COARSE GEOMETRY OF FOLIATIONS

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ABSTRACT

We give a survey with many details of some of the recent work relating the coarse geometry of the leaves of foliations with their dynamics, index theory and spectral theory.

Contents

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1 Introduction

Coarse geometry is an approach for exploring the "structure at infinity" of open manifolds. Compact manifolds look like points in coarse geometry, and continuity is replaced by global Lipshitz estimates on maps. It is remarkable that any geometry can survive in such an environment, but it is now clear that this is precisely the framework for developing a deeper understanding of the geometry of leaves of foliations, of foliation dynamics, and the K-theory invariants of foliated spaces.

There has been a tremendous amount of research activity in geometry, group theory, dynamics and foliation theory which can be categorized as part of coarse geometry. This survey cannot do justice to so many areas, hence will primarily focus on the rôle of coarse geometry in foliation theory. Fortunately, there are many good surveys and sources available for the interested reader wishing to pursue a balanced coarse diet: Gromov has written a survey exposition of coarse properties of groups, which has appeared as a book [47]; Roe's monograph on coarse cohomology [88] is an excellent introduction to the ideas of coarse index theory; there are numerous survey texts now on the coarse geometry of hyperbolic groups – for example the text by Ghys and de la Harpe [45]; and Block and Weinberger are preparing a survey of coarse geometry and homology theories [11] (cf. also [5, 10]). In addition, there are very close ties between the ideas of coarse geometry and the methods of controlled surgery theory – the references to [20] give the background on this topic.

The contents of these note expand on three lectures given by the author at the International Symposium on the *Geometric Theory of Foliations*:

I – Beyond Volume Growth
which discussed the ideas of coarse geometry, introduced the entropy of metric spaces and applied these ideas to recurrence properties of the leaves of foliations (§§2,3 and 4).

II – Dimensions of Ends
which discussed coarse cohomology, coronas and coarse entropy (§§5,6 and 7).

III – Coarse Families Produce Fine Invariants
which discussed the construction of foliation fundamental classes, and their application to spectral geometry and the proof of the Foliation Novikov Conjecture (§§8,9 and 10).

From the author's perspective, these notes omit two important topics:

IV – Coarse geometry of secondary characteristic classes

V – Rigidity of group actions and foliations

The omission is due primarily to questions of maintaining our focus in these notes, as well as for reasons of length, but definitely not due to lack of interest or importance! The interested reader can consult the papers [18, 54, 62, 73, 61, 43] for the former topic, and the papers [101, 104, 105, 106, 64, 68, 72, 77] for the latter topic.

The goal of the notes in any case is to expose some of the ideas of coarse geometry, and suggest some of the syntheses now emerging as coarse methods are applied in a wide variety of areas, including problems in the geometric theory of foliations.

Lecture I - Beyond Volume Growth

2 Coarse geometry of leaves

In this section we introduce the basic constructions and definitions in the coarse geometry of foliations.

2.1 Metric properties of the holonomy groupoid

A *coarse metric* on a set X is a symmetric pairing $\langle \cdot, \cdot \rangle : X \times X \to [0, \infty)$ satisfying the triangle inequality

$$\langle x, z \rangle \leq \langle x, y \rangle + \langle y, z \rangle \qquad \text{for all } \ x, y, z \in X$$

A map $f : X_1 \to X_2$ is said to be *quasi-isometric* with respect to coarse metrics $\langle \cdot, \cdot \rangle_i$ if there exists constants $d_1, d_2, d_3 > 0$ so that for all $y, y' \in X_1$

$$d_1 \cdot (\langle y, y' \rangle_1 - d_3) \leq \langle f(y), f(y') \rangle_2 \leq d_2 \cdot (\langle y, y' \rangle_1 + d_3) \tag{1}$$

A subset $\mathcal{N} \subset X$ is ϵ-*dense* for $\epsilon > 0$ if for each $x \in X$ there exists $n(x) \in \mathcal{N}$ so that $\langle x, n(x) \rangle \leq \epsilon$. An ϵ-*net* is a collection of points $\mathcal{N} = \{x_\alpha \mid \alpha \in \mathcal{A}\} \subset X$ so that $\mathcal{N}$ is ϵ-dense, and there exists $c > 1$ so that distinct points of $\mathcal{N}$ are at least distance ϵ / c apart. The net $\mathcal{N}$ inherits a coarse metric from X.

DEFINITION 2.1 *A map $f : X_1 \to X_2$ is said to be a* coarse isometry *with respect to coarse metrics $\langle \cdot, \cdot \rangle_i$ if f is quasi-isometric and the image $f(X_1)$ is ϵ-dense in X_2 for some $\epsilon > 0$.*

Coarse geometry is the study of geometric properties of a complete metric space which are invariant under coarse isometries. The *fundamental property* of coarse geometry is that the inclusion of a net, $\mathcal{N} \subset X$, is a coarse isometry. The usual example to illustrate this phenomenon is that for a connected Lie group G, a cocompact lattice $\Gamma \subset G$ with the word metric is coarsely isometric to G with the left invariant Riemannian path-length metric: the integers $\mathbf{Z}$ are coarsely isometric to the real line $\mathbf{R}$. Thus, coarse geometry detects only global metric properties of a space, and ignores local properties. For further discussions of coarse geometry for metric spaces, see Gromov [46, 47] or Roe [88].

2.2 Topological foliations

A *topological foliation* $\mathcal{F}$ of a paracompact manifold M^m is a continuous partition of M into tamely embedded submanifolds (the leaves) of constant dimension p and codimension q. We require that these leaves be locally given as the level sets (plaques) of local coordinate charts. We specify this local defining data by fixing:

1. a uniformly locally-finite covering $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ of M; that is, there exists a number $m(\mathcal{A}) > 0$ so that for any $\alpha \in \mathcal{A}$ the set $\{\beta \in \mathcal{A} \mid U_\alpha \cap U_\beta \neq \emptyset\}$ has cardinality at most $m(\mathcal{A})$
2. local coordinate charts $\phi_\alpha : U_\alpha \to (-1,1)^m$, so that each map ϕ_α admits an extension to a homeomorphism $\tilde{\phi}_\alpha : \tilde{U}_\alpha \to (-2,2)^m$ where $\tilde{U}_\alpha$ contains the closure of the open set U_α
3. for each $z \in (-2,2)^q$, the preimage $\tilde{\phi}_\alpha^{-1}((-2,2)^p \times \{z\}) \subset \tilde{U}_\alpha$ is the connected component containing $\tilde{\phi}_\alpha^{-1}(\{0\} \times \{z\})$ of the intersection of the leaf of $\mathcal{F}$ through $\phi_\alpha^{-1}(\{0\} \times \{z\})$ with the set $\tilde{U}_\alpha$.

The extensibility condition in (2) is made to guarantee that the topological structure on the leaves remains tame out to the boundary of the chart ϕ_α. The collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ is called a *regular foliation atlas* for $\mathcal{F}$.

The inverse images

$$\mathcal{P}_\alpha(z) = \phi_\alpha^{-1}((-1,1)^p \times \{z\}) \subset U_\alpha$$

are topological discs contained in the leaves of $\mathcal{F}$, called the *plaques* associated to this atlas. One thinks of the plaques as "tiling stones" which cover the leaves in a regular fashion.

We will always insist that our foliation atlas also be *good*:

4. An intersection of plaques $\mathcal{P}_{\alpha_1}(z_1) \cap \cdots \cap \mathcal{P}_{\alpha_d}(z_d)$ is either empty, or a connected set.

This condition can be guaranteed by requiring that each open set U_α be convex.

The plaques are indexed by the *complete transversal*

$$\mathcal{T} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$$

associated to the given covering, where $\mathcal{T}_\alpha = (-1, 1)^q$. The charts ϕ_α define tame embeddings

$$t_\alpha = \phi_\alpha^{-1}(\{0\} \times \cdot) : \mathcal{T}_\alpha \to U_\alpha \subset M$$

We will implicitly identify the set $\mathcal{T}$ with its image in M under the maps t_α, though it may be that the union of these maps is only finite-to-one.

The foliation $\mathcal{F}$ is said to be C^r if the foliation charts $\{\phi_\alpha \mid \alpha \in \mathcal{A}\}$ can be chosen to be C^r-diffeomorphisms.

2.3 The holonomy groupoid

A pair of indices α and β is *admissible* if $U_\alpha \cap U_\beta \neq \emptyset$. For each admissible pair α, β define

$$\mathcal{T}_{\alpha\beta} = \{z \in \mathcal{T}_\alpha = (-1, 1)^m \text{ such that } \mathcal{P}_a(z) \cap U_\beta \neq \emptyset\}.$$

Then there is a well-defined transition function $\gamma_{\alpha\beta} : \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha}$, which for $x \in \mathcal{T}_{\alpha\beta}$ is given by

$$\gamma_{\alpha\beta}(x) = \phi_\beta \left(S_\beta(\phi_\alpha^{-1}(\mathcal{D}^m \times \{x\}) \cap U_\beta) \cap \mathcal{T}_\beta \right) \in \mathcal{T}_{\beta\alpha}$$

The continuity of the charts ϕ_α implies that each $\gamma_{\alpha\beta}$ is continuous; in fact, one can see that $\gamma_{\alpha\beta}$ is a local homeomorphism from $\mathcal{T}_{\alpha\beta}$ onto $\mathcal{T}_{\beta\alpha}$.

A *leafwise path* γ is a continuous map $\gamma : [0, 1] \to M$ whose image is contained in a single leaf of $\mathcal{F}$. Suppose that a leafwise path γ has initial point $\gamma(0) = t_\alpha(z_0)$ and final point $\gamma(1) = t_\beta(z_1)$, then γ determines a local holonomy map h_γ by composing the local holonomy maps $\gamma_{\alpha\beta}$ along the plaques which γ intersects. h_γ is a local homeomorphism from a neighborhood of z_0 to a neighborhood of z_1. More generally, if the initial point $\gamma(0)$ lies in the plaque $\mathcal{P}_\alpha(z_0)$ and $\gamma(1)$ lies in the plaque $\mathcal{P}_\beta(z_1)$, then γ again defines a local homeomorphism h_γ. Note that the holonomy of a concatenation of two paths is the composition of their holonomy maps. We say that two leafwise paths γ_1 and γ_2 with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ have the same holonomy if h_{γ_1} and h_{γ_2} agree on a common open set about z_0.

Define an equivalence relation on pointed leafwise paths by specifying that $\gamma_1 \sim_h \gamma_2$ if γ_1 and γ_2 have the same holonomy. The *holonomy groupoid* $\mathcal{G}_{\mathcal{F}}$ is the set of $\sim_h$ equivalence classes of pointed leafwise paths for $\mathcal{F}$, equipped with the topology whose basic sets are generated by "neighborhoods of leafwise paths" (cf. section 2, [98]). The manifold M embeds into $\mathcal{G}_{\mathcal{F}}$ by associating to $x \in M$ the constant path $*x$ at x.

The *fundamental groupoid* $\Pi_{\mathcal{F}}$ of $\mathcal{F}$ is the set of endpoint-fixed homotopy equivalence classes of leafwise paths for $\mathcal{F}$, equipped with the topology whose basic sets are generated by "neighborhoods of leafwise paths". Two paths which are endpoint-fixed homotopy equivalent have the same holonomy, so there is a natural map of groupoids $\Pi_{\mathcal{F}} \to \mathcal{G}_{\mathcal{F}}$.

There are natural continuous maps $s, r : \mathcal{G}_{\mathcal{F}} \to M$ defined by $s(\gamma) = \gamma(0)$ and $r(\gamma) = \gamma(1)$. For a point $x \in M$, the pre-image $s^{-1}(x) = \tilde{L}_x$ is the *holonomy cover* of the leaf L_x of $\mathcal{F}$ through x; that is, the image of a closed curve $\gamma \subset \tilde{L}_x$ always has trivial holonomy as a curve in M. We use the source map s to view the groupoid $\mathcal{G}_{\mathcal{F}}$ as a parametrized family of open manifolds (the holonomy covers of leaves of $\mathcal{F}$) over the base M.

Define the *transversal groupoid* $\mathcal{T}_{\mathcal{F}} \subset \mathcal{G}_{\mathcal{F}}$ to be the preimage of $\mathcal{T} \times \mathcal{T}$ under the map

$$s \times r : \mathcal{G}_{\mathcal{F}} \to M \times M.$$

That is, $\mathcal{T}_{\mathcal{F}}$ consists of all the equivalence classes of paths in $\mathcal{G}_{\mathcal{F}}$ which start and end at points in the complete transversal $\mathcal{T}$. For each $x \in \mathcal{T}$ the fiber $(s|\mathcal{T}_{\mathcal{F}})^{-1}(x) \subset \tilde{L}_x$ is a net in the holonomy cover $\tilde{L}_x$, so that $\mathcal{T}_{\mathcal{F}}$ can be considered as a (locally) continuous selection of nets for the fibers of $s : \mathcal{G}_{\mathcal{F}} \to M$.

The topological manifold structure on $\mathcal{G}_{\mathcal{F}}$ may not be Hausdorff: suppose there exists a leafwise closed path γ with basepoint x which has non-trivial holonomy of infinite order, but so that there is a family $\{\gamma_s \,|1 \leq s \geq 0\}$ of closed paths, $\gamma_0 = \gamma$, and which are the transverse "push-off" of γ so that each γ_s has trivial holonomy for $s > 0$. Then every iterate of the path γ is arbitrarily close to the push-offs γ_s for s small. That is, the path $\{\gamma_s \mid s > 0\}$ intersects every neighborhood of the iterates of γ. This property of paths that there are nearby paths for which the holonomy degenerates is typical of the non-Hausdorff aspect of $\mathcal{G}_{\mathcal{F}}$. This was formalized by Winkelnkemper in the following result:

PROPOSITION 2.2 (Proposition 2.1, [98]) *$\mathcal{G}_{\mathcal{F}}$ is Hausdorff if and only if, for all $x \in M$ and $y \in L_x$ the holonomy along two arbitrary leafwise paths γ_1 and γ_2 from x to y are already the same if they coincide on an open subset U of their common domain, whose closure $\overline{U}$ contains x.*

For example, if the holonomy of every leaf has finite order, or is analytic, or is an isometry for some transversal metric, then $\mathcal{G}_{\mathcal{F}}$ will be Hausdorff. In contrast, one knows that the holonomy of the compact leaf in the Reeb foliation of S^3 fails this criterion, so its foliation groupoid is not Hausdorff at the compact leaf.

Let $\mathcal{G}_{\mathcal{F}}^{nh} \subset \mathcal{G}_{\mathcal{F}}$ be the union of the paths for which there exists another path which has the holonomy property of Proposition 2.2. Then $\mathcal{G}_{\mathcal{F}}^{h} = \mathcal{G}_{\mathcal{F}} \setminus \mathcal{G}_{\mathcal{F}}^{nh}$ is a Hausdorff space.

Let $\mathcal{F}_i$ be a topological foliation of M_i for $i = 1, 2$. Let $f : M_1 \to M_2$ be a continuous map which sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$. Then the assignment $\gamma \mapsto f(\gamma)$ induces a map $\mathcal{G}f : \mathcal{G}_{\mathcal{F}_1} \to \mathcal{G}_{\mathcal{F}_2}$. It is clear from the definition that $s(\mathcal{G}f(\gamma)) = f(s(\gamma))$ and similarly for the range map r. Thus, $\mathcal{G}f$ maps the fibers of s over M_1 into the fibers of s over M_2. We let $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ denote the restriction of $\mathcal{G}f$ from the fiber of s over $x \in X_1$ to the fiber of s over $f(x) \in X_2$.

Let $\mathcal{F}_i$ be a topological foliation of M_i for $i = 1, 2$, $f_0, f_1 : M_1 \to M_2$ be continuous maps which sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$. We say that f_0 is *leafwise homotopic* to f_1 if there exists a continuous map $F : M_1 \times [0, 1] \to M_2$ such that

- $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$ for all $x \in M_1$
- F maps the leaves of $\mathcal{F}_1 \times [0, 1]$ into the leaves of $\mathcal{F}_2$, where $\mathcal{F}_1 \times [0, 1]$ is the foliation of $M_1 \times [0, 1]$ with typical leaf $L \times [0, 1]$ for L a leaf of $\mathcal{F}_1$.

