o(y^2);\ d = \psi_{yy}(0,0);\ \delta = \tfrac{1}{2}\,\psi_{yyy}(0,0).
\end{aligned}
$$

Given $v \in V(\delta(\epsilon_1))$, the restriction of v to the manifold W^s_p is given in coordinates, after a rescaling, by $(1,\tilde{v}_p(y))$, where we expand $\tilde{v} = \tilde{v}_p$ locally:

$$(3.11)\qquad \tilde{v}(y) = \tilde{k}_p + \tilde{\ell}_p y + \tilde{\tau}_p(y);\ \tilde{\tau}_p \text{ a } C^1\text{-function with } \tilde{\tau}_p'(0) = 0.$$

The hypothesis (3.2.ii) implies $|k_p| < \delta(\epsilon_1)$. The vector field E^u can similarly be expressed in coordinates about p, after suitable rescaling, as $(1, v_p(y))$, where v_p has a local expansion

$$(3.12)\qquad v_p(y) = \ell_p \cdot y + \tau_p(y);\ \tau_p'(o) = 0.$$

The hypothesis (3.2.iii) implies $|\tilde{\ell}_p - \ell_p| < \delta(\epsilon_1)$.

For the remainder of the proof of Proposition 3.3, we will suppress the dependence on p in (3.11) and (3.12).

We calculate the image $\mathfrak{F}v$ in Ψ_{fp} coordinates using the specialization of (3.7):

$$(3.13)\quad DF(1,\tilde{v}(y)) = (\lambda + \varphi_x(0,y) + \tilde{v}(y)\varphi_y(0,y), \lambda^{-1}\tilde{v}(y) + \psi_x(0,y) + \tilde{v}(y)\psi_y(0,y),$$

which after rescaling becomes $(1,w(z))$, where

$$w(z) = \frac{\lambda^{-1}\tilde{v}(y) + \psi_x(0,y(z)) + \tilde{v}(y(z))\psi_y(0,y(z))}{\lambda + \varphi_x(0,y(z)) + \tilde{v}(y(z)) \cdot \varphi_y(0,y(z))} \tag{3.14}$$

As D_s is simply differentiation in the y-variable,

$$(D_s\tilde{v})(y) = \tilde{\ell} + \tilde{\tau}'(y).$$

Our assumption means that

$$|\tau'(y)| \leq K \cdot |y| \cdot \left|\log |y|\right|. \tag{3.15}$$

because the coordinate, y, coincides with the length parameter on W_p^s. We need to calculate

$$(D_s w)(z) = \frac{dw}{dz}(z).$$

Our strategy will be to calculate the constant terms in $D_s w$ and to estimate the rest by a uniform constant times $|z|$. Let us denote

$$\begin{aligned} A(z) &= \varphi_x(0,y(z)) \\ B(z) &= \varphi_y(0,y(z)) \\ C(z) &= \psi_x(0,y(z)) \\ D(z) &= \psi_y(0,y(z)) \\ u(z) &= \tilde{v}(y(z)). \end{aligned} \tag{3.16}$$

From (3.16) and (3.9), A, B, C and D are all of order $0(|z|)$. Then (3.14) can be

rewritten using (3.16), and differentiated to obtain:

$$\frac{dw}{dz}(z) = \frac{\lambda^{-2}u'(z) + \lambda^{-1}C'(z) + \lambda^{-1}u(z)D'(z)}{(1 + \lambda^{-1}A(z) + \lambda^{-1}u(z)B(z))} \tag{3.17}$$

$$- \frac{(\lambda^{-2}u(z) + \lambda^{-1}C(z) + \lambda^{-1}u(z) \cdot D(z))(\lambda^{-1}A'(z) + \lambda^{-1}u'(z)B(z) + \lambda^{-1}u(z)B'(z))}{(1 + \lambda^{-1}A(z) + \lambda^{-1}u(z)B(z))^2}$$

Let us calculate the quantities involved in (3.17) up to linear terms in z. We use (3.9), (3.10) and (3.16):

$$\begin{aligned}
A(z) &= a\lambda z + 0(z^2) = 0(|z|) \\
A'(z) &= a\lambda + (2\alpha\lambda^2 - ad\lambda^2)z + 0(z^2) = a\lambda + 0(|z|) \\
B(z) &= 0(|z|) \\
B'(z) &= \lambda b + 0(|z|) \\
C(z) &= 0(|z|) \\
C'(z) &= c\lambda + 0(|z|) \\
D(z) &= 0(|z|) \\
D'(z) &= d\lambda + 0(|z|) \\
u(z) &= \tilde{k} + \tilde{\ell}\lambda z + o(z) = \tilde{k} + 0(|z|) \\
u'(z) &= \tilde{\ell}\lambda + \tilde{\tau}'(\lambda z - \frac{\lambda^3 d}{2} z^2 + 0(z^2)) \cdot (\lambda - \lambda^3 dz + 0(z^2)) - \lambda^3\tilde{\ell}dz + o(|z|) \\
&= \tilde{\ell}\lambda + \lambda\tilde{\tau}'(\lambda z - \frac{\lambda^3 d}{2} z^2 + o(z^2)) + 0(|z|).
\end{aligned}$$

Substituting these expressions into (3.17) we obtain

$$\frac{dw}{dz}(z) = \frac{\lambda^{-1}\tilde{\ell} + \lambda^{-1}\tilde{\tau}'(y(z)) + c + \tilde{k}d + 0(|z|)}{1 + 0(|z|)} - \frac{(\lambda^{-2}\tilde{k} + 0(|z|))\,(a + \tilde{k}b + 0(|z|))}{1 + 0\,(|z|)}$$

$$(3.18) \qquad \frac{dw}{dz}(z) = c + \tilde{k}d + \lambda^{-1}\tilde{\ell} - \lambda^{-2}a\tilde{k} - \lambda^{-2}b\tilde{k}^2 + \lambda^{-1}\tilde{\tau}'(y(z)) + 0(|z|).$$

Thus, we derive the simple expression

$$(3.19) \qquad \frac{dw}{dz}(z) - \frac{dw}{dz}(o) = \lambda^{-1}\tau'(y(z)) + 0(|z|).$$

Now we use (3.15) and (3.9) to obtain the estimate

$$(3.20) \qquad |\lambda^{-1}\tau'(y(z))| + 0(|z|) \leq K\lambda^{-1}|y(z)| \cdot \left|\log |y(z)|\right| + 0(|z|)$$

$$= K \cdot (|z| + 0(|z|^2)) \cdot \left|\log|z| + \log(\lambda - \frac{\lambda^3 d}{2}|z| + o(|z|))\right|$$

Since $\log |z|$ is negative, $\log \lambda$ is positive and $\left|\log |z|\right| \gg \log \lambda$ if $\epsilon_1 \ll 1$, the expression on the right-hand side can be rewritten

$$(3.21) \qquad K \cdot (|z| + 0(z^2)) \cdot (\left|\log |z|\right| - \log \lambda + 0(|z|))$$

$$= K \cdot |z| \cdot \left|\log |z|\right| - (K(\log \lambda)|z| + o(|z|)) + 0(|z|).$$

Now the last term $0(|z|)$ is uniform in p, and does not depend on the choice of K. Thus, for a suitably large choice of K and sufficiently small $\epsilon_1 > 0$, $-(K \cdot (\log \lambda)|z|) + 0(|z|) < 0$ uniformly in $p \in M$ and $|z| < \epsilon_1$. Thus,

$$\left|\frac{dw}{dz}(z) - \frac{dw}{dz}(0)\right. < K \cdot |z| \cdot \left|\log |z|\right|$$

proving Proposition 3.3. #

A variant of the proof of Proposition 3.3, taking into account the actual linear terms in the expansion of (3.17), yields a sharper result, Theorem 0.1 of the introduction. The key change is reflected in:

Proposition 3.4. *Given* $K > 0$, *there exists* $\epsilon_1 > 0$ *and* $C(K) > 0$ *so that if* $v \in V(\delta(\epsilon_1))$, $d = \mathrm{dist}(p,q) < \epsilon_1$ *and*

$$\left|(D_s v)(q) + (D_s v)(-q) - 2(D_s v)(p)\right| \leq K \cdot d, \tag{3.22}$$

then for $d' = \mathrm{dist}(fp,q')$,

$$\left|(D_s \mathfrak{F} v)(q') + (D_s \mathfrak{F} v)(-q') - 2(D_s \mathfrak{F} v)(fp)\right| \leq K \cdot d' \cdot (1 + C(K)d'). \tag{3.23}$$

Proof: The first step is to obtain a better estimate for $\frac{dw}{dz}(z)$ than (3.18). Since the Poincaré map F is C^3, the functions A, A′, B, B′, C, C′, D, D′ all have expansions into first-order Taylor series. After substituting these expansions in (3.17), we obtain a linear expansion for $\frac{dw}{dz}$, for constants c_0 and c_1:

$$\frac{dw}{dz}(z) = c_0 + \lambda^{-1}\tilde{\tau}'(y(z)) + c_1 z + 0(|z|^2). \tag{3.24}$$

The key point is again that c_0, c_1 and $0(|z|^2)$ are uniform in $p \in M$. In particular, by (3.9)

$$c_1\left|y(z) + y(-z) - 2y(0)\right| = 0(|z|^2) \tag{3.25}$$

uniform in p. Combining (3.24) and (3.25), the left-hand side at (3.23) is estimated by

$$(3.26) \qquad \lambda^{-1}K\left|\tilde{\tau}'(y(z)) + \tilde{\tau}(y(-z)) - 2\tilde{\tau}'(0)\right| + 0(|z|^2)$$

$$\leq K\left||z| + \frac{\lambda^2 \cdot \psi_{yy}}{2}|z|^2\right| + 0(|z|^2),$$

$$\leq K \cdot |z| \cdot (1 + C(K) \cdot |z|),$$

where C(K) is chosen so that

$$(3.27) \qquad K \cdot C(K) \cdot |z|^2 \geq \tfrac{1}{2} \cdot K \cdot \lambda^2 \cdot \psi_{yy} \cdot |z|^2 + 0(|z|^2), \text{ for all } p \in M. \qquad \#$$

Note that C(K) can be chosen monotone decreasing in K; i.e., if $K' > K$, then $C(K')$ can be chosen $C(K') < C(K)$. In particular, there is a uniform choice of C(K) if K is bounded away from zero.

We use Proposition 3.4 to prove an analog of Theorem 3.1.

<u>Proposition</u> <u>3.5</u>. *There exists* $K > 1$ *and* $\epsilon_0, \delta_0 > 0$ *so that for any* $0 < \epsilon < \epsilon_0$, *there exists* $T(\epsilon)$ *such that for any* $t \geq T(\epsilon)$ *and* $v \in V(\delta_0)$, *if* $q \in W^s_p$ *and* $\epsilon \leq d = \text{dist}(p,q) \leq \epsilon_0$, *then*

$$(3.28) \qquad \left|(D_s\mathfrak{F}_t v)(q) + (D_s\mathfrak{F}_t v)(-q) - 2(D_s\mathfrak{F}_t v)(p)\right| \leq K \cdot d.$$

<u>Proof</u>. Only the changes from the proof of Theorem 3.1 above will be indicated. Choose ϵ_0, $\delta_0 > 0$ as before, except we insist that, for $d = \text{dist}(p,q)$

$$(3.29) \qquad \left|(D_s v)(q) + (D_s v)(-q) - 2(D_s v)(p)\right| \leq d$$

for all $q \in W^s_p$ with $\epsilon_0 \geq d \geq \epsilon_0^2$, $v \in V(\delta_0)$. Given $p \in M$ and $q \in W^s_p$, define $\tilde{p} = F^{-T}p$ and $\tilde{q} = F^{-T}q$, where $T = T(p,q)$ and so $\epsilon_0 \geq \mathrm{dist}(\tilde{p},\tilde{q}) \geq \epsilon_0^2$. Set $d_n = \mathrm{dist}(F^n\tilde{p},F^n\tilde{q})$. Then (3.29) holds for $\tilde{p},\tilde{q}$, and we can repeatedly apply Proposition 3.4 to $\tilde{p},\tilde{q}$ to obtain for $1 \leq n \leq T(p,q)$

$$(3.30) \qquad \left|(D_s\mathfrak{F}^n v)(q) + (D_s\mathfrak{F}^n v)(-q) - 2(D_s\mathfrak{F}^n v)(p)\right|$$

$$\leq d_n \cdot (1 + Cd_1)(1 + Cd_2) \cdots (1 + Cd_n)$$

$$\leq d \cdot \prod_{i=1}^{n} (1 + C\lambda_0^{-i}).$$

Here, $C = C(1)$ works for each step of the induction, as at stage n we are working with $K_n = \prod_{i=1}^{n} (1 + C\lambda_0^{-i})$ and can use $C(K_n) = C(1)$ for the next application of Proposition 3.4. Now set $K = \prod_{i=1}^{\infty} (1 + C\lambda_0^{-1})$, and (3.28) follows. #

<u>Remark 3.6</u>. As in Remark 3.2, it follows from Proposition 3.5 that E^u is C^{1,Λ_*} along the curves W^s_p, for all $p \in M$. As E^u is C^2 along the weak-unstable manifolds of $\{f_t\}$, we conclude that E^u is C^{1,Λ_*} on M, and Theorem 0.1 is proven. #

§4. The Anosov cocycle and local obstructions to smoothness

In this section we construct a real valued cocycle over the flow $\{f_t\}$ from the action of $\mathfrak{F}_t$ on the 2-jets transverse to the weak-stable foliations. At every periodic point p of the flow with period t_o, there is a real-valued obstruction $A_u^{\Psi}(p,t_o)$, which actually depends only on the periodic orbit to which p belongs, to this cocycle being cohomologous to zero. In the case where $\{f_t\}$ is the suspension of a toral automorphism, $A_u^{\Psi}(p,t_o)$ coincides with Anosov's obstruction to the distribution E^u being C^2 at p ([An], Section 24). This observation of Anosov is generalized by the last result of Part I: at each periodic point p, there is an ω-Hölder semi-norm on D_sE^u, so that $A_u^{\Psi}(p,t_o) \neq 0$ implies $\|D_sE^u\|_p^{\omega} \neq 0$. Consequently, E^u cannot be $C^{1,1}$ at p, and the modulus of continuity of D_sE^u is precisely $\omega(s) = O(s \cdot |\log s|)$.

Note that all of the results of this section have obvious counterparts involving the obstructions $A_s^{\Psi}(p,t_o)$ for the weak-stable distribution E^{ws}.

We continue with the notation of section 3. The basic technical idea of the present section is to analyze how $\mathfrak{F}_t$ acts on the 2-jet or mod $o(|y|^2)$ germ of a vector field $v \in V(\delta)$. Also, in this section care must be taken with the dependence on base points, and the presence of "$\sim$".

For $v \in V(\delta)$, $p \in M$, recall that after rescaling in local coordinates $\Psi_p(x,y)$, v has the form $(1, \tilde{v}_p(y))$ along the y-axis, where $\tilde{v}_p(y) = \tilde{k}_p + \tilde{\ell}_p y + \tilde{\tau}_p(y)$. Define $\tilde{\tau}_p^*(y)$ so that $\tilde{\tau}_p(y) = \tilde{\tau}_p^*(y) \cdot y$. Then $\tilde{\tau}_p^*(y)$ is continuous, and $\tilde{\tau}_p^*(0) = 0$. Note that v is C^2 at p precisely when $\tilde{\tau}_p^*(y)$ is C^1 at $y = 0$.

The vector field E^u is expressed in coordinates about p, after rescaling, as $(1, v_p(y))$, where

$$v_p(y) = \ell_p \cdot y + \tau_p^*(y) \cdot y. \tag{4.1}$$

Our interest is in how $\tilde{\tau}_p^*$ is transformed under $\mathfrak{F}_t$. Let $\mathfrak{F}_t v$ be expressed in coordinates about $f_t p$, after rescaling, as $(1,w(z))$ where by (3.14),

$$w(z) = \frac{\lambda_p^{-1} \cdot u(z) + \lambda_p^{-1}C(z) + \lambda_p^{-1}u(z)D(z)}{1 + \lambda_p^{-1}A(z) + \lambda_p^{-1}u(z)B(z)} \tag{4.2}$$

Combining (3.16), (3.10) and (3.9), we obtain expansions up to order 2 for the terms in (4.2):

$$\begin{aligned}
A(z) &= \lambda az + (\lambda^2\alpha - \tfrac{1}{2} \cdot \lambda^3 ad)z^2 \\
B(z) &= \lambda bz + (\lambda^2\beta - \tfrac{1}{2} \cdot \lambda^3 bd)z^2 \\
C(z) &= \lambda cz + (\lambda^2\gamma - \tfrac{1}{2} \cdot \lambda^3 cd)z^2 \\
D(z) &= \lambda dz + (\lambda^2\delta - \tfrac{1}{2}\lambda^3 d^2)z^2 \\
u(z) &= \lambda\tilde{\ell}_p z + \lambda^{-1}\tilde{\tau}^*(\lambda z) \cdot z + (\tfrac{1}{2}\lambda^3\tilde{\ell}_p d) \cdot z^2,
\end{aligned} \tag{4.3}$$

where all of the coefficient functions are evaluated at p. Substituting (4.3) into (4.2), after simplifying and eliminating terms at order larger than two, we obtain

$$w(z) = (\lambda^{-1}\tilde{\ell}_p + c)z + A_u^{\Psi}(p,t,v)z^2 + \lambda^{-1}\tilde{\tau}^*(\lambda z) \cdot z + o(z^2) \tag{4.4}$$

where

$$A_u^{\Psi}(p,t,v) = \tilde{\ell}_p(\tfrac{1}{2}\lambda d - a\lambda^{-1}) - (ac + \tfrac{1}{2}\lambda^2 cd) + \lambda\gamma. \tag{4.5}$$

Differentiating the identity $\varphi_x\psi_y - \varphi_y\psi_x = 1$ yields $a + \lambda^2 d = 0$, and (4.5) can be rewritten:

$$A_u^{\Psi}(p,t,v) = \tfrac{1}{2}(3\lambda d \cdot \tilde{\ell}_p - 2ac - \lambda^2 dc) + \lambda\gamma. \tag{4.6}$$

We define $A_u^{\Psi}(p,t) = A_u^{\Psi}(p,t,E^u)$. Then substituting (4.1) into (4.6) and expressing in terms of F:

$$A_u^{\Psi}(p,t) = \tfrac{1}{2}\,(3\lambda\psi_{yy} \cdot \ell_p - 2\varphi_{xy}\,\psi_{xy} - \lambda^2\psi_{yy}\psi_{xy}) + \lambda\psi_{xyy}. \tag{4.7}$$

The vector field E^u is the unique invariant field transverse to the manifold W_p^s, so $A_u^{\Psi}(p,t)$ is independent of E^u, as the ℓ_p are uniquely determined by $\{f_t\}$ and Ψ_p. For example from (4.4) we obtain:

$$\ell_{f_t p} = \lambda^{-1} \cdot \ell_p + c, \tag{4.8}$$

and if $p = f_t p$ is a fixed point,

$$\ell_p = \frac{\lambda c}{\lambda - 1}. \tag{4.9}$$

Also, there is the relation

$$\tau^*_{f_t p}(z) = \lambda^{-1}\tau^*_p(\lambda z) + A_u^{\Psi}(p,t) + o(z^2). \tag{4.10}$$

We next study the function $(p,t) \mapsto A_u^{\Psi}(p,t)$.

