A Product Theorem for $\Omega B\Gamma_G$[*][†]

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March 28, 2002

Abstract

This note gives a product decomposition theorem for the space $\Omega B\Gamma_G$ of loops on the classifying space of G-foliations. The proof is based on some observations about the interrelation of G with $F\Gamma_G$, the homotopy fiber of the natural map $\nu : B\Gamma_G \to BG$. Some applications and consequences of the main theorem are given. We make a conjecture, which is confirmed in low codimensions by our results, about the loop space $\Omega B\Gamma_q$ for the classifying space $B\Gamma_q$ of smooth codimansion-q foliations.

[*]to appear in *Topology and its applications*

[†]AMS classification: 57R32, 55R??; Keywords: *classifying space, foliations, loop spaces, characteristics classes*
[‡]Supported in part by NSF Grant 91-03297

1 Main Theorem

A G-foliation $\mathcal{F}$ on a manifold M is a foliation of codimension q whose normal bundle has a G-structure which is invariant under the natural parallelism along the leaves of $\mathcal{F}$, where $G \subset \mathrm{GL}(\mathrm{q}, \mathbf{R})$, (cf. [3, 9]). A special example consists of the *integrable* G -structures, where $\mathcal{F}$ is modeled on a manifold B of dimension q with a G-structure on the tangent bundle TB, and the transverse transition functions of $\mathcal{F}$ preserve this G-structure. Foliations with a transverse G-structure arise in many areas of foliation theory: a Riemannian foliation is an $O(q)$-foliation; a foliation with a closed non-vanishing transverse q-form is an $SL(q, \mathbf{R})$-foliation; a symplectic foliation is an $Sp(q, \mathbf{R})$-foliation.

Let $B\Gamma_G$ denote the classifying space for G-foliations constructed by Haefliger ([3], cf. also Milnor [10]) and let $\nu : B\Gamma_G \to BG$ denote the classifying map for the normal G-vector bundle to the canonical foliated microbundle over $B\Gamma_G$. The homotopy fiber of ν is denoted by $F\Gamma_G$. One of the themes of the "homotopy theory of foliations" is to compare the homotopy types of the spaces $B\Gamma_G$ and BG. Usually, this consists of calculations of the homotopy groups of the fiber $F\Gamma_G$ (cf. [3, 2, 4, 5, 7, 8, 11]). In this note we point out a novel fact about the homotopy type of the loop space of $B\Gamma_G$:

THEOREM 1.1 *Suppose that $F\Gamma_G$ is an N-connected space, and there are subgroups $K_1, \ldots, K_\ell$ of G with $\dim K_j \leq N$ for each $1 \leq j \leq \ell$ such that the map*

$$K_1 \times \cdots \times K_\ell \to G \tag{1}$$

induced by the product of the inclusions, is a homotopy equivalence. Then there is a natural homotopy equivalence

$$\Omega B\Gamma_G \simeq G \times \Omega F\Gamma_G. \tag{2}$$

REMARK 1.2 The decomposition (2) is not an equivalence of H-spaces, if BG is not rationally trivial and $B\Gamma_G$ classifies foliations with transverse differentiability at least C^2. Otherwise, one could "deloop" the equivalence to obtain that $B\Gamma_G \simeq BG \times F\Gamma_G$, which implies that $\nu^* : H^*(BG; \mathbf{Q}) \to H^*(B\Gamma_G; \mathbf{Q})$ is injective. For BG not rationally trivial, the natural map $H^*(BG; \mathbf{R}) \to H^*(BGL(q, \mathbf{R}); \mathbf{R})$ is onto some Pontrjagin class $P \in H^{4k}(BGL(q, \mathbf{R}))$, so by naturality all of the powers P^n would be non-trivial in $H^{4kn}(B\Gamma_G; \mathbf{R})$. This contradicts the Bott Vanishing Theorem [1], which implies that the Pontrjagin classes for the normal bundle to a C^2-foliation vanishes in degrees greater than twice the codimension.