The trace of a leafwise homotopy F is the collection of curves $t \mapsto F(x, t)$ for $x \in M_1$. The special property of a leafwise homotopy is simply that the trace consists of leafwise curves.

A continuous map $f : M_1 \to M_2$ which sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$ is a *leafwise homotopy equivalence* if there exists a continuous map $g : M_2 \to M_1$ which sends leaves of $\mathcal{F}_2$ into leaves of $\mathcal{F}_1$ so that the compositions $g \circ f$ and $f \circ g$ are both leafwise homotopic to the respective identity maps on M_1 and M_2.

2.4 Coarse metrics on holonomy groupoids

We next formulate the coarse metric properties of the foliation groupoid (cf. Plante [84]; or section 1, Hurder & Katok [73].) A coarse metric on $\mathcal{G}_{\mathcal{F}}$ will be a family of coarse metrics

$$\langle \cdot, \cdot \rangle_x : \tilde{L}_x \times \tilde{L}_x \to [0, \infty)$$

parametrized by $x \in M$. It is natural to also require the "coarse continuity" of the family, which is satisfied by the examples presented below.

Given groupoids $s\colon \mathcal{G}_i \to X_i$ equipped with coarse metrics $\langle \cdot, \cdot \rangle_x^i$ for $i = 1, 2$, a groupoid map $F\colon \mathcal{G}_1 \to \mathcal{G}_2$ is a *quasi-isometry* if there exists constants $d_1, d_2, d_3 > 0$ so that for all $x \in X_1$ and $y, y' \in s^{-1}(x)$

$$d_1 \cdot \Big(\langle y, y' \rangle_x^1 - d_3 \Big) \leq \langle F_x(y), F_x(y') \rangle_{f(x)}^2 \leq d_2 \cdot \Big(\langle y, y' \rangle_x^1 + d_3 \Big) \tag{2}$$

where $f\colon X_1 \to X_2$ is the map on objects induced by F. We say that F is a *coarse isometry* if there exists $\epsilon > 0$ so that $F_x(s^{-1}(x)) \subset s^{-1}(f(x))$ is ϵ-dense for all $x \in X_1$.

Fix a regular foliation atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ for $\mathcal{F}$. For $x \in M$ and a leafwise path $\gamma\colon [0,1] \to \tilde{L}_x$, define the *plaque length function* $\mathcal{N}_{\mathcal{T}}(\gamma)$ to be the least number of plaques required to cover the image of γ. Define the *plaque distance function* $D_x(\cdot, \cdot)$ on the holonomy cover $\tilde{L}_x$ using the plaque length function: for $y, y' \in \tilde{L}_x$,

$$D_x(y, y') = \inf \{ \mathcal{N}_{\mathcal{T}}(\gamma) \ \mid \gamma \ \text{ is a leafwise path from } \ y \text{ to } y' \}$$

In other words, $D_x(y, y')$ is the minimum number of plaques in $\tilde{L}_x$ such their union forms a connected open set in $\tilde{L}_x$ containing both y and y'. Note that $D_x(\cdot, \cdot)$ is not a distance function, for $D_x(y, y') = 1$ if and only if y and y' lie on the same plaque $\mathcal{P}_\alpha(z)$. It is immediate from the definitions that the pairings D_x satisfy the triangle inequality, hence:

LEMMA 2.3 *The family D_x is a coarse metric for the foliation groupoid $\mathcal{G}_{\mathcal{F}}$.* □

The family of plaque-distance coarse metrics is independent (up to coarse isometry) of the choice of foliation covering of M:

LEMMA 2.4 (Lemma 2.4, [67]) *Suppose that $\mathcal{F}$ is a topological foliation of a compact compact M, and there are given two coverings of M by regular foliation atlases $\{(U_\alpha^i, \phi_\alpha^i) \mid \alpha \in \{1, \ldots, k(i)\}\}$ for $i = 1, 2$, with plaque distance functions D_x^i. Then there exists constants $c_1, c_2 > 0$ so that for all $x \in M$ and $y, y' \in \tilde{L}_x$*

$$c_1 \cdot D_x^1(y, y') \leq D_x^2(y, y') \leq c_2 \cdot D_x^1(y, y') \tag{3}$$

That is, the identity map is a coarse isometry of $\mathcal{G}_{\mathcal{F}}$ endowed with the coarse metrics D_x^1 and D_x^2.

When the foliation $\mathcal{F}$ is at least C^1, then we can give the leaves a Riemannian metric, and define a leafwise Riemannian distance function d_x on $\tilde{L}_x$ by taking the infimum over the lengths of paths in the holonomy cover between y and y'. The family d_x is a coarse metric on $\mathcal{G}_{\mathcal{F}}$ which is coarsely equivalent to the plaque-distance metric:

LEMMA 2.5 (Lemma 2.5 [67]) *Suppose that $\mathcal{F}$ is a C^1-foliation, M is compact, and $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \{1, \ldots, k\}\}$ is a regular foliation atlas with a finite number of open charts. Then there exists constants $c_1, c_2 > 0$ so that for all $x \in M$ and $y, y' \in \tilde{L}_x$*

$$c_1 \cdot (D_x(y, y') - 1) \leq d_x(y, y') \leq c_2 \cdot D_x(y, y') \tag{4}$$

Hence, the identity map is a coarse isometry of $\mathcal{G}_{\mathcal{F}}$ endowed with the metrics D_x and d_x, respectively.

It is expected that a coarse metric on a foliated space should be essentially independent of the choices made, which is the content of the above two lemmas. The more fundamental property of the plaque-distance coarse metric is that continuous maps between foliated manifolds induce controlled maps in this metric:

LEMMA 2.6 *Let M_1 be a compact manifold, and $f : M_1 \to M_2$ be a continuous function which sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$. Then there exists a constant $d_2 > 0$ so that for all $x \in M_1$ and $y, y' \in \tilde{L}_x$, the induced map $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ on holonomy covers satisfies the estimate*

$$D_{f(x)}\left(\mathcal{G}f_x(y), \mathcal{G}f_x(y')\right) \leq d_2 \cdot D_x\left(y, y'\right) \tag{5}$$

A fiberwise map $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ satisfying the condition (5) is said to be *eventually Lipshitz*.

Even if both M_1 and M_2 are assumed to be compact, the induced map $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ need not be a quasi-isometry, or even proper. The first inequality in (1) fails in the following simple example. Let $M_1 = \mathbf{T}^2$ be the 2-torus with $\mathcal{F}_1$ the linear foliation by lines with irrational slope. Let $M_2 = \mathbf{T}^2$ also, with $\mathcal{F}_2$ the foliation having exactly one leaf. The identity map satisfies the estimate (5). On the other hand, the leaves of $\mathcal{F}_1$ contain paths of arbitrarily long length, which map to segments in $\mathbf{T}^2$ which are $\sim_h$ equivalent to a "shortcut" in $\mathbf{T}^2$ of length at most $2\sqrt{2}\pi$, where we assume that each circle factor in $\mathbf{T}^2$ has length 2π. Thus, for this example there is no estimate for the minimum plaque-length of a leafwise path for $\mathcal{F}_1$ in terms of the minimum plaque-length of its image in $\mathcal{F}_2$.

The leafwise homotopy equivalences between foliations are the "natural isomorphisms" of the homotopy category of topological foliations, analogous to isomorphisms for groups. The following basic result asserts that coarse geometry is also preserved by these maps.

PROPOSITION 2.7 *Let $\mathcal{F}_i$ be a topological foliation of a compact manifold M_i for $i = 1, 2$ and $f : M_1 \to M_2$ a leafwise homotopy equivalence. Then there exists constants $d_1, d_2 > 0$ so that for all $x \in M_1$ and $y, y' \in \tilde{L}_x$ with $D_x(y, y') \geq d_3$, the induced map $\mathcal{G}f_x : \tilde{L}_x \to \tilde{L}'_{f(x)}$ satisfies the estimate*

$$d_1 \cdot D_x(y, y') \leq D_{f(x)}(\mathcal{G}f_x(y), \mathcal{G}f_x(y')) \leq d_2 \cdot D_x(y, y') \tag{6}$$

Thus, $\mathcal{G}f : \mathcal{G}_{\mathcal{F}_1} \to \mathcal{G}_{\mathcal{F}_2}$ is a coarse isometry with respect to the coarse metrics D^1_x and D^2_x.

Proof. Choose a leaf-preserving continuous map $g : M_2 \to M_1$ and a leafwise homotopy $F : M_1 \times [0, 1] \to M_1$ between $g \circ f$ and the identity. Let K denote the maximum plaque-lengths of the leafwise traces $t \mapsto F(x, t)$ for $x \in M_1$. Let d'_2 denote the constant for g and d_2 the constant for f given by Lemma 2.6. Given a leafwise path γ between $z = \mathcal{G}f(y)$ and $z' = \mathcal{G}f(y')$, the images $\mathcal{G}g(z)$ and $\mathcal{G}g(z')$ are connected to y and y' by leafwise paths with plaque-lengths at most K each. (This is true for their images in M_1 so by the covering path lifting property also holds for the points in $\tilde{L}_x$.) Applying Lemma 2.6 to g we then obtain

$$D_x(y, y') \leq D_x(\mathcal{G}g_{f(x)}(z), \mathcal{G}g_{f(x)}(z')) + 2K \leq d'_2 \cdot D_{f(x)}(z, z') + 2K$$

hence

$$1/d'_2 \cdot (D_x(y, y') - 2K) \leq D_{f(x)}(z, z')$$

Take $d_3 = 4K$ and $d_1 = 1/(2d'_2)$ and the estimate (6) follows. □

COROLLARY 2.8 *Let $\mathcal{F}_i$ be a topological foliation of a compact manifold M_i for $i = 1, 2$ and $f : M_1 \to M_2$ a leafwise homotopy equivalence. Then $\mathcal{G}f$ is a proper map.* □

Proof. Let $K \subset \mathcal{G}_{\mathcal{F}_2}$ be a compact set. Then there is a finite collections of leafwise paths $\{\gamma_1, \ldots, \gamma_d\}$ for $\mathcal{F}_2$ and a covering of K by basic foliation charts formed from the γ_i. It follows that there is a constant C_K so that K is contained in the diagonal set

$$\Delta(\mathcal{G}_{\mathcal{F}_2}, C_K) = \{y \in \mathcal{G}_{\mathcal{F}_2} \mid D_{s(y)}(y, *s(y)) \leq C_K\}$$

where $*s(y)$ is the canonical basepoint in the fiber $\tilde{L}_{s(y)}$. The inequality (6) implies that the preimage $\mathcal{G}f^{-1}(K)$ is contained in the diagonal set $\Delta(\mathcal{G}_{\mathcal{F}_1}, C_K/d_1)$. Hence $\mathcal{G}f^{-1}(K)$ is a closed set contained in a finite union of basic foliation charts on $\mathcal{G}_{\mathcal{F}_1}$ so is compact. □

Finally, let us observe the fundamental property of coarse geometry in the context of the plaque-distance coarse metric. The transversal groupoid $\mathcal{T}_{\mathcal{F}}$ has an intrinsic *transversal length function* $D_{\mathcal{T}}$, defined analogously to the word length function for groups. (The choice of the transversal $\mathcal{T}$ corresponds to the choice of a generating set for a group.) We say that two points $y \in \mathcal{T}_\alpha$ and $y' \in \mathcal{T}_\beta$ are *adjacent* if their plaques $\mathcal{P}_\alpha(y) \cap \mathcal{P}_\beta(y') \neq \emptyset$. The choice of a path $\gamma_{y,y'} \subset \mathcal{P}_\alpha(y) \cup \mathcal{P}_\beta(y')$ connecting adjacent points y, y' determines a canonical equivalence class $[\gamma_{y,y'}] \in \mathcal{T}_{\mathcal{F}}$. For $[\gamma_y] \neq [\gamma_{y'}] \in \mathcal{T}_{\mathcal{F}}$ define

$$D_{\mathcal{T}}([\gamma_y],[\gamma_{y'}]) = \min \left\{ n > 0 \mid \begin{array}{c} \text{there exists a chain of points } y = y_0, \ldots, y_n = y' \\ \text{with } (y_i, y_{i+1}) \text{ adjacent for each } 0 \leq i < n \text{ and} \\ [\gamma_{y'}] = [\gamma_y] * [\gamma_{y_1,y_2}] * \cdots * [\gamma_{y_{n-1},y_n}] \end{array} \right.$$

and set $D_{\mathcal{T}}([\gamma_y],[\gamma_{y'}]) = \infty$ if no such chain exists, and set $D_{\mathcal{T}}([\gamma_y],[\gamma_y]) = 0$.

PROPOSITION 2.9 *The inclusion* $T\colon \mathcal{T}_{\mathcal{F}} \subset \mathcal{G}_{\mathcal{F}}$ *induces a coarse isometry for the transversal length function* $D_{\mathcal{T}}$ *on* $\mathcal{T}_{\mathcal{F}}$ *and the plaque distance function on* $\mathcal{G}_{\mathcal{F}}$*.*

3 Foliation dynamics

In this section we introduce a few of the basic ideas of topological and measurable dynamics of foliated manifolds. Smale's fundamental paper on smooth dynamical systems [90] concluded with a brief section (Part IV) on the dynamics of Lie group actions, which consisted of more questions than results. Plante's work in the early 1970's investigated growth of leaves and the minimal sets for foliations, and generally developed the paradigm of a foliation on a compact manifold as a generalized dynamical system [84]. Cantwell and Conlon [16, 17, 19] and Hector [52] explored codimension-one foliations as dynamical systems, studying the growth type of leaves and their asymptotic properties, culminating in the proof of the Poincaré-Bendixson theorem for C^2-foliations. Ghys's work has explored many facets of the differential dynamics of foliations [21, 43, 41], and the *geometric entropy* for foliations of Ghys, Langevin, Walczak [34] is a central aspect of their dynamics [63, 78]. Tsuboi has found relations between the behavior of minimal sets for group actions and the homotopy theory of foliation classifying spaces [93, 95, 94]. The author has studied the relation between foliation dynamics and characteristic classes [62, 73]. There is a rapidly developing body of work on the structural stability and rigidity of group actions [40, 42, 44, 64, 77, 48, 1, 38]. The study of the dynamics of group actions and of foliations is fascinating for its complexity, and the added tools that arise from the multi-dimensional nature of the orbits. This paper uses just a few ideas from the dynamics of foliations, which we recall as needed below. A more extensive introduction can be found in the author's monograph [70]. Foliation dynamics is also closely related to the measurable dynamics and ergodic theory of group actions, which has been developed using cocycle theory by Zimmer [101, 102, 100, 106, 104, 105], and also in the context of the classification of von Neumann algebras [37, 81].

3.1 Basic topological dynamics

The most basic questions in topological dynamics address the qualitative properties of orbits of the system, and the nature of the saturated sets. A, $\mathcal{F}$-saturated set $X \subset M$ consists of a union of leaves of $\mathcal{F}$. That is, if L is a leaf with $L \cap X \neq \emptyset$ then $L \subset X$. The *$\mathcal{F}$-saturation* $\mathcal{F}Z$ of a set $Z \subset M$ consists of the union of all leaves which intersect Z. If $Z \subset U_\alpha$ we also define the local saturation

$$\mathcal{F}_\alpha Z = \bigcup_{P_\alpha(x) \cap Z \neq \emptyset} P_\alpha(x)$$

An *exhaustion sequence* for a leaf L is an increasing sequence of connected compact sets

$$K_1 \subset K_2 \subset \cdots K_n \subset \cdots \subset L$$

whose union is all of L. Define the ω-limit set of L to be the intersection

$$\omega(L) = \bigcap_{n=1}^{\infty} \overline{L - K_n}$$

where the closures are formed with respect to the topology on M. We recall some elementary facts:

PROPOSITION 3.1 • *$\omega(L)$ is compact and $\mathcal{F}$-saturated.*

- *$\omega(L)$ is connected if $L - K_n$ is connected for all n.*
- *$\omega(L)$ is independent of the choice of exhaustion sequence.*

This result implies a standard property of generalized dynamical systems. Recall that a compact, non-empty, $\mathcal{F}$-saturated set X is *minimal* for $\mathcal{F}$ if each leaf of X is dense in X. Equivalently, X is minimal with respect to the properties that it be closed, non-empty and $\mathcal{F}$-saturated.

