Pure-point spectrum for foliation geometric operators

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1 Introduction

Let $\mathcal{F}$ be a smooth foliation on a Riemannian manifold V. A smooth differential form on V is said to be *leafwise harmonic* if it is annihilated by the Laplacian acting along the leaves. Let $\mathcal{H}_{\mathcal{F}}$ denote the space of leafwise harmonic smooth forms. When $\mathcal{F}$ is given by the fibers of a fibration $\pi: V \to B$ with compact fibers, then a fiberwise Hodge Theorem implies that $\mathcal{H}_{\mathcal{F}}$ is isomorphic to the space of sections of a vector bundle $H^*(\mathcal{F}) \to B$ with fibers isomorphic to $H^*(L_x; \mathbf{R})$ where $L_x = \pi^{-1}(x)$ is a typical fiber.

If the curvature term of the Weitzenbock formula for the Riemannian metric restricted to the leaves is *non-negative*, then a leafwise harmonic form is also leafwise parallel. This implies that $\mathcal{H}_{\mathcal{F}}$ is finite dimensional when there is a dense leaf. The typical example of this observation is for a linear foliation $\mathcal{F}$ of the n-torus with dense leaves, where a standard calculation shows that $H^*(V, \mathcal{F})$ is finite-dimensional. It has been conjectured that a similar result might hold without assumptions on the curvature of the leaves, if either the foliation is *Riemannian* (that is, V admits a Riemannian metric which is locally projectable along the leaves of $\mathcal{F}$ as in a Riemannian submersion) , or *totally geodesic* (each leaf must be a totally geodesic submanifold of V). Such a result would be the analog of the known finiteness results for the basic and tangential cohomology in ([13, 20, 5]; also, cf. the appendices by V. Sergiescu and G. Cairns in [25].)

The study of leafwise harmonic forms is motivated in part by the question whether the smooth (leafwise) foliated cohomology [16, 27] $H^*(V, \mathcal{F})$ can have infinite-dimension for a Riemannian foliation with dense leaves. By the Hodge Theorem for leafwise L^2 cohomology, $\mathcal{H}_{\mathcal{F}}$ canonically injects into the smooth foliated cohomology $H^*(V, \mathcal{F})$. Thus, if the space of harmonic forms is infinite-dimensional, then the same is true for $H^*(V, \mathcal{F})$.

The aim of this paper is to give the first examples of Riemannian foliations with dense leaves for which $\mathcal{H}_{\mathcal{F}}$ is infinite dimensional. Here is the simplest example:

THEOREM 1.1 *Let M be a compact surface of genus $g \geq 2$ with fundamental group Γ_g. Then for almost all representations $\alpha: \Gamma_g \to \mathbf{T}^q$, the corresponding suspension foliation $\mathcal{F}_\alpha$ has dense leaves and the space of smooth leafwise-harmonic tangential 1-forms $\mathcal{H}^{0,1}$ on M_α is infinite dimensional. Moreover, for each linear closed u-form ψ on $\mathbf{T}^q$, there is an infinite-dimensional space $\psi \wedge \mathcal{H}^{0,1} \subset \mathcal{H}^{u,1}$ of smooth leafwise harmonic $u+1$ forms.*

The proof of this result, given in section 2, involves first constructing leafwise *eigen-distributions* on the ambient manifold V, then using a *diffusion operator* to smooth the eigendistributions.

Our methods are applicable to the more general case of a foliation geometric operator $\mathcal{D}_{\mathcal{F}}: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ acting on the smooth sections of a Hermitian Clifford bundle $\mathbf{E} \to V$. Examples of such include the Euler operator, the signature operator and the Dolbeault operator (cf. section 3, [12]). Assuming that there is a holonomy invariant, transverse measure μ for $\mathcal{F}$, then $\mathcal{D}_{\mathcal{F}}$ admits a unique self-adjoint extension to a densely-defined operator on $L^2(V, \mathbf{E}, \mu)$ and its spectrum $\sigma(\mathcal{D}_{\mathcal{F}}, \mu)$ is a closed subset of the real line. The operator $\mathcal{D}_{\mathcal{F}}$ is not Fredholm in general, so that $\sigma(\mathcal{D}_{\mathcal{F}}, \mu)$ consists of a mixture of pure-point spectrum $\sigma_{pp}(\mathcal{D}_{\mathcal{F}}, \mu)$ corresponding to eigensections in $L^2(V, \mathbf{E}, \mu)$ for $\mathcal{D}_{\mathcal{F}}$, and essential spectrum $\sigma_e(\mathcal{D}_{\mathcal{F}}, \mu)$ represented by approximate eigenvalues. It is interesting to relate these spectral quantities $\sigma_{pp}(\mathcal{D}_{\mathcal{F}}, \mu)$ and $\sigma_e(\mathcal{D}_{\mathcal{F}}, \mu)$ to the geometry of the foliation − the pure-point spectrum roughly corresponds to a transversally isometric part of the foliation, while the continuous spectrum to transverse dynamical mixing (cf. [36]).

PROBLEM 1.2 *Find geometric conditions on a foliation $\mathcal{F}$ which imply that $\sigma_{pp}(\mathcal{D}_{\mathcal{F}}, \mu)$ is non-empty with eigenvalues of infinite multiplicity. Given $\lambda \in \sigma_{pp}(\mathcal{D}_{\mathcal{F}}, \mu)$, when is its eigenspace spanned by smooth eigensections?*

The two steps used in the proof Theorems 1.1 and 2.1 suggests a reformulation of Problem 1.2:

PROBLEM 1.3 *Let $\mathcal{D}_{\mathcal{F}}$ be a foliation geometric operator for a smooth foliation F.*

1. *Given $\lambda \in \mathbf{R}$, find geometric conditions on $\mathcal{F}$ which imply the existence of non-zero distributional section $\xi \in C^{-\infty}(\mathbf{E})$ with $\mathcal{D}_{\mathcal{F}}\xi = \lambda\xi$.*
2. *Given a non-zero eigendistribution $\xi \in C^{-\infty}(\mathbf{E})$ for $\mathcal{D}_{\mathcal{F}}$, find conditions on F which imply that ξ is represented by a smooth section.*

A complete answer to Problem 1.3.2 is possible for a class of Riemannian foliations:

THEOREM 1.4 *Let $\mathcal{F}$ be a totally geodesic Riemannian foliation on V. Then every eigendistribution ξ for the leafwise Laplacian is a sum of smooth eigenforms on V.*

The proof is given in sections 3 and 4.

The first part of Problem 1.3.1 is the more difficult. In section 5 we develop a general criteria (Corollary 5.3) using the finite-propagation speed methods of Cheeger, Gromov and Taylor [7], which suffices to construct pure-point spectrum of infinite multiplicity. Unfortunately, the hypotheses of Corollary 5.3 are very stringent so that Problem 1.3.1 has to be considered an open question, to which we contribute some partial progress.

Section 6 develops C^*-algebras which apply in great generality to the study of the *essential spectrum* of $\mathcal{D}_{\mathcal{F}}$ acting on $C^{\infty}(\mathbf{E})$. Here is a typical result:

THEOREM 1.5 *Let $\mathcal{F}$ be a smooth, totally geodesic Riemannian foliation of a compact Riemannian manifold V. Suppose that the leafwise Euler operator $\mathcal{D}_L$ admits a λ-eigensection, $\phi \in L^2(\mathcal{A}^{0,v}|L)$, for some leaf L. Then the spectrum of $\mathcal{D}_{\mathcal{F}}$ on $\mathcal{A}^{0,v}$ is completely pure-point, and λ is an essential point.*

We can use the index theorem for foliations to guarantee the existence of harmonic L^2-eigensections along the leaves, so this theorem is very useful for constructing examples with 0 in the essential spectrum (cf. Corollaries 6.12 and 6.13).

The last section recalls some of the interesting examples of Riemannian foliations for which the C^*-methods of section 6 apply. These foliations have very rich dynamics, and our results are just a first step in understanding their very complicated spectral geometry.

Recent joint work of the first author with Gilbert Hector [3] has introduced many new techniques to construct foliations for which the space of leafwise harmonic forms is infinite dimensional.

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2 Pure-point spectrum for suspensions

Set $\Gamma = \mathbf{Z}^q$ for a fixed $q \geq 1$. Let M be a compact Riemannian manifold without boundary, and $\tilde{M}_\Gamma \to M$ a normal covering with Galois group Γ. For a representation $\alpha\colon \Gamma \to \mathbf{T}^q$, form the quotient manifold

$$M_\alpha = \left(\tilde{M}_\Gamma \times \mathbf{T}^q\right) / \left\{(\vec{m}x, \vec{\theta}) \sim (x, \alpha(\vec{m})\vec{\theta}) \text{ for } \vec{m} \in \Gamma\right\} \tag{1}$$

with Riemannian foliation $\mathcal{F}_\alpha$ whose leaves are the images of the sheets $\tilde{M}_\Gamma \times \{\theta\}$. The projection $\tilde{\pi}\colon \tilde{M}_\Gamma \times \mathbf{T}^q \to \tilde{M}_\Gamma$ descends to a fibration $\pi\colon M_\alpha \to M$, and the restriction $\pi|L \to M$ to a leaf L of $\mathcal{F}_\alpha$ is a covering map, whose Galois covering group is a quotient group of Γ.

Let $\mathcal{D}_M\colon C^\infty(\mathbf{E}_M) \to C^\infty(\mathbf{E}_M)$ be a first order geometric operator on $\mathbf{E}_M \to M$. For each leaf L of $\mathcal{F}_\alpha$, there is a natural lift of $\mathcal{D}_M$ to the covering $\pi|L \to M$. These lifts combine to yield a smooth foliation geometric operator $\mathcal{D}_{\mathcal{F}_\alpha}\colon C^\infty(\mathbf{E}_\alpha) \to C^\infty(\mathbf{E}_\alpha)$, where $\mathbf{E}_\alpha = \pi^!\mathbf{E}_M \to M_\alpha$. We also introduce the "universal" covering operator, $\mathcal{D}_\Gamma\colon C_c^\infty(\mathbf{E}_\Gamma) \to C_c^\infty(\mathbf{E}_\Gamma)$ for $\mathbf{E}_\Gamma = \pi_\Gamma^!\mathbf{E}_M$, on compactly supported smooth sections over the covering space $\tilde{M}_\Gamma$.

Both $\mathcal{D}_{\mathcal{F}_\alpha}$ and $\mathcal{D}_\Gamma$ are symmetric operators, and hence are essentially self-adjoint with respect to the Hilbert space inner products defined using the smooth Lebesgue measure. Let $\sigma_{pp}(\mathcal{D}_\Gamma)$ denote the pure-point spectrum of the densely-defined operator $\mathcal{D}_\Gamma$ on $L^2(\mathbf{E}_\Gamma)$ and likewise let $\sigma_{pp}(\mathcal{D}_{\mathcal{F}_\alpha})$ denote the pure-point spectrum of the densely-defined operator $\mathcal{D}_{\mathcal{F}_\alpha}$ on $L^2(\mathbf{E}_\alpha)$.

THEOREM 2.1 *Assume there is given a smooth λ-eigensection $\phi \in L^2(\mathbf{E}_\Gamma)$. Then for almost all representations $\alpha\colon \Gamma \to \mathbf{T}^q$, there is a corresponding infinite-dimensional space $\mathcal{H}_\phi \subset C^\infty(\mathbf{E}_\alpha)$ of λ-eigensections for $\mathcal{D}_{\mathcal{F}_\alpha}$. In particular, there is an inclusion of spectral subsets $\sigma_{pp}(\mathcal{D}_\Gamma) \subset \sigma_{pp}(\mathcal{D}_{\mathcal{F}_\alpha})$.*

The surprising aspect of this theorem is that $\mathcal{F}_\alpha$ has the same holonomy group action on S^1 as the foliation by planes $\mathbf{R}^q$ on $\mathbf{T}^{q+1}$, for which each eigenspace is explicitly one-dimensional for generic α. The underlying idea of Theorem 2.1 which distinguishes it from the planar case is best seen by assuming that $\phi \in L^2(\mathbf{E}_\Gamma)$ is compactly supported. Then ϕ determines a distribution on $\tilde{M}_\alpha$ which has support in a compact subset $K_\phi \subset L_x$ of a leaf of $\mathcal{F}_\alpha$. Each such compact set K_ϕ admits a foliated product neighborhood $K_\phi \subset U_\phi$. The restriction of $C^\infty(\mathbf{E}_\alpha)$ to U_ϕ is a tensor product of leafwise sections with normal sections, and the product structure respects the action of $\mathcal{D}_{\mathcal{F}_\alpha}$. Thus, the compactly supported section generates a "local fibration structure" within the spectral analysis of $\mathcal{D}_{\mathcal{F}_\alpha}$, and this property gives rise to an infinite dimensional space of λ-eigensections for $\mathcal{D}_{\mathcal{F}_\alpha}$ independent of the global holonomy of $\mathcal{F}_\alpha$. (This remark is formalized in section 3.) An eigensection ϕ may not be compactly supported, but we show that for almost all α there is a quotient distribution which generates an infinite dimensional space of λ-eigensections for $\mathcal{D}_{\mathcal{F}_\alpha}$ for essentially the same reason.

Theorem 1.1 of the introduction is easily derived from Theorem 2.1. Take M to be a Riemann surface of genus $g > 1$, with fundamental group Γ_g. A representation $\alpha\colon \Gamma_g \to \mathbf{T}^q$ can be factored $\Gamma_g \to \Gamma \to \mathbf{T}^q$. There are at most countably many homomorphisms $\Gamma_g \to \Gamma$, so it suffices to prove that for a fixed Γ-covering $\tilde{M}_\Gamma \to M$, the conclusion is true for almost every representation $\alpha\colon \Gamma \to \mathbf{T}^q$.