Lemma 4.1. *Let* $\{f_t\}$ *be a volume-preserving* C^4 *Anosov flow. Then the function* $A_u^{\Psi} : M \times R \to R$ *is* C^1.

Proof: All of the quantities in (4.7) depend upon the basepoint p and time t in a C^1 way. #

A coordinate system Ψ is said to be special adapted coordinates if Ψ satisfies (3.1), and for each $p \in M$, the vector field E^u is given in local coordinates, after rescaling, as $(1, v_p(y))$ where $v_p(y) = \tau_p^*(y) \cdot y + o(y^2)$. That is, $\ell_p \equiv 0$.

The existence of special adapted coordinates follows from knowing that the distributions E^u and E^s are C^1.

Lemma 4.2. *Let* Ψ *be special adapted coordinates. Then for each* t *and* $p \in M$, *in coordinates* $\Psi_p(x,y)$ *and* $\Psi_{f_t}(x^1,z)$, *we have* $\psi_{xy}(0,0) = 0$.

Proof: By assumption, $\ell_p = 0 = \ell_{f_t p}$. Then by (4.8) we conclude $0 = c = \psi_{xy}(0,0)$#

Corollary 4.3 *Let* Ψ *be special adapted coordinates. Then the invariants* $A_u^{\Psi}(p,t)$ *have the simplified form*

$$A_u^{\Psi}(p,t) = \lambda \cdot \psi_{xyy}. \tag{4.11}$$

Proof: Given $\psi_{xy} = 0 = \ell_p$, we obtain (4.11) from (4.7). #

Proposition 4.4. *Let Ψ be special adapted coordinates. For* $p \in M$ *and* $r,s \in R$,

$$A_u^{\Psi}(p,t) + A_u^{\Psi}(f_t p,s) = A_u^{\Psi}(p,t+s). \tag{4.12}$$

Proof: Let $DF_t = \begin{bmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{bmatrix}_{(x,y)}$ denote the differential from T_p to $T_{f_t p}$,

$DF_s = \begin{bmatrix} \bar{\varphi}_x & \bar{\varphi}_y \\ \bar{\psi}_x & \bar{\psi}_y \end{bmatrix}_{F_t(x,y)}$ denote the differential from $T_{f_t p}$ to $T_{f_{t+s}\, p}$, and

$DF_{t+s} = \begin{bmatrix} \tilde{\varphi}_x & \tilde{\varphi}_y \\ \tilde{\psi}_x & \tilde{\psi}_y \end{bmatrix}_{(x,y)}$ the differential from T_p to $T_{f_{t+s}P}$. The chain rule

yields $\tilde{\psi}_x(0,y) = \bar{\psi}_x(0,\psi(0,y) \cdot \varphi_x(0,y) + \bar{\psi}_y(0,\psi(0,y) \cdot \psi_x(0,y)$. Differentiate this twice with respect to y, set $y = 0$, use $\psi_{xy} = \bar{\psi}_{xy} = 0$, and $\bar{\varphi}_x\bar{\psi}_y = \varphi_x\psi_y = 1$ to conclude $\tilde{\psi}_{xyy} = \bar{\psi}_{xyy} \cdot \psi_y + \bar{\psi}_y \cdot \psi_{xyy}$, or $\tilde{\varphi}_x\tilde{\psi}_{xyy} = \bar{\varphi}_x\varphi_x \cdot \tilde{\psi}_{xyy} = \bar{\varphi}_x\bar{\psi}_{xyy} + \varphi_x\psi_{xyy}$.

This last equation is equivalent to (4.12), using (4.11). #

Actually, Formula (4.12) holds for all adapted coordinates Ψ, not necessarily special. We leave it to the reader to check this. The interpretation of (4.12) is that $A_u^{\Psi} : M \times R \to R$ is a cocycle over the flow $\{f_t\}$. In part II, we analyze the consequences of this cocycle being a coboundary.

toral automorphism (Lemma 24.1, [An]).

<u>Proposition</u> <u>4.5</u>. *Let* $\{f_t\}$ *be a volume-preserving* C^3*-Anosov flow on* M. *Suppose that* E^u *is* C^2 *at* p, *and* p *is a periodic point with period* t_o.

Then $A_u^{\Psi}(p,t_o) = 0$, *or*

$$(4.16) \qquad (4\lambda^2 - \lambda^3)\psi_{xy}\psi_{yy} - (\lambda - 1)\varphi_{xy}\psi_{xy} + 2(\lambda^2 - \lambda)\psi_{xyy} = 0.$$

<u>Proof</u>: It's given that $\tau_p^*(y) = \tau^*_{f_{t_o}P}(y)$ is a C^1 function, so (4.10) implies $A_u^{\Psi}(p,t_o) = 0$. We then substitute (4.9) into (4.7) and simplify. #

<u>Corollary</u> <u>4.6</u>. *Let* p *be a periodic point for the flow* $\{f_t\}$ *with period* t_o. *If* $A_u^{\Psi}(p,t_o) \neq 0$, *then* E^u *is not* C^2 *at* p.

This last corollary is not sharp, and we conclude this section with a general, local result which is sharp. Let $F : (-\epsilon,\epsilon)^2 \to R^2$ be a volume-preserving C^3-Anosov embedding satisfying

$$(4.17) \qquad \begin{aligned} &F(x,0) = (\lambda x + \varphi(x,0),0),\ \lambda > 1 \\ &F(0,y) = (0,\ \lambda^{-1}y + \psi(0,y)) \\ &0 = \varphi_x(0,0) = \varphi_y(0,0) = \psi_x(0,0) = \psi_y(0,0) \\ &(\lambda + \varphi_x(x,y))\,(\lambda^{-1} + \psi_y(x,y)) - \varphi_y(x,y)\psi_x(x,y) \equiv 1. \end{aligned}$$

Let E^u and E^s denote the unstable and stable invariant line fields for DF, and suppose that $E^u(x,y) = (1,v(x,y))$ for $|x| < \epsilon$, $|y| < \epsilon$. The field E^u is C^1 by [HPS],

so there is an expansion $v(0,y) = \ell y + \tau(y)$, with $\tau'(0) = 0$. Define A_F^+ by the formula (4.7), using $\ell = \ell p$.

For $\omega(s) = s \cdot |\log s|$, $s > 0$, set

$$|E^u|_0^{1,\omega} = \limsup_{|y|<\epsilon} \frac{|\tau'(y)|}{\omega(|y|)} \tag{4.18}$$

The semi-norm $|E^u|_o^{1,\omega}$ is finite by Theorem 3.1.

<u>Theorem 4.7</u>. *Let* $F : (-\epsilon,\epsilon)^2 \to R^2$ *be a* C^3*-Anosov map satisfying* (4.17). *Suppose that* $A_F^+ \neq 0$. *Then*

$$|E^u|_o^{1,\omega} \neq 0.$$

Consequently, the restriction to the y*-axis of the derivative field* $D_sE^u = (0, \frac{\partial v}{\partial y})$ *had modulus of continuity at* $y = 0$ *exactly* $\omega(s) = O(s \cdot |\log s|)$, *and is, in particular, not Lipshitz at* $y = 0$.

The following corollary of Theorem 4.7 coincides with part (i) of Theorem 0.2.

<u>Corollary 4.8</u>. *Let* $\{f_t\}$ *be a volume-preserving* C^3*-Anosov flow on a 3-manifold* M. *Let* p *be a periodic point of period* t_o, *and suppose* $A_u^\Psi(p,t_o) \neq 0$ *for some coordinates* ψ. *Then the weak-unstable distribution of* $\{f_t\}$ *has class exactly* $C^{1,\omega}$ *at* p *and in particular is not* $C^{1,1}$ *at* p.

<u>Proof of Theorem 4.7</u>: Via a C^3-volume-preserving change of variables, we can

assume $F(0,y) = (0,\lambda^{-1}y)$ and $F(x,0) = (\lambda x,0)$. From $\psi(0,y) = 0$, we deduce $\psi_{yy}(0,0) = 0$, and differentiating the last equation of 4.17 yields $\varphi_{xy}(0,0) = 0$. Our hypothesis is thus that

$$A_F^+ = \lambda \cdot \psi_{xyy}(0,0) \neq 0.$$

In the new coordinates, restrict E^u to the y-axis and renormalize to obtain the vector field $(1,v(y))$. The germ of $v(y)$ at 0 has an expansion

$$v(y) = \ell \cdot y + \tau^*(y) \cdot y + o\,(|y|^2).$$

Applying DF to E^u and renormalizing to obtain $(1,w(z))$, the invariance at E^u implies $v(z) = w(z)$, so for the germ at $y = 0$ we deduce

$$\tau^*(\lambda^{-1}y) = \tau^*(y) + A_F^+ \cdot (\lambda^{-1}y) + o(|y|). \tag{4.20}$$

Now use (4.20) inductively to obtain

$$\tau^*(\lambda^{-n}y) = \lambda^{-n}\tau^*(y) + n \cdot \lambda^{-n}A_F^+ \cdot y + E(y,n) \tag{4.21}$$

$$E(y,n) = \sum_{j=0}^{n} \lambda^{j-n}o(\lambda^j y). \tag{4.22}$$

Now to estimate $\limsup\limits_{|y|<\epsilon} \dfrac{\tau'(y)}{y\cdot\log y}$, fix $0 < y < \epsilon$ and $n > 0$. By the mean-value Theorem, there is a point z_n with $0 < z_n < \lambda^{-n}y$ so that

$$\tau'(z_n) = \frac{\tau(\lambda^{-n}y)}{\lambda^{-n}y} = \tau^*(\lambda^{-n}y). \tag{4.23}$$

Without loss, we can assume $\epsilon < e^{-1}$, so that $z \cdot |\log z|$ is monotone increasing for $0 < z < \epsilon$, and thus

$$\frac{|\tau'(z_n)|}{z_n|\log z_n|} \geq \frac{|\tau^*(\lambda^{-n}y)|}{(\lambda^{-n}y)\,|\log(\lambda^{-n}y)|}$$

which by (4.21) is estimated by

$$\text{(4.24)} \qquad \frac{|\tau^*(y)|}{y \cdot (n \log \lambda - \log y)} + \frac{n \cdot |A_F^+|}{(n \cdot \log \lambda - \log y)} + \frac{E(y,n)}{y(n \cdot \log \lambda - \log y)}$$

The quantity in (4.24) has limit $\frac{A_F^+}{\log \lambda}$, as $n^{-1} \cdot E(y,n)$ is seen from (4.22) to have limit 0. Thus, $\left| E^u \right|_0^{1,\omega} \geq \frac{A_F^+}{\log \lambda} > 0.$ #

<u>Remark 4.9</u>. It is natural to ask whether a local estimate on the Zygmund norm of $v'(y)$ at $y = 0$ can also be given using (4.21). The answer is undetermined, but we make an observation. The estimate hinges on the behavior of the error term $E(y,n)$ of (4.22), with the relevant question being to obtain an estimate for

$$y^{-1} \cdot \sum_{j=1}^{n} \lambda^j \{o(\lambda^{-j}y) + o(-\lambda^{-j}y)\} \text{ for n large.}$$

This is similar to a question implicit in our proof of Proposition 3.5: If the expression in (3.29) is required to be less than $\epsilon \cdot d$, for $\epsilon > 0$, can the resulting constant $K = K(\epsilon)$ be chosen arbitrarily small as $\epsilon \to 0$? Again, the answer is dictated by the accumulation of the estimate $0(|z|^2)$ of (3.26), which is basically the same error function appearing in $E(y,n)$.

II. Rigidity for Anosov Flows

§5. Vanishing criteria for the Anosov cocycle.

In Section 4 we defined the unstable Anosov cocylce A_u^{Ψ} over the flow $\{f_t\}$. In this section, we state two theorems of Livshitz [L1], [L2] (see also [GK]), which give effective criteria for A_u^{Ψ} to be a coboundary. Recall that a cocycle $A : M \times R \to R$ over $\{f_t\}$ is a C^1-coboundary if there is a C^1-function $\xi : M \to R$ such that $A(p,t) = \xi(f_t p) - \xi(p)$, $p \in M$, $t \in R$. Two cocycles over $\{f_t\}$ are cohomologous if their difference is a C^1-coboundary. $H^1(\{f_t\};R)$ denotes the group of cocycles modulo C^1-coboundaries. We record here a result implicit in Section 4:

Proposition 5.1. *Let Ψ and Ψ' be two sets of adapted coordinates. Then the corresponding Anosov cocycles A_u^{Ψ} and $A_u^{\Psi'}$ are cohomologous.*

We define $A^+ = [A_u^{\Psi}] \in H^1(\{f_t\};R)$, the cohomology class of A_u^{Ψ} for some adapted coordinates Ψ. Let A^- denote the corresponding class obtained from reversing the roles of stable and unstable foliations.

Now we have the statement of Theorem 0.3 with C^1 conjugacy replaced by C^3 conjugacy.

Corollary 5.2. *The cohomology classes A^+ and A^- are well-defined invariants of $\{f_t\}$, i.e. if the flows $\{f_t\}$ and $\{g_t\}$ are C^3 conjugate then the classes A^+ and A^- are carried over by the conjugating diffeomorphism.*

To finish the proof of Theorem 0.3 we need the following result of [Ka].

Theorem 5.3. *If two volume preserving* C^∞ *Anosov flows* $\{f_t\}$ *and* $\{g_t\}$ *on a compact 3-manifold are* C^1 *conjugate then they are* C^∞ *conjugate.*

Basic Problem. *What information about the* C^∞*-conjugacy class at* $\{f_t\}$ *can be derived from the invariants* A^+ *and* A^-*?*

A more specific question along that same line has been formulated in Section 0 (Problem 0.18).

The crucial point for Part II is to show $A^+ = 0$ implies that the weak-unstable distribution E^{wu} of $\{f_t\}$ is C^∞. For a toral automorphism, $A^+ = 0$ or $A^- = 0$ will imply $\{f_t\}$ is C^∞-conjugate to a linear map and a somewhat weaker but similar result will follow for geodesic flows. Thus, we need to establish criteria for the vanishing of A^+ and A^-. These are provided by two theorems of Livshitz, which we now cite with their application to A^+ and A^-.

Theorem 5.4. ([L1], [L2]). *Let* $A : M \times R \to R$ *be a* C^1*-cocycle over a transitive (e.g. volume-preserving) Anosov flow* $\{f_t\}$. *Then* A *is a* C^1*-coboundary if and only if for every periodic point* $p \in M$ *with period* t_o,

$$A(p,t_o) = 0. \tag{5.1}$$

Furthermore, any two continuous solutions of the equation $A(p,t) = \xi(f_t p) - \xi(p)$ *differ by a constant.*

Corollary 5.5. *Let* $\{f_t\}$ *be a volume-preserving transitive* C^4*-Anosov flow on a 3-manifold* M. *Then the Anosov cocycle* A_u^{Ψ} *is a* C^1*-coboundary if and only if* $A_u^{\Psi}(p,t_o) = 0$ *for every periodic point* p *with period* t_o.

Combining Theorem 4.7 with this corollary we obtain:

Corollary 5.6. *If* $A^+ \neq 0$, *then the weak-unstable distribution* E^{wu} *of* $\{f_t\}$ *is not* $C^{1,1}$ *at some periodic point of* $\{f_t\}$. *Similarly,* $A^- \neq 0$ *implies that* E^{ws} *is not* $C^{1,1}$ *at some periodic point of* $\{f_t\}$. *This establishes the statement of Theorem* 0.4 *in one direction.*

There is a second criterion due to Livshitz for the vanishing of a cocycle over $\{f_t\}$.

Theorem 5.7 (Theorem 9, [L2]). *Let* $A : M \times R \to R$ *be a* C^1*-cocycle over a transitive Anosov flow* $\{f_t\}$. *Suppose there exists a measurable function* $\xi : M \to R$ *with*

$$A(p,t) = \xi(f_t p) - \xi(p), \text{ a.e. } p \in M, \text{ all } t \geq 0. \tag{5.2}$$

Then there is a C^1*-function* $\tilde{\xi} : M \to R$ *satisfying* (5.2) *for all* $p \in M$, *and* $\tilde{\xi}(p) = \xi(p)$ *a.e.* $p \in M$.