COROLLARY 3.2 *Every closed $\mathcal{F}$-saturated non-empty subset $X \subset M$ contains a closed minimal set $Z \subset X$.*

Proof: The collection of closed $\mathcal{F}$-saturated subsets of X is closed under intersections, hence by Zorn's Lemma contains a minimal element Z. For each leaf $L \subset Z$, $\omega(L) \subset Z$ is a closed $\mathcal{F}$-saturated subset, hence must equal Z. □

The minimal set $Z \subset X$ need not be unique. For example, if $\mathcal{F}$ is a foliation with all leaves compact, then a minimal set for $\mathcal{F}$ consists of a single leaf, so that every closed $\mathcal{F}$-saturated set with more than one leaf contains more than one minimal set. (There are also much more sophisticated examples of non-uniqueness.)

We can associate to each leaf L the collection $\{Z \subset \omega(L) \mid Z \text{ is minimal }\}$. These are the invariant sets for $\mathcal{F}$ onto which the leaf L "spirals" as we go to infinity. In very special contexts [83, 17, 35], there are generalizations of the Poincaré-Bendixson Theory which give a relation between the global geometry of $\mathcal{F}$ and the structure of the minimal sets for $\mathcal{F}$, but very little seems to be known beyond these facts.

A leaf L is *proper* if the inclusion $L \hookrightarrow M$ induces from M the metric topology on L. Every compact leaf is proper, while a non-compact leaf is proper exactly when $L \cap \omega(L) = \emptyset$.

An *end* ϵ of a non-compact manifold L is determined by a choice of an open neighborhood system of ϵ, which is a collection $\{U_\alpha\}_{\alpha \in A}$ such that

- each U_α is an unbounded open subset of L,
- each finite intersection $U_{\alpha_1} \cap \ldots \cap U_{\alpha_q}$ is nonempty,
- the infinite intersection $\cap_1^\infty U_{\alpha_i} = \emptyset$.

Given an open neighborhood system $\{U_\alpha\}_{\alpha \in A}$ of ϵ, the ϵ-limit set of L is

$$\epsilon - \lim(L) = \bigcap_{\alpha \in A} \overline{U}_\alpha$$

Clearly, for each end ϵ, we have $\epsilon - \lim(L) \subset \omega(L)$. But $\omega(L)$ may include more points than just the union of the ϵ-limit sets of L. An end ϵ of L is *proper* if L is not contained in $\epsilon - \lim(L)$, and ϵ is *totally proper* if $\epsilon - \lim(L)$ is a union of proper leaves.

A leaf L' is said to be the *asymptote* of a leaf L if $\omega(L) = L'$. Note this implies that $\omega(L') = \emptyset$ and hence L' must be compact.

Zorn's Lemma implies that for each end ϵ of L, there is a minimal set contained in $\epsilon - \lim(L)$.

3.2 Expansion rate and entropy

The *expansion rate* and *geometric entropy* of a foliation provide some of the most effective dynamical invariants of foliations in higher codimensions.

The coarse length $|\gamma|$ of leafwise path $\gamma : [0, 1] \to L \subset M$ is the plaque-distance between the endpoints for a lift of γ to the holonomy cover of the leaf. A leafwise path γ with initial point $\gamma(0) = t_\alpha(z_0) \in \mathcal{T}_\alpha$ and final point $\gamma(1) = t_\beta(z_1) \in \mathcal{T}_\beta$ on transversals to $\mathcal{F}$ is said to be *subordinate* to the transversal $\mathcal{T}$. A subordinate path induces local holonomy maps $h_\gamma: U_\alpha \to U_\beta$ for open sets $t_\alpha(z_0) \in U_\alpha \subset \mathcal{T}_\alpha$ and $t_\beta(z_1) \in U_\beta \subset \mathcal{T}_\beta$

Let $D: M \times M \to [0,1]$ be a metric with diameter 1. Define metrics $D_\alpha: \mathcal{T}_\alpha \times \mathcal{T}_\alpha \to [0,1]$ by restriction. For each integer $R > 0$ we define a metric on $\mathcal{T}$ by setting, for $x, y \in \mathcal{T}_\alpha$

$$d_R(x,y) = \max\{D_\beta(h_\gamma(x), h_\gamma(y)) \text{ such that } \mid \gamma \mid \leq R \;\&\; \gamma \text{ subordinate to } \mathcal{T}\}$$

Extend this to a metric d_R on all of $\mathcal{T}$ by setting $d_R(x,y) = 1$ for x and y on distinct transversals. The metrics d_R strongly depend upon the choice of the foliation covering.

For $0 < \epsilon < 1$ and $R > 0$, we say that a finite subset $\{x_1, \ldots, x_\ell\} \subset \mathcal{T}$ is (ϵ, r)-*spanning* if for any $x \in \mathcal{T}$ there exists x_i such that $d_R(x, x_i) < \epsilon$. Let $H(\mathcal{F}, \epsilon, R)$ denote the minimum cardinality of an (ϵ, R)-spanning subset of $\mathcal{T}$. The ϵ-*expansion growth* of $\mathcal{F}$ is the growth class of the function $R \mapsto H(\mathcal{F}, \epsilon, R)$. This function is one of the basic measures of the "transverse dynamics" of a foliation (cf. § 3 [34], and for a detailed discussion see [36]).

Let $Z \subset M$ be an $\mathcal{F}$-saturated set. The restricted spanning function $H(Z|\mathcal{F}, \epsilon, R)$ equals the minimum cardinality of an (ϵ, R)-spanning subset of $\mathcal{T} \cap Z$. Clearly, $H(Z|\mathcal{F}, \epsilon, R) \leq H(\mathcal{F}, \epsilon, R)$.

Note the two properties:

$$\epsilon' < \epsilon \text{ implies } H(\mathcal{F}, \epsilon', R) \geq H(\mathcal{F}, \epsilon, R) \text{ for all } R > 0$$

$$R' > R \text{ implies } H(\mathcal{F}, \epsilon, R') \geq H(\mathcal{F}, \epsilon, R) \text{ for all } \epsilon > 0$$

The *geometric entropy* of Ghys, Langevin and Walczak [34] is the limit

$$h(\mathcal{F}) = \lim_{\epsilon > 0} \; h(\mathcal{F}, \epsilon) \quad \text{where} \quad h(\mathcal{F}, \epsilon) = \limsup_{R \to \infty} \frac{\log H(\mathcal{F}, \epsilon, R)}{R} \tag{7}$$

The limit (7) is finite for a transversally, C^1-foliation [34], but may be infinite for topological foliations.

3.3 Structure theory for topological foliations

A key structure property of foliated manifolds is the *Product Neighborhood Theorem*, which is a direct generalization of the foliated neighborhood theorem for a compact leaf with finite holonomy (cf. Haefliger [51]). For $K \subset M$ and $\epsilon > 0$, let $\mathcal{N}(K, \epsilon)$ be the open neighborhood consisting of points which lie within ϵ of K.

THEOREM 3.3 ([65]) *Let L be a leaf with holonomy covering $\tilde{L}$. Given a compact subset $K \subset \tilde{L}$ and $\epsilon > 0$, there exists a foliated immersion $\Pi: K \times (-1,1)^q \to M$ so that the restriction $\Pi: K \times \{0\} \to L \subset M$ coincides with the restriction to K of the covering map $\pi: \tilde{L} \to L$, and $\Pi(K \times (-1,1)^q) \subset \mathcal{N}(\pi(K), \epsilon)$.*

This result is a mild generalization of the usual proof of of the Reeb Stability Theorem. The importance of this property is that it relates, at a basic level, the dynamics of the leaves (*nb.* the compact set K can be chosen arbitrarily large in $\tilde{L}$) with the transversal structure of $\mathcal{F}$ (the image of the tubular neighborhood $K \times (-1,1)^q$ is an open "shadow neighborhood" of L).

For codimension-one foliations there is much more global structure theory, even in the general case of C^0-foliations. (Hector and Hirsch [53] is an excellent general reference for their structure theory.) We assume that $\mathcal{F}$ is transversely orientable, and fix a topological foliation $\mathcal{N}$ of dimension 1 transverse to $\mathcal{F}$. Let U be an open set in M saturated by $\mathcal{F}$. The *completion* $\hat{U}$ of U is a manifold with boundary equipped with

- a codimension 1 C^0-foliation $\hat{\mathcal{F}}$ tangent to the boundary,
- a continuous map $i: \hat{U} \to M$ which restricts to a homeomorphism from the interior of $\hat{U}$ onto U, so that
- the restriction of $\hat{\mathcal{F}}$ to the interior of $\hat{U}$ agrees with $i^*\mathcal{F}$.

THEOREM 3.4 (Dippolito [31]) *Under the preceding conditions, there is a compact submanifold with boundary and corners K of $\hat{U}$ so that $\partial K = \partial^{tg} \cup \partial^{tr}$ with*

i. $\partial^{tg} \subset \partial\hat{U}$

ii. ∂^{tr} is saturated by the foliation $i^*\mathcal{N}$.

iii. The complement of the interior of K in $\hat{U}$ is the finite union of non-compact submanifolds B_i with boundary and corners homeomorphic to $S_i \times [0,1]$ by a homeomorphism $\phi_i : S_i \times [0,1] \to B_i$ so that $\phi_i(\{*\} \times [0,1])$ is a leaf of $i^*\mathcal{N}$.

The foliation restricted to B_i is defined by suspension of a representation of the fundamental group of S_i into the group of homeomorphisms of the interval $[0,1]$.

3.4 Invariant measures

A *transversal* $T \subset M$ to $\mathcal{F}$ is a Borel subset which intersects each leaf in at most a countable set. A cross-section is thus a special case of a transversal, and one can show (via a Borel selection process) that every transversal is a countable union of local cross-sections.

A *transverse measure* μ for $\mathcal{F}$ is a locally-finite measure on transversals, whose *measure class* is invariant under the transverse holonomy transformations. The transverse measure class defined by μ is equivalently specified by giving finite Borel measures μ_α on each set X_α, so that each local holonomy map $\gamma_{\alpha\beta}$ pulls the *measure class* of $\mu_\beta | X_{\beta\alpha}$ back to that of $\mu_\alpha | X_{\alpha\beta}$. A local cross-section $Z \subset U_\alpha$ is said to have μ-measure zero if and only if $\mu_\alpha(\phi_\alpha(\mathcal{F}_\alpha Z \cap T_\alpha)) = 0$.

We say that μ is an *invariant transverse measure* for $\mathcal{F}$ if the local holonomy preserves the measure; that is, $\gamma^*_{\alpha\beta}\mu_\beta = \mu_\alpha$ for all admissible α, β. The measure of a local cross-section $Z \subset U_\alpha$ is defined as

$$\mu(Z) = \mu_\alpha(\phi_\alpha(\mathcal{F}_\alpha Z \cap T_\alpha))$$

This is extended as a countably additive measure to all transversals $Z \subset M$: use a selection lemma to decompose

$$Z = \bigcup_{\alpha=1}^{k} \bigcup_{i=1}^{\infty} Z_{\alpha,i}$$

where each $Z_{\alpha,i} \subset U_\alpha$ is a local cross-section, then define

$$\mu(Z) = \sum_{\alpha=1}^{k} \sum_{i=1}^{\infty} \mu_\alpha(Z_{\alpha,i})$$

The holonomy invariance of the measures μ_α implies that $\mu(Z)$ is independent of the choice of the decomposition.

DEFINITION 3.5 *A* measured foliation *is a triple* $(M, \mathcal{F}, \mu)$ *where* $(M, \mathcal{F})$ *is a foliated manifold and* μ *is an invariant transverse measure for* $\mathcal{F}$.

The *support* of a transverse measure μ consists of the smallest closed saturated subset $s(\mu) \subset M$ so that $\mu(Z) = 0$ for any transversal $Z \subset M \setminus X$. Note that the pair $(s(\mu), \mathcal{F}|s(\mu))$ formed by the support a transverse measure μ and the restriction of $\mathcal{F}$ forms a foliated space (cf. [82, 73]) and the triple $(s(\mu), \mathcal{F}|s(\mu), \mu)$ is a foliated measure space.

4 Open manifolds of positive entropy

Our first application of the ideas of coarse geometry is to give examples of open complete Riemannian manifolds of bounded geometry which are not quasi-isometric to leaves of any C^1 foliation of a compact manifold. It is well-understood that a leaf of a foliation of a compact manifold has "recurrence", so that an open complete manifolds without "recurrence" cannot be a leaf. The problem is to quantify this idea of recurrence in the coarse geometry of an open manifold, to obtain an obstacle to its being a leaf. This was done in a joint work with Oliver Attie [6], which studied the relation between *bg* surgery theory on open manifolds and the properties of leaves. Here is the precise result:

THEOREM 4.1 (Attie-Hurder [6]) *There exists an uncountable set of quasi-isometry types Riemannian manifolds of bounded geometry and exponential volume growth, none of which is quasi-isometric to a leaf of a C^0-foliation whose expansion growth is less than $[2^{b^r}]$ for all $b > 1$. In particular, these manifolds cannot be quasi-isometric to leaves of C^1-foliations of any codimension.*

Gromov has observed every complete open manifold of bounded geometry is a leaf of a compact "foliated space" X, though X need not be a manifold. The idea of the theorem is to continue this viewpoint, and define *growth complexity function* of open manifolds and the entropy of the open manifold which is derived from it. These yield invariants of the quasi-isometry class. An open manifold of exponential growth which is quasi-isometric to a leaf must have "zero entropy", due to a well-known estimate of the *expansion growth function* of a C^1-foliation [36]. We then show how to construct open manifolds of bounded geometry with positive entropy. This is a special case of a construction of a class of open manifolds indexed by the points of a Markov process (see [71]) – and for almost every process, the corresponding open manifold has positive entropy.

4.1 Leaf entropy

The following definitions expand on a remark of Gromov:

DEFINITION 4.2 *Fix $\epsilon, R > 0$. An (ϵ, R) quasi-tiling of a complete Riemannian manifold M is a collection $\{K_1, \ldots, K_d\}$ of a compact metric spaces with diameters at most R and a countable set of homeomorphisms into $\{f_i : K_{\alpha_i} \to M \mid i \in \mathcal{I}\}$ with:*

- *Each f_i is a quasi-isometry onto its image with $\lambda(f_i) \leq \epsilon$ and $D(f_i) \leq \epsilon$.*
- *For any set K of diameter at most R, there exist $i \in \mathcal{I}$ so that $K \subset f_i(K_{\alpha_i})$.*

REMARK 4.3 • *The integer d is called the cardinality of the quasi-tiling.*

• *The images $f_i(K_{\alpha_i})$ have diameter at most $\epsilon(R+1)$.*

DEFINITION 4.4 *For $\epsilon \geq 0$, the ϵ-growth complexity function of M is*

$$H(M, \epsilon, R) = \min\{d \mid \textit{there exists an } (\epsilon, R) \textit{ quasi-tiling of } M \textit{ of cardinality } d\}$$

If no (ϵ, R) quasi-tiling exists, then set $H(M, \epsilon, R) = \infty$.

The following remarks are obvious from the definition:

PROPOSITION 4.5 *Given a quasi-isometry $f: M \to M'$, for $\epsilon \gg 0$ there exists $\epsilon' > 0$ so that for all $R > \epsilon$*

$$H(M', \epsilon', R - \epsilon) \leq H(M, \epsilon, R) \leq H(M', \epsilon', R + \epsilon)$$

PROPOSITION 4.6 *For $\epsilon' > \epsilon$, $H(M, \epsilon', R) \leq H(M, \epsilon, R)$*

For a complete open manifold M, define

$$\mathcal{V}(M, R) = \sup\{\text{vol}(B(x, R)) \mid x \in M\}$$

where $B(x, R) \subset M$ denotes the ball of radius R centered at x. We can then set

DEFINITION 4.7 *The entropy of an open complete manifold is*

$$h(M) = \lim_{\epsilon \to \infty} \limsup_{R \to \infty} \frac{\ln(H(M, \epsilon, R))}{\mathcal{V}(M, R)}$$

When L is a leaf of a foliation of a compact manifold, endowed with a metric restricted from one on M, then we call $h(M)$ the *leaf entropy* of L.