Take $\mathcal{D}_M = d + \delta$ to be the Euler operator on forms on M and let $\mathcal{D}_\Gamma$ be the lift to an operator acting on the smooth L^2 forms on $\tilde{M}_\Gamma$. The Atiyah index theorem for coverings [4] implies that $\mathcal{D}_\Gamma$ has Γ-index equal to the Euler characteristic of M, which is negative. Therefore, the Γ-dimension of the space of harmonic L^2 1-forms on $\tilde{M}_\Gamma$ is non-zero, and so $0 \in \sigma_{pp}(\mathcal{D}_\Gamma)$.

A representation $\alpha\colon \Gamma \to \mathbf{T}^q$ corresponds to a choice of vector $\vec{\theta} = (\theta_1, \ldots, \theta_q)$ with angles $0 \leq \theta_1, \ldots, \theta_q < 2\pi$. For all totally irrational θ, the suspension foliation $\mathcal{F}_\alpha$ has dense leaves. Then by Theorem 2.1, for almost every $\vec{\theta}$ (possibly a proper subset of the irrational θ), the space of smooth leafwise-harmonic 1-forms for the suspension foliation $\mathcal{F}_\alpha$ on M_α is infinite dimensional. $\square$

The proof of Theorem 2.1 occupies the remainder of this section, and has two steps. First, we show that each smooth λ-eigensection $\phi \in L^2(\mathbf{E}_\Gamma)$ gives rise to an infinite-dimensional space of λ-eigensections for the densely-defined operator $\mathcal{D}_{\mathcal{F}_\alpha}$ on $L^2(\mathbf{E}_\alpha)$. The second step is then to show that these L^2-eigensections are smooth on M_α (which is not immediate, for the operator $\mathcal{D}_{\mathcal{F}_\alpha}$ is not elliptic on M_α.)

For each representation $\rho\colon \Gamma \to \mathbf{U(1)}$, introduce the Hermitian flat bundle

$$\mathbf{C}_\rho = \left(\tilde{M}_\Gamma \times \mathbf{C}\right) / \{(\vec{m}\cdot x, z) \sim (x, \rho(\vec{m})z) \ \text{ for } \vec{m} \in \Gamma\}$$

Given $\phi \in L^2(\mathbf{E}_\Gamma)$ define the formal sum

$$\phi_\rho = \sum_{\vec{\ell}\in\Gamma} \rho(-\vec{\ell})\phi(\vec{\ell}\cdot x) \tag{2}$$

If $\phi \in L^2(\mathbf{E}_\Gamma)$ is integrable, then (2) converges for every representation ρ. However, for $\tilde{M}_\Gamma$ non-compact, a typical $\phi \in L^2(\mathbf{E}_\Gamma)$ need not be integrable. The main technical result of this section, Proposition 2.6 below, asserts that the sum in (2) converges for almost every ρ. The key idea is to use that the Fourier transform for abelian groups extends from $L^1(\Gamma)$ to $L^2(\Gamma)$. We need to introduce some preliminary notions first.

Each point $\vec{\theta} \in \mathbf{T}^q$ determines a representation $\rho(\vec{\theta})\colon \Gamma \to \mathbf{U(1)}$, where $\vec{m} = (m_1, \ldots, m_q)$ is mapped to

$$\rho(\vec{\theta})(\vec{m}) = \exp(2\pi i \vec{m}\cdot\vec{\theta}) = \exp\left(2\pi i m_1\theta_1 + \ldots + 2\pi i m_q\theta_q\right). \tag{3}$$

We let $\mathbf{C}_{\rho(\vec{\theta})}$ denote the corresponding flat line bundle over M.

The space $L^2(\Gamma)$ is naturally a module over the group Γ via the translational action on itself. There is a dual action on $C^\infty(\mathbf{T}^q)$: given $g \in C^\infty(\mathbf{T}^q)$ then for $\vec{m} \in \Gamma$ and $\vec{m}\cdot g$, define $\vec{m}\cdot\vec{g}$ to be the function

$$\vec{m}\cdot\vec{g}(\vec{\theta}) = \exp(2\pi i \vec{m}\cdot\vec{\theta})\vec{g}(\vec{\theta}) = \rho(\vec{\theta})(\vec{m})\cdot\vec{g}(\vec{\theta})$$

Combine this action of Γ on $C^\infty(\mathbf{T}^q)$ with the covering translation action on the lifted bundle $\mathbf{E}_\Gamma \to \tilde{M}_\Gamma$ to obtain an action

$$\begin{array}{rcl} \Gamma \times \{\mathbf{E}_\Gamma \otimes C^\infty(\mathbf{T}^q)\} & \to & \{\mathbf{E}_\Gamma \otimes C^\infty(\mathbf{T}^q)\} \\ \vec{m}(s\otimes\vec{g})(x) & = & s(-\vec{m}\cdot x)\otimes \vec{m}\cdot\vec{g} \end{array} \tag{4}$$

Form the quotient Frechet bundle

$$\mathbf{E}_\Gamma \otimes C^\infty(\mathbf{T}^q)/\Gamma \to M$$

then recall an observation from ([23]; cf. also section 7, [12]):

LEMMA 2.2 *For each smooth section $\hat{\phi} \in C^\infty\left(\mathbf{E}_\Gamma \otimes C^\infty(\mathbf{T}^q)/\Gamma\right)$ and $\vec{\theta} \in \mathbf{T}^q$, evaluating $\hat{\phi}$ at $\vec{\theta}$ defines a section $\hat{\phi}_{\vec{\theta}} \in C^\infty(\mathbf{E}_M \otimes \mathbf{C}_{\rho(\vec{\theta})})$.*

Proof: A section $s \in C^\infty(\mathbf{E}_M \otimes \mathbf{C}_{\rho(\vec{\theta})})$ corresponds to a section $\vec{s}\colon \tilde{M}_\Gamma \to \mathbf{E}_\Gamma$ such that

$$\vec{s}(\vec{m}x) = \rho(\vec{\theta})(\vec{m})\vec{s}(x). \tag{5}$$

On the other hand, by (4) the section $\hat{\phi}$ lifts to a section $\vec{\phi}: \tilde{M}_\Gamma \to \mathbf{E}_\Gamma \otimes C^\infty(\mathbf{T}^q)$ so that

$$\vec{\phi}(\vec{m}x)(\vec{\theta}) = \exp(2\pi i \vec{m} \cdot \vec{\theta})\vec{\phi}(x)(\vec{\theta}) \tag{6}$$

If we now fix $\vec{\theta}$ and set $\hat{\phi}_{\vec{\theta}}(x) = \vec{\phi}(x)(\vec{\theta})$, then by (6) $\hat{\phi}_{\vec{\theta}}(x)$ satisfies the defining equation (5) for a section of $C^\infty(\mathbf{E}_M \otimes \mathbf{C}_{\rho(\vec{\theta})})$. $\square$

PROPOSITION 2.3 *Given a countable set $\Phi = \{\phi_1, \phi_2, \ldots\} \subset L^2(\mathbf{E}_\Gamma)$, there exists a set $G_\Phi \subset \mathbf{T}^q$ of full measure such that for every $\vec{\theta} \in G_\Phi$ the sum (2) is well-defined for*

- *each $\phi_i \in \Phi$,*
- *each representation $\rho = \rho(\vec{m}\vec{\theta})$ where $\vec{m} \in \Gamma$ with $m_i \neq 0$ for all $1 \leq i \leq n$,*
- *and almost every $x \in \tilde{M}_\Gamma$.*

Moreover, $(\phi_i)_\rho = \sum_{\vec{\ell} \in \Gamma} \rho(\vec{m}\vec{\theta})(-\vec{\ell})\phi_i(\vec{\ell} \cdot x)$ descends to an L^2-section $\widehat{(\phi_i)}_\rho \in L^2(\mathbf{E}_M \otimes \mathbf{C}_\rho)$.

Proof: Define a map

$$FT: C_c^\infty(\mathbf{E}_\Gamma) \to C^\infty\left(\mathbf{E}_\Gamma \otimes C^\infty(\mathbf{T}^q)/\Gamma\right) \tag{7}$$

which assigns to a compactly supported section ϕ the section $FT(\phi)(x, \vec{\theta}) = \hat{\phi}_{\vec{\theta}}(x)$ defined by (2). $FT(\phi)(x, \vec{\theta})$ defines a smooth section of $(\mathbf{E}_\Gamma \otimes C^\infty(\mathbf{T}^q)/\Gamma)$ by the proof of Lemma 2.2. The coefficients $\rho(-\vec{\ell})$ in (2) are uniformly bounded, so the map (7) extends to $L^1(\mathbf{E}_\Gamma)$.

The right-hand-side of (7) decomposes into a Hilbert space product

$$L^2\left(\mathbf{E}_\Gamma \otimes L^2(\mathbf{T}^q)/\Gamma\right) \cong L^2(\mathbf{E}_M) \hat{\otimes} L^2(\mathbf{T}^q)$$

The map (7) is simply the Fourier transform on the second factor with respect to this Hilbert space product, so extends from $L^1(\mathbf{E}_\Gamma)$ to an L^2-isometric isomorphism (cf. Sunada [33])

$$\begin{aligned} FT: L^2(\mathbf{E}_\Gamma) \cong L^2(\mathbf{E}_M)\hat{\otimes}L^2(\Gamma) \quad &\cong \quad L^2(\mathbf{E}_M)\hat{\otimes}L^2(\mathbf{T}^q) && (8) \\ &\cong \quad L^2(M \times \mathbf{T}^q, \mathbf{E}_M) && (9) \end{aligned}$$

For each $\phi_i \in \Phi$, the section $FT(\phi_i) \in L^2(M \times \mathbf{T}^q, E_M)$ is defined almost everywhere on $M \times \mathbf{T}^q$. The Fubini theorem implies that the set

$$G_i = \{\vec{\theta} \in \mathbf{T}^q \mid (\widehat{\phi}_i)_{\vec{\theta}} \in L^2(\mathbf{E}_M \otimes \mathbf{C}_\theta)\}$$

has full measure. Let $B_i = \mathbf{T}^q \setminus G_i$ be the set of bad angles for ϕ_i and $B_\Phi = \cup_{i=1}^\infty B_i$ the union of all the bad sets, which also has measure zero.

The vector space of rational vectors $\mathbf{Q}^q$ acts on $\mathbf{T}^q$, and we form the countable union

$$\hat{B}_\Phi = \bigcup_{\vec{p} \in \mathbf{Q}^q} \vec{p} \cdot B_\Phi$$

which has measure zero. Then set $G_\Phi = \mathbf{T}^q \setminus \hat{B}_\Phi$. Clearly, for each $\vec{\theta} \in G_\Phi$ the sum $\hat{\phi}_{\vec{\theta}} \in L^2(\mathbf{E}_M \otimes \mathbf{C}_\alpha)$. It remains to check that this property is stable under forming integer multiples of the representation θ. Suppose that $\vec{m}\vec{\theta} \in B_\Phi$ where $\vec{m}^{-1} \in \mathbf{Q}^q$ is defined, and hence $\vec{\theta} \in \vec{m}^{-1}B_\Phi \subset \hat{B}_\Phi$. Thus, for all $\vec{\theta} \in G_\Phi$, $\alpha = \vec{m}\vec{\theta} \notin B_\Phi$ so that the sum $\hat{\phi}_\alpha \in L^2(\mathbf{E}_M \otimes \mathbf{C}_\alpha)$. $\square$

Note that there is no relation between the L^2-norm of $\phi \in L^2(\mathbf{E}_\Gamma)$, and that of $\hat{\phi}_{\vec{\theta}} \in L^2(E_M \otimes \mathbf{C}_{\vec{\theta}})$.

For each $\vec{\theta} \in \mathbf{T}^q$, extend $\mathcal{D}_M$ to an operator $\mathcal{D}_{\vec{\theta}}$ on sections of the twisted bundle $\mathbf{E}_M \otimes \mathbf{C}_{\vec{\theta}} \to M$.

LEMMA 2.4 *Let $\phi \in L^2(\mathbf{E}_\Gamma)$be a λ-eigensection for $\mathcal{D}_\Gamma$. Suppose that for a fixed $\vec{\theta} \in \mathbf{T}^q$, the sum (2)*

$$\sum_{\vec{m}\in\Gamma} \alpha(-\vec{m})\phi(\vec{m}\cdot x)$$

converges for almost every $x \in \tilde{M}_\Gamma$ and descends to an L^2-section $\hat{\phi}_{\vec{\theta}} \in L^2(\mathbf{E}_M \otimes \mathbf{C}_{\vec{\theta}})$. Then $\hat{\phi}_{\vec{\theta}}$ is a smooth λ-eigensection for $\mathcal{D}_{\vec{\theta}}$.

Proof: $\hat{\phi}_{\vec{\theta}}$ is a weak solution of the elliptic equation $\mathcal{D}_{\vec{\theta}}\psi = \lambda\psi$ for $\psi \in L^2(\mathbf{E}_M \otimes \mathbf{C}_{\vec{\theta}})$, hence is smooth by elliptic regularity theory. □

For each $\lambda \in \sigma_{pp}(\mathcal{D}_\Gamma)$ choose an orthonormal basis $\Phi_\lambda \subset L^2(\mathbf{E}_\Gamma)$ for the solutions of $\mathcal{D}_\Gamma\phi = \lambda\phi$. The union of these sets is an orthonormal basis for the pure-point spectrum of $\mathcal{D}_\Gamma$, which we order

$$\Phi = \bigcup_{\lambda\in\sigma_{pp}(\mathcal{D}_\Gamma)} \Phi_\lambda = \{\phi_1, \phi_2, \ldots\}$$

where $\lambda_i \in \sigma_{pp}(\mathcal{D}_\Gamma)$ is the eigenvalue for ϕ_i. Lemma 2.4 shows that each $\phi \in \Phi_\lambda$ is a smooth eigensection $\hat{\phi}_{\vec{\theta}}$ for the operator $\mathcal{D}_{\vec{\theta}}$ on $C^\infty(\mathbf{E}_M \otimes \mathbf{C}_{\vec{\theta}})$ when $\vec{\theta} \in G_{\Phi_\lambda}$.