The following corollary of this criterion implies statement (ii) of Theorem 0.2.

Corollary 5.8 *Let* $\{f_t\}$ *be a volume-preserving transitive* C^4*-Anosov flow on a 3-*

manifold M. *If* $A^+ \neq 0$, *then for almost every* $p \in M$ *the first derivative,* $v'_p(y)$, *of the local angle function of* E^u *at* p *does not have a derivative in* y.

Proof. Let us choose a special adapted coordinate system. The set of points $p \in M$ where $v'_p(y)$ has a y-derivative of p is measurable and flow-invariant. Since the flow is ergodic that set has either measure zero or full measure. In the latter case for almost every $p \in M$ the function $\tau^*_p(y)$ (cf. (4.1)) has y-derivative at $y = p$. Let q(p) be the value of that derivative so that $\tau^*_p(y) = q(p)y + o(|y|)$. By (4.10) for almost every $p \in M$ and all $t > 0$

$$q(f_t p) - q(p) = A^{\Psi}_u(p,t) \tag{5.3}$$

Then by Theorem 5.7 q(p) can be extended to a C^1 function on M such that (5.3) still holds. Thus $A^+ = 0$. #

§6. Vanishing of A^+ and C^∞-rigidity.

Assume there exists a C^1-function $\xi : M \to R$ with $A_u^\Psi(p,t) = \xi(f_t p) - \xi(p)$ for all $p \in M$, $t \in R$. For $\{f_t\}$ a C^∞-flow on a compact manifold M, we will show this forces E^u to be a C^∞ vector field on M. When $\{f_t\}$ is a C^k-flow, $k \geq 4$, our proof can be adapted to show E^u is $C^{k-\upsilon}$, for appropriate $\upsilon > 0$. We leave the details of these intermediate results to the reader.

Let $V(\delta)$ be the convex cone of C^1-vector fields on M introduced in Section 3. Choose a C^∞-vector field $v \in V(\delta)$, for δ small, and for each $s > 0$ set $v(s) = \mathfrak{F}_s v$. It is easy to show that the family $\{v(s) | s > 0\}$ converges exponentially fast in the C^1-topology to E^u. The main point is to show this family limits to a C^2-vector field, when restricted to the stable manifolds W_p^s. From this, we use Livshitz Theorem to deduce that E^u is C^3 along stable submanifolds, and can then invoke a bootstrapping process to show E^u is uniformly C^∞ along the stable submanifolds. As E^u is known to be uniformly C^∞ along the weak-unstable manifolds, we are in a position to apply Theorem 2.1 to conclude that E^u is C^∞ on M.

Fix $p \in M$, introduce coordinates Ψ_p, and let $v(s)$ restricted to the stable manifold W_p^s be given in these coordinates by

$$v(s)\big|_{(0,y)} = (1, \tilde{v}(p,s;y)) \tag{6.1}$$

Expand $\tilde{v}(p,s;y)$ into its second order Taylor series

$$\tilde{v}(p,s;y) = \tilde{k}(p,s) + \tilde{\ell}(p,s) \cdot y + \tilde{q}(p,s) \cdot y^2 + o(y^2). \tag{6.2}$$

Lemma 6.1. *Given* $p \in M$ *and* $t > 0$, *let* a, b, c, d, γ *and* λ *denote the constants defined in Sections* 3 *and* 4 *for the Poincaré map* $F_s : T_{f_{-s}p} \to T_p$. *Then the following hold:*

$$\tilde{k}(p,s) = \lambda^{-2} \cdot \tilde{k}(f_{-s}p) \tag{6.3}$$

$$\tilde{\ell}(p,s) = c + \lambda^{-1}\tilde{\ell}(f_{-s}p) + \tilde{k}(f_{-s}p) \cdot (d - \lambda^{-2}a + \lambda^{-2}b\tilde{k}(f_{-s}p)) \tag{6.4}$$

$$\tilde{q}(p,s) = \tilde{q}(f_{-s}p) + A_u^{\Psi}(f_{-s}p,s,v) \tag{6.5}$$

where $\tilde{k}(f_{-s}p) = \tilde{k}(f_{-s}p,0)$; $\tilde{\ell}(f_{-s}p) = \tilde{\ell}(f_{-s}p,0)$, $\tilde{q}(f_{-s}p) = \tilde{q}(f_{-s}p,0)$, *and*

$$A_u^{\Psi}(f_{-s}p,s,v) = \lambda\gamma + \left[(\tfrac{1}{2}\lambda d - a\lambda^{-1})\, \tilde{\ell}(f_{-s}p) - \tfrac{1}{2}\lambda^2 dc - ac\right] \tag{6.6}$$

Proof: (6.3) follows from (3.14), (6.4) from (3.18) and (6.5) from (4.4). #

Let $\ell(f_{-s}p)$ denote the linear part of the invariant field E^u at $f_{-s}p$, as in (3.12). Comparing (4.6) and (4.7), we obtain

$$A_u^{\Psi}(f_{-s}p,s,v) - A_u^{\Psi}(f_{-s}p,s) = (\tfrac{1}{2}\, \lambda d - a\lambda^{-1})\, (\tilde{\ell}(f_{-s}p) - \ell(f_{-s}p)). \tag{6.7}$$

Lemma 6.2. *There are constants* $C_1 > 0$, $\lambda_0 > 1$ *so that* $|\tilde{k}(p,s)| \le C_1 \cdot \lambda_0^{-2s}$ *for all* $p \in M$ *and* $s > 0$.

Proof: This follows from (6.3) using that $|\tilde{k}(q)|$ is continuous in q, hence bounded by some C_1 on M, and that there is a constant $\lambda_0 > 1$ such that $\lambda = \lambda(f_{-s}p,s) > \lambda_0^s$ for all $p \in M$ and $s > 0$. #

Lemma 6.3. *There are constants* $C_2 > 0$, $\lambda_0 > 1$ *so that* $\left|\tilde{\ell}(p,s) - \ell(p)\right| \leq C_2 \cdot \lambda_0^{-s}$ *for all* $p \in M$ *and* $s > 0$.

Proof: The invariance of E^u yields the relation $\ell(p) = c + \lambda^{-1} \cdot \ell(f_{-s}p)$, so that

$$(6.8) \quad \tilde{\ell}(p,s) - \ell(p) = \lambda^{-1} \cdot \{\tilde{\ell}(f_{-s}p) - \ell(f_{-s}p)\} + \tilde{k}(f_{-s}p,s) \cdot \{d + \lambda^{-2} \cdot (b \cdot \tilde{k}(f_{-s}p) - a)\}.$$

It is straightforward to show that the terms in both brackets in (6.8) have uniform bounds on M for $s > 0$, so that (6.8) yields

$$\begin{aligned} |\tilde{\ell}(p,s) - \ell(p)| &\leq \lambda_0^{-1} \cdot C' + \tilde{k}(f_{-s}p,s) \cdot C'' \\ &\leq \lambda_0^{-1} \cdot C_2 \end{aligned}$$

using Lemma 6.1. #

The crucial estimate for this section is the next:

Lemma 6.4. *There is a constant* C_3 *so that* $|\tilde{q}(p,s)| \leq C_3$ *for all* $p \in M$ *and* $s > 0$.

Proof: By the hypothesis of this section, there is a C^1-function ξ with $A_u^{\Psi}(p,s) = \xi(f_s p) - \xi(p)$. From (6.5) we then obtain

$$\begin{aligned} (6.9) \qquad \tilde{q}(p,s) &= \tilde{q}(f_{-s}p) + A_u^{\Psi}(f_{-s}p,s,v) \\ &= \tilde{q}(f_{-s}p) + \xi(p) - \xi(f_{-s}p) + \{A_u^{\Psi}(f_{-s}p,s,v) - A_u^{\Psi}(f_{-s}p,s)\}. \end{aligned}$$

The lemma is thus a consequence of the next estimate:

Lemma 6.5. *There is a constant* $C_4 > 0$ *so that*

$$\left|A_u^{\Psi}(f_{-s}p,s,v) - A_u^{\Psi}(f_{-s}p,s)\right| \le C_4 \tag{6.10}$$

for all $p \in M$ *and* $s > 0$.

Proof: *Using* (6.7) *and Lemma* 6.3, *it suffices to show there exists* C_4 *so that for all* p *and* $t > 0$, $\left|\frac{1}{2}\lambda d - a\lambda^{-1}\right| \le C_4$. *First, observe that* $\lambda d = \lambda^{-1}a$, *so it suffices to show* $|\lambda^{-1}a| \le C_4$. *Next, recall that* $a = \varphi_{xy}(0,0)$ *and* $\lambda = \varphi_x(0,0)$, *so*

$$\lambda^{-1}a = \frac{d}{dz}(\log \varphi_x(0,z))\Big|_{z=f_{-s}p} = \frac{d}{dz}(\log \lambda(z,s))\Big|_{z=f_{-s}p}. \tag{6.11}$$

It suffices to consider $s = N$ an integer, so that

$$\lambda^{-1}a = \frac{d}{dz}\sum_{i=0}^{N-1} \log \lambda(F^i z,1)\Big|_{z=f_{-N}p} = \sum_{i=0}^{N-1} \frac{d}{dr}\log \lambda(r,1)\Big|_{r=f_{i-N}p} \cdot \frac{d}{dz}F^i\Big|_{z=f_{-N}p}.$$

There is a bound $C_5 \ge \left|\frac{d}{dr}\log \lambda(r,1)\right|_{r=p}$, for all $p \in M$, so we obtain

$$|\lambda^{-1}a| \le C_5 \sum_{i=1}^{N-1} \lambda(f^{-N}p,i)^{-1} \le C_5 \sum_{i=1}^{\infty} \lambda^{-i}. \qquad \#$$

Proposition 6.6. *For each* $p \in M$, *the restriction* $E^u|W_p^s$ *is* C^3 *near* p.

Proof: For each $q \in W_p^s$, in the coordinates Ψ_q through q, let $E^u|_{(0,y)} \sim (1, v_q(y))$. For given $v \in V(\delta)$ a C^∞ field, Lemma 6.2 shows that $\tilde{v}(q,s;0)$ converges to $v_q(0)$ exponentially fast in s, and uniform in q. Likewise, Lemma 6.3 implies $\tilde{v}'(q,s;0) = \frac{d}{dy}(\tilde{v}(q,s;y))\Big|_{y=0}$ converges to $v_q'(0)$, uniformly in q. Finally, Lemma 6.4 implies that the second derivative

$$\tilde{v}''(q,s;0) = \frac{d^2}{dy^2}(\tilde{v}(q,s;y))\Big|_{y=0} \tag{6.12}$$

is uniformly bounded in $s > 0$ and $q \in M$. By the chain rule, we can convert these pointwise results, for varying q, into estimates for $v_p(y)$ for p fixed with y variable.

<u>Lemma 6.7</u>. Let ξ be real with $|\xi| < \epsilon$, and set $q = \Psi_p(0,\xi) \in W^s_p$. Then the k-jet of $\tilde{v}_p(z + \xi)$ about $z = 0$ depends uniformly in p and q on the k-jet of $\tilde{v}_q(z)$ about $z = 0$. Conversely, the k-jet of $\tilde{v}_q(z)$ depends unformly on the k-jet of $\tilde{v}_p(z + \xi)$.

<u>Proof</u>: Let $\Gamma_{pq} : (-\epsilon,\epsilon)^2 \to R^2$ be the holonomy map for the flow $\{F_t\}$ from T_p to T_q, in terms of the coordinates Ψ_p and Ψ_q. Then note $\Gamma_{pq}(0,y) = (0,y + \xi)$, and

$$D\Gamma_{pq}(1, \tilde{v}_p(z + \xi)) \sim (1, \tilde{v}_q(z)). \tag{6.13}$$

Thus, $\tilde{v}_p(z + \xi)$ is determined by $D\Gamma_{pq}$ and $\tilde{v}_q(z)$, and so the k-jet of $\tilde{v}_p(z + \xi)$ at $z = 0$ is determined by the k-jet of $\tilde{v}_q(z)$ at $z = 0$ and the $(k + 1)$ — jet of Γ_{pq}. The coordinates Ψ_p and Ψ_q depend C^∞ on p and q, so the holonomy map Γ_{pq} is C^∞ in p and q also. Thus, for $|\xi| < \epsilon$ and $q = \Psi_p(0,\xi)$, there are bounds on the $(k + 1)$-jet of Γ_{pq}, uniform in p and such q. The first assertion of the lemma now follows from the chain rule. The second is a consequence of the uniform invertibility of Γ_{pq}, where $\Gamma^{-1}_{pq} = \Gamma_{qp}$. #

By Lemma 6.7, the functions $\{\tilde{v}'(p,s;y) | s > 0\}$ converge uniformly C^1-to $v'_p(y)$, and moreover, the family of second derivatives $\{\tilde{v}''(p,s;y) | s > 0\}$ is uniformly bounded in $s > 0$ and y. Therefore, the limiting derivative function $v'_p(y)$ is

Lipshitz in y, and hence absolutely continuous. This implies that for a.e. $|y| < \epsilon$, the second derivative $v_p''(y)$ exists and v_p'' is integrable to the function v_p'.

Next note that by Lemma 6.7 again, the second derivative $v_q''(0)$ exists for a.e. $q \in W_p^s$, with $q = \Psi_p(0,y)$, $|y| < \epsilon$. Thus, by the Fubini theorem, $v_q''(0)$ exists for a.e. $q \in M$.

Define a function $q : M \to R$ a.e. by requiring that $q(p)$ be the quadratic part of the expansion of $v_p(y)$ at $y = 0$, where it exists (so $q(p) = \frac{1}{2} v_p''(0)$). The invariance of E^u under $\mathfrak{F}_t$ yields

$$A_u^{\Psi}(p,t) = q(f_t p) - q(p), \text{ a.e. } p \in M,\ t > 0. \tag{6.14}$$

By our hypothesis that (6.14) has a C^1-solution and Theorem 5.7, the function q is a.e. equal to a C^1-function. We conclude, using Lemma 6.7, that v_p'' is a.e. equal to a C^1-function. As v_p' is absolutely continuous, this implies v_p' is itself a C^2-function, so that v_p is C^3 as asserted in Proposition 6.6. #

<u>Proposition 6.8</u>. *For* $n \geq 3$, *suppose that for all* $p \in M$, $v_p(y)$ *is* C^n *at* $y = 0$, *and the* n-*jet has a uniform estimate in* p. *Then* $v_p(y)$ *is* C^{n+1} *with a uniform estimate on the* $(n + 1)$-*jet.*

<u>Proof</u>: We introduce notation modifying that of §§3 and 4. For each $p \in M$, consider the expansions up to order n:

$$v_p(y) = \sum_{i=1}^{n} v_i(p) \cdot y^i + o(y^n) \tag{6.15}$$

$$\varphi_x(y) = \lambda(p,t) + \sum_{i=1}^{n} a_i(p,t) \cdot y^i + o(y^n) \tag{6.16}$$

$$\varphi_y(y) = \sum_{i=1}^{n} b_i(p,t)\, y^i + o(y^n) \tag{6.17}$$

$$\psi_x(y) = \sum_{i=1}^{n} c_i(p,t) y^i + o(y^n) \tag{6.18}$$

$$\psi_y(y) = \lambda(p,t)^{-1} + \sum_{i=1}^{n} d_i(p,t) y^i + o(y^n) \tag{6.19}$$

Then we deduce from the invariance of E^u under F_t that

$$v_{f_t p}(y) = \left.\frac{\sum_{i=1}^{n} c_i y^i + (\lambda^{-1} + \sum_{i=1}^{n} d_i y^i)(\sum_{i=1}^{n} v_i y^i)}{(\lambda + \sum_{i=1}^{n} a_i y^i) + (\sum_{i=1}^{n} b_i y^i)(\sum_{i=1}^{n} a_i y^i)}\right|_{(p,t)} + o(y^n).$$

$$= \left[\sum_{i=1}^{n} c_i y^i + \lambda^{-1} \cdot \sum_{i=1}^{n} v^i y^i + \sum_{i,j=1}^{n} v_i d_j y^{i+j}\right]\Big|_{(p,t)} \cdot D^{-1} + o(y^n) \tag{6.20}$$

where D is the denominator of the above, and

$$D^{-1} = \lambda^{-1} \cdot \left[1 + \sum_{\ell=1}^{n} (-1)^{\ell} \lambda^{-\ell} \left[\sum_{i=1}^{n} (a_i + \sum_{j+k=i} (a_j b_k) y^i)\right]^{\ell}\right]_{(p,t)} + o(y^n) \tag{6.21}$$

$$= \lambda^{-1}\left[1 - \lambda^{-1} e_1 + \lambda^{-2} e_2 - \ldots + (-1)^n \lambda^{-n} e_n\right] + o(y^n).$$

Now recall that $z = \lambda^{-1} y + \varphi(0,y) = \lambda^{-1} y + \sum_{j=2}^{n} j^{-1} \cdot a_{j-1}\, y^j$, and solving for y gives an expression

$$y = \lambda z + \sum_{j=2}^{n} g_j z^j + o(z^n). \tag{6.22}$$

Substituting (6.22) into (6.20) and extracting the coefficient of z^n yields

$$v_n(f_t p) = \lambda(p,t)^{n-2} \cdot v_n(p) + \varsigma_n(p,t) \tag{6.23}$$

where $\varsigma_n(p,t)$ is a polynomial in the coefficients $\{a_i, b_i, c_i, d_i, e_i | i = 1, 2, \cdots n\}$ and $\{v_1, \cdots, v_{n-1}\}$, but not involving v_n. Consequently, the function $\varsigma_n(p,t)$ is C^1 in both p and t. Now rewrite (6.23) as

$$v_n(p) = \lambda(p,t)^{2-n} \cdot \{v_n(f_t p) + \varsigma_n(p,t)\}. \tag{6.24}$$

Set $t = 1$, let $\varsigma_n(p) = \varsigma_n(p,1)$, and recursively substitute (6.24) into itself to obtain an expression

$$v_n(p) = \{\sum_{j=0}^{k} \lambda(p,j+1)^{2-n} \varsigma_n(f_j p)\} + \lambda(p,k+1)^{2-n} \cdot \{v_n(f_k p) + \varsigma_n(f_k p)\}. \tag{6.25}$$

Both $v_n(p)$ and $\varsigma_n(p)$ are bounded functions on M, and $n \geq 3$, so letting $k \to \infty$ in (6.25) yields

$$v_n(p) = \left\{\sum_{j=0}^{\infty} \lambda(p, j+1)^{2-n} \varsigma_n(f_j p)\right\}. \tag{6.26}$$

By the hypothesis of Proposition 6.9, and invoking an argument similar to the proof of Proposition 6.7, we conclude that $\varsigma_n(f_j p)$ has a uniform bound on its stable derivative:

$$(6.27) \qquad \left|D_s\zeta_n(f_j p)\right| \le C_6.$$

From the expansion $\lambda(p, j+1) = \sum_{i=0}^{j} \lambda(f_i p, 1)$, we also obtain a uniform bound

$$(6.28) \qquad \left|D_s\{\lambda(p, j+1)^{2-n}\}\right| \le C_7 \cdot \lambda(p, j+1)^{2-n};\ j \ge 1,\ p \in M.$$

Combining (6.26), (6.27) and (6.28) shows first that $D_s v_n$ exists everywhere on M, and also provides the estimate

$$(6.29) \qquad \left|D_s v_n\right|_p \le \sum_{j=0}^{\infty} C_6 \cdot C_7 \cdot \lambda(p, j+1)^{2-n}$$

$$\le C_8 \cdot \sum_{j=0}^{\infty} \lambda_0^{(2-n)j},$$

where $\lambda(p,1) > \lambda_0 > 1$ for all $p \in M$.