The growth complexity function for a leaf is related to the entropy of the foliation. Here is the key technical observation. Let $\mathcal{F}$ be a C^0 codimension-q foliation of a compact manifold V. Fix the path-length metric on V associated to a Riemannian metric on TV. Choose local foliation coordinate charts $\phi_\alpha : U_\alpha \to (-1, 1)^m$ as in section 2.2, for which we can then define the function $H(\mathcal{F}, \epsilon, R)$ which is the maximal cardinality of an (ϵ, R)-spanning subset of the transversal space $\mathcal{T}$.

PROPOSITION 4.8 ([6]) *Let $L \subset V$ be a simply connected leaf of a C^0-foliation. For each $R > 0$ there exists an open covering $\{\mathcal{V}_\beta \mid \beta \in \mathcal{B}\}$ of V so that*

(1) the cardinality of $\mathcal{B}$ is at most $H(L|\mathcal{F}, 1/2, R)$;

(2) *each $\mathcal{V}_\beta$ is a foliated product;*

(3) *for each leaf $L' \subset \overline{L}$ the restriction of the covering $\{\mathcal{V}_\beta\}$ to L' has Lebesgue number at least $R - 3$.*

In particular, the cardinality of $\mathcal{B}$ is at most $H(\mathcal{F}, 1/2, R)$, independently of L.

Proof: Choose a $(1/2, R)$-spanning subset $\{x_1, \ldots, x_{d(R)}\} \subset \mathcal{T} \cap L$ of cardinality $d(R) = H(L|\mathcal{F}, 1/2, R)$. Let α_i be the index for which $x_i \in \mathcal{T}_{\alpha_i}$. Let $K_i \subset L$ denote the union of the plaques in L which can be reached from x_i by a leafwise path of length at most $R-1$. Then each point in the intersection $K_i \cap \mathcal{T}$ can be joined to x_i by a leafwise path of length at most R.

Let $B_R(x_i, 3/4) \subset \mathcal{T}_{\alpha_i}$ be the ball centered at x_i of radius $3/4$ in the metric d_R restricted to the transversal. Define $\mathcal{V}_i$ to be the union of all plaques of $\mathcal{F}$ which can be joined to $t_{\alpha_i}(B_R(x_i, 3/4))$ by a leafwise path of length at most $R-1$.

We show that the collection $\{\mathcal{V}_\beta \mid 1 \leq \beta \leq d(R)\}$ is a covering of $\overline{L}$. Let $x \in \overline{L}$, then $x \in U_\alpha$ for some α and so lies on a plaque $\mathcal{P}_\alpha(z_x)$ for some $z_x \in \mathcal{T}_\alpha$. The metric d_R is quasi-isometric to the Riemannian metric on V, so there exists $z_x^* \in L \cap \mathcal{T}_\alpha$ so that $d_R(z_x, z_x^*) < 1/4$. The 1/2-spanning property then implies there exists $x_i \in \mathcal{T}_\alpha$ with $d_R(z_x^*, x_i) < 1/2$, hence $x \in \mathcal{P}_\alpha(z) \subset \mathcal{V}_i$.

The proof of Theorem 3.3 shows that we can choose the homeomorphisms $\Pi_i \colon K_i \times (-1,1)^q \to \mathcal{V}_i$ to satisfy:

• the leafwise restriction $\Pi_i \colon K_i \times \{0\} \to L \subset V$ is the inclusion; and

• the transverse restriction $\Pi_i \colon \{x_i\} \times (-1,1)^q \to t_{\alpha_i}(B_R(x_i, 1/2))$ is a homeomorphism onto.

Finally, let $Z \subset L' \subset \overline{L}$ be a connected compact subset of diameter at most $R-3$ in a leaf L'. Let $\hat{Z}$ be the union of the plaques with non-empty intersection with Z; then $\hat{Z}$ has diameter at most $R-1$. Choose a point $z \in \hat{Z} \cap \mathcal{T}$ and a point $x_i \in \mathcal{T}_\alpha$ so that $d_R(z, x_i) < 1/2$. Then clearly $\hat{Z} \subset \mathcal{V}_i$. □

The leaf L is itself an $\mathcal{F}$-saturated set, so we can define the growth function of L.

THEOREM 4.9 *Let L be a simply-connected leaf of a C^0-foliation $\mathcal{F}$ of a compact manifold V. Then*

$$H(L, 1, R-3) \leq H(L|\mathcal{F}, 1/2, R) \qquad \textit{for all} \quad R > 3$$

Proof: Let $\{\mathcal{V}_\beta \mid 1 \leq \beta \leq d(R)\}$ be a covering associated to a $(1/2, R)$-spanning set as above, with local homeomorphisms onto $\Pi_i \colon K_i \times (-1,1)^q \to \mathcal{V}_i$. For each $z \in L \cap \mathcal{T}_{\alpha_i}$ define a homeomorphism $f_{i,z} = \Pi_i(\cdot, z) \colon K_i \to L$. As all plaques have diameter at most 1, each $f_{i,z}$ is a quasi-isometry onto its image with distortion $D(f_{i,z}) \leq 1$. So by Proposition 4.8.3 above, the collection $\{K_1, \ldots, K_{d(R)}\}$ with maps $\{f_{i,z}\}$ forms a $(1, R-3)$-quasi-tiling of L of cardinality at most $H(L|\mathcal{F}, 1/2, R)$. □

A geometric estimate based on the mean value theorem estimates yields the following estimate:

LEMMA 4.10 ([34]) *Let $\mathcal{F}$ be a codimension-q, transversally Lipshitz foliation of a compact manifold M. Then for each leaf $L \subset M$ and all $\epsilon > 0$, there is a constant $C(L, \epsilon) > 0$ so that*

$$H(L|\mathcal{F}, \epsilon, R) < C(L, \epsilon) \cdot \exp\{qR\}$$

We combine Proposition 4.5, Theorem 4.9 and Lemma 4.10 to obtain:

COROLLARY 4.11 *Let L be a complete open manifold such that $\mathcal{V}(L, R)$ has exponential growth, which is quasi-isometric to a leaf of a codimension-q, transversally C^1-foliation $\mathcal{F}$ of a compact manifold M. Then the entropy $h(L) = 0$.*

4.2 A construction of non-leaves

In this section, we recall the construction from [6] of open manifolds $\mathcal{M}$ with exponential growth type such that there are constants $a, b > 1$ and $H(\mathcal{M}, \epsilon, R)$ has ϵ-growth type $[a^{b^R}]$ for all $\epsilon > 0$. Hence, $h(\mathcal{M}) > 0$ so that by Corollary 4.11, $\mathcal{M}$ is not quasi-isometric to a leaf of any C^α-foliation of a compact manifold. Our construction will connect-sum an infinite number of copies of $S^4 \times S^2$ onto the hyperbolic n-space $\mathcal{M}_0 = \mathbf{H}^6$, chosen so that we force every quasi-tiling to have maximum growth rate. The role of hyperbolic space can be replaced by the universal cover $\tilde{B}$ of any compact 6-manifold B whose fundamental group Γ has exponential growth.

The construction we give below is reasonably complicated and based on a combinatorial idea, as it must be. But it has a simple guiding framework: we propose to wire the base manifold $\mathcal{M}$ with a pattern of light sockets, and then into each socket we have a choice of what color light bulb to install. The task is to wire the manifold $\mathcal{M}$ so that each pattern can be distinguished by a quasi-isometric homeomorphism. Moreover, each pattern will be repeated enough times to allow all possible bulb patterns to be realized (within a range of choices of colors). The number of patterns grows exponentially with R, due to the basic volume estimate on the number of balls of a given radius $r > 0$ in a ball of radius $R \gg r$ in hyperbolic space. Hence the number of substitution patterns grows super-exponentially. Of course, we confuse matters in the construction below by replacing the bulbs and their colors with surgered copies of $S^4 \times S^2$ having differing Pontrjagin classes (the colors) and noting that one can see these colors under a homeomorphism. The basic idea is more general, and reflects a "Markovian" property at infinity.

Let $B(x, R)$ denote the ball of radius R centered at $x \in \mathbf{H}^6$. Our construction is based on the following property of manifolds of uniformly exponential growth:

PROPOSITION 4.12 *There exist a constant $c > 1$ so that each $x \in \mathbf{H}^6$ and $R > r > 0$, the ball $B(x, R)$ contains at least $\lfloor c^{R-r} \rfloor$ pairwise disjoint balls of radius r.*

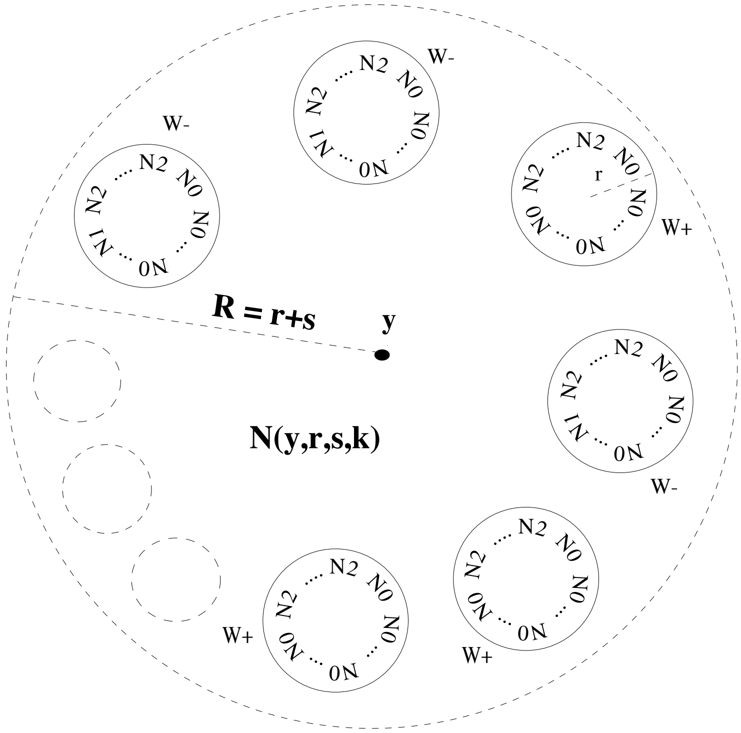
Given $x \in \mathbf{H}^6$ and $r > 0$, choose $d = \lfloor c^r \rfloor$ points $\{x_1, \ldots, x_d\} \subset B(x, r)$ such that the balls $\{B(x_i, 1) \mid 1 \leq i \leq d\}$ are contained in $B(x, r)$ and are pairwise disjoint.

Next, fix model manifolds N_ℓ for $0 \leq \ell \leq 2$, each homotopy equivalent to $S^4 \times S^2$, with $p_1(N_\ell) = \ell \in H^4(S^4 \times S^2; \mathbf{Z}) \cong \mathbf{Z}$. Fix a Riemannian metric on N_ℓ with injectivity radius at least 1/2, and choose a disk of radius 1/2 in N_ℓ which will be the center for a connected sum operation.

For each integer $1 \leq k < d$ construct a manifold $W^+(x, r, k)$ with boundary the sphere $S(x, r)$ of radius r: for $i \leq k$, connect sum N_2 to the ball $B(x_i, 1/2)$; and for for $k < i < d$, connect sum N_0 to the ball $B(x_i, 1/2)$. Note that $W^+(x, r, d)$ has a standard collar neighborhood of radius 1/2 about its boundary.

Modify this construction to define $W^-(x, r, d)$, where we now attach N_1 to the ball $B(x_d, 1/2)$ in $W^+(x, r, d)$.

We repeat this procedure a second time, where for $y \in \mathbf{H}^6$ and $R > s$ we choose points $\{y_1, \ldots, y_D\} \subset B(y, R)$ where $D = \lfloor c^{R-s} \rfloor$, so that the balls $B(y_i, s)$ are contained in $B(y, R)$ and are pairwise disjoint. Assume that $s \geq r$ and set $R = r + s$ so that $D \geq d = \lfloor c^{r-1} \rfloor$, and choose a sequence $\vec{k} = \{k_1, \ldots, k_d\}$ with each $k_i \in \{\pm\}$. For each $1 \leq i \leq d$, surger in a copy of $W^{k_i}(y_i, r, i)$ in place of the ball $B(y_i, r)$. Label the resulting manifold $N(y, r, s, \vec{k})$. Again, note that the boundary of $N(y, r, s, \vec{k})$ is a sphere of radius R about y and admits a product neighborhood. Here is a diagram of the basic building block:



The purpose of this complicated construction of the modified disks $N(y, r, s, \vec{k})$ of radius R in $\mathbf{H}^6$ is to create a set of standard "models" which have distinct quasi-isometry types. There are 2^d choices of the sequences $\vec{k} = \{k_1, \ldots, k_d\}$, hence an equivalent number of manifolds $N(y, r, s, \vec{k})$. Let $\widehat{N}(y, r, s, \vec{k})$ be the result of attaching $N(y, r, s, \vec{k})$ to $\mathbf{H^6}$ in place of the ball $B(y, r+s)$.

PROPOSITION 4.13 *Let* $h\colon N(y, r, s, \vec{k}) \rightarrow \widehat{N}(z, r, s, \vec{\ell})$ *be a quasi-isometric homeomorphism with* $\lambda(h) \leq \epsilon$ *and* $D(h) \leq \epsilon$. *If* $s > 2\epsilon(2r+1)$, *then* $\vec{k} = \vec{\ell}$.

Proof: Let us show that $k_i = \ell_i$. Let $x_1 \in W^{k_i}(y_i, r, i)$ be the first point in the construction of this set. Then the image of the set $W^{k_i}(y_i, r, i)$ under the map h must be contained in the ball $B(h(x_1), \epsilon(2r+1))$. The point $h(x_1)$ must lie in one of the sets $W^{\ell_a}(z_a, r, a)$ used to construct $\widehat{N}(z, r, s, \vec{\ell})$. By the choice of s, the intersection $B(h(x_1), \epsilon(2r+1)) \cap W^{\ell_b}(z_b, r, b)$ is empty unless $a = b$. It follows that $W^{k_i}(y_i, r, i)$ must be mapped quasi-isometrically onto $W^{\ell_a}(z_a, r, a)$.

We can now count the total number of summands of $S^4 \times S^2$ in $W^{\ell_a}(z_a, r, a)$ with positive even Pontrjagin class to obtain that $i = a$. Finally, if $k_i =$ "$-$" then there must also be a summand of $S^4 \times S^2$ in $W^{\ell_i}(z_i, r, i)$ with positive odd Pontrjagin class, hence $\ell_i =$ "$-$". Otherwise, $\ell_i =$ "$+$". This proves the proposition.

Choose a geodesic curve $g\colon (-\infty, \infty) \rightarrow \mathbf{H}^6$. We observe that g is a "straight" curve in the sense of Gromov; that is, the distance $d_{\mathbf{H}^6}(g(r), g(s)) = |r - s|$.

For each integer $i > 0$, set $w_i = g(i!)$.

We are now in a position to inductively define the manifold $\mathcal{M}$ which is not a leaf. Set $\mathcal{M}(0) = \mathbf{H}^6$. Fix $n > 0$ and assume that $\mathcal{M}(n-1)$ has been defined. There are 2^d choices of the manifolds $N(y, n, \mu n, \vec{k})$, where $d = \lfloor c^n \rfloor$ and μ is a positive integer. For each $1 \leq \mu \leq n^2$, attach these 2^d choices onto a subset of the points $\{w_i \mid i > n\}$ which have not been modified in a previous step. This produces $\mathcal{M}(n)$. (That is, we are essentially implementing a diagonalization procedure in order to list all of the choices of these manifolds, spaced out along the increasingly distant points $\{w_i\}$.) Let $\mathcal{M}$ be the direct limit manifold obtained by this inductive procedure.