The obstruction to (2) being an H-space equivalence manifests itself also in the non-trivial action of G on $F\Gamma_G$, which was studied in [6]. This non-trivial action of G on $F\Gamma_G$ is used there to construct (in the homotopy theoretic sense) framed foliated manifolds with non-trivial secondary classes. □

The usefulness of Theorem 1.1 derives from the fact that $F\Gamma_G$ is always $(q-1)$-connected [5], and further that $F\Gamma_q \equiv F\Gamma_{\mathrm{GL}(\mathrm{q},\mathbf{R})}$ is $(q+1)$-connected [2, 11]. We will give several applications of Theorem 1.1 in the next section. Let us state here two general consequences:

COROLLARY 1.3 *Let $G \subset GL(q, \mathbf{R})$ be a Lie group such that the hypotheses of Theorem 1.1 hold. Then for all $m > 0$ there are split epimorphisms:*

$$\nu_{\#} : \pi_m(B\Gamma_G) \to \pi_m(BG). \qquad \square$$

COROLLARY 1.4 *Let $G \subset GL(q, \mathbf{R})$ be a Lie group such that the hypotheses of Theorem 1.1 hold. Suppose that M is a space with the homotopy type of a suspension of a finite CW complex X. Then every map $f : M \to BG$ lifts to a map $\tilde{f} : M \to B\Gamma_G$. If, in addition, M is an open manifold, then every q-plane field $\mathbf{E} \subset TM$ with a G-reduction (on the bundle of frames of $\mathbf{E}$), is homotopic to the normal bundle of a codimension-q G-foliation on M.*

Proof. We are given that $M \simeq \Sigma X$. Form the adjoint $f^* : X \to \Omega BG$ of f. Theorem 1.1 implies that there is a lift $\tilde{f^*} : X \to \Omega B\Gamma_G$, whose adjoint defines the lift $\tilde{f}$. The second part of the corollary follows from the lifting property above and Haefliger's general theory on the existence of G-foliations [3]. $\square$

2 Proof of Main Theorem

Consider the sequence of fibrations:

$$\Omega F\Gamma_G \to \Omega B\Gamma_G \stackrel{\Omega\nu}{\to} \Omega BG \stackrel{\delta}{\to} F\Gamma_G \to B\Gamma_G \stackrel{\nu}{\to} BG. \tag{3}$$

Given the assumptions of Theorem 1.1, we show that the map $\delta : \Omega BG \to F\Gamma_G$ is homotopic to a constant. The fibration $\Omega\nu : \Omega B\Gamma_G \to \Omega BG$ is then trivial, as it is the pull-back under δ of the path fibration $\mathcal{P}F\Gamma_G \to F\Gamma_G$. This implies that $\Omega B\Gamma_G \simeq \Omega BG \times \Omega F\Gamma_G$.

We work with pointed CW complexes. The base-point for a Lie group will always be the identity element. Let CX denote the pointed cone on the pointed space X. Let $K_1, \ldots, K_\ell$ be the given subgroups of G, and let $K_1 \times \cdots \times K_\ell \to G$ be induced from the group multiplication applied to the inclusion maps on each of the factors. By the connectivity assumption on $F\Gamma_G$, the composition $K_1 \subset G \stackrel{\delta}{\to} F\Gamma_G$ is homotopic to the constant map to the base-point in $F\Gamma_G$. If $K_1 \simeq G$, then we are done.

For $\ell \geq 2$, first extend the inclusion of K_1 to a map on the cone, $CK_1 \to F\Gamma_G$. We then require a well-known result:

LEMMA 2.1 *The fibration $F\Gamma_G \to B\Gamma_G$ has a natural fiber-preserving action (up to homotopy) of the H-space G. In particular, there is a canonical (up to homotopy) action*

$$G \times F\Gamma_G \to F\Gamma_G. \tag{4}$$

Proof. Our definition of $F\Gamma_G$ as the homotopy fiber of ν endows it with the action of the H-space ΩBG, via the Puppe sequence for fibrations (cf. [12]). We observe that the inclusion $G \subset \Omega BG$ induces a homotopy equivalence of H-spaces, which then defines the H-space action on the fibers $F\Gamma_G$. $\square$

Now use (4) to define an extension of the map on the first two factors,

$$CK_1 \times K_2 \to F\Gamma_G. \tag{5}$$

The composition $K_2 \subset G \to F\Gamma_G$ is contractible, so the extension (5) is also homotopic to a constant map onto the base-point of $F\Gamma_G$. We can therefore extend it to a map $C(CK_1 \times K_2) \to F\Gamma_G$.