Via an argument similar to the proof of Lemma 6.7, the estimate (6.29) implies the functions $v_p(y)$ are C^{n+1} at $y = 0$, and also provides a uniform bound on $v_{n+1}(p)$. This proves Proposition 6.8. #

Now we can establish the remaining half of Theorem 0.4.

<u>Theorem</u> <u>6.9</u>. *Let $\{f_t\}$ be a volume preserving, C^∞-Anosov flow on a compact 3-manifold M. Suppose that the Anosov invariant $A^+ = 0$. Then the weak-unstable foliation of $\{f_t\}$ is C^∞. If $A^- = 0$, a similar result holds for the weak-stable foliation.*

Proof: $A^+ = 0$ implies that $E^u|W^s_p$ is C^3 for each $p \in M$ by Proposition 6.6. Then via the inductive result, Proposition 6.8, the restrictions $E^u|W^s_p$ are C^n, with uniform estimates in p for all n. Along the unstable manifolds W^u_p, Hirsch-Pugh-Shub [HPS] show that $E^u|W^u_p$ is C^∞ with uniform estimates in p. Then by Theorem 2.1, we conclude that $E^u|T_p$ is C^∞ for each p. As the submanifolds T_p vary C^∞ with p, and E^u is C^∞ along the flow $\{f_t\}$, we conclude that E^u is C^∞ on M. The weak-unstable distribution is spanned by E^u and the vector field generating $\{f_t\}$, so Theorem 6.9 is proven. #

§7. Rigidity of toral automorphisms

In this section we complete the proof of Theorem 0.6.

Theorem 7.1. *Let* $F : T^2 \to T^2$ *be an area-preserving* C^∞ *Anosov diffeomorphism of the 2-torus. Suppose that either stable or unstable Anosov invariant vanishes:* $A^+ = 0$ or $A^- = 0$. *Then* F *is* C^∞ *conjugate to a linear automorphism of* T^2.

Proof: Suppose $A^+ = 0$. The other case is completely similar. By Theorem 6.10 applied to the suspension over F, if $A^+ = 0$ then the unstable foliation W^u for F is C^∞. This foliation is uniquely ergodic; i.e., there exists a unique (up to a constant multiple) measure μ^u defined and finite on Borel subsets of piecewise smooth compact transversals to W^u, and which is invariant under its holonomy. We first prove that in our case, this measure is represented by a closed, non-vanishing smooth 1-form ν on T^2. Choose a closed oriented C^∞-curve C in T^2 which is transverse to W^u and intersects every leaf. The holonomy map $\varphi^u_C : C \to C$ for W^u is a C^∞ diffeomorphism, as W^u is C^∞. The rotation number of φ^u_C is a quadratic irrational, which follows from the fact that any Anosov diffeomorphism of T^2 is topologically conjugate to a linear hyperbolic automorphism. Thus, by a fundamental theorem of M. Herman [He], the map φ^u_C is C^∞-conjugate to an irrational rotation of a circle, and consequently it has exactly one invariant probability Borel measure which is obtained by integrating a C^∞-1-form ν on C. The holonomy of W^u diffuses ν to a C^∞-closed 1-form on T^2, which is also denoted by ν. Integrating ν along transversals defines a transverse invariant Borel measure for W^u. By unique ergodicity this smooth measure must be a constant multiple of μ^u.

Observe that F^*_ν also defines a transverse invariant Borel measure for W^u, so there is a constant $\lambda > 1$ with

$$F^*\nu = \lambda^{-1} \cdot \nu. \tag{7.1}$$

Choose a C^∞-non-vanishing 1-form ω on T^2 which vanishes on the tangents of W^s, and such that $\omega \wedge \nu = dx \wedge dy$. The stable foliation W^s is F-invariant, so $F^*\omega = f \cdot \omega$ for some non-vanishing function f. By (7.1) and the invariance of the volume form $dx \wedge dy$, we conclude $f = \lambda$.

Choose vector fields u, s on T^2 with $u(x) \in T_x(T^2)$ tangent to W^u_x, $s(x) \in T_x(T^2)$ tangent to W^s_x and so that

$$\omega(u(x)) = 1 \quad ; \nu(x(x)) = 1. \tag{7.2}$$

Then for each x, (u(x), s(x)) is a positively oriented unit 2-frame field satisfying

$$\begin{aligned} DF(u(x)) &= \lambda \cdot u(F(x)) \\ DF(s(x)) &= \lambda^{-1} \cdot s(F(x)) \end{aligned} \tag{7.3}$$

<u>Lemma</u> <u>7.2</u>. *The flows of the vector fields* u *and* s *commute.*

<u>Proof</u>: It suffices to show the Lie bracket $[u, s] = 0$. Observe that $F_*[u, s] = [F_*u, F_*s] = [u, s]$ by (7.3). Thus, the bracket $[u, s]$ is a continuous invariant line field for F. The only such field for an Anosov diffeomorphism is the zero one. □

Corollary 7.3. *Each point* $x \in T^2$ *has an open neighborhood with a Euclidean structure defined by the* C^∞*-local coordinates*

$$(a, b) \mapsto \exp(a \cdot u(x)) \cdot \exp(b \cdot s(x)) \cdot x. \tag{7.4}$$

Moreover, the coordinate systems defined by (7.4) *at nearby points* x *and* y *differ by translation.* □

Theorem 7.1 now follows, for Corollary 7.3 asserts there is a global C^∞-flat metric structure on T^2 in which, by formulae (7.3), F is linear. Conjugate this flat structure to a standard one on T^2, and F will be algebraic in the new coordinate system. □

Remark 7.4. The key to the proof of Theorem 7.1 was the construction of the vector fields {u, s} on T^2 which are expanded at a constant rate by F. This used Herman's Theorem in a crucial way. It is tempting to apply this proof to the case of flows on 3-manifolds, but one lacks a result corresponding to Herman's Theorem. For some partial results overcoming this lack, see the excellent paper by Ghys [G1].

§8. Rigidity for geodesic Anosov flows

Let S be a closed orientable Riemann surface with metric g, and assume the curvature k_g is everywhere negative. Let $M = T^1S$ denote the unit tangent bundle for S and ξ_g the geodesic unit vector field on M for the metric g, and $\{f_t(g)\}$ the resulting flow. It is classical that $\{f_t(g)\}$ is a volume-preserving transitive Anosov flow, so the results of Sections 3–6 apply (e.g. see [An], [Eb]). In this section, we show how a theorem of Ghys can be applied to give a sufficient condition for the weak unstable foliation for the geodesic flow $\{f_t\}$ to be C^∞-conjugate to the corresponding geodesic-horocycle foliation for a metric with constant negative curvature. We will show later, in Section 11, that this forces g to also have constant curvature, so that a very strong rigidity result holds.

Consider M as the principal SO(2)-bundle of orthonormal frames of TM. Then there is a canonical framing of TM by vector fields ξ, h, w satisfying the Lie bracket identities (cf. [GK]):

$$
\begin{aligned}
&[w,\xi] = h \\
(8.1) \qquad &[w,h] = -\xi \\
&[\xi,h] = -k_g \circ \pi,
\end{aligned}
$$

where $\pi : M \to S$ is the projection along the fibers. Let $\{\xi^*, w^*, h^*\}$ denote the corresponding dual framing of T^*M. Then w^* is the connection 1-form for g, and $\{\xi^*, h^*\}$ are the Sölder forms [BC]. The invariant Liousville measure $\lambda\{f_t\}$ is given by the 3-form $dvol = w^* \wedge \xi^* \wedge h^*$. The vector field w is tangent to the fibers of π, and is also denoted by $\partial/\partial\theta$ where θ is the unit speed parameter on $SO(2) \cong S^1$.

Equation (8.1) implies w infinitesimally rotates the geodesic field ξ into the conjugate field h.

The canonical contact structure $\varnothing$ for $\{f_t\}$ is defined by the identities:

$$\varnothing(\xi) = 1;\ \varnothing(w) = 0 = \varnothing(h). \tag{8.2}$$

Equations (8.2) imply $\varnothing$ is invariant under the flow, so the stable and unstable vector fields h^-, h^+ lie in $\ker(\varnothing)$. Thus h^+ and h^- can be written in terms of our framing:

$$\begin{aligned} h^+ &= \mu^+ \cdot (h + H^+ w) \\ h^- &= \mu^- \cdot (h + H^- w) \end{aligned} \tag{8.3}$$

for functions μ^+, μ^-, H^+, H^- on M. As h and w are C^∞ vector fields, these functions are C^{1,Λ_*}. The weak-unstable distribution E^{wu} is spanned by $\{\xi, h^+\}$, and E^{ws} by $\{\xi, h^-\}$. Integrability follows from the infinitesimal Anosov condition:

$$\begin{aligned} [\xi, h^+] &= \log \lambda^+ \cdot h^+, \quad \lambda^+ > \lambda_0 > 0, \quad \lambda_0 \text{ constant} \\ [\xi, h^-] &= \log \lambda^- \cdot h^-, \quad \lambda^- < -\lambda_0^{-1} < 1. \end{aligned} \tag{8.4}$$

where λ^+ and λ^- are continuous functions on M. Expanding (8.4) in terms of the framing of TM, and using (8.1) yields a differential equation on M which H^+ and H^- must satisfy:

$$\xi \cdot H^+ + (H^+)^2 + k \circ \pi = 0 \tag{8.5}$$

$$\xi \cdot H^- + (H^-)^2 + k \circ \pi = 0.$$

This is just the Ricatti equation of the geodesic flow discussed in Section 0, with H^+, H^- the unique positive, respectively negative, solutions defined on all at M. We now state the main result of this section. It is a rigidity theorem similar to Theorem 7.1. The role of Herman's theorem used in Section 7 is now played by the rigidity theorem of Ghys (Theorem III.4.2, [Gh]).

<u>Theorem</u> <u>8.1</u>. *Let* S *be a closed surface and* g *a* C^∞*-Riemannian metric with* k_g *everywhere negative. Suppose the Anosov invariant* $A^+ = 0$, *and the solution* H^+ *of* (8.5) *satisfies a differential inequality at every point of* M:

$$2 \cdot (w \cdot H^+)^2 = H^+ \cdot (w \cdot (w \cdot H^+)) + (H^+)^2 > 0. \tag{8.6}$$

Then there is a C^∞ *diffeomorphism* $\Phi : M \to \tilde{M}$ *conjugating the weak-unstable foliation* $\mathfrak{F}^u(f_t(\tilde{g}))$ *on* $\tilde{M}$, *where* $\tilde{g}$ *on* S *has constant negative curvature, and* $\tilde{M}$ *is the corresponding unit tangent bundle.*

<u>Proof</u>: Theorem 6.10 and $A^+ = 0$ implies that $\mathfrak{F}^u(f_t(g))$ is a C^∞-foliation. Recall that SO(2) acts smoothly on M, via rotation of the unit tangent vectors. Let $R(\theta)$ denote rotation by θ. Then $R(\pi/2)$ conjugates $\mathfrak{F}^u(f_t(g))$ into $\mathfrak{F}^s(f_t(g))$, so the stable foliation is also C^∞.

In order to apply Theorem III. 4.2 of Ghys [Gh], it is sufficient to exhibit a Godbillon-Vey form for $\mathfrak{F}^u(f_t(g))$ which is nowhere vanishing.

A 1-form on M vanishing on E^{wu} is provided by

$$N^* = w^* - H^+ \cdot h^*, \tag{8.7}$$

and we calculate using the identities (8.1):

$$\begin{aligned} dN^* &= dw^* - dH^+ \wedge h^* - H^+ dh^* \\ &= - k \circ \pi \cdot \xi^* \wedge h^* - dH^+ \wedge h^* + H^* \cdot w^* \wedge \xi^* \\ &= N^* \wedge \eta, \end{aligned} \tag{8.8}$$

$$\eta = i(w)dN^* = -(w \cdot H^+)h^* + H^+ \cdot \xi^*. \tag{8.9}$$

The Godbillon-Vey form is then

$$\eta \wedge d\eta = \{-2(w \cdot H^+)^2 - (H^+)^2 + H^+ \cdot (w \cdot (wH^+))\}\, w^* \wedge \xi^* \wedge h^*. \tag{8.10}$$

and so (8.6) is equivalent to $\eta \wedge d\eta$ nowhere-vanishing. #

<u>Question</u> <u>8.2</u>. *Is it possible that* (8.6) *always holds; i.e., is* (8.6) *a property of the unique solution* H^+ *of the Ricatti equation?*

III. Godbillon-Vey Classes

§9. Godbillon-Vey classes for $C^{1+\alpha}$-foliations.

In this section we define the Godbillon-Vey invariant for foliations of differentiability $C^{1+\alpha}$, $\alpha > \frac{1}{2}$, and study the invariance under diffeomorphism of this extension of the usual Godbillon-Vey invariant.

Fix a codimension-one C^k-foliation $\mathfrak{F}$ on M, $k \geq 1$, and assume that $\mathfrak{F}$ is transversally orientable. Then there exists a C^k 1-form N^* on M whose kernel is precisely the tangential distribution $T\mathfrak{F}$ to $\mathfrak{F}$. By the Frobenius theorem, there is a C^{k-1} 1-form, η, so that $dN^* = N^* \wedge \eta$. For $k \geq 2$, the Godbillon-Vey class of $\mathfrak{F}$ is defined in [GV] to be the de Rham cohomology class

$$GV(\mathfrak{F}) = [\eta \wedge d\eta] \in H^3_{deR}(M), \tag{9.1}$$

of the closed 3-form $\eta \wedge d\eta$. When M is a closed oriented 3-manifold, we define the Godbillon-Vey invariant by $gv(\mathfrak{F}) = \int_M \eta \wedge d\eta$.

A foliation chart for $\mathfrak{F}$ is an open set $U \subset M$ and a diffeomorphism onto, $\varnothing : U \to (-a,a)^3$, where $a > 0$ and $(-a,a) = \{x \in R \mid -a < x < a\}$, such that the connected components of $\mathfrak{F} \mid U$ correspond one-to-one to the level sets $P_x = \varnothing^{-1}(\{x\} \times (-a,a)^2)$, for $-a < x < a$, called the plaques of $\mathfrak{F} \mid U$. The defining projection of $\mathfrak{F} \mid U$ is the map $p = \pi_1 \circ \varnothing$, where $\pi_1 : R^3 \to R^1$ is projection onto the first factor. The chart $(\varnothing, U)$ is regular if for some $\epsilon > 0$, there is an open set V_ϵ containing U so that $\varnothing$ extends to a foliation chart $\varnothing_\epsilon : V_\epsilon \to (-a-\epsilon, a+\epsilon)^3$.

Definition 9.1. *A regular foliation covering of* M *is a collection* $\{(\varnothing_i, U_i) \mid i \in \mathfrak{A}\}$ *which satisfies*

i) $\{U_i \mid i \in \mathfrak{A}\}$ *is a locally finite covering of* M, *such that* $U_i \cap U_j$ *is always connected, though possibly empty;*

ii) *Each* $(\varnothing_i, U_i)$ *is a regular foliation chart;*

iii) *The intersection of a plaque in* $\mathfrak{F}|U_i$ *with a plaque in* $\mathfrak{F}|U_j$ *is always connected (possibly empty);*

iv) *For some orientation on the normal bundle to* $\mathfrak{F}$, *the local defining projections* $p_i : U_i \to R$ *are orientation preserving.*

In (9.1.ii), we will assume all of the charts have image $(-1, 1)^3$. The existence of a regular foliation covering is standard (cf. [P2]).

