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Differentiability, rigidity and Godbillon-Vey classes for Anosov flows.

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This is an excellent paper on volume-preserving Anosov flows on 3-manifolds.

The authors show that the regularity of the weak-stable and weak-unstable foliations of such a flow (of class C^∞) is always $C^{1,\mathrm{Zygmund}}$, i.e., C^1 and the first transverse derivative belongs to the Zygmund class. A real-valued function f is in the Zygmund class if $f(x+h)+f(x-h)-2f(x)=O(h)$. "Zygmund" is weaker than "Lipschitz" but stronger than "α-Hölder" ($\alpha<1$). They also show that, if one of the foliations is of class $C^{1,\mathrm{Lipschitz}}$, then both of the foliations are of class C^∞. (In fact they show it under a weaker hypothesis.) This is shown by defining an obstruction class (Anosov class) for the smoothness in the first cohomology of the reals $\mathbf{R}$ with values in the functions on M considered as an $\mathbf{R}$-module by the action of the flow. (It was recently shown by É. Ghys that if a volume-preserving Anosov flow has C^∞ weak-stable and weak-unstable foliations, then it is C^∞ conjugate to an algebraic Anosov flow after a time change.)

On the other hand, the authors extend the domain of definition of the Godbillon-Vey invariant for codimension 1 foliations originally defined for foliations of class C^2. They show that it is defined as a 3-dimensional cohomology class of the manifold for codimension 1 foliations of $C^{1,\alpha\text{-Hölder}}$ class ($\alpha>\frac{1}{2}$) and that this class is invariant under conjugations of $C^{1,\beta\text{-Hölder}}$ class ($\alpha+\beta>1$). Since the weak-stable and weak-unstable foliations of volume-preserving Anosov flows are sufficiently regular, as is mentioned, there are two well-defined Godbillon-Vey invariants. They are shown to be invariant under transversely absolutely continuous conjugations. In particular, for the geodesic flow of a surface with a metric of strictly negative curvature, the authors apply their theory to show the following: The Godbillon-Vey invariants of the weak-stable and weak-unstable foliations coincide to give rise to the Godbillon-Vey invariant of the geodesic flow (or the metric). This invariant varies continuously and nontrivially. It attains its maximum if and only if the curvature is constant. This is obtained by reformulating a formula of Mitsumatsu.

Reviewed by *Takashi Tsuboi*