Continuing in the above manner, we obtain a map

$$K_1 \times \cdots \times K_\ell \subset C(C \cdots (CK_1 \times K_2) \cdots \times K_\ell) \to F\Gamma_G \tag{6}$$

which extends the map $K_1 \times \cdots \times K_\ell \subset G \to F\Gamma_G$. The composition is homotopic to a constant as the middle space in (6) is contractible, so that $G \to F\Gamma_G$ is also homotopic to a constant, as was to be shown. □

3 Applications

The first application of Theorem 1.1 is to the classifying space $B\Gamma_q^+ \equiv B\Gamma_{\mathrm{GL^+(q,\mathbf{R})}}$ of codimension-q smooth foliations with orientable normal bundles.

PROPOSITION 3.1 *For $q \leq 4$, $\Omega B\Gamma_q^+ \simeq SO(q) \times \Omega F\Gamma_q$.*

Proof. The space $F\Gamma_q \equiv F\Gamma_{GL(q,\mathbf{R})}$ is $(q+1)$-connected [2, 11], as noted above, and the special orthogonal group $SO(q) \simeq GL^+(q,\mathbf{R})$ has dimension at most q for $q \leq 3$, so we can apply Theorem 1.1. For $q = 4$, we note that $GL^+(4,\mathbf{R}) \simeq SO(4) \cong S^3 \times SO(3)$, where each factor has dimension 3, and observe that $F\Gamma_4$ is 5-connected. □

Let us apply Corollary 1.3 in this case, noting that $SO(3)$ is doubly covered by S^3 and $SO(4)$ is doubly covered by $S^3 \times S^3$:

COROLLARY 3.2 *For all $m \geq 1$, there are split surjections:*

$$\begin{array}{rcl} \pi_{m+1}(B\Gamma_3) & \to & \pi_m(S^3) \\ \pi_{m+1}(B\Gamma_4) & \to & \pi_m(S^3 \times S^3) \end{array}$$

Proposition 3.1 and Corollary 1.4 also have applications to the existence of foliations:

COROLLARY 3.3 *Let M^n be an open n-manifold with the homotopy type of a suspension. Then every m-plane field $E \subset TM$ of codimension $q = (n - m) \leq 4$ is homotopic to an integrable distribution on M, and hence to the tangent field of a foliation $\mathcal{F}$ of codimension q on M.*

We speculate that the conclusion of Proposition 3.1 is true in all dimensions:

CONJECTURE 3.4 *$\Omega B\Gamma_q^+ \simeq SO(q) \times \Omega F\Gamma_q$ for all $q \geq 1$.*

An equivalent formulation of the Conjecture is to ask whether there exists a lifting $\tilde{g}$ of the map g in the diagram:

$$\begin{array}{ccc} & & B\Gamma_q^+ \\ & \overset{\tilde{g}}{\nearrow} & \downarrow \nu \\ \Sigma SO(q) & \overset{g}{\longrightarrow} & BSO(q) \end{array}$$

where g is the adjoint of the natural map $SO(q) \to \Omega BSO(q)$. One could hope to exhibit such a lift $\tilde{g}$ by a direct geometric construction.

The second example where we apply Theorem 1.1 is to Riemannian foliations, which are those foliations with a "transverse Riemannian metric" which is invariant under the natural transverse parallelism (or linear holonomy). We assume the foliation is transversally oriented, so that these are the G-foliations with $G = SO(q)$. The classifying space $F\Gamma_{SO(q)}$ is then $(q-1)$-connected, by a theorem of the author [5].