The following estimate now completes the proof of Theorem 4.1:

PROPOSITION 4.14 *There exists $b > 0$ so that for all $\epsilon > 0$, $H(\mathcal{M}, \epsilon, R) \geq 2^{b^R}$ for $R \gg 0$.*

Proof: Fix $\epsilon > 1$ and an integer $R = n > 10\epsilon^2$. Let $\{K_1, \ldots, K_\nu\}$ be an (ϵ, R) quasi-tiling of $\mathcal{M}$ with countable set of homeomorphisms into $\{f_i : K_{\alpha_i} \to \mathcal{M}\}$ so that:

- Each f_i is a quasi-isometry onto its image with $\lambda(f_i), D(f_i) \leq \epsilon$.
- $\{f_i(K_{\alpha_i})\}$ is an open covering of L with Lebesgue number at least R.

Set $\xi = 4(n+1)\epsilon^2$. Distinct submanifolds $N(y, n, \xi, \vec{k})$ and $N(z, n, \xi, \vec{\ell})$ of $\mathcal{M}$, each of diameter $\xi + n$, are separated by a distance at least $(n-1)! - 2(\xi + n) > \epsilon(n+1)$. The diameter of each set $f_i(K\alpha_i)$ is at most $\epsilon(n+1)$, so the image of the quasi-isometry f_i which contains a set $N(y, n, \xi, \vec{k})$ will intersect no other set of this type.

Assume there are two such maps defined on a common K_{α_i}, with $N(y, n, \xi, \vec{k}) \subset f_i(K_{\alpha_i})$ and $N(z, n, \xi, \vec{\ell}) \subset f_j(K_{\alpha_i})$. Then $f_j \circ f_i^{-1}$ restricts to a quasi-isometry from $N(y, n, \xi, \vec{k})$ to $N(z, n, \xi, \vec{\ell})$ with $\lambda(f_j \circ f_i^{-1}) \leq 2\epsilon$ and $D(f_j \circ f_i^{-1}) \leq \epsilon^2$. Apply the above proposition to conclude that $\vec{k} = \vec{\ell}$. In particular, $\nu \geq 2^d$ where $d = \lfloor c^{n-1} \rfloor$. Take $1 < b < c$ and the proposition follows.

Lecture II - Dimensions of Ends

5 Coarse cohomology

It is natural to consider the coarse geometry of a complete metric space L as measuring only the "relative size" and "position" of objects in L, which is essentially the idea behind the examples of the last section. It is surprising, both at first glance and upon continued reflection, that coarse geometry also captures global cohomology invariants of L – using the coarse cohomology theory of John Roe [86, 88]. We give a brief definition and introduction to this theory, in terms suitable for the other ideas to be presented. The interested reader is strongly advised to read [88] for a proper treatment!

5.1 Coarse cohomology for manifolds and nets

Let L be a complete Riemannian manifold of bounded geometry. A *multi-diagonal* Δ_d for L is a set $\{(x, \ldots, x) \in L^d \mid x \in L\}$ for some $d > 1$. A *uniform tube* about Δ_d is a set

$$\mathcal{U}_\epsilon = \{(x_1, \ldots, x_d) \mid d_L(x_i, x_j) < \epsilon\}$$

for some $\epsilon > 0$.

DEFINITION 5.1 (Roe [88]) *The coarse cohomology $HX^*(L; \mathbf{R})$ is the cohomology of the subcomplex of the Alexander-Spanier cochains on L whose supports intersect each uniform tube around a multi-diagonal in a compact set.*

This definition was inspired from the work of Connes and Moscovici on index theory [28, 29]. In fact, many of the results of coarse cohomology theory are closely related to properties of index theory of open complete manifolds. In spite of the simplicity of the above definition of coarse cohomology, the calculation of $HX^*(L; \mathbf{R})$ is far from obvious in most cases: Theorem 3.14 [88] gives the basic structure theorem relating it to usual cohomology theories, while the works [87, 57, 59]) develop various Mayer-Vietoris techniques for calculating it.

Roe established several basic properties of coarse cohomology , which begin to explain the interest in the theory. The first is invariance under coarse isometries:

THEOREM 5.2 (Corollary 3.35 [88]) *Let L and L' be complete Riemannian manifolds, and suppose there exists coarse isometry $f\colon L \to L'$. Then f induces an isomorphism $f^*\colon HX^*(L'; \mathbf{R}) \to HX^*(L; \mathbf{R})$.*

In order to convey the ideas of how coarse geometry is applied, we adopt an *ersatz* coarse cohomology theory which is much more transparent in its properties, and yet captures the basic flavor of the subject. Let $H^*_c(L; \mathbf{R})$ denote the Čech cohomology with compact supports on L, which has a natural map into the usual cohomology theory $H^*(L; \mathbf{R})$ without restrictions on support.

DEFINITION 5.3 *The* ersatz coarse cohomology *of L is defined to be the kernel*

$$HX^*_{er}(L;\mathbf{R}) = \ker\{H^*_c(L;\mathbf{R}) \longrightarrow H^*(L;\mathbf{R})\}$$

A second main result in coarse cohomology theory is the existence of the character map:

PROPOSITION 5.4 (section 2.10,[88]) *There is a natural character map*

$$c: HX^*(L;\mathbf{R}) \to HX^*_{er}(L;\mathbf{R})$$

Roe proves that for manifolds which have a very strong control over their structure in a neighborhood of infinity, either in terms of a uniform contraction mapping (Proposition 3.39 [88]) or the manifold is quasi-isometric to a metric cone over a compact metric space (Proposition 3.49 [88]), then $HX^*_{er}(L;\mathbf{R}) = HX^*(L;\mathbf{R})$. In almost all applications of coarse cohomology theory to connected open manifolds, the approximation $HX^*_{er}(L;\mathbf{R})$ reveals the key intuitive insights.

The definition of coarse cohomology actually makes sense for discrete metric spaces as well as manifolds. Recall that a subset $\mathcal{N} \subset L$ is a *net* if there exists constants $0 < c_1 < c_2$ so that for any two points $x, y \in \mathcal{N}$ we have $d_L(x,y) > c_1$ yet for any point $z \in L$ there exists a point $x \in \mathcal{N}$ with $d_L(z,x) < c_2$. It is elementary to construct a net for a complete open manifold, using a simple induction procedure. We have the remarkable corollary of Theorem 5.2:

COROLLARY 5.5 *The inclusion $\mathcal{N} \subset L$ induces an isomorphism*

$$HX^*(\mathcal{N};\mathbf{R}) \cong HX^*(L;\mathbf{R})$$

There is no restriction on the constants c_1 and c_2 in the definition of a net. One usually tends to think of them as small quantities, but here we consider them as possibly very large numbers. For large values, the coarse cohomology of a net, $HX^*(\mathcal{N};\mathbf{R})$, is obviously seen to be a combinatorial invariant of the coarse geometry of L.

5.2 Two basic examples

Here are two basic examples of the calculation of coarse cohomology for open manifolds, both due to J. Roe [88]. The examples begin to give an intuitive feel for the theory.

PROPOSITION 5.6 (Proposition 2.25 [88]) *Let L be a complete open connected metric space, and $\epsilon(L)$ the topological space of ends for L. There is an exact sequence*

$$0 \longrightarrow \mathbf{R} \longrightarrow \check{H}^0(\epsilon(L);\mathbf{R}) \longrightarrow HX^1(L;\mathbf{R}) \longrightarrow 0$$

Hence, the degree one coarse cohomology $HX^1(L;\mathbf{R})$ can be calculated from the most basic topological property of the "space at infinity" for L, its end-space.

PROPOSITION 5.7 *Let L be a complete open manifold which is diffeomorphic to the interior of a compact manifold M with boundary ∂M. There is a natural surjection $H^*(\partial M;\mathbf{R}) \to HX^{*+1}(L;\mathbf{R})$ with kernel the image of the restriction map $H^*(M;\mathbf{R}) \to H^*(\partial M;\mathbf{R})$.*

Proof: A slight modification of the proof of Lemma 3.29 [88] shows that the character map $c\colon HX^*(L;\mathbf{R}) \to HX^*_{er}(L;\mathbf{R})$ is an isomorphism in this case, so it suffices to identify $H^*(\partial M;\mathbf{R})$ with the ersatz cohomology of L. But this follows from the commutative diagram:

$$\begin{array}{ccccccc} H^*(M;\mathbf{R}) & \longrightarrow & H^*(\partial M;\mathbf{R}) & \longrightarrow & H^{*+1}(M,\partial M;\mathbf{R}) & \longrightarrow \\ & & \downarrow & & \downarrow\cong & \\ 0 & \longrightarrow & HX^*_{er}(L;\mathbf{R}) & \longrightarrow & H^{*+1}_c(M;\mathbf{R}) & \longrightarrow \end{array}$$

Proposition 5.7 holds more generally for connected metric spaces which admit a "metric product neighborhood" at infinity. The prime examples of these are the metric cones described in section 6.4 below.

5.3 Coarse cohomology for foliations

The definition of coarse cohomology extends very naturally to the case of foliations [55]. The basic idea is to consider a subcomplex of the Alexander-Spanier cochains on the holonomy groupoid $\mathcal{G}_{\mathcal{F}}$, with a support condition for their restriction to uniform tubes around the fiberwise diagonals. The actual condition is more delicate, but the general idea is just that.

The ersatz coarse theory is easier to define, under the assumption that $\mathcal{G}_{\mathcal{F}}$ is Hausdorff:

$$HX^*_{er}(\mathcal{F};\mathbf{R}) = \ker\{H^*_c(\mathcal{G}_{\mathcal{F}};\mathbf{R}) \longrightarrow H^*(\mathcal{G}_{\mathcal{F}};\mathbf{R})\}$$

and there is again a natural character map $c\colon HX^*(\mathcal{F};\mathbf{R}) \to HX^*_{er}(\mathcal{F};\mathbf{R})$.

The key property of coarse cohomology for foliations is that $HX^*(\mathcal{F};\mathbf{R})$ is an invariant of the coarse isometry type of $\mathcal{F}$. In particular, by Proposition 2.7 we have the basic result:

PROPOSITION 5.8 ([55]) *Let $f\colon M_1 \to M_2$ be a leafwise homotopy equivalence between topological foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of a compact manifolds M_1 and M_2, respectively. Then f induces an isomorphism*

$$f^*\colon HX^*(\mathcal{F}_2;\mathbf{R}) \cong HX^*(\mathcal{F}_2;\mathbf{R})$$

It is a very open problem to calculate the groups $HX^*(\mathcal{F};\mathbf{R})$ for some model classes of foliations (cf. section 6.5 below). A natural first (and accessible) case is to determine the ersatz groups for codimension-one foliations of 3-manifolds.

6 Coronas everywhere

Proposition 5.7 suggests investigating the relation between the coarse cohomology $HX^*(L;\mathbf{R})$ – or for an open connected manifold L, the ersatz groups $HX^*_{er}(L;\mathbf{R})$ – and the transgressed cohomology of some sort of boundary for L. In general, it is a fundamental problem

to define a "good" compact boundary ∂L for a complete metric space L (cf. § 2, [47]). A "good boundary theory" should have the property that a coarse isometry of metric spaces induces a homeomorphism of their boundaries – in particular, ∂L should depend only on the coarse isometry class of L. The usual cohomology groups $H^*(\partial L;\mathbf{R})$ of the boundary would then be invariants of the coarse isometry type of L. One can then ask if there is a natural map $\delta^*\colon H^*(\partial L;\mathbf{R}) \to HX^*(L;\mathbf{R})$, and under what conditions is it an isomorphism.

Higson and Roe observed in 1988 that the analytic construction of Higson of a boundary for complete open manifolds in (section 3, [56]) provided exactly the sought-after "good boundary". Roe adapted the definition to complete metric spaces (Chapter 5, [88]) and constructed a character map in this generality. . The *Higson corona* $\partial_h L$ of L is defined as the spectrum of a certain commutative C^*-algebra associated to the metric. The corona is a coarse isometry invariant of L, almost by its definition. Its most fundamental property, however, is the existence of a canonical pairing with the operator K-theory of the Roe algebra of L – which is the K-theory equivalent of the existence of a character map. Each boundary K-theory class in $K^*(\partial_h L)$ thus yields an index invariant of the open manifold L. All of these ideas have their counterpart for foliations. In this section, we give an overview of the construction of coronas and some of their basic properties.

6.1 Coronas for manifolds

First, consider the case where L is a C^1-manifold with a complete Riemannian metric [56]. Let $C_h(L)$ denote the C^*-algebra closure (in the sup norm on functions) of the functions on L whose *gradients* tend to zero at infinity. The algebra of continuous functions which vanish at infinity, $C_0(L)$, is a closed C^*-subalgebra of $C_h(L)$. The Higson corona of L, denoted by $\partial_h L$, is defined to be the spectrum of the quotient C^*-algebra $C_h(L)/C_0(L)$.

There is an inclusion of closed C^*-algebras, $C_0(L) \subset C_h(L) \subset C(L)$, so that $\partial_h L$ is an intermediate boundary between the maximal Stone-$\check{C}$ech compactification $\check{L} = spec(C(L))$ and the one-point compactification $L \cup \infty = spec(C_0(L))$. One can show that if the coarse metric on L is not bounded, then $\partial_h L$ is non-separable.

One motivation for introducing the algebra $C_h(L)$ is that the vanishing gradient condition is exactly what is required to obtain a well-defined index pairing between the K-theory groups $K_*(C_h(L))$ and first order geometric operators on L with "bounded geometry". Roe abstracted Higson's construction to complete metric spaces, replacing the decay condition on the gradient with a decay condition on the variation function (cf. Definition 6.1 below).

6.2 Coronas for foliations

The construction of the corona for a foliation uses the Higson-Roe construction fiberwise on the foliation groupoid. We define a C^*-subalgebra of the uniformly continuous functions on $\mathcal{G}_{\mathcal{F}}$ via a *uniform* leafwise decay condition on their variations along the holonomy covers of the leaves. There is a subtlety in the groupoid case, in that the space of continuous functions on $\mathcal{G}_{\mathcal{F}}$ is closed under pointwise multiplication of functions only if $\mathcal{G}_{\mathcal{F}}$ is Hausdorff. For this reason, we only discuss the Hausdorff case here.

Equip $\mathcal{G}_{\mathcal{F}}$ with the plaque-distance coarse metric. Let $C(\mathcal{G}_{\mathcal{F}})$ denote the topological vector space of continuous functions on the groupoid $\mathcal{G}_{\mathcal{F}}$, with the uniform norm topology obtained from the sup-norm on functions:

$$\sup\ |h| = \sup_{y \in \mathcal{G}_{\mathcal{F}}} |h(y)|$$

Denote by $C_u(\mathcal{F}) = C_u(\mathcal{G}_{\mathcal{F}}) \subset C(\mathcal{G}_{\mathcal{F}})$ the closed subspace consisting of uniformly continuous functions, and $C_0(\mathcal{G}_{\mathcal{F}}) \subset C_u(\mathcal{F})$ the closure of the subspace spanned by finite sums of continuous functions supported in basic open sets in $\mathcal{G}_{\mathcal{F}}$.

DEFINITION 6.1 *For $x \in M$ and $r > 0$, define the* fiberwise variation function

$$\begin{aligned} V_s(x,r) : C(\tilde{L}_x) &\rightarrow [0,\infty) \\ V_s(x,r)(h)(y) &= \sup\{|h(y') - h(y)| \text{ such that } D_x(y,y') \leq r\} \end{aligned}$$

We say that $f \in C(\mathcal{G}_{\mathcal{F}})$ has *uniformly vanishing variation at infinity* if there exists a function $D : (0,\infty) \rightarrow [0,\infty)$ so that if $D_x(y,*x) > D(\epsilon)$ then $V_s(x,r)(i_x^* f)(y) < \epsilon$. Let $C_h(\mathcal{F}) \subset C_u(\mathcal{F})$ denote the subspace of uniformly continuous functions which have uniformly vanishing variation at infinity.

