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Cyclic cocycles, renormalization and eta-invariants.
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Let M be a compact, oriented, odd-dimensional Riemannian manifold and D an operator of Dirac type on M. Let E be a trivialized Hermitian vector bundle over M equipped with a flat connection with holonomy $\alpha\colon\pi_1(M)\to \mathrm{U}(n)$. The relative η-invariant of D with coefficients in (E,α) is a real number which measures the incompatibility of the trivialization and the flat structure. Now let V be the principal $\mathrm{U}(n)$-bundle of frames of E. V is equipped with a foliation $\mathcal{F}_\alpha$ coming from the flat structure, and this foliation has a transverse measure coming from Haar measure on $\mathrm{U}(n)$. The operator D lifts naturally to a leafwise operator D_α on $\mathcal{F}_\alpha$, and the "Toeplitz" operator, defined by D_α and the unitary multiplier $V\to \mathrm{U}(n)$ coming from the trivialization of E, has a real-valued index. It turns out that the topological formula for this index obtained from Connes' measured foliation index theorem is the same as the topological formula for the relative η-invariant given by Atiyah, Patodi and Singer. The article under review provides an analytic explanation of this "coincidence".

The proof proceeds by relating both analytic invariants to a third one, the transverse index of D_α considered as an operator transverse to the action of $G=\mathrm{U}(n)$ on V. This transverse index (defined by Connes) is an odd cyclic cocycle over the foliation algebra for the G-action, which is itself a module over the convolution algebra of smooth functions on G. The transverse index is related to the leafwise index by a procedure that the authors term "renormalization"; roughly speaking, the Haar measure on G (which is used in the construction of the leafwise index) is represented by the leading term in the asymptotic expansion of the scalar heat kernel on G, and this heat kernel is then multiplied into the transverse index-cocycle. A technical problem in doing this arises from the different notions of "parametrix" appropriate to leafwise and transverse operators.

The transverse index is related to the η-invariant by harmonic analysis on G. Given a character

χ, the ordinary Toeplitz index on the space of χ-isotypical functions can be interpreted as a spectral flow. The authors show that the relative η-invariant can be evaluated as an average spectral flow, where the averaging takes place over an exhaustion sequence for the space $\widehat{G}$ of irreducible representations of G; this uses uniform estimates on the η-invariant due to Cheeger and Gromov. They then show that the renormalized transverse index can be identified with this "average" Toeplitz index, using the fact that the heat kernel on G defines an exhaustion sequence in $\widehat{G}$.

An announcement of the results has previously been published [the authors, Bull. Amer. Math. Soc. (N.S.) **21** (1989), no. 1, 83–87; MR 90g:58128].

Reviewed by *John Roe*