Each $\vec{\theta} \in \mathbf{T}^q$ also defines a representation $\alpha(\vec{\theta})\colon \Gamma \to \mathbf{T}^q$ by setting

$$\alpha(\vec{\theta})(\vec{m}) = \vec{m}\cdot\vec{\theta} = (m_1\theta_1, \ldots, m_q\theta_q)$$

The next step in the proof of Theorem 2.1 is to embed the smooth eigensections $\hat{\phi}_{\vec{\theta}}$ for the operator $\mathcal{D}_{\vec{\theta}}$ into the pure-point spectrum of $\mathcal{D}_{\mathcal{F}_\alpha}$ acting on $C^\infty(\mathbf{E}_\alpha)$ for $\alpha = \alpha(\vec{\theta})$.

Let $(x, \vec{\theta}) \mapsto x\vec{\theta}$ denote the right action of $\mathbf{T}^q$ on M_α. The bundle $\mathbf{E}_\alpha$ is invariant for this action, so the Hilbert space $L^2(\mathbf{E}_\alpha)$ admits an orthogonal decomposition with respect to the spectrum $\Gamma \cong \hat{\mathbf{T}}^q$ of the group $\mathbf{T}^q$,

$$L^2(\mathbf{E}_\alpha) \cong \bigoplus_{\vec{m}\in\Gamma} L^2(\mathbf{E}_\alpha)^{\vec{m}} \tag{10}$$

where $s \in L^2(\mathbf{E}_\alpha)^{\vec{m}}$ transforms by the rule $s(x\vec{\theta}) = s(x)\cdot\vec{\theta}(\vec{m})$.

Define a correspondence between $C^\infty(\mathbf{E}_M \otimes \mathbf{C}_{\vec{m}\vec{\theta}})$ and $C^\infty(\mathbf{E}_\alpha)^{\vec{m}}$: given a section $s_M \in C^\infty(\mathbf{E}_M \otimes \mathbf{C}_{\vec{m}\vec{\theta}})$, it lifts to s_Γ on $\tilde{M}_\Gamma \times \mathbf{T}^q$ satisfying

$$s_\Gamma(x, \vec{\phi}) = \rho(\vec{m}\vec{\theta})(\vec{\ell})s_\Gamma(x, \vec{\phi}).$$

Multiply s_Γ by the character $\rho(\phi)(\vec{m}) = \exp(2\pi i\vec{m}\cdot\vec{\phi})$ to obtain a section $\tilde{s}_{\vec{m}}$

$$\tilde{s}_{\vec{m}}(x, \vec{\phi}) = s_\Gamma(x)\rho(\phi)(\vec{m}) \tag{11}$$

Note that $\tilde{s}_{\vec{m}}$ is invariant under the action of Γ on $\tilde{M}_\Gamma \times \mathbf{T}^q$:

$$\begin{aligned}
\vec{\ell}\cdot\tilde{s}_{\vec{m}}(x, \vec{\phi}) &= \tilde{s}_{\vec{m}}(-\vec{\ell}\cdot x, \alpha(\vec{\theta})(\vec{\ell})\cdot\vec{\phi}) \\
&= s_\Gamma(-\vec{\ell}\cdot x)\cdot\rho(\alpha(\vec{\theta})(\vec{\ell})\cdot\vec{\phi})(\vec{m}) \\
&= \rho(\vec{m}\vec{\theta})(-\vec{\ell})s_\Gamma(x, \vec{\phi})\rho(\vec{m}\vec{\theta})(\vec{\ell})\rho(\vec{\phi})(\vec{m}) = \tilde{s}_{\vec{m}}(x, \vec{\phi})
\end{aligned}$$

so descends to a section $s_{\vec{m}} \in C^\infty(\mathbf{E}_\alpha)^{\vec{m}}$. The correspondence $s_M \mapsto s_{\vec{m}}$ induces a Hilbert space isomorphism between $L^2(\mathbf{E}_M \otimes \mathbf{C}_{\vec{m}\vec{\theta}})$ and $L^2(\mathbf{E}_\alpha)^{\vec{m}}$, and shows that the smooth sections $C^\infty(\mathbf{E}_\alpha)^{\vec{m}} \subset L^2(\mathbf{E}_\alpha)^{\vec{m}}$ are dense.

LEMMA 2.5 (cf. Proposition 7.5, [12]) *The operator $\mathcal{D}_{\mathcal{F}_\alpha}$ leaves the space $C^\infty(\mathbf{E}_\alpha)^{\vec{m}}$ invariant, and the restriction*

$$\mathcal{D}_{\mathcal{F}_\alpha} : C^\infty(\mathbf{E}_\alpha)^{\vec{m}} \to C^\infty(\mathbf{E}_\alpha)^{\vec{m}}$$

is Hermitian isomorphic to the operator $\mathcal{D}_\rho : C^\infty(\mathbf{E}_M \otimes \mathbf{C}_\rho) \to C^\infty(\mathbf{E}_M \otimes \mathbf{C}_\rho)$ for $\rho = \rho(\vec{m}\vec{\theta})$.

Proof: The lift of $\mathcal{D}_{\mathcal{F}_\alpha}$ to $\tilde{M}_\Gamma \times \mathbf{T}^q$ acts as the operator $\mathcal{D}_\Gamma \otimes \mathrm{Id}$ on $C^\infty(\mathbf{E}_\Gamma) \otimes C^\infty(\mathbf{T}^q)$, and thus the defining equation (11) for sections of $L^2(\mathbf{E}_\alpha)^{\vec{m}}$ is invariant under the action of $\mathcal{D}_\Gamma$. Hence, $C^\infty(\mathbf{E}_\alpha)^{\vec{m}}$ is invariant under $\mathcal{D}_{\mathcal{F}_\alpha}$.

Next, a section $s \in C^\infty(\mathbf{E}_\alpha)^{\vec{m}}$ lifts to $\tilde{s}$ satisfying (11). The action of Γ on $\tilde{M}_\Gamma \times \mathbf{T}^q$ induces a transformation of the lift $\tilde{s}$ given by the equation (5) for the representation determined by $\vec{m}\alpha$. That is, s canonically corresponds to a section of the bundle $\mathbf{E}_M \otimes \mathbf{C}_\rho$. This correspondence implements an isomorphism $C^\infty(\mathbf{E}_\alpha)^{\vec{m}} \cong C^\infty(\mathbf{E}_M \otimes \mathbf{C}_\rho)$.

Finally, we note that the restriction of the lift of $\mathcal{D}_{\mathcal{F}_\alpha}$ to the lifted sections of $C^\infty(\mathbf{E}_\alpha)^{\vec{m}}$ acts as $\mathcal{D}_\Gamma \otimes \mathrm{Id}$, while $\mathcal{D}_\rho$ lifted to $\tilde{M}_\Gamma$ acts as the operator $\mathcal{D}_\Gamma \otimes \mathrm{Id}$ on $C^\infty(\mathbf{E}_\Gamma) \otimes \mathbf{C}$. These two are clearly the same. $\square$

We have now established the following result which implies Theorem 2.1:

PROPOSITION 2.6 *Let $\mathcal{D}_M$ be a geometric operator on $C^\infty(\mathbf{E}_M)$, and let Φ be an orthonormal basis for the pure-point spectrum of $\mathcal{D}_\Gamma$ as above. Then for each $\phi_i \in \Phi$ and $\vec{\theta} \in G_\Phi$ there is a countably infinite orthonormal collection of smooth λ_i-eigensections for $\mathcal{D}_{\mathcal{F}_\alpha}$ acting on $L^2(\mathbf{E}_\alpha)$:*

$$\{\widehat{(\phi_i)}_{\rho(\vec{m}\vec{\theta})} \in C^\infty(\mathbf{E}_M \otimes \mathbf{C}_{\rho(\vec{m}\vec{\theta})}) \cong C^\infty(\mathbf{E}_{\alpha(\vec{\theta})})^{\vec{m}} \subset L^2(\mathbf{E}_{\alpha(\vec{\theta})}) \mid \vec{m} \in \Gamma \text{ with } m_1 \neq 0, \ldots, m_q \neq 0\}$$

There is an alternate proof that the L^2-sections $\widehat{(\phi_i)}_\alpha$ are smooth on M_α. Let $\Delta_{\mathbf{T}^q} : C^\infty(\mathbf{T}^q) \to C^\infty(\mathbf{T}^q)$ denote the Laplacian associated to the bi-invariant Riemannian metric on $\mathbf{T}^q$. The action of $\alpha = \alpha(\vec{\theta})$ on $\mathbf{T}^q$ preserves the metric, so $\Delta_{\mathbf{T}^q}$ induces an operator $\Delta_T : C^\infty(M_\alpha) \to C^\infty(M_\alpha)$ along the fibers of $M_\alpha \to M$. This extends to an operator on $C^\infty(\mathbf{E}_\alpha)$ as the bundle $\mathbf{E}_\alpha$ is the lift of a bundle from M. Note that the leafwise operator $\mathcal{D}_\alpha$ commutes with Δ_T as this is true for their lifts to $\tilde{M}_\Gamma \times \mathbf{T}^q$. Form the elliptic, second order operator $\Delta = \mathcal{D}_\alpha \mathcal{D}_\alpha + \Delta_T$ acting on sections $C^\infty(\mathbf{E}_\alpha)$. The operators $\mathcal{D}_\alpha$ and Δ_T commute, so we can calculate for $\phi_i \in \Phi$ with corresponding $\widehat{(\phi_i)}_{\rho(\vec{m}\vec{\theta})} \in L^2(\mathbf{E}_M)$,

$$\Delta\left(\widehat{(\phi_i)}_{\rho(\vec{m}\vec{\theta})}\right) = \left(\lambda_i^2 + 4\pi^2(m_1^2 + \cdots m_q^2)\right)\widehat{(\phi_i)}_{\rho(\vec{m}\vec{\theta})}$$

Thus, $\widehat{(\phi_i)}_{\rho(\vec{m}\vec{\theta})}$ is an L^2-eigenvector of the elliptic operator Δ so by regularity must be smooth.

The Laplacian approach to smoothness is clearly equivalent to the approach via the Fourier decomposition with respect to the action of $\mathbf{T}^q$, as the Laplacian Δ_T commutes with the fiber action of $\mathbf{T}^q$ on M_α. However, we see in the next section that the Laplacian method generalizes to foliation contexts where there is no group action.

3 Spectral propagation

The distribution space $C^{-k}(\mathbf{E})$ is the topological dual to the space of C^k-sections $C^k(\mathbf{E})$ for $k \geq 0$. A λ-eigendistribution for the leafwise operator $\mathcal{D}_{\mathcal{F}}$ is a distribution $\xi \in C^{-k}(\mathbf{E})$ such that

$$\langle \xi, \mathcal{D}_{\mathcal{F}} \psi \rangle = \lambda \langle \xi, \psi \rangle$$

for all $\psi \in C^{k+1}(\mathbf{E})$. In the proof of Theorem 2.1, the Plancherel theorem was used to produce a 0-eigendistribution $\xi \in L^2(\mathbf{E})$. It was then observed that ξ is an eigendistribution for an elliptic second order operator on $C^\infty(\mathbf{E})$, hence is smooth. In this section, we assume there is given a second-order elliptic differential operator Δ which commutes with $\mathcal{D}_{\mathcal{F}}$ as operators on $C^\infty(\mathbf{E})$, and show there is heat-kernel regularization technique to "diffuse" an eigendistribution into a family of smooth eigensections, exactly analogous to the well-known diffusion process on forms (cf. Sullivan [32]). For example, an L^1-eigensection ξ_L on an individual leaf L of $\mathcal{F}$ determines an eigendistribution, and the diffusion process smears the support of this section to an open set of leaves. *Spectral propagation* refers to this aspect of the diffusion process.

We assume in this section that $\mathcal{F}$ is a smooth foliation of codimension-q on a compact Riemannian manifold V without boundary, and that $\mathcal{F}$ admits a smooth invariant transverse volume form. That is, ν is a smooth closed q-form ν on V whose kernel $ker(\nu)$ is the tangent distribution to $\mathcal{F}$. By a conformal change of the Riemannian metric on V we can assume that ν is the transverse measure associated to the Riemannian metric.

Let $\mathcal{D}_{\mathcal{F}}: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ denote a first order geometric operator along the leaves of $\mathcal{F}$, acting on the smooth sections of the Hermitian bundle $\mathbf{E} \to V$ (cf. section 3, [12]). The operator $\mathcal{D}_{\mathcal{F}}$ is symmetric, and Lemma 2.1 of Chernoff [8] implies that $\mathcal{D}_{\mathcal{F}}$ and each of its powers is essentially self-adjoint. Hence, there is a unique closure to a densely-defined self-adjoint operator on $L^2(\mathbf{E})$. Let $\sigma(\mathcal{D}_{\mathcal{F}}) \subset \mathbf{R}$ denote the spectrum of $\mathcal{D}_{\mathcal{F}}$, and $\sigma_{pp}(\mathcal{D}_{\mathcal{F}})$ the pure-point spectrum.

The main result of this section replaces the geometric hypothesis of Theorem 2.1 that there is a transverse action of $\mathbf{T}^q$ which commutes with the foliation global holonomy, with the analytic hypotheses that there exists a positive second order elliptic operator Δ commuting with the given leafwise operator:

THEOREM 3.1 *Let $\mathcal{F}$ and $\mathcal{D}_{\mathcal{F}}$ be as above, and $\phi \in C^{-k}(\mathbf{E})$ a λ-eigendistribution for $\mathcal{D}_{\mathcal{F}}$. Suppose there exists a symmetric, second order, elliptic geometric operator $\Delta: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ which commutes with $\mathcal{D}_{\mathcal{F}}$. Then ϕ "propagates" to an subspace $\mathcal{H}_\phi \subset C^\infty(\mathbf{E})$ such that each $\psi \in \mathcal{H}_\phi$ is a smooth λ-eigensection for $\mathcal{D}_{\mathcal{F}}$. Moreover, if ϕ is singular then $\mathcal{H}_\phi$ will be an infinite-dimensional subspace.*

For each leaf $L \subset V$, let $\mathbf{E}_L \to L$ denote the restriction of the bundle $\mathbf{E}$ to L and $\mathcal{D}_L: C_c^\infty(\mathbf{E}_L) \to C_c^\infty(\mathbf{E}_L)$ the restriction of $\mathcal{D}_{\mathcal{F}}$ to L, which is is elliptic and essentially self-adjoint (cf. Chapter 5 of [30]).