LEMMA 6.2 *$C_h(\mathcal{F})$ is a commutative C^*-algebra. $C_0(\mathcal{F})$ is a closed C^*-subalgebra of $C_h(\mathcal{F})$.* □

DEFINITION 6.3 *Let $\mathcal{F}$ be a topological foliation of a paracompact manifold M equipped with a regular foliation atlas. The* corona*, $\partial_h \mathcal{F}$, of $\mathcal{F}$ is the spectrum of the quotient C^*-algebra $C_h(\mathcal{F})/C_0(\mathcal{F})$.*

We also define the closure $\overline{\mathcal{G}_{\mathcal{F}}} = spec(C_h(\mathcal{F}))$. The uniform continuity of the functions in $C_h(\mathcal{F})$ ensures that $\overline{\mathcal{G}_{\mathcal{F}}}$ "fibers" over the total space of the foliation (though the fibers corresponding to non-compact leaves $\tilde{L}$ need not be homeomorphic):

PROPOSITION 6.4 ([67]) .

1. *The source projection extends to a continuous map $\overline{s}\colon \overline{\mathcal{G}_{\mathcal{F}}} \longrightarrow M$.*
2. *For each $x \in M$ there is an inclusion $\iota_x\colon \overline{\tilde{L}_x} = spec(C_h(\tilde{L}_x)) \hookrightarrow \overline{\mathcal{G}_{\mathcal{F}}}$.*
3. *For each $x \in M$ there is an inclusion $\partial\iota_x\colon \partial_h \tilde{L}_x \hookrightarrow \partial_h \mathcal{F}$, where $\partial_h \tilde{L}_x$ is the Higson corona of $\tilde{L}_x$.*

6.3 Functorial properties of the corona

The foliation corona has very nice functorial properties [67]:

PROPOSITION 6.5 *Let M_1 be a compact manifold, and $f : M_1 \to M_2$ be a continuous function which sends leaves of $\mathcal{F}_1$ into leaves of $\mathcal{F}_2$ and induces a proper map of groupoids $\mathcal{G}f\colon \mathcal{G}_{\mathcal{F}_1} \to \mathcal{G}_{\mathcal{F}_2}$. Then there is an induced map*

$$\overline{f} : \overline{\mathcal{G}_{\mathcal{F}_1}} \to \overline{\mathcal{G}_{\mathcal{F}_2}}.$$

PROPOSITION 6.6 *Let $\mathcal{F}$ be a topological foliation of a compact manifold M and $f : M \to M$ be a leaf-preserving continuous map which is leafwise-homotopic to the identity map. Then $\partial_h f : \partial_h\mathcal{F} \to \partial_h\mathcal{F}$ is homotopic to the identity map.*

COROLLARY 6.7 *For $i = 1, 2$, let $\mathcal{F}_i$ be a topological foliation of a compact manifold M_i. Then a leafwise homotopy equivalence $f : M_1 \to M_2$ induces a homotopy equivalence $\partial_h f : \partial_h\mathcal{F}_1 \cong \partial_h\mathcal{F}_2$.*

When the map f is a homeomorphism, we can strengthen the conclusion of Proposition 6.6:

PROPOSITION 6.8 *For $i = 1, 2$, let $\mathcal{F}_i$ be a topological foliation of a compact manifold M_i. Then a leaf-preserving homeomorphism $f : M_1 \to M_2$ induces a homeomorphism*

$$\partial_h f : \partial_h\mathcal{F}_1 \xrightarrow{\cong} \partial_h\mathcal{F}_2$$

Finally, there is a character map:

THEOREM 6.9 *There is a natural map $\delta\colon H^*(\partial_h\mathcal{F}; \mathbf{R}) \to HX^*(\mathcal{F}; \mathbf{R})$ so that the composition*

$$H^*(\partial_h\mathcal{F}; \mathbf{R}) \to HX^*(\mathcal{F}; \mathbf{R}) \to H^*_c(\mathcal{G}_{\mathcal{F}}; \mathbf{R})$$

is the boundary map of the pair $(\overline{\mathcal{G}_{\mathcal{F}}}, \partial_h\mathcal{F})$.

The corona for a foliation seems to capture new topological aspects of the foliated manifold M. One notes that $\partial_h\mathcal{F}$ is a generalized fibration over M, so the coboundary terms in the Leray spectral sequence for the map $f\colon \partial_h\mathcal{F} \to M$ measure some quantitative aspects of $\mathcal{F}$. It is an open problem to investigate the significance of these invariants of a foliation.

6.4 The endset and Gromov-Roe coronas

The foliation corona $\partial_h \mathcal{F}$ of a topological foliation with non-compact leaves of a compact manifold is non-separable, and is a truly enormous space. The problem is that the criteria for a function to be in $C_h(\mathcal{G}_\mathcal{F})$ imposes no restrictions on the *rate of decay* of its variation, so the corona captures more of the Stone-Cech compactification of the metric space $\mathcal{G}_\mathcal{F}$ than is perhaps intended. For this reason, one introduces alternate definitions of coronas which are separable quotient spaces of the corona.

A *separable corona* (L, q) for $\mathcal{F}$ is a separable compact space L equipped with a continuous surjection $q: \partial_h \mathcal{F} \to L$. A separable corona (L, q) determines a separable subalgebra

$$\mathcal{A}_L = \{f \in C_h(\mathcal{F}) \text{ such that } f|\partial_h \mathcal{F} = g \circ q \text{ for some } g \in C(L)\}$$

Conversely, given a separable C^*-subalgebra $\mathcal{A} \subset C_h(\mathcal{F})$ containing $C_0(\mathcal{F})$ there is a natural map

$$q: \partial_h \mathcal{F} \to spec(\mathcal{A}_L) \equiv L_\mathcal{A}$$

which defines a separable corona for $\mathcal{F}$. A natural way to obtain a separable corona for $\mathcal{F}$ is to construct such a subalgebra $\mathcal{A}$ which is generated by functions in $C_h(\mathcal{F})$ satisfying a "rate-of-decay" condition on their variations.

The *endset*, or *Freudenthal*, compactification of $\mathcal{G}_\mathcal{F}$ is obtained by requiring that the variations of the functions vanish outside some compact set. Let $C_\epsilon(\mathcal{F}) \subset C_h(\mathcal{F})$ be the closed topological subalgebra generated by the functions which are constant outside a compact set. That is, $h \in C_h(\mathcal{F})$ is in $C_\epsilon(\mathcal{F})$ if and only if there is a compact subset $K_h \subset \mathcal{G}_\mathcal{F}$ so that the restriction of h to $C_\epsilon(\mathcal{F}) \setminus K_h$ is constant. Note that $C_0(\mathcal{F}) \subset C_\epsilon(\mathcal{F})$.

DEFINITION 6.10 *The* endset *of a foliation $\mathcal{F}$ is the compact topological space $\epsilon(\mathcal{F})$ defined as the spectrum of the unital topological algebra $C_\epsilon(\mathcal{F})/C_0(\mathcal{F})$.*

PROPOSITION 6.11 *$\epsilon(\mathcal{F})$ is a corona for $\mathcal{F}$.*

Proof. A point in the spectrum of $C_h(\mathcal{F})/C_0(\mathcal{F})$ can be identified with an evaluation

$$\hat{\epsilon} : C_h(\mathcal{F})/C_0(\mathcal{F}) \to \mathbf{C},$$

which naturally restricts to an evaluation $\hat{\epsilon} : C_\epsilon(\mathcal{F})/C_0(\mathcal{F}) \to \mathbf{C}$. Thus, there is a natural map $\partial_h \mathcal{F} \to \epsilon(\mathcal{F})$. $C_\epsilon(\mathcal{F})/C_0(\mathcal{F})$ has a unit so $\epsilon(\mathcal{F})$ is compact. There is a countable base for the space of the functions which are constant outside a compact set, hence $\epsilon(\mathcal{F})$ is separable. Finally, let us show that $\epsilon(\mathcal{F})$ is the Freudenthal compactification for $\mathcal{G}_{\mathcal{F}}$. A function which is constant outside of a compact set in $\mathcal{G}_{\mathcal{F}}$ extends continuously to the Freudenthal compactification, hence $C_\epsilon(\mathcal{F})/C_0(\mathcal{F})$ is contained in the continuous functions on the Freudenthal compactification. The functions in $C_\epsilon(\mathcal{F})/C_0(\mathcal{F})$ separate the ends on $\mathcal{G}_{\mathcal{F}}$, so by the Stone-Weierstrass Theorem it must equal the standard end compactification. (We are indebted to John Roe for pointing out this last trick.) □

Let us next introduce a family of foliation coronas, parametrized by a real number $\tau > 0$. For $f \in C(\mathcal{G}_{\mathcal{F}})$, we say that the variation of f has *uniform τ-decay* if for each $r > 0$ there exists $C(f, k, r) > 0$ and a uniform estimate

$$V_\tau(x, r)(i_x^* f)(y) < C(f, k, r)\,[D_x(y, *x) + 1]^{-\tau} \text{ for each } x \in M \text{ and all } y \in \tilde{L}_x \tag{8}$$

The τ-decay condition is especially useful when $\tau > 1$ for it then implies an estimate on the change in the value of f along paths in the fibers (cf. the proof of Proposition 6.15).

Let $C_\tau(\mathcal{F}) \subset C_h(\mathcal{F})$ be the closed topological subalgebra generated by the functions whose variations have uniform τ-decay.

DEFINITION 6.12 *Let $\mathcal{F}$ be a topological foliation of a compact manifold M. For $\tau > 0$ the* τ-boundary $\partial_\tau \mathcal{F}$ *of $\mathcal{F}$ is the spectrum of the quotient C^*-algebra $C_\tau(\mathcal{F})/C_0(\mathcal{F})$.*

The variation of f has *uniformly rapid decay* if it has uniform τ-decay for all $\tau > 0$. Let $C_\infty(\mathcal{F}) \subset C_h(\mathcal{F})$ be the closed topological subalgebra generated by the functions whose variations have uniformly rapid decay. Roe proved that for a complete metric space L which is hyperbolic in the sense of Gromov, the spectrum of the algebra of functions with rapid decay is homeomorphic to the geodesic compactification of L (Proposition 2.3, [87]). This boundary is well-defined for any metric space, so we propose:

DEFINITION 6.13 *Let $\mathcal{F}$ be a topological foliation of a compact manifold M. The* Gromov-Roe boundary $\partial_\infty \mathcal{F}$ *of $\mathcal{F}$ is the spectrum of the quotient C^*-algebra $C_\infty(\mathcal{F})/C_0(\mathcal{F})$.*

There is an important class of examples of foliations for which the above boundaries can be effectively described – those with cone-like holonomy groupoids. Assume there is given:

- a compact CW-complex Z and a fibration $\Pi\colon Z \to M$,
- a fiberwise metric $\Re_x\colon Z_x \times Z_x \to [0, 1]$ which varies continuously with x,
- a continuous "weight" function $\Phi\colon M \times [0, \infty) \to [0, \infty)$ with $\Phi(M \times \{0\}) = 0$ and each restriction $\Phi_x\colon [0, \infty) \to [0, \infty)$ is monotone-increasing and unbounded.

The *parametrized cone* determined by the map Π is the fibration $C\Pi\colon C(Z,\Pi) \to M$, where for each $x \in M$ the fiber $CZ_x \equiv C\Pi^{-1}(x)$ over x is the cone with vertex x and base $Z_x = \Pi^{-1}(x)$. The additional data $\Re$ and Φ determines a fiberwise metric $C_\Phi\Re$ on $C(Z,\Pi)$, where the fiber CZ_x has the cone metric determined by Φ_x and $\Re_x$ (cf. section (3.46) of [88]). The data $\{C\Pi\colon C(Z,\Pi) \to M, C\Re\}$ is called the *parametrized metric cone* on $\{\Pi\colon Z \to M, \Re, \Phi\}$.

DEFINITION 6.14 *A foliation $\mathcal{F}$ is* cone-like with base $\Pi\colon Z \to M$ *if there exists*

- *a parametrized metric cone $\{C\Pi\colon C(Z,\Pi) \to M, C\Re\}$*
- *a fiber-preserving map $C\mathcal{F}\colon C(Z,\Pi) \to \mathcal{G}_\mathcal{F}$ which covers the identity on M,*
- *constants d_1, d_2, d_3, ϵ so that for each $x \in M$ the restriction $C\mathcal{F}_x\colon CZ_x \to \tilde{L}_x$ is a coarse isometry with respect to these constants (cf. Definition 2.1).*

PROPOSITION 6.15 *Let $\mathcal{F}$ be a cone-like foliation with base $\Pi\colon Z \to M$. Then there are fiber-preserving continuous surjections*

$$\partial_h\mathcal{F} \xrightarrow{\partial C\mathcal{F}} Z \xrightarrow{\partial_\tau C\mathcal{F}} \partial_\tau\mathcal{F}$$

for $1 < \tau \leq \infty$ such that the composition is the canonical map $\partial_h\mathcal{F} \to \partial_\tau\mathcal{F}$. In particular, $\partial_\tau\mathcal{F}$ is a separable corona for $1 < \tau \leq \infty$.

REMARK 6.16 The ∂_τ-boundary for a cone-like space need not be homeomorphic to the cone, as it may collapse "flats at infinity". For example, when all leaves of $\mathcal{F}$ are metrically Euclidean of dimension greater than 1, then it is a nice exercise to show that each fiber of $\partial_\tau\mathcal{F} \to M$ is a point for $\tau > 1$. So in general, the surjection $\partial_\tau C\mathcal{F}\colon Z \longrightarrow \partial_\tau\mathcal{F}$ need not be a homeomorphism. However, when the leaves of $\mathcal{F}$ admit metrics of uniformly negative curvature, the arguments of Roe (cf. the proof of Proposition 2.3, [87]) show that $\partial_\infty\mathcal{F}$ is a fibration over M with fibers S^{p-1}. The Gromov-Roe boundary $\partial_\infty\mathcal{F}$ is a very interesting object for further study.

6.5 Coronas for special classes of foliations

We give some examples examples of foliations whose coronas fiber over the base M. We first establish a general result, then consider geometric special cases to illustrate it.

DEFINITION 6.17 *A foliation $\mathcal{F}$ is said to be* coarsely geodesic *if*

- *$\mathcal{G}_\mathcal{F}$ is a Hausdorff space, with $s\colon \mathcal{G}_\mathcal{F} \to M$ a fibration.*

- *For each $x \in M$ there exists an open neighborhood $x \in U \subset M$ and a trivialization $T_U: s^{-1}U \to \tilde{L}_x \times U$, so that for each $y \in U$ the restriction $T_{U,y}: s^{-1}(y) \to \tilde{L}_x \times \{y\}$ is a coarse isometry, with uniform constants independent of $y \in U$.*

A coarsely geodesic foliation $\mathcal{F}$ has a "typical leaf" $\tilde{L}$ which is a complete metric space, and for all $x \in M$ the holonomy cover $\tilde{L}_x$ is diffeomorphic and coarsely isometric to $\tilde{L}$. This property is analogous to one enjoyed by totally geodesic foliations [75], hence the terminology. This uniform metric property has a strong consequence for the structure of the corona:

PROPOSITION 6.18 *Let $\mathcal{F}$ be a coarsely geodesic foliation. Then the corona of $\mathcal{F}$ fibers*

$$\partial_h \tilde{L} \longrightarrow \partial_h \mathcal{F} \xrightarrow{\partial s} M$$

Recall the construction of the class of suspension foliations (cf. Chapter 5, [15]). Let X denote a compact topological manifold, and Γ isomorphic to the fundamental group $\pi_1(B, b_0)$ of a compact manifold B. Let Γ act on the universal covering $\tilde{B} \to B$ by deck translations on the left. Given a continuous action $\varphi : \Gamma \times X \to X$, form the product of the deck action with φ to obtain an action of Γ on $\tilde{B} \times X$. Introduce the quotient compact topological manifold,

$$M_\varphi = \Gamma \setminus (\tilde{B} \times X).$$

The product foliation on $\tilde{B} \times X$, with typical leaf $L = \tilde{B} \times \{x\}$ for $x \in X$, descends to a topological foliation on M_φ denoted by $\mathcal{F}_\varphi$. The projection onto the first factor map, $\tilde{B} \times X \to \tilde{B}$, descends to a map $\pi : M_\varphi \to B$, and π restricted to the leaves $\mathcal{F}_\varphi$ is a covering map. A Riemannian metric on TB lifts via π to a leafwise metric on $T\mathcal{F}_\varphi$, so that the foliation always carries a leafwise Riemannian distance function (even though $\mathcal{F}_\varphi$ need only be a topological foliation).

