Rigidity for Cartan Actions of Higher Rank Lattices

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1 Introduction

The natural action of the determinant-one, integer $n \times n$ matrices $SL(n, \mathbf{Z})$ on $\mathbf{R}^n$ preserves the integer lattice $\mathbf{Z}^n$, hence for each subgroup $\Gamma \subset SL(n, \mathbf{Z})$ there is an induced "standard action" on the quotient n-torus, $\varphi : \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$. It is a basic question whether for every Anosov action of a higher rank lattice Γ, is there a subgroup of finite index $\Gamma' \subset \Gamma$ so that the restricted action to Γ' is smoothly conjugate to a standard linear action (cf. [27, 28, 6])? Such a conclusion would be a type of non-linear version of the Margulis arithmeticity theorem for lattices.

In this paper the authors prove that *volume-preserving Cartan* actions of a higher rank lattice must be smoothly conjugate to an affine action. We also prove that a perturbation of a volume-preserving Cartan action preserves a smooth volume form, so the hypotheses of our main theorem are stable under perturbation. The standard actions for many subgroups of $SL(n, \mathbf{Z})$ are Cartan (cf. Example 1 below), so these results imply their smooth rigidity under C^1-perturbations.

A finitely-generated group Γ is a said to be a *higher rank lattice* if Γ is a discrete subgroup of a connected semi-simple algebraic $\mathbf{R}$-group G, with the $\mathbf{R}$-split rank of each factor of G at least 2, G has finite center and $G^0_{\mathbf{R}}$ has no compact factors, so that G/Γ has finite volume.

A C^r-action $\varphi : \Gamma \times X \to X$ of a group Γ on a compact manifold X is said to be *Anosov* (cf. [9, 7]) if there exists at least one element, $\gamma_h \in \Gamma$, such that $\varphi(\gamma_h)$ is an Anosov diffeomorphism of X (cf. [1, 23]). We then say that γ is *hyperbolic* for the action. (Recall that for the standard action of a subgroup $\Gamma \subset SL(n, \mathbf{Z})$ on the torus $\mathbf{T}^n$, the action of $\gamma \in SL(n, \mathbf{Z})$ is Anosov exactly when γ is a hyperbolic matrix; that is, one with no eigenvalues on the unit circle.)

A C^r-action $\varphi : \Gamma \times X \to X$ is said to be *Cartan* (cf. [9]) if there exists an abelian subgroup $\mathcal{A} \subset \Gamma$ generated (not necessarily freely!) by $\Delta = \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$, such that each $\varphi(\gamma_i)$ is an Anosov diffeomorphism of X, and the strongest stable foliations of the diffeomeorphisms $\{\varphi(\gamma_1), \ldots, \varphi(\gamma_n)\}$ form a trellis for X. (See section 2 for further details.)

THEOREM 1 *Let $\varphi : \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ be a Cartan C^r-action on the n-torus $\mathbf{T}^n$, for $r = 1, \infty$ or $r = \omega$ in the real analytic case. Suppose that Γ is a higher rank lattice and the subgroup $\mathcal{A} \subset \Gamma$ generated by Δ is a cocompact lattice in a maximal $\mathbf{R}$-split torus in G. If the action φ preserves an absolutely continuous probability measure on $\mathbf{T}^n$, then φ is C^r-conjugate to an affine action.*

Our second result states that the volume-preserving hypothesis is stable under C^1-perturbation, with the weaker hypothesis that the action is Anosov:

THEOREM 2 *Let $\varphi : \Gamma \times X \to X$ be an Anosov C^r-action on a compact n-manifold X without boundary, for $r = 1, \infty$ or ω. Suppose that Γ satisfies Kazhdan's property T, and the action φ preserves an absolutely continuous probability measure on X. Then there exists $\epsilon > 0$ so that if φ_1 is ϵ-C^1-close to φ, then φ_1 also preserves a C^r-volume form on X.*

Proposition 2.17 of [9] proved that the Cartan condition is stable under C^1-perturbations, and D. Stowe's Theorem A, [24] implies that the origin is a stable fixed-point for the standard action of a higher rank lattice. So together with Theorems 1 and 2 we obtain:

COROLLARY 1 *Let $\varphi : \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ be a Cartan C^r-action on the n-torus $\mathbf{T}^n$, for $r = 1, \infty$ or $r = \omega$ in the real analytic case. Suppose that Γ is a higher rank lattice and the subgroup $\mathcal{A} \subset \Gamma$ generated by the Δ is a cocompact lattice in a maximal $\mathbf{R}$-split torus in G. If the action φ preserves an absolutely-continuous probability measure on $\mathbf{T}^n$, then there exists $\epsilon > 0$ so that if φ_1 is ϵ-C^1-close to φ, then φ_1 is C^r-conjugate to the action φ.*

The section 7 of examples in [9] gives an extensive list of higher rank lattice groups whose standard actions on $\mathbf{T}^n$ are Cartan, and examples of affine, non-standard Cartan actions are given in [8]. We extract a short list of these groups to illustrate the corollary.

EXAMPLE 1 *Let $\varphi : \Gamma \times \mathbf{T}^n \to \mathbf{T}^n$ be a standard action, and suppose that either:*

1. $\Gamma \subset SL(n, \mathbf{Z})$ *is a subgroup of finite index for* $n \geq 3$*; or*
2. $\Gamma \subset Sp(n, \mathbf{Z}) \subset SL(2n, \mathbf{Z})$ *is a subgroup of finite index of the group of integer symplectic matrices* $Sp(n, \mathbf{Z})$*, for* $n \geq 2$*; or*
3. $\Gamma \cong \Gamma_0 \times \cdots \Gamma_d \subset SL(n, \mathbf{R})$*, where each factor group* Γ_i *satisfies one of the two above cases, and* Γ *contains a hyperbolic element.*

Then φ is C^r-rigid for $r = 1, \infty$ and for $r = \omega$.

The development of the rigidity theory for lattice actions on tori began with work of R. Zimmer [29], who proved *infinitesimal rigidity* for ergodic actions on locally homogeneous spaces by higher-rank, cocompact lattices (cf. also [27, 28].) The Thesis of J. Lewis [13] showed that for $n \geq 7$, the standard action of $SL(n, \mathbf{Z})$ on $\mathbf{T}^n$ is infinitesimally rigid.

Rigidity of lattice actions under continuous deformation was proved next by S. Hurder in [7, 9]. This work also established that abelian "Cartan" actions with constant exponents are affine (see Theorem 4 below) - a key step in the proof of Theorem 1 above.

A. Katok and J. Lewis proved in [12] that the standard action of a subgroup $\Gamma \subset SL(n, \mathbf{Z})$ of finite index is rigid for real rank at least 3, or $n \geq 4$. Their methods also extend to cover the case of actions of the finite-index symplectic lattices in $Sp(2n, \mathbf{Z})$ for $n \geq 2$.