COROLLARY 3.2 *Let $\mathcal{F}$ and $\mathcal{D}_{\mathcal{F}}$ be as above. Given a leaf L, let $\phi \in L^2(\mathbf{E}_L)$ be an λ-eigensection for $\mathcal{D}_L$ such that $|\phi|$ is integrable on L. Suppose, also, that there exists a symmetric, second order, elliptic geometric operator $\Delta: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ which commutes with $\mathcal{D}_{\mathcal{F}}$. Then ϕ determines an infinite-dimensional subspace $\mathcal{H}_\phi \subset C^\infty(\mathbf{E})$ of smooth λ-eigensections for $\mathcal{D}_{\mathcal{F}}$.*

Proof of Corollary 3.2 Let $\phi \in L^2(\mathbf{E}_L)$ be a λ-eigensection for $\mathcal{D}_L$ which is integrable. Then ϕ determines a distribution $c_\phi: C^\infty(\mathbf{E}) \to \mathbf{C}$ via restriction: for $\psi \in C^\infty(\mathbf{E})$ set

$$c_\phi(\psi) = \int_L \langle \phi, \psi | L \rangle_{\mathbf{E}_L} d\chi_L$$

where $\langle \cdot, \cdot \rangle_{\mathbf{E}_L}$ denotes the Hermitian inner product on $\mathbf{E}_L \to L$ and $d\chi_L$ is the leafwise Riemannian volume form. Self-adjointness of $\mathcal{D}_L$ implies that the distribution c_ϕ is a λ-eigensection for $\mathcal{D}_{\mathcal{F}}$. $\square$

Proof of Theorem 2.1 The operator Δ is elliptic, which implies that $\exp\{-t\Delta\}: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ induces a map on the space of distributions, for all $k \geq 0$

$$\exp\{-t\Delta\}: C^{-k}(\mathbf{E}) \to C^\infty(\mathbf{E}) \tag{12}$$

Thus, for all $t > 0$ we obtain a smooth section $\phi_t = \exp\{-t\Delta\}(c_\phi)$ on V. Since $\mathcal{D}_{\mathcal{F}}$ commutes with $\exp\{-t\Delta\}$ on smooth sections, it is immediate that

LEMMA 3.3 *$\mathcal{D}_{\mathcal{F}}\phi_t = \lambda\phi_t$ for all $t > 0$.* $\square$

We let $\mathcal{H}_\phi \subset C^\infty(\mathbf{E})$ be the subspace spanned by the sections $\{\phi_t \mid t > 0\}$. The proof of Theorem 3.1 is completed by:

LEMMA 3.4 *If ϕ is singular, then $\mathcal{H}_\phi \subset C^\infty(\mathbf{E})$ has infinite dimension.*

Proof: The distribution c_ϕ cannot be written as a finite sum of smooth sections in $C^\infty(\mathbf{E})$ as ϕ is singular. So choose an orthonormal basis $\{\xi_i \mid 1 = 1, 2, \ldots\}$ of $L^2(\mathbf{E})$ consisting of smooth eigensections for Δ. Note that for each $\lambda \in \sigma(\Delta)$ the span of the λ-eigensections has finite dimension, so if λ_i is the eigenvalue for ξ_i, we can assume the sections are ordered so that $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$.

The distribution c_ϕ lies in the Sobolev spaces $W^{-k}(\mathbf{E})$ for $k > \dim(\mathrm{V})$. The set $\{\xi_i\}$ is an orthogonal basis for $W^{-k}(\mathbf{E})$, as Δ is symmetric, though each ξ_i need not have length one in the Sobolev (-k)-norm. We can thus expand the distribution

$$c_\phi = \sum_{i=1}^{\infty} a(\phi)_i \cdot \xi_i$$

where we identify ξ_i with the distribution $\hat{\xi}_i(\psi) = \langle \hat{\xi}_i, \psi \rangle_V$. Use this expansion to calculate

$$\phi_t \quad = \quad \exp\{-t\Delta\}(c_\phi) \quad = \quad \sum_{i=1}^{\infty} a(\phi)_i \exp\{-t\lambda_i\} \cdot \xi_i \tag{13}$$

The set of non-zero coefficients $\{a(\phi)_i \neq 0\}$ is infinite, hence we can choose an infinite subset $\Lambda \subset \{i \mid a(\phi)_i \neq 0\}$ such that $\ell \neq k \in \Lambda$ implies $\lambda_\ell \neq \lambda_k$.

Use (13) to evaluate the linear functionals $\{\hat{\xi}_\ell \mid \ell \in \Lambda\}$ applied to the space $\mathcal{H}_\phi$:

$$\hat{\xi}_\ell(\phi_t) = a(\phi)_\ell \exp\{-t\lambda_\ell\}$$

The collection of functions $\{\exp\{-t\lambda_\ell\} \mid \ell \in \Lambda\}$ spans an infinite dimensional space, as the coefficients in the exponential functions are all distinct. Therefore, the set of distributions $\{\hat{\xi}_\ell \mid \ell \in \Lambda\}$ restricts to an infinite-dimensional space on $\mathcal{H}_\phi$, which must also have infinite-dimension. $\square$

4 Laplacians for totally geodesic Riemaniann foliations

A foliation $\mathcal{F}$ is said to be *totally geodesic* if there exists a smooth Riemannian metric on TV so that each leaf is a totally geodesic submanifold (cf. [5, 19]; and Appendix C by G. Cairns, in [25].) Johnson and Whitt [19] observed that a totally geodesic foliation has the remarkable property that the *leafwise* Riemannian metric tensor g is transversally parallel. That is, for any vector field $\vec{Y}$ everywhere orthogonal to $\mathcal{F}$, the covariant derivative $\nabla_{\vec{Y}} g$ vanishes on any vector tangent to $\mathcal{F}$. This implies the leaves of $\mathcal{F}$ are locally isometric (Proposition 1.4, [19]).

A foliation $\mathcal{F}$ is *Riemannian* if there exists a Riemannian metric on the normal bundle $\mathcal{Q}$ which is invariant under the transverse linear holonomy along the leaves (cf. [25].) We show in this section that the transverse Laplacian for a totally geodesic Riemannian foliation commutes with the leafwise Laplacian on forms. As a consequence, the leafwise signature and Euler operators on such foliations have the spectral propagation property of Theorem 3.1.

Let $\mathcal{Q} \subset TV$ be the orthogonal bundle to the tangent bundle $T\mathcal{F}$, and let $\mathcal{Q}^*$ and $T\mathcal{F}^*$ denote their dual bundles, respectively. The space of smooth differential forms on V has an induced bigrading

$$\mathcal{A}^{u,v} \quad = \quad C^\infty\left(\wedge^u \mathcal{Q}^* \otimes \wedge^v T\mathcal{F}^*\right)$$

where $\mathcal{A}^{0,*}$ is the complex of leafwise forms. The horizontal forms $\mathcal{A}^{*,0}$ are canonically associated to $\mathcal{F}$, as they are also determined by the condition that $\alpha \in \mathcal{A}^{u,0}$ if and only if $i_{\vec{X}}\alpha = 0$ for all vector fields $\vec{X}$ tangent to the leaves of $\mathcal{F}$. The induced filtration $F^u\mathcal{A} = \mathcal{A}^{u,0} \wedge \mathcal{A}$ of $\mathcal{A}$ is also canonically associated to $\mathcal{F}$.

The deRham derivative decomposes as a sum of bihomogeneous components,

$$d = d_{0,1} + d_{1,0} + d_{2,-1}$$

where the double index denotes the corresponding bidegree. The term $d_{-1,2}$ vanishes because $T\mathcal{F}$ is completely integrable, and the corresponding term $d_{2,-1}$ vanishes if and only if $\mathcal{Q}$ is completely integrable. Equating the bihomogeneous terms of the expansion of $d^2 = 0$ yields relations

$$d_{0,1}^2 = 0 \qquad d_{0,1}d_{1,0} + d_{1,0}d_{0,1} = 0 \qquad d_{1,0}^2 + d_{2,-1}d_{0,1} + d_{0,1}d_{2,-1} = 0 \tag{14}$$

From the decomposition of d we get a decomposition of the co-derivative,

$$\delta = \delta_{0,-1} + \delta_{-1,0} + \delta_{-2,1}$$

satisfying similar properties. On $\mathcal{A}$, we now define the operators

$$\begin{array}{rclc}
\mathcal{D}_{\mathcal{F}} & = & d_{0,1} + \delta_{0,-1} & \text{the leafwise Euler operator} \\
\Delta_T & = & \delta_{-1,0}d_{1,0} + d_{1,0}\delta_{-1,0} & \text{the transverse Laplacian} \\
\Delta_{\mathcal{F}} & = & \mathcal{D}_{\mathcal{F}}^2 & \text{the leafwise Laplacian} \\
\Delta & = & \Delta_{\mathcal{F}} + \Delta_T & \text{the total Laplacian}
\end{array}$$

Note the total Laplacian Δ coincides with the metric Laplacian on V if and only if $\mathcal{Q}$ is totally integrable. In any case, both operators have the same leading symbol since $d_{2,-1}$ and $\delta_{-2,1}$ are both of order zero.

THEOREM 4.1 *Let $\mathcal{F}$ be a totally geodesic Riemannian foliation on V. Then the total Laplacian Δ of homogeneous bidegree (0,0) acting on the smooth forms $\mathcal{A}^{*,*}$ commutes with the leafwise Laplacian $\Delta_{\mathcal{F}}$.*

Proof. A form $\alpha \in \mathcal{A}^{u,0}$ is *basic* if the Lie derivative $\Theta_{\vec{X}}\alpha = 0$ for all vector fields $\vec{X}$ on V tangent to the leaves of $\mathcal{F}$. Let $\mathcal{A}_b^{u,0} \subset \mathcal{A}^{u,0}$ denote the subspace of basic forms. Since $\mathcal{F}$ is Riemannian, for $\alpha \in \mathcal{A}_b^{u,0}$ we have [1, 2, 13]

$$\delta_{0,-1}(\alpha \wedge \beta) = (-1)^u \alpha \wedge \delta_{0,-1}\beta \tag{15}$$

For any smooth vector field $\vec{X}$ on V, the Lie derivative $\Theta_{\vec{X}}$ decomposes as a sum of homogeneous components $(\Theta_{\vec{X}})_{i,-i}$ (where the double index again denotes the bi-degrees.) If $\vec{X}$ is an infinitesimal transformation of $\mathcal{F}$, then $\Theta_{\vec{X}}$ preserves the filtration $F^u\mathcal{A}$, and thus $(\Theta_{\vec{X}})_{i,-i} = 0$ for $i \neq 0$. In this case we get

$$\left[(\Theta_{\vec{X}})_{0,0}, d_{0,1}\right] = 0 \qquad \text{on} \quad \mathcal{A}^{0,*} \tag{16}$$

by comparing bi-degrees in the formula $[\Theta_{\vec{X}}, d] = 0$. If $\vec{X}$ is also orthogonal to the leaves then (16) also implies that

$$\left[(\Theta_{\vec{X}})_{0,0}, \delta_{0,-1}\right] = 0 \qquad \text{on} \quad \mathcal{A}^{0,*} \tag{17}$$

since $\mathcal{F}$ is totally geodesic.