We denote by (i,j) two indices $i, j \in \mathfrak{A}$ such that $U_i \cap U_j \neq \varnothing$, and say {i, j} are admissible. We will write (i, j, k) if $U_i \cap U_j \cap U_k \neq \varnothing$. For each (i, j), define $I_{ij} = p_i(U_i \cap U_j)$, a connected subinterval of $(-1, 1)$, and define the transition function

$$\gamma_{ij} : I_{ij} \to I_{ji} \tag{9.2}$$

$$\gamma_{ij}(x) = p_j(p_i^{-1}(x) \cap U_j).$$

Let $\omega : (0, \infty) \to (0, \infty)$ denote a continuous increasing function with $\omega(0) = 0$.

Definition 9.2. *Let* k *be a positive integer and* ω *as above. We say that* $\mathfrak{F}$ *has transverse differentiability class* $C^{k+\omega}$ *(and total differentiability class* $C^{\infty,\, k+\omega}$*) if the leaves of* $\mathfrak{F}$ *are* C^{∞} *submanifolds of* M, *and there is a regular foliation covering* $\{(\varnothing_i, U_i) \mid i \in \mathfrak{A}\}$ *of* M *satisfying:*

i) *Each* $\varnothing_i : U_i \to (-1,1)^3$ *is a* C^k*-map, and the restriction of* $\varnothing_i$ *to each plaque of* $\mathfrak{F} \mid U_i$ *is* C^{∞};

ii) *For each* (i,j), *the transition function* γ_{ij} *is* C^k, *and the* k-th *derivative* $\gamma_{ij}^{(k)} : I_{ij} \to R$ *has modulus of continuity* $K \cdot \omega$, *for some constant* K.

When $\omega(s) = cs^{\alpha}$ for some $c > 0$, $0 < \alpha < 1$, we say that $\mathfrak{F}$ is $C^{k+\alpha}$.

Assume that $\mathfrak{F}$ is $C^{1+\alpha}$, and let $\{(\varnothing_i, U_i) \mid i = 1,\ldots,N\}$ be a $C^{1+\alpha}$ regular foliation chart for $\mathfrak{F}$. By (9.1.iv) the derivative $\gamma'_{ij}(x) = \frac{d}{dx}\gamma_{ij}(x)$ is positive for all $x \in I_{ij}$, and we define for each (i, j):

$$\psi_{ij}(x) = \log \gamma'_{ij}(x);\ x \in I_{ij} \tag{9.3}$$

$$\xi_{ij}(y) = \psi_{ij} \circ p_i(y);\ y \in U_{ij} = U_i \cap U_j. \tag{9.4}$$

The collection of functions $\{\xi_{ij} : U_{ij} \to R \mid (i,j)\}$ is called the additive Radon-Nikodyn (RN)-cocycle associated to the foliation covering, and satisfies the cocycle law for (i, j, k):

$$\xi_{ij} + \xi_{jk} = \xi_{ik} \text{ on } U_i \cap U_j \cap U_k. \tag{9.5}$$

Note that the ξ_{ij} are $C^{(k-1)+\alpha}$-functions if $\mathfrak{F}$ is $C^{k+\alpha}$.

For $k \geq 1$, there is a natural transformation from C^k-cocycles $\{\xi_{ij}\}$ over the foliation cover $\{(\varnothing_i, U_i)\}$ of M into "Godbillon-Vey" classes. This first appeared in the thesis of H. Shulman [Sh] for the RN-cocycle above, and was later developed and expanded by several authors [BSS], [CC], [Hu1]. The idea behind the results of this section are based upon an extension of this transformation to C^α-cocycles, for $\alpha > \frac{1}{2}$. As the results needed to carry this out are not well-known, we will briefly, but completely, describe the construction of this transformation.

A continuous function $\lambda : M \to R$ is of class $C^{k;a}$ if λ is C^k when restricted to the leaves of $\mathfrak{F}$, and the k-jets of these restrictions are absolutely continuous in the transverse variable. In local coordinates, the interpretation of $C^{k;a}$ is this: set $\bar{\lambda}(x,y,z) = \lambda \circ \varnothing_i^{-1}(x,y,z)$. Then for any $\mu + \upsilon \leq k$, $\left[\frac{\partial}{\partial y}\right]^\mu \left[\frac{\partial}{\partial z}\right]^\upsilon (\bar{\lambda})$ is absolutely continuous in x for fixed (y_o, z_o). The corresponding $C^{k;a}$-topology on functions is defined by taking the supremum over M of the leafwise k-jet norms, and by requiring L^2-convergence of all expressions $\frac{\partial}{\partial x}\left[\frac{\partial}{\partial y}\right]^\mu \left[\frac{\partial}{\partial z}\right]^\upsilon (\bar{\lambda})$, $\mu + \upsilon \leq k$. A form on M is $C^{k;a}$ if all its coefficient functions, in local coordinates, are $C^{k;a}$.

Let M be any Riemannian manifold with a $C^{k;a}$ foliation $\mathfrak{F}$. For λ a $C^{k;a}$-function, with $k \geq 1$, define a $C^{(k-1);a}$-1-form $d_{\mathfrak{F}}\lambda$ on M by first restricting λ to the leaves of $\mathfrak{F}$, applying the exterior differential, and then extending via the orthogonal projection $TM \to T\mathfrak{F}$. Note that if λ is a C^1 function on M, then for any plaque $P \subset U_i$ of $\mathfrak{F}$, $(d\lambda)|P = (d_{\mathfrak{F}}\lambda)|P$.

Let $\{\lambda_i \mid i = 1,\ldots,N\}$ be a $C^{1;a}$ partition-of-unity (p.o.u.) subordinate to the

given foliation cover. For each i, set $N_i^* = p_i^*(dx)$, where dx is the standard coordinate 1-form on R. Then define a 1-form N^* on M by setting, on U_i:

$$N^* \mid U_i = \exp(\sum_{(i,j)} \lambda_j \xi_{ij}) \cdot N_i^*. \tag{9.6}$$

The compatibility requirement for N^* to be well-defined is that $N^* \mid U_i = N^* \mid U_j$ on $U_i \cap U_j$, and this is a consequence of (9.5). It is an immediate consequence of (9.1.iv) that N^* is nowhere-vanishing, and its kernel distribution is precisely $T\mathfrak{F}$, the tangential distribution to $\mathfrak{F}$.

Define a $C^{0;a}$ 1-form on M by setting

$$\eta \mid U_i = \sum_{(i,j)} (d_{\mathfrak{F}}\lambda_i) \cdot \xi_{ij}. \tag{9.7}$$

Clearly, $dN^* = \eta \wedge N^*$ as distributions, so that if N^* is a C^2-form on M, then the Godbillon-Vey form is represented by $\eta \wedge d\eta$. For M oriented, the Godbillon-Vey invariant has the expansion

$$\begin{aligned} gv(\mathfrak{F}) &= \int_M \eta \wedge d\eta \\ &= \int_M \Big(\sum_{i=1}^N \lambda_i\Big) \eta \wedge d\eta \\ &= \sum_{i=1}^N \int \lambda_i(\eta \wedge d\eta \mid U_i) \end{aligned}$$

$$= \sum_{i=1}^N \sum_{(i,j,k)} \int_{U_i} \lambda_i d\lambda_j \wedge d\lambda_k \wedge \xi_{ij} \wedge d\xi_{ik} \tag{9.8}$$

We will extend formula (9.8) to cocycles $\{\xi_{ij}\}$ of class C^{α}, for $\alpha > \frac{1}{2}$.

<u>Theorem</u> <u>9.3</u>. *Let $\mathfrak{F}$ be a codimension-one $C^{1+\alpha}$-foliation on a closed oriented 3-manifold* M. *If $\alpha > \frac{1}{2}$, then there is a well-defined Godbillon-Vey invariant,* $gv(\mathfrak{F})$, *which agrees with* (9.8) *when $\mathfrak{F}$ is* C^2.

<u>Proof</u>: Let $\{\lambda_i\}$ be a $C^{1;a}$-p.o.u. subordinate to the given foliation covering of M. For each (i,j,k) and $x \in I_{ij} \cap I_{ik}$, set

$$c_{ijk}(x) = \iint_{\{x\}\times(-1,1)^2} \lambda_i \circ \varnothing_i^{-1} \cdot d(\lambda_j \circ \varnothing_i^{-1}) \wedge d(\lambda_k \circ \varnothing_i^{-1}). \tag{9.9}$$

Let $m_i : (-1,1) \to [0,1]$ be a C^{∞}-function with compact support and which is 1 on the projection of $spt(\lambda_i)$ to $(-1,1)$.

The first lemma follows immediately from definitions:

<u>Lemma</u> <u>9.4</u>. *For each* (i,j,k), *the function* c_{ijk} *is absolutely continuous with compact support in* $I_{ij} \cap I_{ik}$. *For each* x, $c_{ijk}(x)$ *is anti-symmetric in* i, j *and* k.

The goal is to define, by analogy with (9.8), the invariant $gv(\mathfrak{F})$ via

$$gv(\mathfrak{F}) = \sum_{(i,j,k)} I(c_{ijk}\psi_{ij}, m_i \cdot \psi_{ik}) \tag{9.10}$$

where $I : C_c^{\alpha}(-1,1) \times C_c^{\alpha}(-1,1) \to R$ is a skew-symmetric bilinear form extending the natural pairing

$$I(f,g) = \int_{-1}^{1} f \cdot dg.$$

for C^1-functions f and g with compact support in $(-1,1)$. (Note that ψ_{ij} and ψ_{ik} are C^α functions, and c_{ijk} is absolutely continuous so $c_{ijk} \cdot \psi_{ij}$ is again C^α.) The key result is the following lemma, whose proof is taken from Stein (page 139, [St]). We are indebted to J. Roe for bringing this result to our attention.

<u>Lemma 9.5</u>. *Let* $f : S^1 \to R$ *be a continuous function on the circle of Hölder class* $C^{\alpha+\epsilon}$, *for some* $\epsilon > 0$ *with* $0 < \alpha + \epsilon < 1$. *Let* $f(\theta) = \sum_{n\in Z} a_n e^{in\theta}$ *be the Fourier expansion of* f. *Then for* $\delta > 0$ *small,*

$$\sum_{n\in Z} |a_n|^2 \, |n|^{2\alpha} \leq c(\alpha,\epsilon,\delta)^2 \cdot \mathrm{Lip}(\alpha + \epsilon, \delta; f)^2 \tag{9.11}$$

where we define

$$\mathrm{Lip}(\alpha + \epsilon, \delta; f) = \sup_{\theta\in S^1} |f(\theta)| + \sup_{0<|\theta_1-\theta_2|<\delta} \frac{|f(\theta_1) - f(\theta_2)|}{|\theta_1-\theta_2|^{\alpha+\epsilon}} \quad . \tag{9.12}$$

$$c(\alpha,\epsilon,\delta) = \left\{\frac{1-\alpha}{2\pi\epsilon}\right\}^{\frac{1}{2}} \cdot \{\delta^{\epsilon+\alpha-1} + O(\delta)\}. \tag{9.13}$$

<u>Proof</u>: For $L = \mathrm{Lip}(\alpha + \epsilon, \delta; f)$ the uniform estimate $|f(\theta + t) - f(\theta)| \leq L \cdot |t|^{\alpha+\epsilon}$ for $0 < |t| < \delta$ implies for the same range of t that

$$\sum_{n\in Z} |a_n|^2 \left| 1-e^{2\pi int}\right|^2 = \int_{S^1} |f(\theta + t) - f(\theta)|^2 \, d\theta$$

$$\leq 2\pi \cdot L^2 \cdot |t|^{2\alpha+2\epsilon}.$$

From this we deduce that

$$\sum_{n\in Z} |a_n|^2 \cdot \int_0^\delta \frac{|1 - e^{2\pi i n t}|^2}{t^{1+2\alpha}}\, dt \leq 2\pi \cdot L^2 \cdot \frac{\delta^{2\epsilon}}{2\epsilon}. \tag{9.14}$$

Via the substitution $s = nt$, the integral in (9.14) is evaluated:

$$\int_0^\delta |1-e^{2\pi i n t}| \cdot t^{-(1+2\alpha)} dt = |n|^{2\alpha} \cdot \int_0^\delta \{(2\pi s)^2 + O(s^4)\} s^{-1-2\alpha} ds$$

$$= |n|^{2\alpha} \cdot (2\pi)^2 \cdot \left\{\frac{\delta^{2-2\alpha}}{2-2\alpha} + O(\delta^2)\right\}$$

and collecting terms on the right side yields the estimate (9.11). #

Let $C^{\alpha,\delta}(S^1)$ denote the Banach space at α-Hölder functions, with the norm defined by $\mathrm{Lip}(\alpha,\delta;f)$. For all α, Let $H_2^\alpha(S^1)$ denote the α-Sobolev space on S^1 with the norm of an element $\{a_n \mid n \in Z\}$ defined by

$$\|\{a_n\}\|_\alpha^2 = \sum_{n\in Z} |a_n|^2 \cdot (1 + |n|^{2\alpha}).$$

An immediate corollary of Lemma 9.5 is then:

<u>Corollary</u> <u>9.6</u>. *For ϵ, $\alpha > 0$ and $\alpha + \epsilon < 1$, and $\delta > 0$ sufficiently small, there is a continuous inclusion*

$$C^{\alpha+\epsilon,\delta}(S^1) \to H_2^{\alpha}(S^1)$$

with norm bounded by $C(\alpha,\delta,\epsilon)$.

Define $C_c^{\alpha}(-1,1)$ to be the Banach space of C^{α}-functions with compact support in $(-1,1)$, and norm

$$|f|_{\alpha} = \sup_x |f(x)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}}.$$

<u>Proposition 9.7</u>. *For* $\alpha,\beta > 0$, *with* $\alpha + \beta > 1$, *there is a compactly continuous skew-symmetric pairing*

$$I : C_c^{\alpha}(-1,1) \times C_c^{\alpha}(-1,1) \to R$$

which is defined on the dense subalgebra $C_c^1(-1,1)$ *by*

$$I(f,g) = \int_{-1}^{1} f dg$$

<u>Proof</u>: Let $\theta \to e^{i\theta}$ define an inclusion of $(-1,1)$ into S^1. For each $f,g \in C_c^{\alpha}(-1,1)$, we thus obtain $f, g \in C^{\alpha,\delta}(S^1)$ for $\delta > 0$ sufficiently small. By Corollary 9.6, for all $\epsilon > 0$ the Fourier expansions $f(\theta) = \Sigma\, a_n e^{in\theta}$ and $g(\theta) = \Sigma\, b_n e^{in\theta}$ define classes $\{f\} = \{a_n\} \in H_2^{\alpha-\epsilon}(S^1)$ and $\{g\} = \{b_n\} \in H_2^{\beta-\epsilon}(S^1)$. Then $\{dg\} = \{n \cdot b_n\} \in H_2^{\beta-1-\epsilon}(S^1)$ and we use the natural pairing of Sobolev spaces to define

$$I(f,g) = \langle\{f\}, \{dg\}\rangle = \sum_{n \in Z} n \cdot a_n \cdot \bar{b}_{-n}. \tag{9.15}$$

This can be rewritten as the L^2-inner product of the series, for $\alpha + \beta - 2\epsilon > 1$,

$$(9.16) \qquad I(f,g) = \langle\{n^{\alpha-\epsilon}a_n\}, \{n^{1-\alpha+\epsilon}b_n\}\rangle,$$

from which the continuity of I follows: Let $\{f_n\}$, $\{g_n\}$ be two Cauchy sequences, and suppose there is a c, $0 < c < 1$, for which all of the functions f_n and g_n have support in $(-c,c)$. Then for a uniform $\delta > 0$, all of these functions embed in $H^{\alpha-\epsilon,\delta}(S^1)$ or $H^{\beta-\epsilon,\delta}(S^1)$. Then by Corollary 9.6, formula (9.16) and the Cauchy-Schwartz inequality, we conclude $I(f_n, g_m)$ is Cauchy in n and m. Finally, if f and g are C^1-functions, then the identification of (9.15) with $\int_{-1}^{1} f d\bar{g}$ is standard. #

An immediate corollary of Proposition 9.7 is that formula (9.10) is well-defined, for both $c_{ijk} \cdot \psi_{ij}$ and $m_i \cdot \psi_{ik}$ are C^α-functions, for $\alpha > \frac{1}{2}$ and all i,j,k, so can be paired. Moreover, this pairing of η with $d\eta$ has an extension that will be useful. Consider the class $\mathfrak{A}(\alpha)$ of 1-forms on M with the property

(9.17) $A \in \mathfrak{A}(\alpha)$ if for each foliation chart $\varnothing : U \to (-1,1)^3$, there is a C^α-function $a_\varnothing : (-1,1) \to R$ and a $C^{1;a}$-form $A_\varnothing$ on $(-1,1)^3$ so that

$$(\varnothing^{-1})^* A\big|_{(x,y,z)} = a_\varnothing(x) \cdot A_\varnothing\big|_{(x,y,z)}.$$

We give the space $\mathfrak{A}(\alpha)$ the topology of uniform C^1-convergence on leaves and uniform C^α-convergence transversally.