J. Lewis and R. Zimmer announced in [31], among other rigidity results, that the cocycle super-rigidity theorem ([25], and also Theorem 5.2.5 of [26]) and techniques of Anosov diffeomorphisms yields the C^∞-rigidity of the standard action of $\Gamma \subset SL(n, \mathbf{Z})$ on $\mathbf{T}^n$ for $n \geq 3$.

The principal idea of this paper is that by combining the super-rigidity theorem of Zimmer, with the classification of Cartan actions in terms of their exponents from [9], one obtains an invariant affine structure for the action of a higher rank lattice, and hence a classification of such actions. The greater program is to use the combinations of techniques from differentiable dynamics with the algebraic techniques inherent in super-rigidity theory to produce global rigidity for lattice actions on manifolds. For further results in this direction, see the papers [8, 10, 11].

2 Preliminaries

Let Γ be a finitely-generated group, and choose a set of generators $\{\delta_1, \ldots, \delta_d\}$. Let X be a compact Riemannian manifold of dimension n without boundary. $\varphi : \Gamma \times X \to X$ will denote a C^r-action of Γ on X. We will assume either that $r = 1$ or ∞ for a differentiable action, or set $r = \omega$ if the action is real analytic. All of the results of this paper have counterparts for C^r-actions with $1 < r < \infty$, with an appropriate loss of differentiability due to Sobolev regularity theory. For reasons of exposition we omit discussion of the intermediate differentiability cases.

Recall first the definition of the C^1-topology on the space of C^r-actions on X. Given $\epsilon > 0$, two C^r-actions $\varphi_0, \varphi_1 : \Gamma \times X \to X$ are *ϵ-C^1-close* if for each generator δ_i of Γ, the diffeomorphism $\varphi_0(\delta_i)$ of X is ϵ-close to $\varphi_1(\delta_i)$ in the uniform C^1-topology on maps. The ϵ-C^1-ball about φ is the set of all C^r-actions φ_1 which are ϵ-C^1-close to φ.

Given $\epsilon > 0$, an ϵ-C^1*–perturbation* of φ is a C^r-action $\varphi_1 : \Gamma \times X \to X$ with φ_1 contained in the ϵ-C^1-ball about φ.

An ϵ-C^1-perturbation $\{\varphi_1\}$ of φ is *differentiably trivial* if there is a C^r-diffeomorphism, $H_1 : X \to X$, such that for each $\gamma \in \Gamma$ we have

$$H_1^{-1} \circ \varphi_1(\gamma) \circ H_1 = \varphi(\gamma) \tag{1}$$

When φ is an analytic action, there is the corresponding notion of *analytically trivial* perturbation, where we require that H_1 be an analytic diffeomorphism.

DEFINITION 1 *Let φ be a C^r-action of Γ on X, for $r = 1, \infty$. We say that φ is C^r-*rigid *if there exists $\epsilon > 0$ so that every ϵ-C^1-perturbation of φ is differentiably trivial. If φ is a C^ω-action, then we say it is C^ω-*rigid *if every analytic ϵ-C^1-perturbation of φ is analytically trivial.*

Let G be a locally compact group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation on a separable Hilbert space $\mathcal{H}$. Given $\epsilon > 0$ and a compact $K \subset G$, a unit vector $\xi \in \mathcal{H}$ is (ϵ, K)-invariant if

$$\sup \{||\pi(g)\xi - \xi|| \mid g \in K\} < \epsilon \tag{2}$$

If we fix a compact $K \subset G$ which generates G, then (2) defines the *Fell topology* on the space of unitary representations on the fixed Hilbert space $\mathcal{H}$.

We say that π has almost invariant vectors if, for all (ϵ, K), there exists an (ϵ, K)-invariant unit vector. A locally compact group G has *Property T* if every representation of G which has almost invariant vectors, has a non-zero invariant vector. G is a *Kazhdan group* if it has property T.

We need the following consequence of property T (cf. Proposition 16, page 15 of [4]):

PROPOSITION 1 *Let Γ be a Kazhdan group generated by a finite set Γ_0, and let δ be a number such that $0 < \delta \leq 2$. There exists a number $\epsilon > 0$ with the property: for every unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ which possesses a unit vector ξ that is (ϵ, Γ_0)-invariant, then there exists a $\pi(\Gamma)$-invariant unit vector $\eta \in \mathcal{H}$ such that $||\eta - \xi|| < \delta$.* $\square$

Let us next introduce some special dynamical properties that we will use in our study of group actions in this paper. The first is a special property of hyperbolic elements, which corresponds in the linear case to $\gamma \in SL(n, \mathbf{Z})$ having a unique maximally contracting direction.

DEFINITION 2 *An Anosov diffeomorphism f has a* one-dimensional strongest stable *distribution if there exists a Df-invariant, 1-dimensional vector subbundle $E^{ss} \subset E^-$ which satisfies an exponential dichotomy: that is, there exists*

- *a Finsler on TX,*
- *a continuous splitting of the tangent bundle into Df-invariant subbundles, $TX \cong E^{nss} \oplus E^{ss}$,*
- *constants $\lambda > 1$ and $1 > \epsilon > 0$*

such that for all positive integers m,

$$\begin{aligned} \| D(f^m)(v) \| &> (\lambda - \epsilon)^{-m} \cdot \| v \| ; \quad 0 \neq v \in E^{nss} \\ \| D(f^m)(v) \| &< (\lambda + \epsilon)^{-m} \cdot \| v \| ; \quad 0 \neq v \in E^{ss}. \end{aligned} \tag{3}$$

The strongest stable distribution E^{ss} is necessarily integrable, and the leaves of the resulting foliation $\mathcal{F}^{ss}$ are C^r-immersed 1-dimensional submanifolds (cf. [5, 3]). While the foliation $\mathcal{F}^{ss}$ need not even be C^1, the immersions of its leaves vary Hölder continuously in the C^r-topology on immersions into X (cf. [op cit]).

The 1-dimensional foliations associated to a collection of Anosov diffeomorphisms f with one-dimensional strongest stable distributions, are used to construct a very special additional geometric structure on X, a *trellis*, which is the technical key to obtaining the differentiable regularity results of [9] and the results of this paper.