Choose locally defined orthonormal vector fields $\{\vec{e}_1, \ldots, \vec{e}_q\}$ which are sections of $\mathcal{Q}$ and infinitesimal transformations of $\mathcal{F}$, and let $\{\omega_1, \ldots, \omega_q\}$ be the local dual frame of $\mathcal{Q}^*$. From the formula

$$i_{\vec{X}} d_{1,0} = (\Theta_{\vec{X}})_{0,0}$$

on $\mathcal{A}^{0,*}$ for $\vec{X} \in C^\infty(\mathcal{Q})$, we deduce that

$$d_{1,0}\beta = \sum_{i=1}^{q} \omega_i \wedge (\Theta_{\vec{e}_i})_{0,0}\beta \tag{18}$$

for $\beta \in \mathcal{A}^{0,*}$. Then (15),(17) and (18) imply

$$d_{1,0} \circ \delta_{0,-1} = \delta_{0,-1} \circ d_{1,0} \tag{19}$$

on $\mathcal{A}^{0,*}$. Also, (15) implies that (19) holds on all of $\mathcal{A}$, because each $\mathcal{A}^{u,v}$ is (locally) generated by the exterior products $\mathcal{A}^{0,v}$ with basic u-forms. Taking the adjoint of the identity (19), we obtain

$$d_{0,1} \circ \delta_{-1,0} = \delta_{-1,0} \circ d_{0,1} \tag{20}$$

Note that $\mathcal{D}_\mathcal{F}$ and $\Delta_\mathcal{F} = \mathcal{D}_\mathcal{F}^2$ commute, so by equations (14), (19) and (20):

$$\begin{aligned}
\Delta_T\mathcal{D}_\mathcal{F} - \mathcal{D}_\mathcal{F}\Delta_T &= (\delta_{-1,0}d_{1,0} + d_{1,0}\delta_{-1,0})(d_{0,1} + \delta_{0,-1}) - (d_{0,1} + \delta_{0,-1})(\delta_{-1,0}d_{1,0} + d_{1,0}\delta_{-1,0}) \\
&= (\delta_{-1,0}d_{1,0}d_{0,1} + d_{1,0}\delta_{-1,0}d_{0,1}) + (\delta_{-1,0}d_{1,0}\delta_{0,-1} + d_{1,0}\delta_{-1,0}\delta_{0,-1}) \\
&\quad -(d_{0,1}\delta_{-1,0}d_{1,0} + d_{0,1}d_{1,0}\delta_{-1,0}) - (\delta_{0,-1}\delta_{-1,0}d_{1,0} + \delta_{0,-1}d_{1,0}\delta_{-1,0}) \\
&= (\delta_{-1,0}d_{1,0}d_{0,1} - d_{0,1}\delta_{-1,0}d_{1,0}) + (d_{1,0}\delta_{-1,0}d_{0,1} - d_{0,1}d_{1,0}\delta_{-1,0}) \\
&\quad +(\delta_{-1,0}d_{1,0}\delta_{0,-1} - \delta_{0,-1}\delta_{-1,0}d_{1,0}) + (d_{1,0}\delta_{-1,0}\delta_{0,-1} - \delta_{0,-1}d_{1,0}\delta_{-1,0}) \\
&= 0
\end{aligned}$$

from which $\mathcal{D}_\mathcal{F}\Delta = \Delta\mathcal{D}_\mathcal{F}$ follows. □

The leafwise Laplacian has an additional property, that its eigenspaces are modules over the algebra of basic forms $\mathcal{A}_b^{*,0}$ for $\mathcal{F}$:

COROLLARY 4.2 *Let $\xi \in \mathcal{A}_b^{u,0}$ be a non-zero basic form for $\mathcal{F}$. If $\phi \in \mathcal{A}$ is a λ-eigenform for the leafwise Laplacian $\Delta_\mathcal{F}$, then $\phi \wedge \xi \in \mathcal{A}$ is a λ-eigenform for $\Delta_\mathcal{F}$.*

Proof: Calculate using (15)

$$\Delta_\mathcal{F}(\phi \wedge \xi) = (d_{0,1} + \delta_{0,-1})^2(\phi \wedge \xi) = \{(d_{0,1} + \delta_{0,-1})^2\phi\} \wedge \xi = \lambda\phi \wedge \xi \qquad \square$$

COROLLARY 4.3 *The leafwise Euler operator $\mathcal{D}_\mathcal{F}\colon \mathcal{A}_b^{u,0} \otimes \mathcal{A}^{0,*} \to \mathcal{A}_b^{u,0} \otimes \mathcal{A}^{0,*}$ on a totally geodesic Riemannian foliation $\mathcal{F}$ commutes with the total Laplacian on forms Δ.* □

5 Existence of eigendistributions

The L^2-index theorem for foliation operators gives topological criteria for the existence of L^2 0-eigensections along a set of leaves, but there is no similar theorem for a leafwise operator $\mathcal{D}_{\mathcal{F}}: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$. In the next two sections, we establish two approaches for the existence of global eigendistributions for $\mathcal{D}_{\mathcal{F}}$. Corollary 5.3 gives an existence criterion, using the finite-propagation speed methods of Cheeger, Gromov and Taylor (cf. proof of Propositions 1.1, [7]), but with stringent hypotheses. In the next section, we use methods of C^*-algebras which apply in more generality, but with less exact conclusions.

A point $\lambda \in \sigma_{pp}(\mathcal{D}_L)$ is said to be *isolated* if there exists an interval (a,b) with $a < \lambda < b$ so that $\sigma(\mathcal{D}_L) \cap (a,b) = \{\lambda\}$.

PROPOSITION 5.1 *Let L be a complete Riemannian manifold with bounded geometry, and $\mathcal{D}_L: C_c^\infty(\mathbf{E}_L) \to C_c^\infty(\mathbf{E}_L)$ a geometric operator with $\lambda \in \mathbf{R}$ an isolated point in the spectrum $\sigma_{pp}(\mathcal{D}_L)$. Then there exists a λ-eigensection $\phi \in L^2(\mathbf{E}_L)$ so that the pointwise norm $|\phi(y)|$ has rapid decay.*

Proof: Let $(a,b) \subset \mathbf{R}$ be an interval such that $\sigma(\mathcal{D}_L) \cap (a,b) = \lambda$. Choose a smooth function $f: \mathbf{R} \to [0,1]$ such that the support of f is contained in (a,b) and $f(\lambda) = 1$. Then $f(\mathcal{D}_L)$ is the projection operator onto the λ-eigenspectrum of $\mathcal{D}_L$. The existence of some $\psi \in L^2(\mathbf{E}_L)$ such that $f(\mathcal{D}_L)\psi = \psi$ implies that there exists a compactly supported smooth section $\psi_c \in C_c^\infty(\mathbf{E}_L)$ so that $\phi = f(\mathcal{D}_L)\psi_c \neq 0$. (For example, $f(\mathcal{D}_L)$ is a bounded operator, so we can take ψ_c to be the compression of ψ to a sufficiently large compact set in L.) The image $\phi \in L^2(\mathbf{E}_L)$ is then a λ-eigensection for $\mathcal{D}_L$.

The wave operator method of [7] implies that for any $\psi \in L^2(\mathbf{E})$ with support in a compact set $K \subset L$, the image $\phi = f(D)\psi$ has rapid decay. Let $\hat{f}(\xi) = \int_{\mathbf{R}} f(r) \exp\{-ir\xi\} dr$ denote the Fourier transform of f, and $\exp\{i\xi\mathcal{D}_L\}$ the wave operator on $L^2(\mathbf{E}_L)$ associated to $\mathcal{D}_L$. Recall that $\exp\{i\xi\mathcal{D}_L\}$ is a unitary operator. Then

$$f(\mathcal{D}_L) = \frac{1}{2\pi} \int_{\mathbf{R}} \hat{f}(\xi) \exp\{i\xi\mathcal{D}_L\} d\xi \tag{21}$$

The assumption f is smooth implies that for each power $p > 0$ the function $\xi^p \hat{f}(\xi)$ is uniformly bounded in ξ (cf. Chapter 5, [30]). If ψ is supported in a compact set $K \subset L$, then by the unit propagation speed of the wave operator $\exp\{i\xi\mathcal{D}_L\}$, the section $\exp\{i\xi\mathcal{D}_L\}\psi$ is supported in the compact set

$$\mathrm{Pen}(K, |\xi|) = \{x \in L \mid \ \mathrm{Dist}_L(x, K) \leq |\xi|\}$$

Thus,

$$\begin{aligned}
|\phi(y)| &= \left| \int_{|\xi| \geq \mathrm{Dist}_L(y,K)} \hat{f}(\xi) \cdot \exp\{i\xi\mathcal{D}_L\}\psi d\xi \right| \\
&\leq \int_{|\xi| \geq \mathrm{Dist}_L(y,K)} |\hat{f}(\xi)| \cdot || \exp\{i\xi\mathcal{D}_L\}\psi||_L d\xi \\
&\leq \int_{|\xi| \geq \mathrm{Dist}_L(y,K)} |\hat{f}(\xi)| d\xi
\end{aligned}$$

For a function $|\hat{f}(\xi)|$ with rapid decay, the integral function $r \mapsto \int_{|\xi| \geq r} |\hat{f}(\xi)| d\xi$ also has rapid decay, and the Proposition follows. □

REMARK 5.2 It should be possible to replace the spectral gap hypotheses of Proposition 5.1 with less stringent estimates on the spectral density of $\mathcal{D}_{\mathcal{F}}$ near $\lambda \in \sigma_{pp}(\mathcal{D}_L)$ along the lines of the Gromov-Shubin invariants of differential operators [15]. It is an interesting problem to determine the exact conditions on spectral density which suffices.

Let L be a complete Riemannian manifold with bounded geometry. Define an equivalence relation on functions $f, g\colon [0,\infty) \to [0,\infty)$ by declaring $f \sim g$ if there exists constants $a, b, c > 0$ so that

$$f(x)/b - a \le g(x) \le b \cdot f(x) + a \qquad \text{for all } \; x \ge c.$$

The *growth type* of L about a compact set $K \subset L$ is the class of the volume function

$$\mathrm{gr}_L(K, r) = \mathrm{Vol}_L(\mathrm{Pen}(K, \mathrm{r})) \tag{22}$$

The growth type of L is independent of the choice of K for any manifold with bounded geometry. (For L a leaf of a foliation this is proved in Plante [26]; for more general manifolds, see section 1 of Januszkiewicz [18].) The growth type is *polynomial of degree at most* p if there exists constants $a, b > 0$ and exponent $p > 0$ such that

$$\mathrm{gr}_L(K, r) \le a r^p + b$$

At the other extreme, the growth type is *exponential* if

$$\liminf_{r\to\infty} \frac{\log\{\mathrm{gr}_L(K, r)\}}{r} \;\; > \;\; 0.$$

COROLLARY 5.3 *Let L be a complete Riemannian manifold with bounded geometry and polynomial growth type, and $\mathcal{D}_L\colon C_c^\infty(\mathbf{E}_L) \to C_c^\infty(\mathbf{E}_L)$ a geometric operator with $\lambda \in \mathbf{R}$ an isolated point in its spectrum. Then there exists a λ-eigensection $\phi \in L^2(\mathbf{E}_L) \cap L^1(\mathbf{E}_L)$.*

Proof: The volume of the sets

$$\mathrm{Ann}(K, r) = \mathrm{Pen}_L(K, r + 1) \setminus \mathrm{Pen}_L(K, r)$$

are bounded by a polynomial function in r, so by Proposition 5.1 and a straightforward estimate, the summand in

$$\int_L |\phi| \; d\chi_L = \sum_{\ell=1}^{\infty} \left\{ \int_{\mathrm{An}(K,\ell)} |\phi| \; d\chi_L \right\}$$

decreases faster than any polynomial in ℓ, and hence converges. $\square$

6 Operator algebras and spectral propagation

This section investigates C^*-algebraic techniques for constructing eigendistributions. The main result, Theorem 6.7, develops a spectral transfer from the leaves to the ambient space, and seems to yield the strongest possible conclusion for the given hypotheses. We begin with a preliminary discussion of the action of the foliation groupoid (cf. §2 [12]).

Given a foliation $\mathcal{F}$ of V, the *holonomy groupoid* $\mathcal{G}_{\mathcal{F}}$ is the set of equivalence classes $[\gamma]$ of pointed leafwise paths $\gamma\colon([0,1],0,1) \to (V,\gamma(0),\gamma(1))$ for $\mathcal{F}$, equipped with the topology whose basic open sets are generated by "neighborhoods of leafwise paths" (cf. section 2, Winkelnkemper [34]). The topology on $\mathcal{G}_{\mathcal{F}}$ need not be Hausdorff, so that the continuous functions on $\mathcal{G}_{\mathcal{F}}$ are defined as the uniform closure of the finite sums of continuous functions, each supported on a basic open set.

The space $\mathcal{G}_{\mathcal{F}}$ has the structure of a topological groupoid: Concatenation of paths induces a groupoid product $\mathcal{G}_{\mathcal{F}}^2 \to \mathcal{G}_{\mathcal{F}}$ which is associative up to homotopy. The manifold V embeds into $\mathcal{G}_{\mathcal{F}}$ as the subgroupoid of units, where the constant path $*x$ is associated to $x \in V$. The *source* and *range* maps $s, r\colon \mathcal{G}_{\mathcal{F}} \to V$ are defined by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$, respectively. The pre-image $s^{-1}(x) = \tilde{L}_x$ of a point $x \in V$ is called the *holonomy cover* of the leaf L_x through x. The characterizing property of $\tilde{L}_x$ is that the image of a closed curve $\gamma \subset \tilde{L}_x$ always has trivial holonomy as a curve in V. Via the source map $s\colon \mathcal{G}_{\mathcal{F}} \to V$, the space $\mathcal{G}_{\mathcal{F}}$ is parametrized as a family of open manifolds – the holonomy covers of leaves of $\mathcal{F}$ – over the base V. The source map $s\colon \mathcal{G}_{\mathcal{F}} \to V$ is "fibration-like" when restricted to compacts subsets of $\mathcal{G}_{\mathcal{F}}$, but it need not be a fibration.

A *transverse measure* μ for $\mathcal{F}$ is a Radon measure on the Borel subsets of the transversals to $\mathcal{F}$, which takes finite value on compact subsets [26]. A transverse measure is *quasi-invariant* if, for every transversal Z with μ-measure zero, all holonomy transports of Z also have μ-measure zero. A transverse measure is *invariant* if the μ-measure of a transverse set Z does not change under holonomy transport. A transverse measure for $\mathcal{F}$ is said to be *non-atomic* if it has no atoms. That is, it assigns measure zero to each countable transverse set Z. Conversely, if μ is supported on a countable collection of compact leaves, then we say μ is *atomic*.

Fix a Riemannian metric on the tangential distribution to $\mathcal{F}$, then the leafwise volume forms for define a leafwise Haar system $dv_{\mathcal{F}}$ (cf. [10, 11, 28]). Let $C_c(\mathcal{G}_{\mathcal{F}})$ denote the convolution algebra generated by the compactly supported continuous functions on $\mathcal{G}_{\mathcal{F}}$: the product of $f, g \in C_c(\mathcal{G}_{\mathcal{F}})$ is given by

$$f * g(\gamma) = \int_{\delta \in \tilde{L}_x} f(\delta) g(\delta^{-1}\gamma) dv_{\mathcal{F}}(\delta)$$

and the *-involution is given by $f^*(\gamma) = \overline{f(\gamma^{-1})}$.