Let $K_\varphi \subset \Gamma$ denote the subgroup of elements which act trivially on X under φ, let $\Gamma_\varphi = \Gamma / K_\varphi$ denote the quotient group and $\tilde{B}_\varphi$ the covering of B corresponding to Γ_φ. Then Γ_φ is isomorphic to a subgroup of Homeo(X), called the *global holonomy group* $\mathcal{H}_{\mathcal{F}_\varphi} \subset$ Homeo(X).

The action φ is *effective* if for all open subsets $U \subset X$ and all $\gamma \in \Gamma$, if $\varphi(\gamma)$ restricts to the identity on U, then $\varphi(\gamma)$ acts as the identity on X. Winkelnkemper showed that the holonomy groupoid of the suspension of an effective action is Hausdorff, and there is a homeomorphism

$$\mathcal{G}_{\mathcal{F}_\varphi} \cong \Gamma \setminus \left(\tilde{B} \times X \times \tilde{B}_\varphi\right) \tag{9}$$

PROPOSITION 6.19 *Let $\mathcal{F}_\varphi$ be the suspension foliation associated to an effective continuous action φ. Then the foliation endset $\epsilon(\mathcal{F}_\varphi)$ fibers over M_φ with fiber homeomorphic to the endset $\epsilon(\Gamma_\varphi)$ of the global holonomy group.*

A much stronger is possible: The deck translations act via isometries on $\tilde{B}_\varphi$ so induce a continuous action on the compactification $\overline{\tilde{B}_\varphi} = \tilde{B}_\varphi \cup \partial_h \tilde{B}_\varphi$. There is a Γ-equivariant homeomorphism of boundaries $\partial_h \tilde{B}_\varphi \cong \partial_h \Gamma$, so by the identification (9) and an application of the Proposition 6.18 we obtain:

PROPOSITION 6.20 *Let $\varphi : \Gamma \times X \to X$ be an effective on a compact topological manifold X. Then the foliation corona is homeomorphic to the suspension fibration obtained from the induced action of Γ on the Higson corona of the global holonomy group Γ_φ*

$$\partial_h \mathcal{F} \cong \Gamma \setminus \left(\tilde{B} \times X \times \partial_h \Gamma_\varphi\right)$$

Foliations defined by locally free Lie group actions provide another class of examples where the corona has additional structure. Let G be a connected Lie group. A topological action $\varphi : G \times M \to M$ is *locally-free* if for all $x \in M$ the isotropy subgroup $G_x \subset G$ is a finite subgroup. The action is *effective* if g must be the identity element whenever there is an open set $U \subset M$ so that $\varphi(g)$ restricts to the identity on U.

LEMMA 6.21 *Let $\varphi : G \times M \to M$ be a locally-free effective C^1-action. Then the orbits of the action φ define a C^1-foliation $\mathcal{F}_\varphi$ of M, and there is a natural homeomorphism*

$$\mathcal{G}_{\mathcal{F}_\varphi} \cong G \times M \tag{10}$$

Choose an orthonormal framing of the Lie algebra of G, which determines a right-invariant Riemannian metric on TG. At each $x \in M$ the left action of G on M induces a framing of the orbit of G through x. The action of G is locally free, so the resulting continuous vector fields on M are linearly independent at each point, hence yields a global framing of the leaves of $\mathcal{F}_\varphi$. Declare this to be an orthonormal framing to obtain a Riemannian metric on the leaves. Note that the identification (10) maps $G \times \{x\}$ to the holonomy cover of the orbit of G through x, which by the essentially free hypotheses is exactly the orbit Gx. The Riemannian manifolds G and Gx are isometric. By the identification (10) and an application of the Proposition 6.18 we obtain:

PROPOSITION 6.22 *Let $\mathcal{F}_\varphi$ be a C^1-foliation of M determined by a locally-free effective C^1-action $\varphi : G \times M \to M$. Then the foliation corona of $\mathcal{F}_\varphi$ is homeomorphic to a product,*

$$\partial_h \mathcal{G}_{\mathcal{F}_\varphi} \cong \partial_h G \times M$$

Riemannian foliations on compact manifolds provide a third geometric class of foliations whose coronas have a fibration structure. This index invariants associated to their coronas should be quite useful, beyond the context of this paper, as the leafwise geometric operators for Riemannian foliations are a generalization of the study of almost-periodic operators. The study of their analysis and index theory is a natural extension of more classical topics, and the corona construction gives an additional topological tool for their investigation.

Recall that a C^1-foliation $\mathcal{F}$ is *Riemannian* [80] if there exists a Riemannian metric on the normal bundle to $\mathcal{F}$ which is invariant under the linear holonomy transport. This has many consequences for the topology of M and the structure of the foliation [80] – for a compact manifold M, there is an open dense set of leaves in a Riemannian foliation which have no holonomy, and the holonomy covers of all of the leaves of $\mathcal{F}$ are homeomorphic. The homeomorphisms are induced by first forming the principal $O(q)$-bundle $P \to M$ of orthogonal frames to the foliation $\mathcal{F}$, where q is the codimension. The foliation lifts to a foliation $\hat{\mathcal{F}}$ without holonomy, and the leaves of $\hat{\mathcal{F}}$ cover those of $\mathcal{F}$. The compact manifold P carries a collection of linearly independent vector fields which span the normal bundle to $\hat{\mathcal{F}}$, whose flows induce leaf preserving homeomorphisms of P and which are transitive on the leaf space of $\hat{\mathcal{F}}$. Thus, given any two leaves of $\mathcal{F}$, there is a homeomorphism of their holonomy covers which is realized by a sequence of homeomorphisms, each the flow associated to a vector field on P. As noted by Winkelnkemper (section3, Corollary [98]), this implies that the foliation groupoid is a fibration over the base M,

$$L \longrightarrow \mathcal{G}_{\mathcal{F}} \xrightarrow{s} M \tag{11}$$

where L is called the "typical" leaf of $\mathcal{F}$ – as almost every leaf of $\mathcal{F}$ is diffeomorphic to L. The explicit construction of the homeomorphisms between the fibers of (11) as the composition of flows on the compact manifold P implies that the fibration transition functions are coarse isometries on fibers, so the typical leaf also has a well-defined coarse isometry type. By the identification (11) and an application of the Proposition 6.18 we obtain:

PROPOSITION 6.23 *Let $\mathcal{F}$ be a Riemannian foliation of a compact manifold M, with typical leaf L. Then the foliation corona of $\mathcal{F}$ fibers*

$$\partial_h L \longrightarrow \partial_h \mathcal{F} \longrightarrow M$$

The other coronas $\partial_\tau \mathcal{F}$ for $\tau > 0$ constructed above also fiber in this way over the base M.

We conclude this discussion of examples with a class of foliations for which there is a canonically associated separable corona (L, q) for $\mathcal{F}$ where L is again a manifold of dimension $2p + q - 1$

PROPOSITION 6.24 *Let $\mathcal{F}$ be a C^2-foliation of a compact manifold M such that the holonomy cover of each leaf is simply connected. Assume there is a Riemannian metric on the tangential distribution to $\mathcal{F}$ so that each leaf has* non-positive *sectional curvatures. Then there exists a separable corona $\pi: \partial\mathcal{F} \to M$, where the fiber $\pi^{-1}(x) \cong S^{p-1}$ is identified with the "sphere at infinity" on the holonomy cover $\tilde{L}_x$.*

Proof. Let $T\mathcal{F} \to M$ be the tangential distribution to the leaves of $\mathcal{F}$. The metric assumption implies that the *leaf exponential map* $\exp_{\mathcal{F}}: T\mathcal{F} \to M \times M$ is a covering map onto each leaf. (The leaf exponential is defined by considering M with a new topology in which each leaf is an open connected component, hence the exponential spray stays inside each leaf. cf. [63, 97].) Thus, we obtain a diffeomorphism $\exp_{\mathcal{F}}: T\mathcal{F} \cong \mathcal{G}_{\mathcal{F}}$. Let $\overline{T\mathcal{F}}^g = T\mathcal{F} \cup \partial\mathcal{F}$ be the compactification of $T\mathcal{F}$ obtained by adding on the sphere at infinity in each fiber. Then $\exp_{\mathcal{F}}^{-1}$ extends to a continuous map of the compactifications

$$\exp_{\mathcal{F}}^{-1}: \overline{\mathcal{G}_{\mathcal{F}}} \longrightarrow \overline{T\mathcal{F}}^g$$

which restricts to a fiber-preserving surjective map $\partial_h \mathcal{F} \to \partial\mathcal{F}$. $\square$

The compactification in Proposition 6.24 is called the geodesic compactification.

7 Manifolds not coarsely isometric to leaves

Coarse cohomology theory and the corona construction associate topological invariants to the "space at infinity" of a complete metric space L. In particular, we can use this to define the (cohomological) dimension of an end of L. In this section, we apply the "dimension of ends" to improve the main result of section 5. Recall that the construction in section 5 of open manifolds with positive entropy used the Pontrjagin classes of the attached "handles" N_ℓ to distinguish the manifolds with boundary $N(y, r, s, \vec{k})$ up to quasi-isometric homeomorphism. A coarse isometry does not preserve any local cohomology data – each handle N_ℓ is equivalent to a point – so an invariant of coarse isometry is needed to distinguish the tiles in the tiling we are to create. The dimensions of ends is exactly right for this task!

The following result shows that there are metric nets which are not coarse isometric to the net of a leaf of a transversally C^1-foliation (or for that matter, to a net given by an orbit of any topological groupoid compactly generated by C^1-maps.)

THEOREM 7.1 *There exists an uncountable set of quasi-isometry types Riemannian manifolds of bounded geometry and exponential volume growth, none of which is coarse isometric to a leaf of a C^0-foliation whose expansion growth is less than $\lceil 2^{b^r} \rceil$ for all $b > 1$. In particular, a* net *in one of these manifolds cannot be coarse isometric to a net in any leaf of a of C^α-foliations of any codimension, for any modulus of continuity $\alpha > 0$.*

We only sketch the proof – the details are in [71]. The first point to make is is to explain the idea in terms of the light bulb patterns of section 4. Recall, that in the earlier construction, we varied the patterns from one set of sockets to another, by changing the colors of the light bulbs. Unfortunately, in coarse geometry there are no colors, as each light bulb is equivalent to a point. The idea is then to replace varying the "color" of the bulb with varying its shape! That is, we will replace the elliptic light bulbs with hyperbolic (or conical) models. The boundaries of these hyperbolic models are coarse invariants. It is not possible to discern the finer differential properties of the boundary construction, but the cohomological dimension is a coarse invariant.

Continuing the analogy with changing lights in a pattern of sockets, the mathematical description of the construction of M is now apparent. In place of the choices of manifolds model manifolds N_ℓ for $0 \leq \ell \leq 2$ which are homotopy equivalent to $S^4 \times S^2$ used in section 4.2, let us introduce manifolds N_ℓ for $0 \leq \ell \leq 2$ where N_ℓ is $\mathbf{R}^{\ell+2} \times S^{4-\ell}$. The corona $\partial_h N_\ell$ has the same cohomology as the sphere in the boundary $S^{\ell+1}$. We can attach the N_ℓ to the "pattern manifold" $\mathcal{M}_0$ by removing a disk of radius $1/2$ in each N_ℓ, then proceeding exactly as in section 4.2.

The key point is that for the resulting inductively constructed manifold M, a coarse isometry to a leaf L will induce a homeomorphism of their coronas, $\partial_h M \cong \partial_h L$. Each "hyperbolic bulb" N_ℓ contributes an asymptotic homology class $S^{\ell+1} \subset \partial_h M$ which is detected by a cohomology class on the corona. One then has to note that the Product Neighborhood Theorem 3.3 can be used to deduce recurrence for the cohomology of $\partial_h L$, using the boundary map Theorem 6.9 in cohomology $H^*(\partial_h L; \mathbf{R}) \to HX^{*+1}_{er}(L; \mathbf{R})$. By this means, one can then work down the ends of L to identify the socket patterns for M with those for L, and deduce once again that $h(L) > 0$, contrary to Corollary 4.11.

Lecture III - Coarse Families Produce Fine Invariants

8 The foliation Novikov conjecture

We next discuss the application of the corona construction to the Foliation Novikov Conjecture. On the surface, this is a completely unrelated topic, as the question revolves around the topological invariance of characteristic classes. However, the deepest approaches to the Novikov Conjecture are based on the homotopy invariance of certain structures at infinity, so the introduction of the methods of coarse geometry are completely natural. The basic idea first showed up in the early work by Gromov and Lawson on positive scalar curvature [50, 49, 89]. Roe developed this application for the coarse Novikov Conjecture for open manifolds [88, 87]. Our treatment of the index theory for foliations formulates this theory for families of open manifolds, which can be used to give a coarse geometry proof of the Novikov Conjecture for compact manifolds [66] parallel to the methods of geometric topology [39, 20]and KK-theory [76].

8.1 Coarse fundamental classes

The compactly-supported fundamental class for $\mathbf{R}^n$ is a generator of the exotic cohomology $HX^n(\mathbf{R}^n; \mathbf{R})$. More generally, for an open complete manifold L, classes in $HX^*(L; \mathbf{R})$ represent "fundamental classes" that naturally pair with the locally-finite homology of L.

The index class of a leafwise elliptic differential operator is a K-theory class in $K_*(C^*(\mathcal{F}))$, whose Chern character can be considered as a cohomology class on the leaf space $M/\mathcal{F}$. A K-theory fundamental class for $\mathcal{F}$ is defined to be homomorphism $Z_* = \langle \cdot, Z\rangle \colon K_*(C^*(\mathcal{F})) \to \mathbf{Z}$ which depends only on the leafwise homotopy class of $\mathcal{F}$. Connes proved that an invariant transverse elliptic operator to $\mathcal{F}$ yields a fundamental class [25]. He later showed that a cyclic cocycle on the smooth convolution algebra $C_c^{\infty}(\mathcal{G}_{\mathcal{F}})$ which satisfies appropriate growth estimates yields a fundamental class [24]. The new observation from coarse geometry is that each K-theory class in $K^{\ell+1}(\partial_h \mathcal{F})$ generates a family of fundamental classes for $\mathcal{F}$.

The index class of the leafwise signature operator with coefficients in a leafwise almost flat bundle $\mathbf{E} \to M$ is a leafwise homotopy invariant (cf. Hilsum and Skandalis [60]), so a K-theory fundamental class Z_* yields a numerical invariant $\langle Ind\left((d_{\mathcal{F}} * - * d_{\mathcal{F}}) \otimes \mathbf{E}\right), Z\rangle$ of the leafwise homotopy class of $\mathcal{F}$. Consequently, for each K-theory class in $K^{\ell+1}(\partial_h \mathcal{F})$, the fundamental class construction yields homotopy invariants for leafwise elliptic operators.

Let $C_r^*(\mathcal{F})$ denote the reduced C^*-algebra associated to the foliation $\mathcal{F}$ with its given leafwise Haar system $dv_{\mathcal{F}}$ (cf. [22, 23, 85].)

The first result is the existence of a boundary map from $K^*(\partial_h\mathcal{F})$ to the parametrized K-theory of $\mathcal{G}_{\mathcal{F}}$ over M, whose image consists of generalized "dual-Dirac" classes for $\mathcal{F}$ in Kasparov bivariant-KK-theory:

THEOREM 8.1 ([67]) *Let $\mathcal{F}$ be a C^2-foliation of a compact manifold M. Then there is a natural map*

$$\rho: K^*(\partial_h\mathcal{F}) \longrightarrow KK_{*+1}(C_r^*(\mathcal{F}), C(M)) \tag{12}$$

whose image consists of generalized foliation dual-Dirac classes.