DEFINITION 3 *Let X be a compact smooth n-manifold without boundary.*
Let $1 \leq r \leq \infty$, or $r = \omega$ for the real analytic case. A C^r-trellis $\mathcal{T}$ *on X is a collection of 1-dimensional, pairwise-transverse foliations $\{\mathcal{F}_i | 1 \leq i \leq n\}$ of X such that*

1. *The tangential distributions have internal direct sum $T\mathcal{F}_1 \oplus \cdots \oplus T\mathcal{F}_n \cong TX$;*
2. *For each $x \in X$ and $1 \leq i \leq n$, the leaf $L_i(x)$ of $\mathcal{F}_i$ through x is a C^r-immersed submanifold of X;*
3. *The C^r-immersions $L_i(x) \hookrightarrow X$ depend* uniformly Hölder *continuously on the basepoint x in the C^r-topology on immersions.*

$\mathcal{T}$ *is a* regular C^r-trellis *if it also satisfies the additional condition:*

4. *Each foliation $\mathcal{F}_i$ is transversally absolutely continuous, with a quasi-invariant transverse volume form that depends smoothly on the leaf coordinates.*

We now recall the precise defintion of a Cartan action (Definition 2.13, [9]):

DEFINITION 4 *Let $\mathcal{A}$ be a free abelian group with a given set of generators $\Delta = \{\gamma_1, \ldots, \gamma_n\}$. (φ, Δ) is a* Cartan C^r-action *on the n-manifold X if:*

- *$\varphi : \mathcal{A} \times X \to X$ a C^r-action on X;*
- *each $\gamma_i \in \Delta$ is φ-hyperbolic and $\varphi(\gamma_i)$ has a 1-dimensional strongest stable foliation $\mathcal{F}_i^{ss}$;*
- *the tangential distributions $E_i^{ss} = T\mathcal{F}_i^{ss}$ are pairwise-transverse with their internal direct sum $E_1^{ss} \oplus \cdots \oplus E_n^{ss} \cong TX$.*

We say that (φ, Δ) is a maximal Cartan action *if φ is a Cartan action, and for each $1 \leq i \leq n$, the* stable *foliation $\mathcal{F}_i$ of the Anosov diffeomorphism $\varphi(\gamma_i)$ is 1-dimensional; hence $\mathcal{F}_i = \mathcal{F}_i^{ss}$.*

We say that (φ, Δ) is an orientable Cartan action *if each trellis foliation $\mathcal{F}_i$ is orientable, and the restricted action of each $\varphi(\gamma)$ on $\mathcal{F}_i$ for $\gamma \in \Delta$ is orientation-preserving.*

We recall some useful properties of a Cartan action.

THEOREM 3 (Theorem 2.16, [9])

1. *For a Cartan C^r-action (φ, Δ), the collection of strongest stable foliations $\mathcal{T} = \{\mathcal{F}_1^{ss}, \ldots, \mathcal{F}_n^{ss}\}$ is a C^r-trellis on X.*
2. *For a maximal Cartan C^r-action (φ, Δ), the collection of stable foliations $\mathcal{T} = \{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ is a regular C^r-trellis on X.*
3. *For a volume-preserving maximal Cartan C^r-action (φ, Δ) with $r \geq 3$, each stable foliation $\mathcal{F}_i$ is transversally $C^{1+\alpha}$ for some $0 < \alpha < 1$.*

For a non-abelian group action, we set:

DEFINITION 5 *Let $\varphi : \Gamma \times X \to X$ be an Anosov C^r-action on a manifold X. We say that φ is a* Cartan action *if there is a subset of* commuting *hyperbolic elements $\Delta = \{\gamma_1, \ldots, \gamma_n\} \subset \Gamma$, which generate an abelian subgroup $\mathcal{A}$, such that the restriction of φ to $\mathcal{A}$ is a Cartan C^r-action on X. We call $(\varphi|\mathcal{A}, \Delta)$ the Cartan subaction for φ.*

The existence of a Cartan subaction for a standard (algebraic) lattice action is easily verified for latices in semi-simple linear subgroups: The theorem of Prasad and Raghunathan [21] implies that every such lattice intersects some maximal **R**-split connected Cartan subgroup in a cocompact abelian lattice. As all such subgroups are conjugate in a semi-simple Lie group G, so it suffices to consider the standard **R**-split diagonal subgroup. Then the Cartan action condition becomes the requirement that the simultaneous eigenvalue decomposition of this subgroup have 1-dimensional eigenspaces, with eigenvalues not on the unit circle.

For a Cartan action φ, let $x \in \Lambda$ be a periodic point, and let $\mathcal{A}_x$ be the isotropy subgroup of x. The linear isotropy representation

$$D_x\varphi : \mathcal{A}_x \to GL(T_xX)$$

has image in a maximal diagonal subgroup. The choice of a trellis $\{\mathcal{F}_i\}$ for the action defines a basis in each tangent space T_xX for which the action is diagonal. Introduce the abelian (multiplicative) diagonal group $\mathbf{R}^+ \oplus \cdots \oplus \mathbf{R}^+$, then we can consider the isotropy representations as homomorphisms $D_x\varphi : \mathcal{A}_x \to \lambda^n$. A Cartan action is said to have *constant exponents* if there exist homomorphisms

$$\lambda_i : \mathcal{A} \to \mathbf{R}^+ \text{ for } 1 \leq i \leq n$$

such that for each $x \in \Lambda$ and $\gamma \in \mathcal{A}_x$,

$$D_x\varphi(\gamma) = \lambda_1(\gamma) \oplus \ldots \oplus \lambda_n(\gamma).$$

The following result is a key ingredient for the proof of Theorem 1:

THEOREM 4 (Theorem 2.20, [9]) *Let (φ, Δ) be a Cartan C^r-action on the n-torus $\mathbf{T}^n$, for $r = 1, \infty$ or ω. If φ has constant exponents, then φ is C^r-conjugate to an affine action of $\mathcal{A}$ on $\mathbf{T}^n$. Hence, there is a subgroup of finite index $\mathcal{A}' \subset \mathcal{A}$ whose action is smoothly conjugate to a standard linear action.*

3 Proofs of Main Theorems

Let us first give the proof of Theorem 2. We can assume without loss that X is an orientable manifold, and each $\varphi(\gamma)$ is orientation preserving.

Ω is an invariant density for the C^1-Anosov diffeomorphism $\varphi(\gamma_h)$, where $\gamma_h \in \Gamma$ is a hyperbolic element. By the Livsic-Sinai Theorem [15], Ω must be a C^1-volume form. The C^∞-regularity theorem (Theorem 1.1, [17]) implies that Ω is a C^∞-volume form for $r = \infty$, and similarly Ω must be C^ω for $r = \omega$ by Theorem 1 of [16].