<u>Proposition</u> <u>9.8</u>. *For α, $\beta > 0$ with $\alpha + \beta > 1$, there is a continuous, bilinear, symmetric pairing*

$$\int : \mathfrak{A}(\alpha) \times \mathfrak{A}(\beta) \to R$$
$$A, B \to \int A \wedge d(B)$$

so that for η a C^{α}-form as above,

$$\int \eta \wedge d(\eta) = \int_M \eta \wedge d\eta$$

as defined by (9.10)

<u>Proof</u>: Choose a foliation covering $\{(\varnothing_i, U_i) \mid i = 1,\ldots,N\}$ and a $C^{1;a}$-p.o.u. $\{\lambda_i\}$ subordinate to it. For each i, choose a C^{∞}-function $m_i : (-1,1) \to [0,1]$ which has compact support, and is 1 on the projection of the support of λ_i to $(-1,1)$. For each i, choose factorizations $(\varnothing_i^{-1})^* A = a_i A_i$ and $(\varnothing_i^{-1})^* B = b_i \cdot B_i$ as given in (9.17). Then define

$$c_i(x) = \int_{(-1,1)^2} \lambda_i \circ \varnothing_i^{-1}(x,y,z) \cdot A_i \wedge B_i\Big|_x \tag{9.18}$$

$$\int A \wedge d(B) = \sum_{i=1}^{N} I(c_i a_i, m_i b_i). \tag{9.19}$$

First, note that (9.19) is independent of the choice of mollifiers $\{m_i\}$, for the pairing I can be calculated using a sequence at C^1-forms converging to a_i and b_i, and for these the independence of $\{m_i\}$ is clear. To show the pairing $\int A \wedge d(B)$ is independent of the choice of covering and p.o.u. is then an easy exercise using the bilinearity of the pairing I. Finally, the stated properties of (9.19) are consequences of Proposition 9.7. #

Remark 9.9. Note that for $A \in \mathfrak{A}(\alpha)$ and $B \in \mathfrak{A}(\beta)$ with $\alpha + \beta > 1$, we use $\int A \wedge d(B)$ to denote the pairing of (9.19), as distinguished from the Lebesgue integral $\int_M A \wedge dB$, which is defined when A and B are C^1-1-forms. When $A = B = \eta$ is C^1, we can identify $\int_M \eta\, d\eta = \int \eta \wedge d(\eta)$. In general they may not agree, for $\int A \wedge d(B)$ is defined by first restricting A and B to plaques of $\mathfrak{F}$, and this operation has a kernel. A useful consequence of this quirk in the definition is that $\int d_{\mathfrak{F}}h \wedge d(B) = \int dh \wedge d(B)$ for any $C^{1;a}$ function h on M.

The remainder of the proof of Theorem 9.3 consists of showing that $gv(\mathfrak{F})$ is independent of the choice of foliation covering and of $C^{1;a}$-p.o.u. $\{\lambda_i\}$. This is closely related to the following problem: Let $f : M \to R$ be a measurable function whose restrictions to the leaves of $\mathfrak{F}$ are C^1, and the leafwise 1-jets depend measurably on the transverse parameter. Let $d_{\mathfrak{F}}f$ denote a 1-form on M such that $d_{\mathfrak{F}}f|P_x = d(f|P_x)$ for all plaques $P_x \subset M$, and so that for η the 1-form of (9.7) and $\tilde{\eta} = \eta + d_{\mathfrak{F}}f$, the 3-form $\tilde{\eta} \wedge d\tilde{\eta}$ is integrable on M. The problem is to find the minimum hypothesis on f so that $\eta \wedge d\eta$ and $\tilde{\eta} \wedge d\tilde{\eta}$ define the same cohomology class. A sufficient condition is provided by:

Lemma 9.10. *Let η and $d_{\mathfrak{F}}f$ be transversally C^{α}-1-forms for some $\alpha > \frac{1}{2}$. Suppose there exists a sequence $\{f_n\}$ of $C^{1;a}$-functions on M such that $\{d_{\mathfrak{F}}f_n\}$ converges to $d_{\mathfrak{F}}f$ in the transverse C^{β}-topology, for some $\beta > 1 - \alpha$. Then*

$$\int_M \eta \wedge d\eta = \int_M \tilde{\eta} \wedge d\tilde{\eta}.$$

Proof: By Proposition 9.8 and the Remark 9.9, we write

$$\int_M \tilde{\eta} \wedge d\tilde{\eta} = \int \tilde{\eta} \wedge d(\tilde{\eta})$$

$$= \int \eta \wedge d(\eta) + \int d_{\mathfrak{F}} f \wedge d(\eta) + \int \eta \wedge d(d_{\mathfrak{F}} f) + \int d_{\mathfrak{F}} f \wedge d(d_{\mathfrak{F}} f).$$

It will suffice to show:

$$\int d_{\mathfrak{F}} f \wedge d(\eta) = 0 \tag{9.20}$$

$$\int \eta \wedge d(d_{\mathfrak{F}} f) = 0 \tag{9.21}$$

$$\int d_{\mathfrak{F}} f \wedge d(d_{\mathfrak{F}} f) = 0. \tag{9.22}$$

Continuity of the pairing (9.19) implies we can calculate the integrals (9.20) and (9.21) via C^{β}-approximations to $d_{\mathfrak{F}} f$, and C^{α}-approximations to η. Choose a sequence of C^2-1-forms $\{\eta_m\}$ converging uniformly to η in the C^{α}-topology, and let $\{f_n\}$ be the given functions with $\{d_{\mathfrak{F}} f_n\}$ converging uniformly to $d_{\mathfrak{F}} f$ in the C^{β}-topology. For these approximations to the integrands of (9.20) and (9.21), the pairing (9.19) reduces to the usual integration of measures, so we can apply Stokes' theorem. The last ingredient used is the property of the pairing pointed out in Remark 9.9:

$$\int d_{\mathfrak{F}} f_n \wedge d(\beta) = \int df_n \wedge d(\beta). \tag{9.23}$$

Now calculate:

$$\int d_{\mathfrak{F}} f \wedge d(\eta) = \lim_{m,n\to\infty} \int d_{\mathfrak{F}} f_n \wedge d(\eta_m) \tag{9.24}$$

$$= \lim_{m,n\to\infty} \int df_n \wedge d(\eta_m)$$

$$= \lim_{m,n\to\infty} \int_M df_n \wedge d\eta_m$$

$$= 0 \quad \text{by Stokes' Theorem.}$$

(9.25)
$$\int \eta \wedge d(d_{\mathfrak{F}} f) = \lim_{m,n\to\infty} \int \eta_m \wedge d(d_{\mathfrak{F}} f_n)$$

$$= \lim_{m,n\to\infty} \int d_{\mathfrak{F}} f_n \wedge d(\eta_m)$$

$$= 0 \quad \text{by the above.}$$

For (9.22), write $(\varnothing_i^{-1})^* (d_{\mathfrak{F}} f) = a_i A_i$, so that $\int d_{\mathfrak{F}} f \wedge d(d_{\mathfrak{F}} f) = \sum_{i=1}^{N} I(c_i a_i, m_i a_i) = 0$ where $c_i = \int_{(-1,1)^2} \lambda_i A_i \wedge A_i = 0$ as A_i is a 1-form. #

Let $\{\lambda_i\}$ and $\{\tilde{\lambda}_i\}$ be two $C^{1;a}$-p.o.u.'s subordinate to the foliation cover $\{(\varnothing_i, U_i)\}$. Define a 1-form $\tilde{\eta}$ on M by

(9.26)
$$\tilde{\eta}|U_i = \sum_{(i,j)} d_{\mathfrak{F}} \tilde{\lambda}_j \cdot \xi_{ij}.$$

We must show $\int \tilde{\eta} \wedge d(\tilde{\eta}) = \int \eta \wedge d(\eta)$.

Define a function $f : M \to R$ by setting, on U_i

$$f|U_i = \sum_{(i,j)} (\tilde{\lambda}_j - \lambda_j) \cdot \xi_{ij}.$$

On $U_i \sim U_j \sim U_k$, $\xi_{ij} + \xi_{jk} = \xi_{ik}$, so $f|U_i$ and $f|U_j$ agree on $U_i \cap U_k$ and f is well-defined. Note that f is C^1 along leaves, and $\tilde{\eta} = \eta + d_{\mathfrak{F}}f$, where $d_{\mathfrak{F}}f = \Sigma(d_{\mathfrak{F}}\tilde{\lambda}_j - d_{\mathfrak{F}}\lambda_j) \sim \xi_{ij}$ is C^{α} transversally. By approximating the functions $\{\xi_{ij}\}$ by C^1-functions $\{\xi^n_{ij}\}$, we get $f_n = \Sigma\ (\tilde{\lambda}_j - \lambda_j) \cdot \xi^n_{ij}$ converges to f in the C^{α}-topology transversally, and the same for $\{d_{\mathfrak{F}}f_n\}$. The equality desired follows from Proposition 9.10.

Next suppose that two foliation coverings $\{(\varnothing'_i, U'_i) \mid i = 1,\ldots,N'\}$ and $\{(\varnothing''_i, U''_i) \mid i = 1,\ldots,N''\}$ are given. Choose a foliation covering $\{(\varnothing_i, U_i) \mid i = 1,\ldots,N\}$ for which $\{U_i\}$ is a common refinement of $\{U'_i\}$ and $\{U''_i\}$. A $C^{1;a}$-p.o.u. $\{\lambda_i\}$ for $\{U_i\}$ can be grouped and partially summed to yield a $C^{1;a}$-p.o.u. $\{\lambda'_i\}$ for $\{U'_i\}$, and a $C^{1;a}$-p.o.u. $\{\lambda''_i\}$ for $\{U''_i\}$. Use these partitions to define 1-forms η' and η'' as in (9.26). We must show that

$$\int \eta' \sim d(\eta') = \int \eta'' \sim d(\eta''), \tag{9.27}$$

and for this it suffices to show that each side is equal to $\int \eta \sim d(\eta)$. We prove one case, with the other following in the same way.

Let $\{\xi'_{ij} : U_i \cap U_j \to R\}$ be the RN-cocycle for the covering $\{(\varnothing'_i, U'_i)\}$. For each (i,j), select (i', j') with $U_i \cap U_j \subset U'_i \cap U'_j$. Let $\xi'_{ij} : U_i \cap U_j \to R$ denote the restriction of $\xi'_{i'j'}$ to $U_i \cap U_j$.

Let $\{\xi_{ij} : U_i \cap U_j \to R\}$ be the RN-cocycle for the covering $\{(\varnothing_i, U_i)\}$. Define 1-forms by

$$\eta \mid U_i = \sum_{(i,j)} d\lambda_j \xi_{ij}$$

(9.28)

$$\eta' \mid U_i = \sum_{(i,j)} d\lambda_j \xi'_{ij},$$

a function $f : M \to R$ by

(9.29) $$f \mid U_i = \sum_{(i,j)} \lambda_j (\xi'_{ij} - \xi_{ij}),$$

and a corresponding 1-form on M by

(9.30) $$d_{\mathfrak{F}} f \mid U_i = \sum_{(i,j)} d_{\mathfrak{F}} \lambda_j \cdot (\xi'_{ij} - \xi_{ij}).$$

Then $\eta' = \eta + d_{\mathfrak{F}} f$, $d_{\mathfrak{F}} f$ is C^{α} and as above, f is the limit of $C^{1;a}$-functions $\{f_n\}$ with $\{d_{\mathfrak{F}} f_n\}$ converging C^{α} to $d_{\mathfrak{F}} f$. Thus, (9.27) follows from applying Proposition 9.10 twice, and noting that the two definitions of η' (and also for η'') agree.

This concludes the proof of Theorem 9.3. #

A fundamental problem is to determine the most general class of diffeomorphisms which leave the Godbillon-Vey number invariant. There are basically two previous results on this problem. The original definition of Godbillon and Vey is easily seen to be C^2-invariant. In a 1981 unpublished note [R], Gilles Rabi showed that for $\mathfrak{F}$ of class C^2 and codimension-one, $gv(\mathfrak{F})$ is invariant under C^1-diffeomorphisms. Independently, but later, the first author discovered a proof of C^1-invariance using semi-simplicial geometric methods valid in all codimensions. In our study of the invariant $gv(\mathfrak{F})$, we prove three extensions of these results.

Theorem 9.11. *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be codimension-one, $C^{1+\alpha}$-foliations on closed oriented 3-manifolds M and M′, respectively. Suppose there exists an orientation preserving $C^{1+\beta}$-diffeomorphism $\Phi : M \to M'$ conjugating $\mathfrak{F}$ to $\mathfrak{F}'$, with $\alpha + \beta > 1$. Then* $gv(\mathfrak{F}) = gv(\mathfrak{F}')$.

Theorem 9.12 *Let $\mathfrak{F}$ and $\mathfrak{F}'$ be C^2, codimension-one foliations on closed oriented 3-manifolds M and M′, respectively. If there exists an orientation-preserving C^1-diffeomorphism $\Phi : M \to M'$ conjugating $\mathfrak{F}$ to $\mathfrak{F}'$, then* $gv(\mathfrak{F}) = gv(\mathfrak{F}')$.

Proof of 9.11: Let $\{(\varnothing_i, U_i) \mid i = 1,\ldots,N\}$ be a $C^{1+\alpha}$-foliation covering for $\mathfrak{F}$ on M. Define $U_i' = \Phi(U_i) \subset M'$, and for each $i = 1,\ldots,N$ choose a map $\varnothing_i' : U_i' \to (-1,1)^3$ so that the collection $\{(\varnothing_i', U_i') \mid i = 1,\ldots,N\}$ is a $C^{1+\alpha}$-regular foliation covering for $\mathfrak{F}'$. Choose a $C^{1;\alpha}$ p.o.u. $\{\lambda_i'\}$ on M′ subordinate to this covering, and use it to define η'. Next , define a $C^{1;\alpha}$-p.o.u. $\{\lambda_i\}$ for the cover $\{U_i\}$ by setting $\lambda_i = \lambda_i' \circ \Phi$, and use this to define η. We must show

$$\int \eta \wedge d(\eta) = \int' \eta' \wedge d(\eta') = \int \Phi^*(\eta') \wedge d(\Phi^*(\eta')). \tag{9.31}$$

The idea behind proving (9.31) is that η and $\Phi^*(\eta')$ are formed from cohomologous RN-cocycles (cf. [H1]) whose coboundary function is C^{β}-continuous, so that we can apply Lemma 9.10.

For each $i = 1,\ldots,N$ and $-1 < x < 1$, set

$$T_i(x) = \frac{d}{dx}\,(p_i' \cdot \Phi \cdot p_i^{-1})\,(x) \tag{9.32}$$

Note T_i is C^δ-continuous, where $\delta = \min(\alpha, \beta)$. The chain rule yields a coboundary formula relating ψ_{ij} with ψ'_{ij} on I_{ij}; and ξ'_{ij} with ξ_{ij} on $U_i \cap U_j$:

$$\psi'_{ij} \circ p'_i \circ \Phi \circ p_i^{-1} = T_j \circ \gamma_{ij} + \psi_{ij} - T_i \tag{9.33}$$

$$\xi'_{ij} \circ \Phi = T_j \circ p_j + \xi_{ij} - T_i \circ p_i \,. \tag{9.34}$$

Define $f : M \to R$ and $d_{\mathfrak{F}} f$ on M by

$$f = \sum_{j=1}^{N} \lambda_j \cdot (T_j \circ p_j) \tag{9.35}$$

$$d_{\mathfrak{F}} f = \sum_{j=1}^{N} d_{\mathfrak{F}} \lambda_j \cdot (T_j \circ p_j) \tag{9.36}$$

Then using the property $d_{\mathfrak{F}} (\sum_{j=1}^{N} \lambda_j) = 0$ and (9.34), we obtain:

$$\Phi^*(\eta') = \eta + d_{\mathfrak{F}} f \tag{9.37}$$

Choose a sequence at C^1-functions $\{T_j^n\}$ converging C^δ-uniformly to T_j, for each j, and set

$$f_n = \sum_{j=1}^{N} \lambda_j \circ (T_j^n \circ p_j)$$

Then $d_{\mathfrak{F}} f_n$ converges C^δ to $d_{\mathfrak{F}} f$, hence (9.31) follows from Lemma 9.10 as $\alpha + \delta > 1$#

As the reader may observe, the hypothesis that Φ is $C^{1+\beta}$ is used in the proof of (9.11) only at the conclusion, in order to construct the sequence of

functions $\{f_n\}$ which have $d_{\mathfrak{F}}f_n$ converging C^{δ} to $d_{\mathfrak{F}}f$. If such a sequence can be shown to exist without requiring that the $\{T_j\}$ be C^{β}-continuous, then the hypothesis of Theorem 9.11 can be weakened to where Φ is just $C^{1;a}$.

<u>Proof of Theorem 9.12</u>: We indicate the changes to be made in the proof of Theorem 9.11 in order to prove 9.12. First, the compositions $p_i' \circ \Phi \circ p_i^{-1}$ and $p_i \circ \Phi^{-1} \circ (p_i')^{-1}$ are C^1, so the transfer functions $T_j : (-1,1) \to R$ are continuous. Define f as in (9.35), $d_{\mathfrak{F}}f$ as in (9.36) and we then have $\Phi^* \eta' = \eta + d_{\mathfrak{F}}f$. Note that both $\Phi^*\eta'$ and $d_{\mathfrak{F}}f$ are transversally C^1. Proposition 9.8 has an obvious extension to a pairing between $\mathfrak{A}(1)$ and $\mathfrak{A}(0)$, where $\mathfrak{A}(1)$ consists of the transversally C^1, 1-forms on M, and $\mathfrak{A}(0)$ are the continuous 1-forms. Lemma 9.10 also extends naturally, and the remainder of the proof is the same as for 9.11. #

<u>Remark 9.13</u>. It is reasonable to expect that with finer analytical techniques, Theorems 9.11 and 9.12 can be proven with the hypothesis that Φ is only transversally absolutely continuous. As the careful reader will observe, such an extension would follow from showing that

$$\int d\eta \wedge d_{\mathfrak{F}}f = 0 = \int d(d_{\mathfrak{F}}f) \wedge \eta,$$

when f is a leafwise smooth, transversally measurable function. We prove in Theorem 10.1 below that if $\mathfrak{F}$ is the weak-stable foliation of an Anosov flow on a circle bundle, then $gv(\mathfrak{F})$ is an invariant of absolute continuity.