The group $SO(2) \cong S^1$ has dimension 2, so Theorem 1.1 yields

$$\Omega B\Gamma_{SO(2)} \simeq S^1 \times \Omega F\Gamma_{SO(2)}. \tag{7}$$

To further understand the homotopy type of $B\Gamma_{SO(2)}$ requires a better understanding of the space $F\Gamma_{SO(2)}$. It is known that the volume form associated with the transverse $SO(2)$-structure induces a fibration,

$$\mathbf{Vol} : F\Gamma_{SO(2)} \to K(\mathbf{R}, 2) \tag{8}$$

where $K(\mathbf{R}, 2)$ denotes the Eilenberg-MacLane space in dimension 2 for the group $\mathbf{R}$ (cf. [7]). It is unknown whether the fiber $\widehat{F\Gamma}_{SO(2)}$ of $\mathbf{Vol}$ is 2-connected. However, $\pi_3(F\Gamma_{SO(2)})$ is highly non-trivial, as it has torsion subgroups which are not finitely generated, and also there are uncountably-generated free $\mathbf{Z}$-summands [5].

The decomposition as in (7) is not valid for $\Omega B\Gamma_{SO(3)}$, as a key step in the proof of Theorem 1.1 fails:

PROPOSITION 3.5 *The map*

$$SO(q) \simeq \Omega BSO(q) \xrightarrow{\delta} F\Gamma_{SO(q)} \tag{9}$$

is homologically essential for $q \geq 3$.

Proof. For $q = 2k+1$, let $\mathbf{S}^{4k-1} \to SO(2k+1)$ be an essential map in real homology. This class is detected by the transgression of the Pontrjagin class $P_k \in H^{4k}(BSO(2k+1); \mathbf{R})$ to $\hat{P} \in H^{4k-1}(SO(4k-1) : \mathbf{R})$. The composition α given by

$$\mathbf{S}^{4k-1} \to SO(2k+1) \simeq \Omega BSO(q) \xrightarrow{\delta} F\Gamma_{SO(q)}$$

determines the foliated microbundle $S^{4k-1} \times \mathbf{R}^{2k+1}$ with the product foliation and the framing twisted by the map $\mathbf{S}^{4k-1} \to SO(2k+1)$. The method of [6] shows that α determines a non-trivial class in $H^{4k-1}(F\Gamma_{SO(2k+1)}; \mathbf{R})$, which proves the proposition in this case.

For $q = 2k+2$, the transgressed class $\hat{P}$ still defines a characteristic class in $H^{4k-1}(F\Gamma_{SO(2k+2)}; \mathbf{R})$ as $4k$ exceeds the codimension for $k > 1$, so the inclusion of the cycle α into the classifying space of one higher codimension is still homologically essential. □

References

[1] R. Bott. On a topological obstruction to integrability. In *Proc. Symp. Pure Math.*, volume 10, pages 127–131. Amer. Math. Soc., Providence, R.I., 1970.

[2] A. Haefliger. Feuilletages sur les variétés ouvertes. *Topology*, pages 183–194, 1970.

[3] A. Haefliger. Homotopy and integrability. pages 133–166, New York and Berlin, 1971. Springer-Verlag. Lecture Notes in Math. vol. 197.

[4] S. Hurder. Dual homotopy invariants for G-foliations. *Topology*, 20:365–387, 1980.

[5] S. Hurder. On the homotopy and cohomology of the classifying space of Riemannian foliations. *Proc. Amer. Math. Soc.*, 81:485–489, 1981.

[6] S. Hurder. On the secondary classes of foliations with trivial normal bundles. *Comment. Math. Helv.*, 56:307–327, 1981.

[7] S. Hurder. Global invariants for measured foliations. *Trans. Amer. Math. Soc.*, 280:367–391, 1983.

[8] S. Hurder. On the classifying space of smooth foliations. *Illinois Jour. Math.*, 29:108–133, 1985.

[9] F. Kamber and Ph. Tondeur. G-foliations and their characteristic classes. *Bulletin Amer. Math. Soc.*, 84:1086–1124, 1978.

[10] J. Milnor. Foliated microbundles. *MIT preprint*, 1969.

[11] W. Thurston. Foliations and groups of diffeomorphisms. *Bulletin Amer. Math. Soc.*, 80:304–307, 1974.

[12] G. W. Whitehead. *Elements of Homotopy Theory*. Springer-Verlag, New York and Berlin, 1978. Graduate Text in Mathematics vol. 61.