Let $\mathcal{H}_{\frac{1}{2}}$ denote the Hilbert space of $\frac{1}{2}$-densities on X. The action φ induces a unitary representation $\hat{\varphi} : \Gamma \to \mathcal{U}(\mathcal{H}_{\frac{1}{2}})$, for which the invariant volume form Ω determines a fixed vector $\hat{\Omega} \in \mathcal{H}_{\frac{1}{2}}$. Fix a finite set of generators Γ_0 for Γ, then give the space of unitary representations on $\mathcal{H}_{\frac{1}{2}}$ the Fell topology. The representation $\hat{\varphi}$ depends continuously on φ with respect to the C^1-topology on actions, so an ϵ-C^1-perturbation φ_1 of the action φ determines a nearby representation $\hat{\varphi}_1$. In particular, $\hat{\Omega}$ determines an (ϵ, Γ_0)-invariant vector for an ϵ_1-C^1-perturbation φ_1 of φ. Thus, by Proposition 1 there exists $\epsilon_1 > 0$ so that if φ_1 is ϵ_1-C^1-close to φ, then there is a fixed vector $\hat{\Omega}_1$ for φ_1. By the compactness of X, the vector $\hat{\Omega}_1$determines an invariant absolutely continuous measure Ω_1 for the action φ_1, which we can assume to have mass 1.

Let $\gamma_h \in \Gamma$ be a hyperbolic element for φ. Choose $\epsilon_1 > 0$ sufficiently small so that every ϵ_1-C^1-perturbation φ_1 preserves an absolutely continuous measure by the above discussion, and so that $\varphi_1(\gamma_h)$ is again Anosov. The measure Ω_1 is invariant for the Anosov C^r-diffeomorphism $\varphi_1(\gamma_h)$, so again by the Livsic-Sinai Theorem and the C^r-regularity theory, Ω_1 is C^r . □

The proof of Theorem 1 will follow from a sequence of Propositions and Lemmas. Our first preliminary result combines the Cartan hypotheses with cocycle super-rigidity:

PROPOSITION 2 *Let Γ be isomorphic to a higher rank lattice. Suppose that (φ, Δ) is an orientable Cartan C^1-action of Γ on a Riemannian n-manifold X which preserves a continuous volume-form Ω on X. Then there exist constants $\{\lambda_1^*, \ldots, \lambda_n^*\}$ and Hölder continuous, integrable vector fields $\{\vec{v}_1, \ldots, \vec{v}_n\}$ on X so that the integral curves of $\vec{v}_i$ are the leaves of the trellis foliation $\mathcal{F}_i$ and for each $\gamma_i \in \Delta$,*

$$D\varphi(\gamma_i)(\vec{v}_i) = \lambda_i^* \cdot \vec{v}_i \tag{4}$$

Proof. For each $1 \leq i < n$, choose a unit vector field $\vec{w}_i$ on X tangent to the trellis foliation $\mathcal{F}_i$, hence $\vec{w}_i$ is Hölder continuous. Choose the last vector field, $\vec{w}_n$ tangent to the trellis foliation $\mathcal{F}_n$ so that the collection $\{\vec{w}_1, \ldots, \vec{w}_n\}$ defines a frame field which has fiber-volume one at every point of X. In other words, we obtain a Hölder continuous section $\sigma : X \to P(X)$ of the right principal $SL(n, \mathbf{R})$-bundle of frames with volume 1 in each fiber of TX.

The C^1-action φ induces a continuous action on frames $D\varphi : \Gamma \times P(X) \to P(X)$. Define a bounded continuous cocycle $\alpha :\ \Gamma \times X \to SL(n, \mathbf{R})$ using the cross-section σ and the rule

$$D\varphi(\gamma)(\sigma(x)) = \sigma(\varphi(\gamma)(x)) \cdot \alpha(\gamma, x)^t \tag{5}$$

where $\alpha(\gamma, x)^t$ denotes the transpose of the matrix $\alpha(\gamma, x)$.

LEMMA 1 *For each $\gamma \in \Delta$, the matrix $\alpha(\gamma, x)$ is diagonal for all $x \in X$.*

Proof. The elements of Δ commute, so for each $\gamma \in \Delta$, the action $\varphi(\gamma)$ preserves the trellis foliations $\mathcal{F}_j$. Hence the differentials $D\varphi(\gamma)$ map the vector fields $\vec{w}_j$ into themselves, which implies that the matrix $\alpha(\gamma, x)$ is diagonal. $\square$

The diagonal entries $\{\mu_1(\gamma, x), \ldots, \mu_n(\gamma, x)\}$ of $\alpha(\gamma, x)$ define cocycles over the abelian group $\mathcal{A}$ with values in the multiplicative group of positive real numbers $\mathbf{R}^*$. The Lyapunov exponents of the element $\gamma \in \Gamma$,

$$\lambda_j(\gamma) = \lim_{p\to\infty} \frac{1}{p} \cdot \log(\mu_j(\gamma^p, x))$$

where the limit exists for almost every $x \in X$ (cf. [20, 22]), and is independent of x by the ergodicity of the Anosov map $\varphi(\gamma)$.

The fact that $\mathcal{F}_i$ is the strongest stable foliation of $\varphi(\gamma, x)$ for some $\gamma \in \Delta$ implies the existence of a real constant $\epsilon > 0$ and negative constants $\{\tilde{\lambda}_1^*, \ldots, \tilde{\lambda}_n^*\}$, and an ordering $\{\gamma_1, \ldots, \gamma_n\}$ of the elements of Δ so that satisfy the exponential dichotomies:

$$\begin{aligned} \lambda_i(\gamma_i) &< \tilde{\lambda}_i^* - \epsilon \\ \lambda_j(\gamma_i) &> \tilde{\lambda}_i^* + \epsilon \quad \text{for } j \neq i \end{aligned} \tag{6}$$

The estimates (6) imply the "cocycle Schur's Lemma" for Cartan actions:

LEMMA 2 *Let (φ, Δ) be a Cartan C^1-action as above. Let $x \mapsto \mathbf{E}(x)$ be a measurable field over X of vector subspaces of constant rank ℓ, with each $\mathbf{E}(x) \subset \mathbf{R}^n$, such that*

$$\mathbf{E}(\varphi(\gamma)(x)) = \mathbf{E}(x) \cdot \alpha(\gamma, x) \quad \text{for all } \gamma \in \Delta \text{ and a.e. } x \in X \tag{7}$$

Then there exists $1 \leq i_1 < \cdots < i_\ell \leq n$ so that $\mathbf{E}(x)$ is spanned by the constant basis vectors $\{\vec{e}_{i_1}, \ldots, \vec{e}_{i_\ell}\}$ for almost all $x \in X$.

Proof. Let $\mathcal{G}(n, \ell)$ denote the Grassmann manifold of ℓ-planes in $\mathbf{R}^n$. The framing $\{\vec{w}_1, \ldots, \vec{w}_n\}$ determines a trivialization $TX \cong X \times \mathbf{R}^n$, so that the measurable field $x \mapsto \mathbf{E}(x)$ determines a map $\Pi : X \to \mathcal{G}(n, \ell)$. The cocycle $\alpha : \mathcal{A} \times X \to GL(n, \mathbf{R})$ defines an action Φ of $\mathcal{A}$ on the set of measurable maps $Maps(X, \mathcal{G}(n, \ell))$ by

$$\Phi_\alpha : \mathcal{A} \times Maps(X, \mathcal{G}(n, \ell)) \longrightarrow Maps(X, \mathcal{G}(n, \ell)) \tag{8}$$

$$(\gamma \cdot f)(x) = f(\varphi(\gamma)(x)) \cdot \alpha(\gamma, x)^{-1} \tag{9}$$

where we use the natural action of $GL(n, \mathbf{R})$ on $\mathcal{G}(n, \ell))$. The hypotheses of the Lemma imply that Π is invariant for this action.