<u>Remark 9.14</u>. The pairing I used to define $gv(\mathfrak{F})$ makes use of an embedding of the integrands of (9.8) into appropriate Sobolev spaces, and pairing there. Extending this pairing to its full domain in Sobolev space gives our extension of $gv(\mathfrak{F})$. There

are other possible schemes for extending the bilinear form $I(f,g) = \int_{-1}^{1} f d\bar{g}$ from the C^1-functions to larger classes: one was considered by Duminy-Sergiescu [DS] j our original idea is a geometric construction described in §10, which explains geometricly the restriction $\alpha > \frac{1}{2}$ used above. Both of these extensions agree with the extension used in this section. Other extensions may be possible using more refined analytical techniques; for example, the pairing I has been studied by A. Connes in [Co] from the viewpoint of cyclic cocycle theory. We conclude this section with two basic questions: Do further extensions of the pairing I exist, and what are their geometric significance? Can I be extended to a pairing $C_c^{\alpha}(-1,1) \times C_c^{\beta}(-1,1) \to R$, for $\alpha + \beta \leq 1$?

§10. Foliations transverse to a circle fibration

In this section we specialize the results of §9 to the case of a foliation on a 3-manifold which fibers over a surface, and with the foliation transverse to the fibers. We first prove that $gv(\mathfrak{F})$ for such $\mathfrak{F}$ is invariant under homeomorphisms which are absolutely continuous transversely, if $\mathfrak{F}$ is the weak-stable foliation of a transitive Anosov flow. Secondly, we recall the Thurston cocycle description of $gv(\mathfrak{F})$, as originally defined for C^2 foliations transverse to a circle fibration. This cocycle has a natural geometric extension to foliations of transverse class $C^{1+\alpha}$, which we describe. This geometric extension agrees with the definition of $gv(\mathfrak{F})$ given in §9. Its interest is because it shows the restriction $\alpha > \frac{1}{2}$ is mediated by considerations of Hausdorff dimension.

Let S be a closed, oriented surface, $\pi : M \to S$ a smooth fibration with circle fibers, and $\mathfrak{F}$ a $C^{1+\alpha}$ codimension-one foliation transverse to the fibers of π. Choose a basepoint $p \in S$ and identify $\pi^{-1}(p) \equiv S^1$. Then $\mathfrak{F}$ determines a global holonomy homomorphism

$$h = h_{\mathfrak{F}} : \Gamma \equiv \pi_1(S, p) \to \mathrm{Diff}^{1+\alpha}(S^1).$$

by lifting paths in S, with base point p, to leaves of $\mathfrak{F}$ with initial point $\theta \in S^1$ and terminal point $h_{\mathfrak{F}}(\theta)$. If $\mathfrak{F}$ is the weak-unstable foliation of a geodesic flow, then $h_{\mathfrak{F}}$ is precisely the action of $\pi_1(S, p)$ on the circle at ∞.

Let $\tilde{S}$ denote the universal cover of S. Then Γ acts naturally on $\tilde{S}$, and $h_{\mathfrak{F}}$ defines an action of Γ on S^1, so we can define the quotient manifold $\hat{M} = \tilde{S} \underset{\Gamma}{\times} S^1$.

There is a natural foliation $\hat{\mathfrak{F}}$ on $\hat{M}$ whose leaves are covered by the Poincaré discs $\tilde{S} \times \{\theta\}$, $\theta \in S^1$. The holonomy homorphism of $\hat{F}$ is $h_{\mathfrak{F}}$ by construction. Furthermore, there is a natural $C^{1+\alpha}$ diffeomorphism $\theta : \hat{M} \to M$ conjugating $\hat{\mathfrak{F}}$ to $\mathfrak{F}$, and which is C^∞ along leaves (cf. [Go], [HH]).

Let $\pi' : M' \to S$ be a second circle fibration with a $C^{1+\alpha}$ foliation $\mathfrak{F}'$ transverse to the fibers of π'. Let $\Phi : M' \to M$ be a homeomorphism conjugating $\mathfrak{F}'$ to $\mathfrak{F}$, and set $p' = \Phi^{-1}(p)$. Define $h_{\mathfrak{F}'} = h'$ using the basepoint $\pi'(p')$. Then Φ determines a homeomorphism $\varnothing : S^1 \to S^1$ such that

$$h'(\delta) = \varnothing \circ h(\delta) \circ \varnothing^{-1}, \ \gamma \in \Gamma.$$

Furthermore, $\varnothing$ has the same properties of continuity and differentiability as does Φ transversally. Thus Φ is transversally absolutely continuous if and only if $\varnothing$ is absolutely continuous. The map $\varnothing$ induces a homeomorphism of suspensions

$$F_{\varnothing} : \hat{M}' \to \hat{M}$$

which is C^∞ along leaves, and is transversally of the same continuity as $\varnothing$. The assignment $\Phi \to \varnothing \to \theta^{-1} \circ F_{\varnothing} \circ \theta' : M \to M$ is a natural transformation which replaces Φ with a homeomorphism that is C^∞ along leaves, and whose transverse continuity class is that of Φ.

<u>Theorem</u> <u>10.1</u>. *Let $\mathfrak{F}$ be the weak-unstable foliation of a transitive, volume-preserving Anosov flow $\{g_t\}$ on a closed, oriented 3-manifold M, where M fibers over a closed Riemann surface S. Let M′ be another oriented 3-manifold with a*

$C^{1+\alpha}$, *codimension-one foliation* $\mathfrak{F}'$. *If there exists an orientation-preserving homemorphism* $\Phi : M \to M'$, *conjugating* $\mathfrak{F}$ *to* $\mathfrak{F}'$ *with* Φ *and* Φ^{-1} *transversally absolutely continuous, then* $gv(\mathfrak{F}) = gv(\mathfrak{F}')$.

Proof: In the proof of Proposition 2.1 of [Gh 2], Ghys shows there is a smooth fibration in circles on M which is transverse to $\mathfrak{F}$. Then Φ carries this fibration into a topological fibration of M′, which can be smoothed to give a smooth fibration in circles of M′ transverse to $\mathfrak{F}'$. Then by the discussion before the theorem, we can assume that Φ is $C^{\infty;a}$.

Next, repeat the proof of Theorem 9.11, with the following modifications. The p.o.u. $\{\lambda_i'\}$ on M′ is chosen to be C^∞. The transfer functions $\{T_j | j = 1,\ldots,N\}$ are defined as in (9.32), but now note that each T_j is only measurable, and the cocycle equation (9.33) continues to hold for a.e. $x \in I_{ij}$. However, the difference $T_j \circ p_j - T_i \circ p_i = \xi_{ji}' \circ \Phi - \xi_{ji}$ is $C^{\infty;a}$, so that for the measurable function $f = \sum_{j=1}^{N} \lambda_j \cdot T_j \circ p_j$, the leafwise differential $d_{\mathfrak{F}}f = \sum_{j=1}^{N} d\lambda_j \cdot T_j \circ p_j$ is C^α transversally, and C^∞ along leaves.

Define a cocycle over the flow $\{g_t\}$ by setting

$$\varnothing(x, s) = \int_0^s (d_{\mathfrak{F}}f)\, dg_t, \tag{10.1}$$

the integral of $d_{\mathfrak{F}}f$ along the path $\gamma_t(x) = \{g_t(x) | 0 \le t \le s\}$. Recall that for any leaf $L \subset M$, $d(f|L) = d_{\mathfrak{F}}f|L$. As $\gamma_t(x)$ lies entirely within a leaf, the integral in (10.1) is exact:

(10.2) $$\varnothing(x,s) = f(g_s(x)) - f(x). \quad \text{a.e. } x \in M$$

Thus, by the measurable Livshitz Theorem 5.7 above, all of the obstructions to solving (10.1) continuously vanish, so there is a continuous function $\tilde{f} : M \to R$ which agrees with f a. e., and also satisfies (10.2) for all x and t. Then by Theorem 1 of [L1], this $\tilde{f}$ is C^{α} transversally and C^{∞} when restricted to the leaves of $\mathfrak{F}$. Both $d_{\mathfrak{F}}f$ and $d_{\mathfrak{F}}\tilde{f}$ are continuous on M. We are now in a position to apply Lemma 9.10, as in the proof of Theorem 9.11, to conclude $gv(\mathfrak{F}) = gv(\mathfrak{F}')$. #

The naturality of the construction of $\hat{\mathfrak{F}}$ and $\hat{M}$ from $\mathfrak{F}$ and M, and the existence of $\theta : M \to \hat{M}$, shows that all of the transverse geometric data for $\mathfrak{F}$ is contained in the homomorphism $h = h_{\mathfrak{F}} : \Gamma \to \mathrm{Diff}^{1+\alpha}(S^1)$. Thus, one expects a formula for $gv(\mathfrak{F})$ in terms of h. For C^2-foliations, this is provided by the Thurston 2-cocycle, as we next recall.

For $h : \Gamma \to \mathrm{Diff}_+S^1$, an action of Γ on S^1 by orientation-preserving C^1-diffeomorphisms, define the additive RN cocycle $\upsilon_h : \Gamma \times S^1 \to R$ by setting

$$\upsilon_h(\gamma)(\theta) = \log\{\mathrm{div}\ h(\gamma)(\theta)\}$$

where div $h(\gamma)$ is the divergence of $h(\gamma)$ with respect to the Haar measure $d\theta$ on S^1 having total mass 2π. When $h(\gamma)$ is C^2, then $\upsilon_h(\gamma)$ is C^1.

<u>Definition</u> <u>10.2</u>. *The Thurston cocycle of a homomorphism* $h : \Gamma \to \mathrm{Diff}^{(2)}S^1$ *is the 2-cocycle on* Γ, $c_h : \Gamma \times \Gamma \to R$, *defined by*

$$c_h(\gamma_1,\gamma_2) = \int_{S^1} \upsilon_h(\gamma_2) \cdot d(\upsilon_h(\gamma_1 \circ \gamma_2)). \tag{10.3}$$

For a proof that (10.3) defines a cocycle, and an explanation of its origin, see Bott [B].

When the action of Γ on S^1 is by $C^{1+\alpha}$ diffeomorphisms, then each $\upsilon_h(\gamma)$ is a C^{α}-function on S^1, so via the pairing I of Proposition 9.7, we can formally extend (10.3) for $\alpha > \frac{1}{2}$ by setting

$$c_h(\gamma_1, \gamma_2) = I\big(\upsilon_h(\gamma_2), \upsilon_h(\gamma_1 \circ \gamma_2)\big). \tag{10.4}$$

It is a standard fact (cf. [B]) that for $\mathfrak{F}$ a C^2-foliation and $h = h_{\mathfrak{F}}$, the cohomology class $[c_h] \in H^2(\Gamma; R)$ is identified with $GV(\mathfrak{F}) \in H^3(M; R)$ under the integration along the circle fibers map, $\pi_* : H^3(M; R) \to H^2(S; R) \cong H^2(\Gamma; R)$. It is not hard to show that this identification also holds for $C^{1+\alpha}$ foliations, using the techniques of §9. We state this precisely, and leave the proof to the reader.

Let $[S] \in H_2(\Gamma; Z)$ be the fundamental class associated to the cycle defined by the oriented surface S. Then there are integers $\{n_i\}$ and elements $\{\gamma_{1,i},\gamma_{2,i}\}$ so that $[S] = \sum_{i=1}^{P} n_i(\gamma_{1,i} \times \gamma_{2,i})$

<u>Proposition</u> <u>10.3</u>. *For $\mathfrak{F}$ a $C^{1+\alpha}$ foliation, $\alpha > \frac{1}{2}$, and $h = h_{\mathfrak{F}}$ as above,*

$$gv(\mathfrak{F}) = c_h[S] = \sum_{i=1}^{P} n_i \cdot c_h(\gamma_{1,i},\gamma_{2,i})$$

Moreover, if $h' = \varnothing^{-1} \circ h \circ \varnothing$, where $\varnothing : S^1 \to S^1$ is a $C^{1+\beta}$ diffeomorphism

with $\alpha + \beta > 1$, *then* $c_{h'}[S] = c_h[S]$.

In the remainder of this section, we show how to define c_h for $C^{1+\alpha}$-actions using a completely geometric approach. Thurston remarked (see page 40 [B]) that $c_h(\gamma_1,\gamma_2)$ is the area inside a planar curve determined by $\upsilon_h(\gamma_2)$ and $\upsilon_h(\gamma_1 \circ \gamma_2)$, for γ_1 and γ_2 C^2-maps. This will be the basis for our extension.

Let f, g : $S^1 \to R$ be C^1-functions. Define $C(f,g) : S^1 \to R^2$ by setting

$$C(f,g)\,(\theta) = (f(\theta),\, g(\theta)),\ \theta \in S^1.$$

Identify S^1 with the unit circle $\{(x,y) \mid x^2 + y^2 = 1\}$, and let $D^2 = \{(x,y) \mid x^2 + y^2 \leq 1\}$ be the interior. Choose a C^1-extension of C(f,g) to a map $D(f,g) : D^2 \to R^2$. Then by Stokes' Theorem,

$$\int_{S^1} f dg = \int_{S^1} C(f,g)^*(x dy) = \int_{D^2} D(f,g)^*(dx \wedge dy),$$

where the last integral represents the algebraic area inside the C^1-curve C(f,g). We will set

$$A(f,g) = \int_{S^1} f dg = \int_{D^2} D(f,g)^*(dx \wedge dy). \tag{10.5}$$

To define A(f,g) for functions which are not C^1, it suffices to have a good definition of the algebraic area inside the curve C(f,g). We will show this exists when $\alpha > \frac{1}{2}$, and f, g are C^α. First, what appears to be a severe limitation to

defining A(f, g) when $\alpha \leq \frac{1}{2}$ is provided by the following standard fact.

<u>Lemma</u> <u>10.4</u>. *Let* f, g : $S^1 \to R$ *be* C^α*-continuous functions, with* $\alpha < 1$. *Then the image of* C(f,g) *in* R^2 *has Hausdorff dimension no greater than the minimum of* $\{2, 3 - 2\alpha\}$.

The estimate in Lemma 10.4 is sharp, although it is not clear that for (f, g) of the form $(\upsilon_h(\gamma_2), \upsilon_h(\gamma_1 \circ \gamma_2))$ that this is sharp. As it seems natural to require the curve C(f,g) have Hausdorff dimension less than 2, in order to define the area inside the curve, this forces $\alpha > \frac{1}{2}$ upon us.

Now represent S^1 as the interval $0 \leq \theta < 2\pi$, with 0 and 2π identified. Let T $\subset S^1$ be a finite set of points, given in increasing order, $0 \leq y_1 < y_2 < \ldots < y_N < 2\pi$, with $y_{N+1} = y_0$ for notational convenience. Let f,g : $S^1 \to R$ be given. Define piecewise-linear functions f_T, g_T : $S^1 \to R$ by requiring $f_T(y_i) = f(y_i)$ and $g_T(y_i) = g(y_i)$ for i = 0,...,N, and f_T, g_T are linear between the points of T. Define

$$C(f,g,T)\ (\theta) = (f_T(\theta), g_T(\theta)),\ \theta \in S^1,$$

a p.l. curve in R^2. The algebraic area inside C(f,g,T) is then well-defined, and we denote it A(f,g,T).

For a given finite point set $T_0 = T \subset S^1$, let T_1 denote the barycentric subdivision of T_0, and then for n > 1, let T_n denote the barycentric subdivision of T_{n-1}.

Proposition 10.5. *Let* $f, g : S^1 \to R$ *be* C^α*-continuous,* $\alpha > \frac{1}{2}$.

1) *For each* T, *the sequence* $\{A(f,g, T_n) | n = 0, 1,\ldots\}$ *is Cauchy*

2) *The limit* $A(f, g) = \lim_{n\to\infty} A(f,g, T_n)$ *is independent of the choice of initial point set* T.

3) *The function* $A : C^\alpha(S^1) \times C^\alpha(S^1) \to R$ *defined by* $f, g \to A(f,g)$ *is bilinear, skew-symmetric, and for* f, g *piece-wise* C^1*-functions, satisfies* $A(f, g) = \int_{S^1} f dg$.

Proof: Fix f and g, and let K be a constant so that f satisfies

$$|f(\theta) - f(\varphi)| < K \cdot |\theta - \varphi|^\alpha, \; 0 \le \theta < \varphi \le 2\pi;$$

and similarly for g.

Define the mesh of $T \subset S^1$, $T = \{y_1,\ldots,y_N\}$, as

$$\text{mesh } (T) = \max_{i=0,\ldots,N} \{\text{dist}(y_i, y_{i+1})\}. \tag{10.6}$$

Let $T' \subset S^1$ be another subset with $T \subset T'$. Assume the points of T' are ordered and labeled as

$$T' = \{z_{i,j} | i = 1,\ldots,N \text{ and } 0 \le j \le p_i\}$$

for integers p_i depending on i, where

$$y_i = z_{i,0} < z_{i,1} < \cdots < z_{i,p_i} < y_{i+1}.$$

Also, let $z_{i,p_i+1} = y_{i+1}$ for notational convenience. The basic fact we need is the elementary estimate:

<u>Lemma</u> <u>10.6</u>. *Assume that* $p_i \leq d$ *for all* i. *(So that there are at most* d *points of* T′ *between any two adjacent points of* T.) *Then*

$$|A(f,g, T') - A(f, g, T)| \leq K^2 \text{ (mesh } T)^{2\alpha} \cdot d \cdot N.$$

<u>Proof</u>: Set $a_i = (f(y_i), g(y_i))$ for $i = 1,\ldots,N$ and $b_{i,j} = (f(z_{ij}), g(z_{ij}))$ for $i = 1,\ldots,N$, $0 \leq j \leq p_i + 1$. Let $\overline{a_i a_{i+1}}$ denote the line segment in R^2 from a_i to a_{i+1}. Then

$$|A(f, g, T') - A(T)| = \left|\sum_{i=1}^{N} E(i)\right| \tag{10.7}$$

where E(i) is the algebraic area bounded by the segment $\overline{a_i a_{i+1}}$ and the polygonal curve joining $b_{i,0}$ to $b_{i+1,0}$ via the points $\{b_{i,1}, b_{1,2},\ldots,b_{i,p_i}\}$.