For each $x \in V$ let

$$R_x\colon C_c(\mathcal{G}_{\mathcal{F}}) \to \mathcal{B}\left(L^2(\tilde{L}_x)\right)$$

denote the *-representation given by, for $f \in C_c(\mathcal{G}_{\mathcal{F}})$, $\xi \in L^2(\tilde{L}_x)$ and $\gamma \in \tilde{L}_x$,

$$\{R_x(f)\xi\}(\gamma) = \int_{\delta \in \tilde{L}_x} f(\gamma^{-1}\delta)\xi(\delta) dv_{\mathcal{F}}(\delta)$$

The C^*-closure of the $*$-algebra $C_c(\mathcal{G}_{\mathcal{F}})$ with respect to the sup-norm defined by the representations R_x yields the *reduced foliation C^*-algebra* $C^*(V, \mathcal{F})$.

For our applications, we need to allow for operators acting on the leafwise L^2-sections of an Hermitian vector bundle $\mathbf{E} \to V$. Assume that $\mathbf{E} \subset V \times \mathbf{C}^N \to V$ is an embedded Hermitian subbundle of the trivial bundle. Let $C_c(\mathcal{G}_{\mathcal{F}}, \mathbf{C}^N) = C_c(\mathcal{G}_{\mathcal{F}}) \otimes \mathrm{End}(\mathbf{C}^N)$ denote the convolution algebra with coefficients in the $N \times N$ matrix algebra. Each representation R_x extends in the natural way, so that $f \in C_c(\mathcal{G}_{\mathcal{F}}, \mathbf{C}^N)$ yields an operator on the leafwise L^2-sections $L^2(\tilde{L}_x, \mathbf{C}^N)$. For each $x \in V$, the fiberwise orthogonal projection $\Pi_{\mathbf{E}}: V \times \mathbf{C}^N \to \mathbf{E}$ induces a projection operator on the L^2-sections over the holonomy cover,

$$\Pi_{\mathbf{E},x}: L^2(\tilde{L}_x, \mathbf{C}^N) \to L^2(\tilde{L}_x, \mathbf{E}|\tilde{L}_x)$$

Define $C_c(\mathcal{G}_{\mathcal{F}}, \mathbf{E})$ to be the subalgebra of $f \in C_c(\mathcal{G}_{\mathcal{F}}, \mathbf{C}^N)$ such that for all $x \in V$,

$$R_x(f) \circ \Pi_{\mathbf{E},x} = R_x(f) = \Pi_{\mathbf{E},x} \circ R_x(f)$$

The *von Neumann algebra* $W^*(\mathcal{F}, \mu)$ of $\mathcal{F}$ can be defined with respect to a quasi-invariant μ transverse measure μ for $\mathcal{F}$ by assembling the leaf-wise representations R_x into a total representation of $C_c(\mathcal{G}_{\mathcal{F}})$ on the measurable field of Hilbert spaces

$$\mathcal{H}_\mu = \int_V L^2(\tilde{L}_x) d\tilde{\mu}(x)$$

The closure of the image of this representation in the weak-* topology on $\mathcal{B}(\mathcal{H}_\mu)$ is the *von Neumann algebra* $W^*(\mathcal{F}, \mu)$. When the measure class of $\tilde{\mu}$ is equivalent to the smooth Riemannian measure class on V, then we write $W^*(V/\mathcal{F})$ for the resulting von Neumann algebra. As before, we write $W^*(V/\mathcal{F}, \mu, \mathbf{E})$ to indicate operators with coefficients in the Hermitian bundle $\mathbf{E} \to V$.

Let $\mathcal{D}_{\mathcal{F}}: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ be a foliation geometric operator. For each bounded Borel function $\phi: \mathbf{R} \to \mathbf{R}$ use the spectral theorem applied to the essentially self-adjoint operators

$$\mathcal{D}_{\tilde{L}_x}: C_c^\infty(\tilde{L}_x, \mathbf{E}|\tilde{L}_x) \to C_c^\infty(\tilde{L}_x, \mathbf{E}|\tilde{L}_x)$$

to define a family of bounded operators $\phi(\mathcal{D}_{\mathcal{F}}) = \{\phi(\mathcal{D}_{\tilde{L}_x}) | \ x \in V\}$ which satisfy for all $x \in V$

$$\mathcal{D}_{\tilde{L}_x} \circ \Pi_{\mathbf{E},x} = \mathcal{D}_{\tilde{L}_x} = \Pi_{\mathbf{E},x} \circ \mathcal{D}_{\tilde{L}_x}$$

Roe (Theorem 2.1, [29]) identified a class of functions for which this leafwise collection of operators is represented by an element of $C_c(\mathcal{G}_{\mathcal{F}}, \mathbf{E})$:

THEOREM 6.1 *Let $\phi \in C_0(\mathbf{R})$ whose Fourier transform is smooth and compactly supported. Then for a foliation geometric operator $\mathcal{D}_{\mathcal{F}}: C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$, the operator $\phi(\mathcal{D}_{\mathcal{F}})$ is represented by a kernel in $C_c^\infty(\mathcal{G}_{\mathcal{F}}, \mathbf{E})$.*

We indicate the main points of the proof, as it introduces important technical aspects of the foliation groupoid and the leafwise spectral constructions from foliation geometric operators. Given $\phi \in C_0(\mathbf{R})$ with smooth, compactly supported Fourier transform, $\phi(\mathcal{D}_{\tilde{L}_x})$ is represented by a distributional kernel $\tilde{k}_x$ on $\tilde{L}_x \times \tilde{L}_x$. The methods of [7, 31] imply that $\tilde{k}_x$ is actually a smooth kernel with support contained in a uniform diameter tube about the diagonal in $\tilde{L}_x \times \tilde{L}_x$. Moreover, $\mathcal{D}_{\tilde{L}_x}$ is the lift of a differential operator along the leaf L_x hence $\tilde{k}_x$ is invariant under the diagonal action of the holonomy group on $\tilde{L}_x \times \tilde{L}_x$, so descends to the quotient $s^{-1}(L_x) \subset \mathcal{G}_{\mathcal{F}}$. A fundamental point in [29] is these functions on the subsets $s^{-1}(L_x)$ combine to give a smooth function on $\mathcal{G}_{\mathcal{F}}$.

Given a bounded Borel set $\mathbf{B} \subset \mathbf{R}$ with characteristic function $\chi_{\mathbf{B}}$, choose a sequence of functions $\{\phi_i \mid i = 1, 2, \ldots\} \subset C_0(\mathbf{R})$ with smooth, compactly-supported Fourier transforms and which converge pointwise to the characteristic function $\chi_{\mathbf{B}}$. For each leaf $L_x \subset V$, the sequence of bounded operators $\phi_i(\mathcal{D}_{\tilde{L}_x})$ converges to $\chi_{\mathbf{B}}(\mathcal{D}_{\tilde{L}_x})$ in the strong topology on $\mathcal{B}(L^2(\tilde{L}_x, \mathbf{E}|\tilde{L}_x))$ by the spectral theorem. Even stronger, one has that for the sequence of continuous functions $\{\phi_i(\mathcal{D}_{\mathcal{F}})\}$ on $\mathcal{G}_{\mathcal{F}}$, the restriction to each $s^{-1}(L_x) \subset \mathcal{G}_{\mathcal{F}}$ converges uniformly on compact subsets to the kernel representing $\chi_{\mathbf{B}}(\mathcal{D}_{\tilde{L}_x})$. To see this, note that for each $x \in V$ the operator $\chi_{\mathbf{B}}(\mathcal{D}_{\tilde{L}_x})$ is represented by a smooth kernel on $\tilde{L}_x \times \tilde{L}_x$, as $\chi_{\mathbf{B}}(\mathcal{D}_{\tilde{L}_x})$ defines a bounded operator from $L^2(\tilde{L}_x, \mathbf{E}|\tilde{E}_x)$ to every Sobolev completion of $C_c^\infty(\tilde{L}_x, \mathbf{E}|\tilde{L}_x)$ so we can apply Lemma 5.6, [30]. The spectral techniques of [31] imply that the sequence of kernels $\phi_i(\mathcal{D}_{\tilde{L}_x})$ considered as functions on $\tilde{L}_x \times \tilde{L}_x$ converge uniformly on compact subsets to the kernel representing $\chi_{\mathbf{B}}(\mathcal{D}_{\tilde{L}_x})$. □

It is not necessarily true that the sequence $\{\phi_i(\mathcal{D}_{\mathcal{F}})\} \subset C_c(\mathcal{G}_{\mathcal{F}})$ converges uniformly on compact sets in $\mathcal{G}_{\mathcal{F}}$. In general, we only know that the limiting kernel $\chi_{\mathbf{B}}(\mathcal{D}_{\mathcal{F}})$ is a Borel function on $\mathcal{G}_{\mathcal{F}}$, and that for each quasi-invariant transverse measure the sequence $\{\phi_i(\mathcal{D}_{\mathcal{F}})\}$ acting on $\mathcal{H}_\mu$ converges to the spectral projection associated to $\mathbf{B}$.

PROPOSITION 6.2 ([9, 10]) *For any bounded Borel function $\phi: \mathbf{R} \to \mathbf{R}$ with compact support, and quasi-invariant transverse measure μ for $\mathcal{F}$, the collection of leafwise operators $\{\phi(\mathcal{D}_{\tilde{L}_x}) | x \in V\}$ define an element $\phi(\mathcal{D}_\mu) \in W^*(V/\mathcal{F}, \mu, \mathbf{E})$.*

An invariant transverse measure μ for $\mathcal{F}$ determines a trace $\mathbf{T}_\mu: C_c(\mathcal{G}_{\mathcal{F}}, \mathbf{E}) \to \mathbf{C}$. The Haar system $dv_{\mathcal{F}}$ on the leaves of $\mathcal{F}$ product with μ defines a locally-finite Borel measure $\tilde{\mu}$ on V. For $f \in C_c^\infty(\mathcal{G}_{\mathcal{F}}, \mathbf{E})$ set

$$\mathbf{T}_\mu(f) = \int_{x \in V} \mathbf{Tr}_{\mathbf{E}}(f)(*x) d\tilde{\mu}(x) \tag{23}$$

where $\mathbf{Tr}_{\mathbf{E}}$ denotes the fiberwise trace on the endomorphisms of the bundle $\mathbf{E}$. Connes proved [10, 11] that $\mathbf{Tr}_{\mathbf{E}}$ extends to a densely defined trace on $W^*(V/\mathcal{F}, \mu, \mathbf{E})$. Normality of the trace (i.e., the dominated convergence property) and standard results of spectral theory imply:

LEMMA 6.3 *Let μ be an invariant transverse measure for $\mathcal{F}$. Given a bounded Borel function $\phi: \mathbf{R} \to \mathbf{R}$ with compact support, let $\{\phi_i \mid i = 1, 2, \ldots\} \subset C_0(\mathbf{R})$ be a sequence of functions which converge pointwise to ϕ, and such that the Fourier transform of each ϕ_i is smooth and compactly supported. Then $\phi(\mathcal{D}_{\mathcal{F}}) \in W^*(V/\mathcal{F}, \mu, \mathbf{E})$ is in the domain of $\mathbf{Tr}_{\mathbf{E}}$, and*

$$\lim_{i \to \infty} \mathbf{T}_\mu(\phi_i(\mathcal{D}_{\mathcal{F}})) \quad = \quad \mathbf{T}_\mu(\phi(\mathcal{D}_{\mathcal{F}})) \tag{24}$$

The von Neumann trace satisfies a positivity property:

LEMMA 6.4 *Let μ be an invariant transverse non-negative measure for $\mathcal{F}$, $\mathcal{D}_{\mathcal{F}}$ a leafwise geometric operator, and $\phi: \mathbf{R} \to \mathbf{R}$ a non-negative bounded Borel function with compact support. Then $\mathbf{T}_\mu(\phi(\mathcal{D}_{\mathcal{F}})) = 0$ if and only if the leafwise operator $\phi(\mathcal{D}_{\tilde{L}_x}) = 0$ for μ-almost every leaf L_x of $\mathcal{F}$.*

Proof: The leafwise operators $\phi(\mathcal{D}_{\tilde{L}_x})$ are selfadjoint, and positive by the assumption that $\phi \geq 0$, so the local densities $\mathbf{Tr}_{\mathbf{E}}(\phi(\mathcal{D}_{\tilde{L}_x}))(*x)$ are non-negative. Given that the integral (24) vanishes, the integrand must vanish almost everywhere. Hence, for almost every leaf of $\mathcal{F}$ the local density $\mathbf{Tr}_{\mathbf{E}}(\phi(\mathcal{D}_{\tilde{L}_x}))(*x) = 0$ which forces $\phi(\mathcal{D}_{\tilde{L}_x}) = 0$ as well. The converse is immediate. □

Given an invariant transverse measure μ for $\mathcal{F}$ let $\tilde{\mu}$ be its Haar extension to a locally-finite Borel measure on V. Let $L^2(V, \mathbf{E}, \mu)$ denote the completion of $C^\infty(V, \mathbf{E})$ with respect to the Hilbert space inner product defined by $\tilde{\mu}$. The resulting Hilbert space depends substantially on the geometric properties of the measure μ. For example, if μ is the transverse Dirac measure associated to a compact leaf L, then $L^2(V, \mathbf{E}, \mu) \cong L^2(L, \mathbf{E}|L)$ is the Hilbert space completion of the smooth sections over L with respect to the smooth leafwise Riemannian measure. While if $\tilde{\mu}$ is equivalent to the smooth Lebesgue measure on V, then $L^2(V, \mathbf{E}, \mu)$ is isomorphic to the Hilbert space completion of the smooth sections over V with the usual Riemannian inner product.

The convolution algebra $C_c(\mathcal{G}_{\mathcal{F}}, \mathbf{E})$ represents on $L^2(V, \mathbf{E}, \mu)$ via

$$\{R_\mu(f)\xi\}(x) = \int_{\delta \in \tilde{L}_x} f(\gamma^{-1}\delta)\xi(r(\delta))d\tilde{\mu}(r(\delta))$$

It is a standard calculation (but tedious) to show that the holonomy invariance of μ implies that R_μ is a *-representation of $C_c(\mathcal{G}_{\mathcal{F}}, \mathbf{E})$.