Lemma 1 and the exponential dichotomies (6) imply that the action of $\Phi_\alpha(\gamma_i)$ on the constant maps in $Maps(X, \mathcal{G}(n, \ell))$ has two closed invariant connected components, whose union contains the ω-limit sets of all orbits: the subvariety of $\mathcal{V}_i \subset \mathcal{G}(n, \ell)$ consisting of ℓ-planes which contain the vector $\vec{e}_i$, and the subvariety $\mathcal{W}_i \subset \mathcal{G}(n, \ell)$ consisting of ℓ-planes in the span of the deleted basis $\{\vec{e}_1, \ldots, \hat{\vec{e}_i}, \ldots, \vec{e}_n\}$.

We now invoke the following elementary observation, which is a much weaker form of one of the key observations by Margulis used in the proof of super-rigidity:

LEMMA 3 *Let (X, μ) be a Borel measure space, equipped with a μ-measure-preserving ergodic action $\phi : \mathbf{Z} \times X \to X$. Let $\alpha : \mathbf{Z} \times X \to GL(n, \mathbf{R})$ be a cocycle, and consider the induced action $\Phi_\alpha : \mathcal{A} \times Maps(X, \mathcal{G}(n, \ell)) \to Maps(X, \mathcal{G}(n, \ell))$. Let $\{V_1, \ldots, V_d\}$ be a finite disjoint collection of closed, invariant, connected subsets of $\mathcal{G}(n, \ell)$ so that the ω-limit sets of all orbits under $\mathbf{Z}$ of the constant maps in $Maps(X, \mathcal{G}(n, \ell))$ take values in the union of these sets. Then for any Φ-invariant map $\Pi \in Maps(X, \mathcal{G}(n, \ell))$, there exists an index $1 \leq j \leq d$ so that the range of Π is almost everywhere contained in V_j.*

Proof. We first assert that the range of the map Π must be everywhere contained in the union $V_1 \cup \cdots \cup V_d$. For let U be an open subset of $\mathcal{G}(n, \ell)$ such that its closure

$$\overline{U} \subset \mathcal{G}(n, \ell) \setminus \{V_1 \cup \cdots \cup V_d\}$$

and suppose there exists a set $Y \subset X$ with positive measure such that $\Pi(Y) \subset U$. Then by ergodicity of the ϕ-action, there exists a μ-conull subset $Y_0 \subset Y$ so that for $x \in Y_0$, there exists a sequence of integers $\{n_1^x, n_2^x, \ldots\}$ tending to infinity such that $\phi(n_i^x)(x) \in Y$ for all $i \geq 1$. Hence, $\Pi(\phi(n_i^x)(x)) \in U$ for all $1 \geq 1$. By the cocycle invariance, this implies that for each $x_0 \in Y_0$, the ω-set of the orbit of the constant map $x \mapsto \Pi(x_0) \in \mathcal{G}(n, \ell)$ contains points in $\overline{U}$, contrary to our assumption.

The space X is decomposed into Borel subsets $X = X_1 \cup \cdots \cup X_d$ so that $\Pi(X_i) \subset V_i$. Each X_i is invariant under ϕ, as the sets V_i are invariant under the action of α. Ergodicity of the action implies there exists an index j such that $X_j \subset X$ is conull. □

The proof of Lemma 2 now follows by applying Lemma 3 to the restriction of the action φ to the $\mathbf{Z}$-actions defined by the powers of the elements γ_i. That is, we take $\phi(n, x) = \varphi(\gamma_i^n, x)$, and note that the ω-limit sets of the orbits of this action are contained in the disjoint subsets $\mathcal{V}_i$ and $\mathcal{W}_i$. □

Note that for each hyperbolic element γ_h and for any finite covering $X' \to X$, the lift of the action of $\varphi(\gamma_h)$ is still Anosov, and hence ergodic. In particular, the full action of Γ' on X' is ergodic for any subgroup $\Gamma' \subset \Gamma$ of finite index which lifts to an action on X'.

Let $H(\alpha) \subset SL(n, \mathbf{R})$ be the algebraic hull of α [26]. Thus, there is a measurable framing $\tilde{\sigma} : X \to P(X)$ so that the action $D\varphi$ with respect to the framing $\tilde{\sigma}$ defines a measurable cocycle $\tilde{\alpha} : \Gamma \times X \to H(\alpha)$; but, there is no measurable framing such that $D\varphi$ takes values almost everywhere in a proper algebraic subgroup of $H(\alpha)$. Introduce the measurable coboundary $\tilde{g} : X \to SL(n, \mathbf{R})$ defined by

$$\tilde{\sigma}(x) = \sigma(x) \cdot \tilde{g}(x)^t \text{ for all } x \in X. \tag{10}$$

and note the coboundary law relating $\tilde{\alpha}$ with α

$$\tilde{\alpha}(\gamma, x) = \tilde{g}(x) \cdot \alpha(\gamma, x) \cdot \tilde{g}(\varphi(\gamma)(x))^{-1} \tag{11}$$

By passing to a finite covering of X, again denoted by X, we can assume that the algebraic hull of $\tilde{\alpha} : \Gamma \times X \to SL(n, \mathbf{R})$ is the Zariski connected component H_0 of H (cf. Proposition 9.2.6, [26]).

Write $H_0 = L_0 \ltimes U$ where L_0 is reductive and U is unipotent. The composition of $\tilde{\alpha}$ with projection of H onto $H/([L_0, L_0] \ltimes U)$ is into an abelian algebraic group whose algebraic hull is the whole group. Since Γ has Property T, it follows from Theorem 9.1.1, [26] that $H/([L_0, L_0] \ltimes U)$ is a compact group, and hence the center $Z(L_0)$ is compact. We then write $L_0 = SZ(L_0)$, where S is a center-free connected semi-simple Lie group.

Let $L_1 \times \cdots \times L_d$ be the product decomposition of a finite covering of S into simple factors.