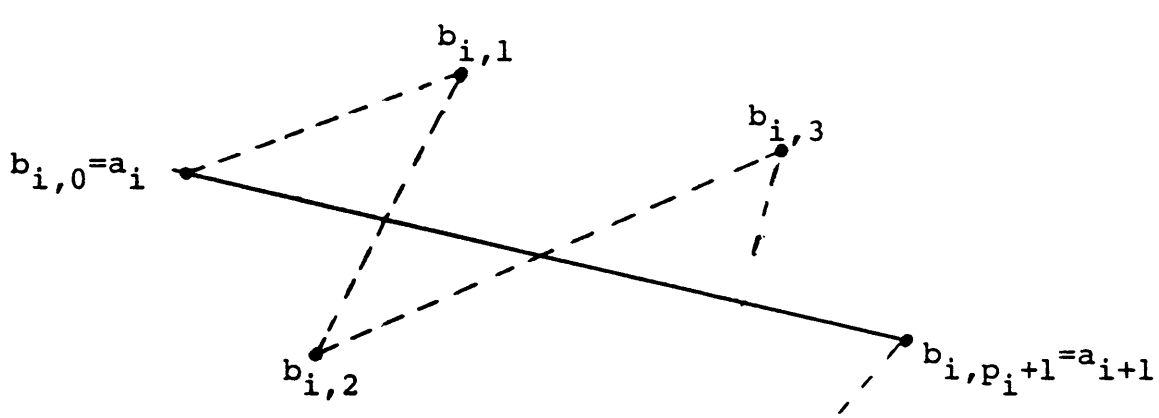

Figure 10.1

Let $\Delta_{i,j}$ be the <u>area</u> of the plane triangle with vertices a_i, $b_{i,j}$ and $b_{i,j+1}$. It is elementary to see that

$$(10.8) \qquad |E(i)| \leq \sum_{j=1}^{p_i} \Delta_{i,j}.$$

We use the Hölder hypothesis to estimate each $\Delta_{i,j}$. First, $y_i \leq z_{i,j} \leq y_{i+1}$, so $\text{dist}(y_i, z_{i,j}) \leq \text{dist}(y_i, y_{i+1}) \leq \text{mesh}\ (T)$.

Thus,

$$(10.9) \qquad |f(z_{i,j}) - f(y_i)| \leq K \cdot \text{mesh}(T)^{\alpha}$$

$$(10.10) \qquad |g(z_{i,j}) - g(y_i)| \leq K \cdot \text{mesh}(T)^{\alpha}$$

and consequently

$$(10.11) \qquad \text{dist}(b_{i,j}, a_i) \leq \sqrt{2} \cdot K \cdot \text{mesh}(T)^{\alpha},\ 0 < j \leq p_i + 1$$

giving the estimate $\Delta_i \leq K^2 \cdot \text{mesh}(T)^{2\alpha}$. Substituting this into (10.8) and (10.7) yields:

$$|A(f,g,T') - A(f,g,T)| \leq \sum_{i=0}^{N} \sum_{j=1}^{p_i} \Delta_i$$

$$\leq \sum_{i=0}^{N} \sum_{j=1}^{p_i} K^2 \cdot \text{mesh}(T)^{2\alpha}$$

$$\leq N \cdot d \cdot K^2 \cdot \text{mesh}(T)^{2\alpha}. \qquad \#$$

Now to prove part 1 of Proposition 10.5, take $T' = T_{n+1}$ and $T = T_n$ in the lemma. We have $d = 2$, $\text{mesh}\ (T_n) = 2^{-n}\ \text{mesh}(T_0)$, and $N = 2^n \cdot p$, where p is the

number of vertices in T_0. Then

$$|A(f,g,T_{n+1}) - A(f,g,T_n)| \leq 2^n \cdot p \cdot 2 \cdot K^2 \cdot 2^{-2n\alpha}$$

$$= 2\, p \cdot K^2 \cdot 2^{n(1-2\alpha)}.$$

For $\alpha > \frac{1}{2}$, this last term is summable in n, hence $\{A(f,g, T_n)\}$ is Cauchy.

To prove part 2, let T_0' be another initial choice of a finite set in S^1. Set $T_0'' = T_0 \cup T_0'$. We claim that for some d, and all $n > 0$, the number of points in T_n'' between any two adjacent points of T_n is bounded by d (and by symmetry, there is a d′ which works for T_n'' and T_n'.) To find d, first note that there is some p so that T_p contains at most one point of T_0'' in any closed interval defined by adjacent points of T_p. Thus, for all $n > 0$, there is at most one point of T_n'' between any two adjacent points of T_{p+n}. But T_{p+n} contains 2^{p-1} points between any two adjacent points of T_n, so we can take $d = 2^{p-1}$. Now apply Lemma 10.4 to T_n'' and T_n to conclude

$$|A(f,g, T_n'') - A(f,g, T_n)| \leq p \cdot d \cdot K^2 \cdot 2^{n(1-2\alpha)}$$

which tends to zero with n. Similarly, we obtain $|A(f,g, T_n'') - A(f, g, T_n')| \to 0$.

Part 3 of the proposition is immediate from the definition of A(f,g). #

The pairing A defined in Proposition 10.3 can be used to give a geometric extension of the Thurston cocycle by setting $c_h^A(\gamma_1, \gamma_2) = A(\upsilon_h(\gamma_2), \upsilon_h(\gamma_1 \circ \gamma_2))$,

and then

$$(10.12) \qquad gv_A(\mathfrak{F}) = c_h^A[S] = \sum_{i=1}^{P} n_i \cdot A(\upsilon_h(\gamma_{2,i}), \upsilon_h(\gamma_{1,i} \circ \gamma_{2,i})$$

In fact, this geometric extension of C_h is identical with the analytical extension using I, as we will show in a moment. Our final comment on geometric extensions as in (10.12) is that this can obviously also be applied for an arbitrary $C^{1+\alpha}$, codimension-one foliation via the embedding $C_c^\alpha(-1,1) \subset C^\alpha(S^1)$, as in §9. However, the correct viewpoint is to redo the construction completely in terms of semi-simplicial methods, and then the geometric construction is seen to extend to all codimensions (with an appropriate restriction on α!). This is developed in [Hu 3].

To show that (10.4) and (10.12) agree, we will show that for $\{f_n = f_{T_n}\}$ and $\{g_n = g_{T_n}\}$ as constructed above, the α-Hölder norms of the differences $(f - f_n)$ and $(g - g_n)$ tend to zero. Then the continuity of I and the fact that A and I agree on piecewise smooth functions will conclude the proof.

<u>Lemma</u> <u>10.7</u>. *Let* $f, g : S^1 \to R$ *be* C^α*-continuous for any* $0 < \alpha < 1$. *Then for all* $n > 0$, *the* α*-Hölder norms of* $(f - f_n)$ *and* $(g - g_n)$ *are dominated by*

$$K_n = 4 \cdot K \cdot (\text{mesh } T_n)^\alpha,$$

where K *is the maximum of the* α*-Hölder norms of* f *and* g.

<u>Proof</u>: For $0 \leq x < y < 2\pi$, we will estimate

$$(10.13) \qquad |(f(y) - f_n(y)) - (f(x) - f_n(x))| \leq |f(y) - f_n(y)| + |f(x) - f_n(x)|.$$

Let θ_x be a point in T_n closest to x, and θ_y a point in T_n closest to y. Then consider

$$\begin{aligned} |f(x) - f_n(x)| &\leq |(f(x) - f_n(x)) + (f(\theta_x) - f_n(\theta_x))| \\ &\leq |f(x) - f(\theta_x)| + |f_n(x) - f_n(\theta_x)| \\ &\leq K \cdot |x - \theta_x|^\alpha + K \cdot |\theta'_x - \theta_x|^\alpha \\ &\leq 2 \cdot K \cdot \mathrm{mesh}(T_n)^\alpha, \end{aligned}$$

where $\theta'_x \in T_n$ is the closest point to x in T_n <u>such that</u> x lies between θ_x and θ'_x. A similar estimate for y yields the Lemma. #

§11. The Mitsumatsu Defect and Rigidity

In this section, we calculate the Godbillon-Vey invariant for the weak-unstable foliation $\mathfrak{F}^u(f_t(g))$ of the geodesic flow $f_t(g)$ in terms of the Mitsumatsu Defect of g.

Recall that S denotes a closed orientable Riemann surface with metric g having everywhere negative curvature $k_g : S \to R$. Let M denote the unit tangent bundle, $\pi : M \to S$ the fibration, $\xi = \xi_g$ the geodesic vector field, $\{f_t(g)\}$ the geodesic flow, $H = H^+ : M \to R$ the unique positive solution of the Ricatti equation (0.9), and $\{\xi, h, w\}$ the canonical frame field on M introduced in §8. Note that H is C^1 on M, so the derivative along the fibers, $w \cdot H$ is continuous on M.

Definition 11.1. *The Mitsumatsu Defect of* g *is*

$$\mathrm{Def}(g) = 3 \cdot \int_M (w \cdot H)^2 \, \mathrm{dvol}.$$

We will show:

Proposition 11.2. *(Formula of Mitsumatsu). Let* g *be a metric of negative curvature on a closed surface* S *with Euler characteristic* $\chi(S)$. *Then the Godbillon-Vey invariant satisfies*

$$\mathrm{gv}(g) \overset{\mathrm{def}}{=} \mathrm{gv}\left[\mathfrak{F}^u(f_t(g))\right] = (2\pi)^2 \cdot \chi(S) - \mathrm{Def}(g). \tag{11.1}$$

When the foliation $\mathfrak{F}^u(f_t(g))$ is C^2, the derivation of (11.1) was given by Mitsumatsu [M]. The point of this Proposition is that the formula continues to hold when the foliation is no longer C^2. This is a critical extension, for our conjecture is that the only cases where $\mathfrak{F}^u(f_t(g))$ is C^2 are those metrics with $\mathrm{Def}(g) = 0$. Before giving the proof, we draw three consequences of the Proposition. The first was observed by Mitsumatsu (Lemma, p. 12, [M]).

<u>Corollary</u> <u>11.3</u>. *For* g *on* S, *and* $\mathfrak{F}^u(f_t(g))$ *on* M *as above, we have* $gv(g) = (2\pi)^2 \cdot \chi(S)$ *if and only if* g *has constant negative curvature.*

<u>Proof</u>: Observe that $\mathrm{Def}(g) = 0$ if and only if $w \cdot H$ vanishes identically on M. Differentiating the Ricatti equation with respect to w, and noting that $w \cdot (k_g \circ \pi) = 0$, we obtain

$$w \cdot (\xi \cdot H) + 2H \cdot wH = 0. \tag{11.2}$$

The Lie identity $[w, \xi] = h$ from (8.1) implies

$$w \cdot (\xi \cdot H) + \xi \cdot (w \cdot H) = h \cdot H. \tag{11.3}$$

Combining (11.2) and (11.3), we see that $\mathrm{Def}(g) = 0$ precisely when $h \cdot H = 0$. Now this implies H is constant along the flow of h, which is ergodic. Thus, H is constant, so by the Ricatti equation $k_g \circ \pi = -H^2$ is also constant. Conversely, k_g constant implies H is constant and $w \cdot H = 0$. #

<u>Corollary</u> <u>11.4</u>. *Let* $\mathfrak{F}^u(f_t(g))$ *be as above. Suppose there exists a* $C^{1;a}$-*homeomorphism conjugating this foliation to a foliation* $\mathfrak{F}^u(f_t(\tilde{g}))$, *where* $\tilde{g}$ *is a*

metric of constant negative curvature on S. *Then* g *also has constant negative curvature.*

Proof: By Theorem 10.1, $gv\left[\mathfrak{F}^u(f_t(g))\right] = gv\left[\mathfrak{F}^u(f_t(\tilde{g}))\right]$, and thus $\mathrm{Def}(g) = \mathrm{Def}(\tilde{g}) = 0$ by Proposition 11.2. Then by Corollary 11.3, g has constant curvature. #

With the additional hypothesis that there is a $C^{1;a}$ homeomorphism conjugating the flow $\{f_t(g)\}$ to the flow $(f_t(\tilde{g})\}$, then Corollary 11.4 has an alternate proof due to the second author [Ka]. The last result is the local solution of the Conjecture formulated in the Introduction.

Corollary 11.5. *Suppose that* $\mathfrak{F}^u(f_t(g))$ *is a* C^2 *foliation, and at all points of* M

$$2(w \cdot H)^2 + H^2 - H \cdot w \cdot (w \cdot H) > 0.$$

Then g *has constant negative curvature.*

Proof: By Theorem 8.1, $\mathfrak{F}^u(f_t(g))$ is C^∞ conjugate to a foliation $\mathfrak{F}^u(f_t(\tilde{g}))$, where $\tilde{g}$ has constant negative curvature. Then by Corollary 11.4, g also has constant curvature. #

We conclude this paper with a proof of Proposition 11.2. The strategy will be to establish an approximate form of equation (8.10). Note that H is C^1, and from (11.2) we obtain $w \cdot H = \frac{-1}{2} \cdot \xi \cdot (\log H)$. Recall that H is characterized as the unique function such that the vector field h^+ in (8.3) satisfies (8.4). As $\{f_t(g)\}$ admits a C^∞ contact 1-form, the vector field h^+ and its Lie bracket $[\xi, h^+]$ are C^1

on M. Thus, $\xi \cdot \log H$ is a C^1-function on M, and we can define $w \cdot (w \cdot H) = -\frac{1}{2}\, w \cdot (\xi \cdot \log H)$.

Thus, choose a sequence $\{H_n\}$ of C^2 functions on M with $\{w \cdot H_n\}$ converging uniformly to $w \cdot H$ in the C^α-topology, for $\alpha > \frac{1}{2}$, and $w \cdot (w \cdot H_n)$ converging uniformly to $w \cdot (w \cdot H)$. The vector field w is differentiation along the S^1 fibers of $\pi : M \to S$, so one way to obtain H_n is to write $H = \sum_{\ell \in Z} h_\ell\, e^{i\ell\theta}$, as a Fourier series with respect to the fibers (cf. [GKI]), with each $h_\ell : S \to R$ a $C^{1,\alpha}$-function. Then choose C^2-approximations $\{h_{\ell,j}\}$ converging $C^{1,\alpha}$ to h_ℓ, and set

$$H_n = \sum_{|\ell| \leq n} h_{\ell,n}\, e^{i\ell\theta}.$$

Define 1-forms $N_n^* = w^* - H_n\, h^*$. Then define a 1-form

$$\eta_n = i(w) \cdot dN_n^* \tag{11.4}$$

$$= -(w \cdot H_n) \cdot h^* + H_n \cdot \xi_*$$

so that

$$d\eta_n = -d(w \cdot H_n) \wedge h^* + (w \cdot H_n)\, w^* \wedge \xi_* + dH_n \wedge \xi^* + H_n \cdot w^* \wedge h^*, \tag{11.5}$$

$$\eta_n \wedge d\eta_n = \{-2(w \cdot H_n)^2 - H_n^2 + H_n \cdot (w \cdot (w \cdot H_n))\}\, w^* \wedge \xi^* \wedge h^*. \tag{11.6}$$

Then the trick observed by Mitsumatsu is that dvol is invariant under the action of SO(2) on M, so that

$$0 = \int_M w \cdot (H_n \cdot wH_n)\, dvol \tag{11.7}$$

$$= \int_M (w \cdot H_n)^2 \, dvol + \int_M H_n(w \cdot (w \cdot H_n)) \, dvol.$$

Combining (11.6) and (11.7) yields

$$\int_M \eta_n \wedge d\eta_n = -\int_M H_n^2 \, dvol - 3\int_M (w \cdot H_n)^2 \, dvol. \tag{11.8}$$

<u>Lemma</u> <u>11.6.</u> $$\lim_{n\to\infty} -\int_M H_n^2 \, dvol = (2\pi)^2 \, \chi(S).$$

<u>Proof:</u>

$$\lim_{n\to\infty} -\int_M H_n^2 \, dvol = -\int_M H^2 \, dvol$$

$$= \int_M \xi \cdot H \, dvol + \int_M k_g \circ \pi \, dvol$$

$$= (2\pi)^2 \, \chi(S)$$

where we use the Ricatti equation $H^2 + \xi \cdot H + k_g \circ \pi = 0$, the invariance of dvol under $\{f_t\}$, and the Gauss-Bonnet Theorem. #

Continuity yields $\lim_{n\to\infty} 3 \int_M (w \cdot H_n)^2 \, dvol = Def(g)$, so that

$$\lim_{n\to\infty} \int_M \eta_n \wedge d\eta_n = (2\pi)^2 \cdot \chi(S) - Def(g).$$

On the other hand, by the remarks above, (8.10) defines the form $\eta \wedge d\eta$ on M, so that $\int_M \eta \wedge d\eta$ coincides with the definition of gv(g) via the distributional pairing (9.13). In particular, continuity of the pairing implies that $\lim_{n\to\infty} \int_M \eta_n \wedge d\eta_n = \int_M \eta \wedge d\eta$. This proves Proposition 11.2. #

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