We note the following consequence of the hyperbolic method of Chernoff:

PROPOSITION 6.5 (Lemma 2.1,[8]) *Let $\mathcal{D}_{\mathcal{F}}\colon C^\infty(\mathbf{E}) \to C^\infty(\mathbf{E})$ be a foliation geometric operator. Then $\mathcal{D}_{\mathcal{F}}$ has a unique densely-defined extension to a closed operator $\mathcal{D}_\mu$ on $L^2(V, \mathbf{E}, \mu)$.*

For any bounded Borel function $\phi\colon \mathbf{R} \to \mathbf{R}$ we define the bounded operator $\phi(\mathcal{D}_\mu) \in \mathcal{B}(L^2(V, \mathbf{E}, \mu))$ using the spectral theorem. For special class of ϕ we can identify this operator in terms of the leafwise operators $\mathcal{D}_{\tilde{L}_x}$:

PROPOSITION 6.6 (cf. [12]) *Let $\phi \in C_0(\mathbf{R})$ whose Fourier transform is smooth and compactly supported. Then $\phi(\mathcal{D}_\mu) = R_\mu(\phi(\mathcal{D}_{\mathcal{F}}))$ for any foliation geometric operator $\mathcal{D}_{\mathcal{F}}$ and invariant transverse measure μ.*

Proof: Introduce the wave operators $\exp\{2\pi\sqrt{-1}\lambda\mathcal{D}_{\mathcal{F}}\} = \{\exp\{2\pi\sqrt{-1}\lambda\mathcal{D}_{\tilde{L}_x}\} \mid x \in V\}$ and $\exp\{2\pi\sqrt{-1}\lambda\mathcal{D}_\mu\}$. Both operators satisfy the same hyperbolic wave equation along the leaves of $\mathcal{F}$, hence

$$\exp\{it\mathcal{D}_\mu\} = R_\mu(\exp\{it\mathcal{D}_{\mathcal{F}}\})$$

Then use the operator Fourier tranforms to calculate

$$\begin{aligned} \phi(\mathcal{D}_\mu) &= \frac{1}{2\pi}\int_{\mathbf{R}} \hat{\phi}(\lambda)\cdot \exp\{it\mathcal{D}_\mu\}dt \\ &= \frac{1}{2\pi}\int_{\mathbf{R}} \hat{\phi}(\lambda)\cdot R_\mu(\exp\{it\mathcal{D}_{\mathcal{F}}\})dt \\ &= R_\mu\left(\frac{1}{2\pi}\int_{\mathbf{R}} \hat{\phi}(\lambda)\cdot \exp\{it\mathcal{D}_{\mathcal{F}}\}dt\right) \\ &= R_\mu\left(\phi(\mathcal{D}_{\mathcal{F}})\right) \quad \square \end{aligned}$$

The following theorem establishes a relation between the spectral projections for a foliation geometric operator acting along leaves, and for its action on the ambient foliated manifold. This is a general form of a *spectral coincidence theorem* (cf. [17, 21, 22]).

THEOREM 6.7 *Let $\mathcal{D}_{\mathcal{F}}$ be a foliation geometric operator, and μ an invariant transverse non-atomic measure such that μ-almost every leaf in supp(μ) has trivial holonomy. Given $a \leq b$, suppose there is a set $Z \subset V$ of positive $\tilde{\mu}$-measure such that the leafwise operator $\chi_{[a,b]}(\mathcal{D}_{\tilde{L}_x})$ has non-trivial range for all $x \in Z$. Then for every $\alpha < a$ and $b < \beta$, the range of $\chi_{(\alpha,\beta)}(\mathcal{D}_\mu)$ in $L^2(V, \mathbf{E}, \mu)$ is infinite dimensional. In particular, if $a = b$ then a is in the essential spectrum of $\mathcal{D}_\mu$.*

Proof: Let $\phi: \mathbf{R} \to [0,1]$ be a smooth function with support in the interval (α, β) with $\phi(x) = 1$ for $a \leq x \leq b$. For each $n > 0$ define a function $\phi_n \in C_0(\mathbf{R})$ to be the inverse Fourier transform of $\hat{\phi}$ restricted to the interval $[-n, n]$. The sequence $\{\phi_n\}$ converges uniformly to ϕ, so there exists $n_0 > 100$ such that for all $n > n_0$, $\sup_x |\phi(x) - \phi_n(x)| < 1/100$.

For each $x \in Z$ the leafwise operator $\chi_{[a,b]}(\mathcal{D}_{\tilde{L}_x})$ has non-trivial range, so choose a Borel field $\{\xi_x \in C^\infty(\tilde{L}_x, \mathbf{E}|L_x) | x \in Z\}$ of unit-length sections in the range of the projections $\chi_{[a,b]}(\mathcal{D}_{\tilde{L}_x})$.

By Proposition 6.6, the sequence of operators $\{\phi_i(\mathcal{D}_\mu)\}$ acting on $L^2(V, \mathbf{E}, \mu)$ converge to $\phi(\mathcal{D}_\mu)$. The strategy now is to "truncate" the family $\{\xi_z \mid z \in Z\}$ so that it "embeds" into $L^2(V, \mathbf{E}, \mu)$, then use Proposition 6.6 to evaluate $\phi(\mathcal{D}_\mu)$ on the embedded sections. We prove that $\phi(\mathcal{D}_\mu)$ has infinite-dimensional range, hence the same is true for $\chi_{(\alpha,\beta)}(\mathcal{D}_\mu)$.

Let $B(x, r') \subset \tilde{L}_x$ denote the closed ball of radius r' about $*x \in \tilde{L}_x$ for $x \in Z$. The key technical point is to choose a radius $\rho > 0$ so that the restrictions $\xi_x | B(x, \rho)$ have "good spectral estimates". First, there exists $r_0 > 100$ and a Borel subset $Z_0 \subset Z$ of positive $\tilde{\mu}$-measure so that for all $x \in Z_0$

$$||\xi_x - \xi_x | B(x, r_0)||_{\tilde{L}_x} < 1/100 \tag{25}$$
$$||\xi_x - \phi(\mathcal{D}_{\tilde{L}_x})(\xi_x | B(x, r_0))||_{\tilde{L}_x} < 1/100 \tag{26}$$

Set $\rho = r_0 + n_0 + 2$.

LEMMA 6.8 *For all $n > n_0$ and $x \in T_x$*

$$\langle \phi_n(\mathcal{D}_{\tilde{L}_x})(\xi_x | B(x, \rho)), \xi_x | B(x, \rho) \rangle_{\tilde{L}_x} \geq 9/10 \tag{27}$$

Proof: Set $\xi_x^+ = \xi_x | B(x, \rho)$ and $\xi_x = \xi_x^+ + \xi_x^-$. The spectral theorem yields the estimate, for $n > n_0$

$$||\phi(\mathcal{D}_{\tilde{L}_x})(\xi_x^+) - \phi_n(\mathcal{D}_{\tilde{L}_x})(\xi_x^+)||_{\tilde{L}_x} \leq \sup_x |\phi(x) - \phi_n(x)| < 1/100 \tag{28}$$

Combine the estimates (25), (26) and (28) with the observation $\phi(\mathcal{D}_{\tilde{L}_x})(\xi_x) = \chi_{[a,b]}(\mathcal{D}_{\tilde{L}_x})(\xi_x) = \xi_x$ to obtain

$$||\xi_x^+ - \phi_n(\mathcal{D}_{\tilde{L}_x})(\xi_x^+)||_{\tilde{L}_x} \leq ||\xi_x^+ - \xi_x|| + ||\xi_x - \phi(\mathcal{D}_{\tilde{L}_x})(\xi_x^+)||_{\tilde{L}_x} + ||\phi(\mathcal{D}_{\tilde{L}_x})(\xi_x^+) - \phi_n(\mathcal{D}_{\tilde{L}_x})(\xi_x^+)||_{\tilde{L}_x} < 3/100$$

The estimate (27) now is immediate, as $99/100 < ||\xi_x^+||_{\tilde{L}_x} \leq 1$. $\square$

We are now ready to construct the approximate eigen-sections in $L^2(V, \mathbf{E}, \mu)$. We can assume that for every $x \in Z_0$ the leaf L_x has no holonomy (as this is true for $\tilde{\mu}$-a.e. leaf) and we identify $\tilde{L}_x = L_x \subset V$ via the covering map r. Choose a point $z \in Z_0$ with $\tilde{\mu}$-density 1, then every transverse disk $D_z \subset V$ through z intersects Z_0 is a set $T_z = D_z \cap Z_0$ of positive μ-measure. We use the product neighborhood theorem

THEOREM 6.9 (cf. Chapter IV, Theorem 2 [6]) *Let L be a leaf with holonomy covering $\tilde{L}$. Given a compact subset $K \subset \tilde{L}$ and $\epsilon > 0$, there exists a foliated immersion $\Pi: K \times (-1,1)^q \to V$ so that the restriction $\Pi: K \times \{0\} \to L \subset V$ coincides with the restriction to K of the covering map $\pi: \tilde{L} \to L$. If L has no holonomy, then Π can be chosen to be an embedding.*

As L_z has no holonomy, there is a foliated embedding $\Pi: B(z,\rho) \times (-1,1)^q \to V$. The Riemannian metric on $T\mathcal{F}|B(x,\rho)$ depends absolutely continuously on the parameter x, so by restricting to $(-t,t)^q \subset (-1,1)^q$ we can also assume that the natural maps $B(x,\rho) \times w \to B(x,\rho) \times w'$ distort distances by at most 1 (for the metrics obtained from the Riemannian metrics induced by Π). Choose $D_z = \Pi(z, (-1,1)^q)$, set $\mathcal{T}_z = \Pi^{-1}(z \times T_z)$ and let

$$W(z,\rho) = \Pi(B(z,\rho) \times \mathcal{T}_z) \subset V$$

For each Borel subset $X \subset T_z$ with $\mu(X) > 0$, we introduce $\xi(X,\rho) \in L^2(V, \mathbf{E}, \mu)$ with support on $W(z,\rho)$ by setting

$$\xi(X,\rho)(w) = \begin{cases} \frac{1}{\sqrt{\mu(X)}} \cdot \xi_x(w) & \text{for } w \in B(x,\rho) \ \& \ x \in X \\ \\ 0 & \text{for } w \notin \bigcup_{x \in X} B(x,\rho) \end{cases}$$

Observe that $99/100 < ||\xi(X,\rho)||_\mu \leq 1$, where $|| \cdot ||_\mu$ is the norm on $L^2(V, \mathbf{E}, \mu)$.

The proof of Theorem 6.7 is completed by

PROPOSITION 6.10 *For each Borel subset $X \subset T_z$ with $\mu(X) > 0$,*

$$9/10 < ||\phi(\mathcal{D}_\mu)\xi(X,\rho)||_\mu \leq 1 \tag{29}$$

Moreover, if $\{X_n \subset T_z \mid n = 1,2,\ldots\}$ is a disjoint collection of Borel subsets with $\mu(X_n) > 0$ for all n, then the subset

$$\{\phi(\mathcal{D}_\mu)\xi(X_n,\rho) \mid n = 1,2,\ldots\} \subset L^2(V, \mathbf{E}, \mu)$$

spans an infinite-dimensional subspace.

Proof: For all $n > n_0$, set $\mathcal{E}_n = \phi(\mathcal{D}_\mu) - \phi_n(\mathcal{D}_\mu)$ then $||\mathcal{E}_n||_\mu \leq 1/100$ by the spectral theorem. By Proposition 6.6 and the choice of ϕ_n

$$\phi_n(\mathcal{D}_\mu) = R_\mu \left(\frac{1}{2\pi} \int_{-n}^{n} \hat{\phi}(\lambda) \cdot \exp\{it\mathcal{D}_\mathcal{F}\} dt \right)$$

Using the unit-speed propagation of the geometric operators $\mathcal{D}_{\tilde{L}_x}$ and the foliation product structure on $W(z,\rho)$ and Lemma 6.8, we have

$$\begin{aligned} \langle \phi_n(\mathcal{D}_\mu)\xi(X,\rho), \xi(X,\rho) \rangle &= \frac{1}{\mu(X)} \cdot \int_X \langle \phi_n(\mathcal{D}_{\tilde{L}_x})(\xi_x|B(x,\rho)), \xi_x|B(x,\rho) \rangle_{L_x} d\mu(x) \\ &\geq \frac{1}{\mu(X)} \cdot \int_X 9/10 \quad d\mu(x) \\ &= 9/10 \end{aligned}$$

This proves the first claim of the proposition. Observe that if X_ℓ and X_m are disjoint, then the supports of $\phi_{n_0}(\mathcal{D}_\mu)(\xi(X_\ell, \rho)$ and $\phi_{n_0}(\mathcal{D}_\mu)(\xi(X_m, \rho)$ are disjoint, so their images are linearly independent and the range of $\phi_{n_0}(\mathcal{D}_\mu)$ contains the infinite dimensional subspace spanned by the collection $\{\phi_{n_0}(\mathcal{D}_\mu)\xi(X_n, \rho) \mid n = 1, 2, \ldots\}$. Now

$$\phi(\mathcal{D}_\mu) = \phi_{n_0}(\mathcal{D}_\mu) + \mathcal{E}_{n_0}$$

where the operator norm $||\mathcal{E}_{n_0}||_\mu < 1/100$ so by the estimate (29) and linear algebra, the collection $\{\phi(\mathcal{D}_\mu)\xi(X_n, \rho) \mid n = 1, 2, \ldots\}$ also spans an infinite dimensional subspace.