LEMMA 4 *Each factor L_i is a non-compact simple Lie group.*

Proof. Let $\ell \subset \mathbf{sl(n, R)}$ be the reductive Lie subalgebra corresponding to the Lie group $L \subset SL(n, \mathbf{R})$, and let $\ell = \ell_1 \oplus \cdots \oplus \ell_d$ be the direct sum decomposition of ℓ into simple Lie algebra factors, where ℓ_j is the Lie algebra of L_j. There exists an internal direct sum decomposition

$$\mathbf{R}^n \cong \mathbf{R}^{n_1} \oplus \cdots \oplus \mathbf{R}^{n_d} \tag{12}$$

which completely reduces the action of ℓ; that is, each simple summand ℓ_j of ℓ acts irreducibly on $\mathbf{R}^{n_j}$, and the other complementary simple summands of ℓ act trivially on $\mathbf{R}^{n_j}$.

Suppose that L_j is compact, for some $1 \leq j \leq d$. The subspace $\mathbf{R}^{n_j} \subset \mathbf{R}^n$ is invariant under the action of the covering group L_j, hence for a finite extension of the cocycle $\tilde{\alpha}$, the product bundle $X \times \mathbf{R}^{n_j} \to X$ is invariant under the action of $\tilde{\alpha}$. The conjugate of this constant field by the coboundary $\tilde{g}^{-1} : X \to SL(n, \mathbf{R})$ yields a measurable field $\mathbf{E}_j$ satisfying the hypotheses of Lemma 2. Hence, for some $1 \leq k \leq n$ the vector field $\vec{w}_k$ takes values in $\mathbf{E}_j$ almost everywhere on X. Recall that $\vec{w}_k$ is the direction of maximal contraction for the action of $\gamma_k \in \Delta$, and the action of $\varphi(\gamma_k)$ on the vector field $\vec{w}_k$ has Lyapunov exponent $\lambda_k(\gamma_k) < 0$. On the other hand, our assumption that the restriction of $\tilde{\alpha}(\gamma, x)$ to $\mathbf{R}^{n_j}$ takes values in a compact Lie group implies the action $p \mapsto \tilde{\alpha}(\gamma_k^p, x)$ is bounded on the invariant subspace $X \times \mathbf{R}^{n_k}$. This implies that the cocycle $p \mapsto \tilde{\alpha}(\gamma_k^p, x)$ has zero exponent when restricted to the vector field $\vec{w}_k$. The cocycles $p \mapsto \tilde{\alpha}(\gamma_k^p, x)$ and $p \mapsto \alpha(\gamma_k^p, x)$ are cohomologous and tempered when restricted to the subspace $X \times \mathbf{R}^{n_j}$, so must have the same Lyapunov spectrum there, which is a contradiction. □

Let $\beta : \Gamma \times X \to S$ be the composition of $\tilde{\alpha}$ with the projection $H_0 \to S$. Then β has algebraic hull S, which is a semi-simple Lie group without compact factors. Apply cocycle super-rigidity (Theorem 5.2.5, [26]) to each of the simple factors of S to obtain, for a subgroup of finite index $\Gamma' \subset \Gamma$, a rational homomorphism $\rho : \Gamma' \to S$ defined over $\mathbf{R}$ which is cohomologous to β. Note that the resulting coboundary $\tilde{h} : X' \to SL(n, \mathbf{R})$ between β and ρ has range contained in the subgroup

$$SL(n_1, \mathbf{R}) \times \cdots \times SL(n_d, \mathbf{R}) \subset SL(n, \mathbf{R}) \tag{13}$$

which preserves the internal direct sum (12).

Let $\hat{\alpha}$ denote the cocycle obtained from conjugating $\tilde{\alpha}$ by $\tilde{h}$, with formula

$$\hat{\alpha}(\gamma, x) = \tilde{h}(x) \cdot \tilde{\alpha}(\gamma, x) \cdot \tilde{h}(\varphi(\gamma)(x))^{-1}.$$

LEMMA 5 *There is a re-ordering of the decomposition (12) so that with respect to the new basis of* $\mathbf{R}^n$*, the subgroup* U *is contained in the triangular unipotent matrices.*

Proof. U is unipotent, so there is a unique maximal flag of subspaces

$$\{0\} = F_0 \subset F_1 \subset \cdots \subset F_p \subset F_{p+1} = \mathbf{R}^n \tag{14}$$

which is invariant under the right action of U on $\mathbf{R}^n$, and the induced action of U on each quotient F_i/F_{i-1} is isometric [2, 18]. L normalizes U so the right action of L must also preserve this flag. It follows that there is a re-ordering of the decomposition (12) so that

$$F_i = \mathbf{R}^{n_1} \oplus \cdots \oplus \mathbf{R}^{n_i} \tag{15}$$

Next note that the action of L_i on the quotient F_i/F_{i-1} normalizes the induced isometric action of U on this quotient. As L_i is non-compact and acts irreducibly, this implies that U commutes with L_i hence the induced action of U is diagonal. As U is unipotent and connected, it must act trivially on the quotient F_i/F_{i-1}, which implies the lemma. □

Re-order the vector fields $\{\vec{w}_1, \ldots, \vec{w}_n\}$ so that $\{\vec{w}_1 \cdot \tilde{g}^t, \ldots, \vec{w}_{n_1} \cdot \tilde{g}^t\}$ spans F_1, $\{\vec{w}_1 \cdot \tilde{g}^t, \ldots, \vec{w}_{n_1+n_2} \cdot \tilde{g}^t\}$ spans F_2, and so forth.

In the formation of ρ, we passed to repeated extensions for the action of Γ on X, resulting finally in an action $\varphi' : \Gamma' \times X' \to X'$ where $X' \to X$ is a finite Galois (normal) covering, and $\Gamma' \subset \Gamma$ is a finite index subgroup. Let $\alpha' : \Gamma' \times X' \to SL(n, \mathbf{R})$ be the corresponding extension of the cocycle α. For each element $\gamma_i \in \Delta$, there is an exponent $p_i > 0$ so that $\gamma_i^{p_i} \in \Gamma'$. The continuous field of matrices $\alpha(\gamma_i, x)$ has a 1-dimensional maximally contracting distribution spanned by the vector field $\vec{w}_i$, hence $\alpha'(\gamma_i^{p_i}, x)$ also has this property for the lifted vector field $\vec{w}'_i$. Let $\vec{u}_i = \vec{w}'_i \cdot \tilde{g}^t \cdot \tilde{h}^t$ the vector field obtained by transforming $\vec{w}'_i$ first with $\tilde{g}$ and then with $\tilde{h}$.

LEMMA 6 *For each* $1 \leq i \leq n$,

$$D\varphi(\gamma_i^{p_i}, x)(\vec{u}_i(x)) = \mu(\gamma_i^{p_i}, x) \cdot \vec{u}_i(\varphi(\gamma_i^{p_i})(x)). \tag{16}$$

Proof.

$$\begin{aligned}
&D\varphi(\gamma_i^{p_i}, x)(\vec{u}_i(x)) \\
&= \vec{u}_i(\varphi(\gamma_i^{p_i})(x)) \cdot \hat{\alpha}(\gamma_i^{p_i}, x)^t \\
&= \left(\vec{w}_i'(\varphi(\gamma_i^{p_i})(x)) \cdot \tilde{g}(\varphi(\gamma_i^{p_i})(x))^t \cdot \tilde{h}(\varphi(\gamma_i^{p_i})(x))^t\right) \cdot \\
&\quad \left(\tilde{h}^t(\varphi(\gamma)(x))^{-1} \cdot \tilde{g}^t(\varphi(\gamma_i^{p_i})(x))^{-1}\alpha(\gamma, x) \cdot \tilde{g}(x)^t \cdot \tilde{h}(x)^t\right) \\
&= \vec{w}_i'(\varphi(\gamma_i^{p_i})(x)) \cdot \alpha(\gamma, x) \cdot \tilde{g}(x)^t \cdot \tilde{h}(x)^t \\
&= \mu(\gamma_i^{p_i}, x) \cdot \vec{w}_i'(\varphi(\gamma_i^{p_i})(x)) \cdot \tilde{g}(x)^t \cdot \tilde{h}(x)^t \\
&= \mu(\gamma_i^{p_i}, x) \cdot \vec{u}_i(\varphi(\gamma_i^{p_i})(x)) \quad \Box
\end{aligned}$$

Express the matrix $\hat{\alpha}(\gamma_i^{p_i}, x) = \rho(\gamma_i^{p_i}) \cdot n(\gamma_i^{p_i}, x)$ in terms of the semi-direct product $L_0 \ltimes U$.

LEMMA 7 *For each* $1 \leq i \leq n$,

$$\vec{u}_i(\varphi(\gamma_i^{p_i})(x)) \cdot n(\gamma_i^{p_i}, x)^t = \vec{u}_i(\varphi(\gamma_i^{p_i})(x)) \tag{17}$$

Proof. Each vector field $\vec{w}_i'$ takes values in a fixed subspace $\mathbf{E}_i = \mathbf{E}^{n_{j_i}}$ for some j_i, and by our choices of coboundaries $\tilde{g}$ and $\tilde{h}$, the field $\vec{u}_i$ also takes values in $\mathbf{E}_i$, and hence in the flag subspace F_{j_i}. The matrix $n(\gamma_i^{p_i}, x)$ acts as the identity on each quotient of the flag spaces (14), so that

$$\vec{u}_i(\varphi(\gamma_i^{p_i})(x)) \cdot n(\gamma_i^{p_i}, x)^t = \vec{u}_i(\varphi(\gamma_i^{p_i})(x)) + \vec{\epsilon}_i(\varphi(\gamma_i^{p_i})(x))$$

where $\vec{\epsilon}_i$ is a vector field with values in the flag subspace F_{j_i-1}. Suppose that $\vec{\epsilon}_i(\varphi(\gamma_i^{p_i})(x)) \neq 0$, then

$$\vec{\epsilon}_i(\varphi(\gamma_i^{p_i})(x)) \cdot \rho(\gamma_i^{p_i}) \in F_{j_i-1}$$

also is not zero. But this contradicts Lemma 6, so $\vec{\epsilon}_i = 0$ which implies (17). $\Box$

Combine Lemmas 6 and 7 to obtain

COROLLARY 2 *For each* $1 \leq i \leq n$, *there exists a constant* λ_i^* *such that* $\mu(\gamma_i^{p_i}, x) = (\lambda_i^*)^{p_i}$. $\Box$

We have now established that $\vec{u}_i$ is a measurable vector field on X' which has a constant exponent of expansion under the action of the volume-preserving Anosov diffeomorphism $\varphi(\gamma^{p_i})$. The Lyapunov spectrum of a tempered cocycle over $p \mapsto \varphi(\gamma^p)$ is independent of cohomology, as long as the resulting cocycle is tempered. Applying this remark to the restriction of $D\varphi(\gamma^{p_i})$ to the spans of $\vec{w}_i'$ and $\vec{u}_i$, we see that λ_i^* must correspond to the exponent of maximal contraction for $\varphi(\gamma^{p_i})$, and hence its direction field is almost everywhere equal to that of $\vec{w}_i'$. That is, we conclude the dynamically obvious fact that $\vec{u}_i$ is almost everywhere tangent to the lifted foliation $\mathcal{F}_i'$ on X'.

In particular, there is a measurable function $c_i : X' \to \mathbf{R}$ so that

$$\vec{u}_i(x) = \exp\{c_i(x)\} \cdot \vec{w}_i'(x) \quad \text{a.e. } x \in X. \tag{18}$$

LEMMA 8 *There exists a Hölder continuous function $\tilde{c}_i : X' \to \mathbf{R}$ so that $c_i(x) = \tilde{c}_i(x)$ for almost every $x \in X'$.*

Proof. Form the cocycle equation

$$F_i(\varphi(\gamma_i^p)(x)) - F_i(x) = \lambda_i(\gamma_i^p, x) - \lambda_i^* \tag{19}$$

over the action $\mathbf{Z} \times X \to X$ induced from the Anosov diffeomorphism $\varphi(\gamma_i^{p_i})$. The vector field $\vec{w}_i'$ is Hölder continuous on X' so the exponent cocycle λ_i is also Hölder continuous.

Corollary 2 and Equation (18) imply that the measurable function c_i is solves the cohomology equation (19). The measurable Livsic Theorem (Theorem 9, page 1298 of [14]) implies that there then exists a unique Hölder continuous function F_i which solves (19), and that any measurable solution of (19) equals a constant multiple of F_i for a.e. $x \in X$. Thus, there exists a constant K_i so that $c_i(x) = F_i(x) + K_i$ for almost every $x \in X'$. We set $\tilde{c}_i = F_i + K_i$. □

By Lemma 6, Corollary 2 and Lemma 8 we obtain a collection of Hölder vector fields $\{\vec{u}_1, \ldots, \vec{u}_n\}$ on the covering X' which are mapped to constant multiples of themselves under the action of the Anosov diffeomorphisms $\{\varphi(\gamma_i^{p_i}), \ldots, \varphi(\gamma_n^{p_n})\}$. Let $\mathcal{G}$ be the Galois group of the covering $X' \to X$. The action of $\mathcal{G}$ preserves the lifted foliations $\mathcal{F}_i'$ and their orientations, and commutes with the lifts of the action φ, so we can average each vector field $\vec{u}_i$ over $\mathcal{G}$ to obtain a $\mathcal{G}$-invariant vector field $\vec{v}_i'$ on X' which satisfies $D\varphi(\gamma_i^{p_i})(\vec{v}_i') = \lambda_i^* \cdot \vec{v}_i'$ for $1 \leq i \leq n$. The vector fields $\vec{v}_i'$ descend to vector fields $\vec{v}_i$ on X which then satisfy the conclusion of Proposition 2. □

The proof of Proposition 2 has essentially linearized the action of the abelian subgroup $\mathcal{A}$, and almost linearized the full action of the lattice Γ, modulo the unipotent factor U. The next result uses the full force of the hypotheses of Theorem 1 to eliminate the ambiguity posed by the unipotent radical.