Finally, when $a = b$ we have that for all $\epsilon > 0$, the range of $\chi_{(a-\epsilon,a+\epsilon)}(\mathcal{D}_\mu)$ is infinite dimensional. This implies $a \in \sigma_e(\mathcal{D}_\mu)$. $\square$

COROLLARY 6.11 *Let $\mathcal{D}_\mathcal{F}$ be a foliation geometric operator, and μ an invariant transverse non-atomic measure such that μ-almost every leaf in supp(μ) has trivial holonomy. Given $a \leq b$, suppose that $\mathbf{T}_\mu(\chi_{[a,b]}(\mathcal{D}_\mathcal{F})) \neq 0$. Then for every $\alpha < a$ and $b < \beta$, the range of $\chi_{(\alpha,\beta)}(\mathcal{D}_\mu)$ in $L^2(V, \mathbf{E}, \mu)$ is infinite dimensional. In particular, if $a = b$ then a is in the essential spectrum of $\mathcal{D}_\mu$.*

Proof: Lemma 6.4 and the hypothesis $\mathbf{T}_\mu(\chi_{[a,b]}(\mathcal{D}_\mathcal{F})) \neq 0$ implies that there is a set $Z \subset V$ with positive $\tilde{\mu}$-measure so that $\chi_{[a,b]}(\mathcal{D}_{\tilde{L}_x}) \neq 0$ for all $x \in Z$. Then proceed as in the proof of Theorem 6.7. $\square$

COROLLARY 6.12 *Let $(\mathcal{D}_\mathcal{F}, \epsilon)$ be a graded foliation geometric operator and μ an invariant transverse non-atomic measure such that μ-almost every leaf in supp(μ) has trivial holonomy. Suppose that the Connes' foliation index $\mathbf{Ind}_\mu(\mathcal{D}_\mathcal{F}, \epsilon)) \neq 0$, then 0 is in the essential spectrum of $\mathcal{D}_\mathcal{F}$ acting on $L^2(V, \mathbf{E}, \mu)$.*

Proof: $\mathbf{Ind}_\mu(\mathcal{D}_\mathcal{F}, \epsilon)) = \mathbf{T}_\mu(\epsilon \circ \chi_{[0,0]}(\mathcal{D}_\mathcal{F}))$, hence $\mathbf{T}_\mu(\chi_{[0,0]}(\mathcal{D}_\mathcal{F})) \neq 0$ and apply Corollary 6.11. $\square$

COROLLARY 6.13 *Let $\mathcal{F}$ be a totally geodesic Riemannian foliation of the compact manifold V, and $\mathcal{D}_\mathcal{F} = d_{0,1} + \delta_{0,-1}$ be the leafwise Euler operator. Suppose there is a there is a leaf with a non-trivial harmonic L^2-form. Then 0 is in the essential spectrum of $\mathcal{D}_\mathcal{F}$ acting on $L^2(V, \mathbf{E})$.*

Proof: The holonomy invariant transverse Riemannian metric for $\mathcal{F}$ defines the transverse invariant measure μ, so that $\tilde{\mu}$ is just the smooth volume density on V. The leaves of a totally geodesic Riemaniann foliation $\mathcal{F}$ are all locally isometric, so have isometric holonomy coverings hence satisfy $\chi_{[0,0]}(\mathcal{D}_{\tilde{L}_x})) \neq 0$. In a Riemannian foliation, the set of leaves with non-trivial holonomy is a countable union of codimension-2 subvarieties, so has Lebesgue measure 0. The conclusion then follows from Theorem 6.7. $\square$

The obvious question raised by Theorem 6.7 is whether $\chi_{[a,b]}(\mathcal{D}_\mathcal{F}) \neq 0$ implies that $\chi_{[a,b]}(\mathcal{D}_\mu) \neq 0$. If we choose a descending sequence of smooth functions

$$\chi_{[a,b]} \leq \cdots \psi_\ell \leq \psi_{\ell-1} \leq \psi_1 = \phi$$

whose limit is $\chi_{[a,b]}$ then we are asking whether the intersection of the ranges $\psi_\ell(\mathcal{D}_\mu)$ is non-trivial. The answer in unknown for the generality of Theorem 6.7.

The converse of Theorem 6.7 is false. The simplest example arises from the linear foliation $\mathcal{F}_\alpha$ of $\mathbf{T}^2$ with irrational slope α. Take $\mathcal{D}_\mathcal{F} = -i\frac{d}{d\ell}$ to be the symmetric first-order operator obtained from differentiating with respect to the leaf length coordinate ℓ. This foliation satisfies all of the geometric hypotheses of the theorem, but the spectrum of each leaf is absolutely continuous, while the operator $\mathcal{D}_\mathcal{F}$ acting on $C^\infty(\mathbf{T}^2, \mathbf{C})$ is completely pure-point with dense spectrum. This operator also illustrates the technical difficulty in answering the above question regarding the range of the limit of the sequence $\{\psi_\ell(\mathcal{D}_\mu)\}$, as it exhibits many properties similar to one with leafwise pure-point spectrum.

7 Examples with pure-point spectrum

This work was inspired by the analysis of the simple examples of section 2, but to conclude we will give examples of totally geodesic Riemannian foliations with far more complicated dynamics and spectral theory for which our theorems apply.

Here is the basic construction: Let M and N be complete Riemannian manifolds of dimensions m and q, respectively. Give the manifold $M \times N$ the product Riemannian metric, and note it naturally has two foliations $\mathcal{F}^M$ and $\mathcal{F}^N$ whose leaves are the manifolds $\{M \times \{y\} \mid y \in N\}$ and $\{\{x\} \times N \mid x \in M\}$, respectively. Let $\alpha\colon \Gamma \to \mathrm{Isom(M \times N)}$ be an isometric action by a finitely-generated group Γ so that:

- the quotient space $V_\alpha = (M \times N) \,/\, \{(x,y) \sim \alpha(\gamma)(x,y) \ \text{ for } \gamma \in \Gamma\}$ is a compact manifold.
- each diffeomorphism $\alpha(\gamma)$ for $\gamma \in \Gamma$ preserves the foliation $\mathcal{F}^M$ (and hence the orthogonal foliation $\mathcal{F}^N$ as well.)

For example, suppose we start with two isometric actions $\alpha_M\colon \Gamma \times M \to M$ and $\alpha_N\colon \Gamma \times N \to N$, then form the product action $\alpha = \alpha_M \times \alpha_N$. If the action α is discrete with compact quotient, then α satisfies the hypotheses above.

The quotient manifold V_α carries two foliations $\mathcal{F}^M_\alpha$ and $\mathcal{F}^N_\alpha$ which are the quotients of $\mathcal{F}^M$ and $\mathcal{F}^N$ respectively. The tangential distributions $T\mathcal{F}^M_\alpha$ and $T\mathcal{F}^N_\alpha$ have induced metrics from $TM \times N$ and $M \times TN$, respectively. This yields a Riemannian metric on TV_α by declaring that $T\mathcal{F}^M_\alpha$ and $T\mathcal{F}^N_\alpha$ are orthogonal, for which both $T\mathcal{F}^M_\alpha$ and $T\mathcal{F}^N_\alpha$ are totally geodesic Riemannian [19].

A geometric operator $\mathcal{D}_M\colon C^\infty_c(\mathbf{E}_M) \to C^\infty_c(\mathbf{E}_M)$ extends to a leafwise operator $\tilde{\mathcal{D}}_M$ on the lifted bundle $\tilde{\mathbf{E}} \to M \times N$. We say that $\tilde{\mathcal{D}}_M$ is *invariant* under α if there is given a unitary action $\alpha_{\tilde{\mathbf{E}}}\colon \Gamma \times \tilde{\mathbf{E}} \to \tilde{\mathbf{E}}$ covering α, and the induced action on leafwise geometric operators leaves $\tilde{\mathcal{D}}^M_{\mathcal{F}_\alpha}$ invariant. We then obtain a foliation geometric operator $\mathcal{D}_\alpha$ along $\mathcal{F}^M_\alpha$.

The Laplacian $\Delta_N\colon C^\infty_c(N) \to C^\infty_c(N)$ extends to an operator

$$\tilde{\Delta}_N\colon C^\infty_c(M \times N, \tilde{\mathbf{E}}) \to C^\infty_c(M \times N, \tilde{\mathbf{E}})$$

which is invariant under α, hence descends to an operator $\Delta_{\mathcal{F}^N_\alpha}$ along $\mathcal{F}^N_\alpha$. Moreover, $\tilde{\mathcal{D}}_M$ and $\tilde{\Delta}^N$ commute as they are locally "uncoupled", so the quotient operators $\mathcal{D}_\alpha$ and $\Delta_{\mathcal{F}^N_\alpha}$ commute. The sum $\mathcal{D}^*_\alpha\mathcal{D}_\alpha + \Delta_{\mathcal{F}^N_\alpha}$ is self-adjoint and elliptic on $C^\infty(\mathbf{E})$, hence has pure-point spectrum. As $\mathcal{D}_\alpha$ commutes with $\mathcal{D}^*_\alpha\mathcal{D}_\alpha + \Delta_{\mathcal{F}^N_\alpha}$ its spectrum must also be completely pure-point.

The first examples are intermediate between fibrations and foliations with all leaves dense:

EXAMPLE 7.1 Let X be a compact manifold with a geometric operator $\mathcal{D}_X\colon C^\infty(\mathbf{E}_X) \to C^\infty(\mathbf{E}_X)$. Let $\Gamma = \pi_1(X) \to \Gamma_q$ be a quotient of the fundamental group of X, and $\mathcal{D}_M$ be the lift of $\mathcal{D}_X$ to the the Galois covering M corresponding to Γ_q. We take α_M to be the action of Γ on M by covering transformations. Let $\alpha_N\colon \Gamma \times N \to N$ be an isometric action on a compact Riemannian manifold, and assume that α_N has a fixed-point $z \in N$. Denote by L_z the leaf of $\mathcal{F}^M_\alpha$ obtained as the quotient of $M \times \{z\}$. Then L_z is compact and isometric to X. Note that the orbits of α_N cannot be dense, as the distance from the fixed-point z is a preserved quantity.

For example, a simple way to obtain such α_N is to choose $N = G$ a compact Lie group, endowed with a bi-invariant metric. Let $\beta\colon \Gamma \to G$ be a representation. Then the Adjoint action of G on itself is an isometry which restricts to an isometric action of $\beta(\Gamma)$. Set $\alpha_N = \mathrm{Ad} \circ \beta$ and note that the identity element $e \in G$ is a fixed-point. Let $L_e \cong X$ be the leaf through e, then every λ-eigensection ϕ on X yields a singular eigendistribution for $\mathcal{D}_\alpha$. Theorem 3.1 implies $\mathcal{H}_\phi$ will be an infinite-dimensional λ-eigenspace of $\mathcal{D}_\alpha$. Thus, there is an inclusion $\sigma(\mathcal{D}_X) \subset \sigma_{pp}(\mathcal{D}_\alpha)$.

EXAMPLE 7.2 Repeat the construction of Example 7.1, but replace α_N with an isometric action with all orbits dense. This implies the isometry group G of N acts transitively, hence N is isometric to a compact symmetric space G/H for some closed subgroup H. The action α_N is equivalent to a representation of Γ into G acting on G/H via left translations. The examples of section 2 are of this form, with $N = \mathbf{T}^q$.

More general examples are obtained by letting Γ' be a uniform (cocompact) lattice of higher rank in a semi-simple Lie group [35], which then acts on the associated symmetric space M with compact orbi-fold quotient. A torsion-free subgroup $\Gamma \subset \Gamma'$ will act freely on M with quotient $X = M/\Gamma$ a manifold. By Margulis' Theorem [24] we can assume Γ is arithmetic, so admits a dense faithful representation into a compact Lie group G. Choose any closed proper subgroup $H \subset G$ and take $N = G/H$.

Every leaf of $\mathcal{F}_\alpha^M$ will be dense in V_α, with the generic leaf isometric to X. Suppose the Euler characteristic of X is not zero, then as the leaves of $\mathcal{F}_\alpha$ cover X, the average Euler characteristic of $\mathcal{F}_\alpha$ will also be non-zero. By Corollary 6.12, 0 is in the essential spectrum of the leafwise Laplacian on forms on V_α.

EXAMPLE 7.3 The product Lie group $SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ has a cocompact irreducible torsion-free lattice subgroup Γ (cf. Chapter 6 [35]). The projection of Γ into each factor $SL(2,\mathbf{R})$ is dense by irreducibility. Each factor $SL(2,\mathbf{R})$ acts on the 2-dimensional hyperbolic plane $\mathbf{H}^2 = SL(2,\mathbf{R})/SO(2)$, so by projecting we obtain two actions α_M and α_N of Γ on $M = N = \mathbf{H}^2$. The product action $\alpha = \alpha_M \times \alpha_N$ is simply the left action of Γ on

$$\mathbf{H}^2 \times \mathbf{H}^2 \cong SL(2,\mathbf{R}) \times SL(2,\mathbf{R})/SO(2) \times SO(2)$$

so has quotient a compact 4-manifold V_α. The product action of $SL(2,\mathbf{R}) \times SL(2,\mathbf{R})$ on $\mathbf{H}^2 \times \mathbf{H}^2$ preserves the two factors, hence α preserves the two foliations $\mathcal{F}^M$ and $\mathcal{F}^N$. The leaves of $\mathcal{F}_\alpha^M$ are all dense and generically isometric to the hyperbolic plane. In particular, the average Euler characteristic is non-zero for $\mathcal{F}_\alpha^M$. The leaves are non-compact so there are no harmonic L^2 0-forms or 2-forms. We conclude that 0 is in the essential spectrum of the leafwise Laplacian on 1-forms on V_α.

REMARK 7.4 Note than in the examples above, the leaves of $\mathcal{F}_\alpha^M$ have exponential growth if Γ is a uniform lattice in a semi-simple Lie group other than $\mathbf{R}^q$. There seems to be no standard methods for analyzing the spectrum of the operator $\mathcal{D}_\alpha$ on $L^2(V_\alpha, \mathbf{E})$ in this case. It would be interesting to have a representation approach to proving that 0 is in the essential spectrum of $\mathcal{D}_\alpha$, as this may shed more light on whether 0 is actually a point of infinite multiplicity in the pure-point spectrum.

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