PROPOSITION 3 *The algebraic hull $H(\alpha) \subset SL(n, \mathbf{R})$ of $\alpha : \Gamma \times X \to SL(n, \mathbf{R})$ is reductive. That is, the unipotent factor U in the semi-direct product $H = L \ltimes U$ is the trivial group.*

Proof. Proposition 2 implies that the cocycle action of the abelian subgroup $\mathcal{A}$ generated by Δ can be diagonalized by a coboundary taking values in the subgroup $L \subset H(\alpha)$. That is, the algebraic hull of the restricted action $\varphi|\mathcal{A}$ is a diagonal subgroup of L. The claim of the proposition will then follow from the following result extracted from the techniques of the paper [30]:

PROPOSITION 4 *Let Γ be a higher rank lattice in a semi-simple connected Lie group G, with $\varphi : \Gamma \times X \to X$ a volume-preserving Anosov action on a compact manifold X. Suppose that $\beta : \Gamma \times X \to H$ is a cocycle over φ, where H is a real algebraic group. Suppose also that there exists an abelian subgroup $\mathcal{A} \subset \Gamma$ which is a cocompact lattice in a maximal $\mathbf{R}$-split torus in G, so that the restriction of β to the action $\varphi : \mathcal{A} \times X \to H$ takes values in a semi-simple abelian subgroup of H. Then the algebraic hull $H(\beta)$ of β is reductive.*

Proof. Induce the action of $\mathcal{A}$ on X up to an action of $\mathbf{R}^n$ by forming the left suspension action

$$\varphi^G : G \times (G \times X)/\Gamma \longrightarrow (G \times X)/\Gamma$$

Also, consider the induced cocycle

$$\beta^G : G \times (G \times X)/\Gamma \longrightarrow H(\beta)$$

and its restriction to β^G to $\mathbf{R}^n$. The restricted action $\varphi^G|\mathbf{R}^n$ acts ergodicly on $(G \times X)/\Gamma$ by Moore's Ergodicity Theorem (cf. Theorem 2.2.15, [26]), and the algebraic hull of $\beta^G|\mathbf{R}^n$ is again diagonal (cf. proof of Theorem 9.4.14).

As in the proof of Lemma 3.1 of [30], if the unipotent radical $\tilde{U} \subset H(\beta)$ is non-trivial, then we obtain a nontrivial action of a Levi factor $L \subset H(\beta)$ on a vector space associated to $\tilde{U}$. We then note that Lemma 3.3 of [30] applies in the present context, so that this is a contradiction. Hence, the unipotent radical must be trivial, and the algebraic hull of α^G is reductive. This implies in turn that the restricted cocycle $\alpha|\Gamma$ also has reductive algebraic hull equal to L (cf. cocycle restriction technique, proof of Theorem 9.4.14, page 182 [26]), so that U is trivial. □

We deduce Theorem 1 from Propositions 2 and 3. Take $X = \mathbf{T}^n$, and note first that the given invariant absolutely-continuous measure must be a C^r-volume form by the first step in the proof of Theorem 2, where $r = 1, \infty$ or ω.

Let $\varphi|\mathcal{A} : \mathcal{A} \times X \to X$ be the Cartan subaction of the given action φ. There exists a finite covering, of order at most 2^n, so that the lift of the trellis foliations to X' are orientable. There is then a finite-index subgroup $\mathcal{A}' \subset \mathcal{A}$ such that the action $\varphi|\mathcal{A}'$ lifts to X'.

Apply Proposition 2 to obtain a basis $\{\gamma_1, \ldots, \gamma_n\}$ of $\mathcal{A}'$ and continuous vector fields $\{\vec{v}_1, \ldots, \vec{v}_n\}$ on X such that the action of $D\varphi(\gamma_i)$ on $\vec{v}_i$ is expansion by a constant λ_i^*. Thus, the restriction of $\varphi|\mathcal{A}'$ is a Cartan C^r-action by an abelian group, and the action has constant exponents at all periodic orbits. It then follows that the Cartan action $\varphi|\mathcal{A} : \mathcal{A} \times X \to X$ on the base also has constant exponents.

By Theorem 4, there is a C^r-diffeomorphism $\Phi : X \to X$ such that the conjugate abelian Cartan action $\Phi^{-1} \circ \varphi|\mathcal{A} \circ \Phi$ is a linear action of $\mathcal{A}$. Define a new action,

$$\tilde{\varphi} = \Phi^{-1} \circ \varphi \circ \Phi : \Gamma \times X \to X$$

The proof of Theorem 1 is completed by the

LEMMA 9 *$\tilde{\varphi}$ is an affine action.*

Proof. Proposition 3 implies that there exists

- a normal subgroup $\Gamma' \subset \Gamma$ of finite index
- a finite Galois covering $X' \to X$
- a lift $\tilde{\varphi}' : \Gamma' \times X' \to X'$ of the action $\tilde{\varphi}$
- measurable vector fields $\{\vec{w}_1, \ldots, \vec{w}_n\}$ on X'

so that the cocycle $D\tilde{\alpha}$ with respect to the framing $\{\vec{w}_1, \ldots, \vec{w}_n\}$ is given almost everywhere by the representation $\rho : \Gamma' \to L \subset SL(n, \mathbf{R})$. The image of the diagonal subgroup $\mathcal{A}'$ in L has uniquely determined 1-dimensional eigenspaces, so by another application of the Livsic theory and basic linear algebra, the vector fields $\vec{w}_i$ must be the lift of linear vector fields on X. (Recall that the action of $\tilde{\varphi}|\mathcal{A}'$ on X is linear.) Therefore, $D\tilde{\varphi}$ is affine with respect to coordinates lifted from X, and hence the action of $\tilde{\varphi}|\Gamma'$ on X is affine.

It remains to establish that $\tilde{\varphi}(\gamma)$ is affine for arbitrary $\gamma \in \Gamma$. Note that $\varphi(\gamma)$ is a C^1-conjugacy between the linear actions on X of the two abelian subgroups of semi-simple matrices, $\mathcal{A}' \subset \Gamma'$ and $\gamma^{-1}\mathcal{A}'\gamma \subset \Gamma'$. The actions of these groups have unique 1-dimensional strong stable linear foliations, so $\varphi(\gamma)$ must conjugate corresponding foliations. It then follows from Taylor's theorem that $\varphi(\gamma)$ is an affine transformation (cf. proof of Proposition 0, [19]). □

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