Compose the map ρ of equation (12) with the KK-external product

$$KK(\mathbf{C}, C_r^*(\mathcal{F})) \otimes KK_{*+1}(C_r^*(\mathcal{F}), C(M)) \to KK(\mathbf{C}, C(M)) \cong K^*(M)$$

to obtain:

COROLLARY 8.2 *Let $k, \ell = 0, 1$ be fixed. Then for each $[u] \in K^\ell(\partial_h\mathcal{F})$ there is an exotic index map*

$$\rho[u]: K_k\left(C^*(\mathcal{F})\right) \longrightarrow K^{k+\ell+1}(M) \tag{13}$$

The exotic index in $K^*(M)$ can be coupled to an elliptic operator on M to obtain numerical invariants; these are the fundamental classes mentioned above:

THEOREM 8.3 *For each $[u] \in K^\ell(\partial_h\mathcal{F})$ and $[\mathcal{D}_M] \in KK(C_0(M), \mathbf{C})$, there is a K-theory fundamental class*

$$Z([u], [\mathcal{D}_M])_*: K_*(C^*(\mathcal{F})) \to \mathbf{Z}$$

The net result is that given a K-homology class on the ambient space, M, and an exotic class "along the leaves", their "cap product" is a transverse fundamental class for $\mathcal{F}$ which can be paired with the indices of leafwise elliptic operators to get the exotic indices.

The exotic index $\rho[u](Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)) \in K^*(M)$ is an "integral" invariant of $\mathcal{D}_{\mathcal{F}}$. This contrasts with the real-valued measured index of a leafwise operator for a foliation admitting a holonomy-invariant transverse measure, which is typically a renormalized index with values in $\mathbf{R}$.

Let $\mathcal{D}_{\mathcal{F}}$ be a leafwise-elliptic, pseudo-differential operator for $\mathcal{F}$. The Connes-Skandalis construction [30] yields a KK-index class $Ind(\mathcal{D}_{\mathcal{F}}) \in KK_*(C_0(M), C^*(\mathcal{F}))$, which via the external KK-product yields a map:

$$\mu(\mathcal{D}_{\mathcal{F}}): K^*(M) \cong KK(\mathbf{C}, C_0(M)) \longrightarrow KK(\mathbf{C}, C^*(\mathcal{F})) \cong K_*(C^*(\mathcal{F})) \qquad (14)$$

The map (14) is a special case of the Baum-Connes "μ-map" whose domain is the K-theory of all leafwise symbols for $\mathcal{F}$ [7, 8].

We say that $\mathcal{F}$ is a *contractable foliation* if the identity map of $\mathcal{G}_{\mathcal{F}}$ is homotopic to the fiberwise projection onto the diagonal, $*s: \mathcal{G}_{\mathcal{F}} \to M \hookrightarrow *M \subset \mathcal{G}_{\mathcal{F}}$. If the homotopy can be chosen to preserve the fibers of s, then we say that $\mathcal{F}$ has *uniformly contractable leaves*. We observed in [67] that for contractable foliations, there is a foliated form of Atiyah's trick in [2] which reduces the calculation of the exotic index pairing to that of determining one K-theory class:

THEOREM 8.4 *Let $\mathcal{F}$ be a contractable foliation of leaf dimension p with Hausdorff holonomy groupoid $\mathcal{G}_{\mathcal{F}}$. For each boundary K-theory class $[u] \in K^{\ell+1}(\partial_h \mathcal{F})$ the composition*

$$\rho[u] \circ \mu(\mathcal{D}_{\mathcal{F}}): K^k(M) \longrightarrow K^{k+\ell+p}(M) \qquad (15)$$

is multiplication by the exotic index class $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u]) = \rho[u](Ind_e(\mathcal{D}_{\mathcal{F}}, \epsilon)) \in K^{\ell+p}(M)$ for p even and $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u]) = \rho[u](Ind_e(\mathcal{D}_{\mathcal{F}})) \in K^{\ell+p}(M)$ for p odd.

Theorem 8.4 yields our best tool for proving the injectivity of the map $\mu(\mathcal{D}_{\mathcal{F}})$. Here is the main result:

COROLLARY 8.5 *Let $\mathcal{F}$ be a contractable foliation of leaf dimension p with Hausdorff holonomy groupoid $\mathcal{G}_{\mathcal{F}}$ and $\mathcal{D}_{\mathcal{F}}$ be a leafwise-elliptic, pseudo-differential operator. Suppose that for each $[\mathbf{e}] \in K^*(M)$, there exists a boundary K-theory class $[u_{\mathbf{e}}] \in K^*(\partial_h \mathcal{F})$ so that $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u_{\mathbf{e}}]) \otimes [\mathbf{e}] \in K^*(M) \otimes \mathbf{Q}$ is non-zero. Then the leafwise index map*

$$\mu(\mathcal{D}_{\mathcal{F}}): K^*(M) \otimes \mathbf{Q} \longrightarrow K_*(C^*(\mathcal{F})) \otimes \mathbf{Q}$$

is injective. In particular, if there exists $[u] \in K^*(\partial_h \mathcal{F})$ so that $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u_{\mathbf{e}}]) \in K^*(M) \otimes \mathbf{Q}$ is invertible, then $\mu(\mathcal{D}_{\mathcal{F}})$ is injective. □

A class $\mathcal{I} \in K^0(M) \otimes \mathbf{R}$ for a connected manifold M is invertible if and only if its virtual dimension is non-zero. That is, the restriction of $\mathcal{I}$ to a point $x \in M$

yields a non-trivial class in $K^0(x) \cong \mathbf{Z}$. In the above context, this implies that if $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$ has even degree and its restriction to a fiber over each connected component of M is non-trivial, then $\mu(\mathcal{D}_{\mathcal{F}})$ is injective.

In the Atiyah formalism of [2], given an hermitian vector bundle $p_{\mathbf{E}}\colon \mathbf{E} \to M$ and an elliptic operator $\mathcal{D}_{\mathbf{E}}$ along the fibers of $p_{\mathbf{E}}$, there is a map $\alpha(\mathcal{D}_{\mathbf{E}})\colon K(\mathbf{E}) \to K(M)$ given by integration along the fibers in K-theory. A key property of this map is that it commutes with the natural $p_{\mathbf{E}}^*$-module action of $K(M)$ on $K(\mathbf{E})$. Tensor product with the Bott class $\beta[\mathbf{E}] \in K(\mathbf{E})$ of the bundle $\mathbf{E}$ defines a map $\beta\colon K(M) \to K(\mathbf{E})$. The $K(M)$-module properties of α and β imply that $\alpha(\mathcal{D}_{\mathbf{E}}) \circ \beta\colon K(M) \to K(M)$ is multiplication by $\mathcal{I}(\beta[\mathbf{E}] \otimes \mathcal{D}_{\mathbf{E}}) \in K(M)$, which is calculated from the index theorem for families.

The constructions of the exotic index bear a strong similarity with the Atiyah approach. In the foliation context, the groupoid "fibration" $s\colon \mathcal{G}_{\mathcal{F}} \to M$ replaces the vector bundle $\mathbf{E} \to M$, and the fiberwise operator $\mathcal{D}_{\mathcal{G}_{\mathcal{F}}}$ replaces $\mathcal{D}_{\mathbf{E}}$. The transgression $\delta[u] \in K^*(\mathcal{G}_{\mathcal{F}})$ of a boundary class $[u] \in K^*(\partial_h \mathcal{F})$ replaces the Bott class $\beta[\mathbf{E}]$. There are generalized α and β maps as well:

$$\alpha(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}})\colon K(\mathcal{G}_{\mathcal{F}}) \to K(M) \tag{16}$$

$$\beta[u]\colon K(M) \to K(\mathcal{G}_{\mathcal{F}}) \tag{17}$$

where $\beta[u]([\mathbf{e}]) = \delta[u] \otimes [s^! \mathbf{e}]$ and $\alpha(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}})[\mathbf{e}] = Ind\,([\mathbf{e}] \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})$. The composition $\alpha(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}}) \circ \beta[u] = \mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$, so that injectivity of $\rho[u] \circ \mu(\mathcal{D}_{\mathcal{F}})$ is equivalent to injectivity of $\alpha(\mathcal{D}_{\mathcal{G}_{\mathcal{F}}}) \circ \beta[u]$.

The corona of Euclidean space $\mathbf{R}^N$ has the same K-theory as S^{N-1}, so for a vector bundle $\mathbf{E} \to M$, there is a unique boundary K-theory class which transgresses to a fiberwise fundamental class for the fibration (just as there is a unique Bott class.) For the more general situation of $s\colon \mathcal{G}_{\mathcal{F}} \to M$, each class $\delta[u] \in K^*(\mathcal{G}_{\mathcal{F}})$ can be used as a "Bott class" and the topological problem is to calculate the range of the index pairings $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$ for the various classes $[u] \in K^*(\partial_h \mathcal{F})$.

8.2 The foliation Novikov conjecture

The composition of groupoids $M \cong *M \subset \Pi_{\mathcal{F}} \subset \mathcal{G}_{\mathcal{F}}$ induces a sequence of classifying maps

$$M \simeq B(*M) \longrightarrow B\Pi_{\mathcal{F}} \longrightarrow B\mathcal{G}_{\mathcal{F}}$$

Haefliger (Corollaire 3.2.4, [51]) proved that for a foliation with uniformly contractable leaves, the composition $M \to B\mathcal{G}_{\mathcal{F}}$ is a homotopy equivalence. As a corollary, we note that the image of the induced map $H^*(\mathcal{G}_{\mathcal{F}}) \to H^*(M)$ equals the image of $H^*(B\Pi_{\mathcal{F}}) \to H^*(M)$.

CONJECTURE 8.6 (Foliation Novikov Conjecture, [8]) *Let* $(M, \mathcal{F})$ *and* $(M', \mathcal{F}')$ *be oriented* C^∞ *foliations with* M, M' *compact. Let* $f: M \to M'$ *be an orientation-preserving leafwise homotopy equivalence. Then for any class* $\omega \in H^*(B\Pi_{\mathcal{F}}; \mathbf{Q})$

$$(B\pi')^*\omega \cup L(TM') = f^*\left((B\pi)^*\omega \cup L(TM)\right) \tag{18}$$

where $L(TM)$ *denotes the Hirzebruch L-polynomial in the Pontrjagin classes of* TM.

The Foliation Novikov conjecture is said to hold for $\mathcal{F}$ if the conclusion (18) is true for all leafwise homotopy equivalences $f: M \to M'$ as above. For a foliation $\mathcal{F}$ with uniformly contractable leaves, Haefliger's theorem implies it suffices to check (18) holds for all $\omega \in H^*(B\mathcal{G}_{\mathcal{F}}; \mathbf{Q}) \cong H^*(M; \mathbf{Q})$.

Baum and Connes proved this conjecture for foliations whose leaves admit a metric with non-positive sectional curvatures, using the "dual Dirac" method [8]. We next show how the exotic index applies to extend their result. First, we need the foliation formulation of an idea introduced by Roe (section 6.2, [88].) Let $T\mathcal{F} \to M$ be the tangent bundle to the leaves of $\mathcal{F}$ and $S\mathcal{F}$ the sphere bundle for $T\mathcal{F}$ considered as a corona for $T\mathcal{F}$. There is a unique class $\Theta \in H^{p-1}(S\mathcal{F})$ whose boundary $\delta\Theta = \mathbf{Th}[T\mathcal{F}] \in H^p_c(T\mathcal{F})$ is the Thom class.

DEFINITION 8.7 *A foliation* $\mathcal{F}$ *on a connected manifold* M *is said to be* ultra-spherical *if there exists a map of coronas* $\sigma: \partial_h\mathcal{F} \to S\mathcal{F}$ *which commutes with the projections onto* M*, and so that* $\sigma^*\Theta \in H^*(\partial_h\mathcal{F})$ *is non-zero.*

THEOREM 8.8 *Let* $\mathcal{F}$ *be an oriented ultra-spherical foliation with uniformly contractable leaves and Hausdorff holonomy groupoid. Then the Foliation Novikov Conjecture is true for* $\mathcal{F}$.

Proof: By the standard reduction of the problem (cf. [8]), it suffices to show that the map $\mu(\mathcal{D}_{\mathcal{F}})$ is injective for the leafwise Dirac operator. By Corollary 8.5, this will follow from proving there exists a boundary K-theory class $[u] \in K^*(\partial_h\mathcal{F})$ so that $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u]) \in K^*(M) \otimes \mathbf{Q}$ is invertible.

Let $\eta \in K(S\mathcal{F})$ with K-theory boundary $\beta[T\mathcal{F}] \in K(T\mathcal{F})$, and set $[u] = \sigma^*\eta$.

LEMMA 8.9 $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$ *is is invertible in* $K^*(M) \otimes \mathbf{Q}$.

Proof: There is a continuous extension of σ to a map of pairs (cf. proof of Lemma 6.3, [88])

$$\overline{\sigma}: (\overline{\mathcal{G}_{\mathcal{F}}}, \partial_h\mathcal{F}) \longrightarrow (\overline{T\mathcal{F}}, S\mathcal{F})$$

which commutes with the projection onto M. By naturality of the boundary map, $\partial[u] = \overline{\sigma}^* \beta[T\mathcal{F}]$, so that

$$\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u]) = Ind\,(\overline{\sigma}^* \beta[T\mathcal{F}] \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}}) \tag{19}$$

The index class $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$ has even dimension, so it suffices to show that $Ind\,(\overline{\sigma}^* \beta[T\mathcal{F}] \otimes \mathcal{D}_{\mathcal{G}_{\mathcal{F}}})$ is non-zero when restricted to any fiber over M. But this follows from the original calculation of Roe, Theorem 6.9 [88]. □

REMARK 8.10 The sequence of hypotheses above have progressed from the least restrictive, "$\mathcal{F}$ is contractable" to the more restrictive, "$\mathcal{F}$ is ultra-spherical" with each assumption yielding further progress towards establishing the foliation Novikov Conjecture for that class of foliations. This is precisely parallel to the development of the proof of the Novikov Conjecture for compact manifolds, where the all current methods of proof seem to require a version of the "ultra-spherical hypotheses" and speculate that the techniques extend to the uniformly contractable case. It is natural to conjecture that the above techniques will show that the map $\mu(\mathcal{D}_{\mathcal{F}})$ is injective for contractable foliations. That is, the problem is to show that all contractable foliations admit a boundary K-theory class $[u] \in K^*(\partial_h \mathcal{F})$ so that $\mathcal{I}(\mathcal{D}_{\mathcal{F}}, [u])$ is a multiplicative unit in $K^*(M)$ for the leafwise signature operator $\mathcal{D}_{\mathcal{F}}$.

EXAMPLE 8.11 A uniformly contractable foliation $\mathcal{F}$ whose leaves have a metric so that their holonomy covers have no conjugate points is ultraspherical.

EXAMPLE 8.12 Let $\mathcal{F}$ be a Riemannian foliation $\mathcal{F}$ whose universal leaf L is ultra-spherical. Then by the proof of Proposition 6.23, $\mathcal{F}$ satisfies the hypotheses of Theorem 8.8.

9 Non-commutative isoperimetric functions

The Fourier transform $\hat{f}$ of a compactly supported continuous function $f \in C_c(\mathbf{R})$ has infinite support. Given a sequence of functions $\{f_n\} \subset C_c(\mathbf{R})$ whose supports tend to a point, then their transforms $\{\hat{f}_n\}$ are a family with vanishing gradients. These are elementary analytic remarks, but we can use the analogy with the ideas of the last section to ask about what aspect of coarse geometry corresponds to the Fourier transforms of the geometric data contained in the ersatz cohomology $H_{er}(L; \mathbf{R})$? The answer is almost flat K-theory, $K^*_{af}(L)$, and while their is no direct connection between the K-theory groups $K^*(\partial_h L)$ and $K^{*+1}_{af}(L)$, they are clearly strongly intuitively related by the above analogy.

The notion of *almost flat vector bundles* was introduced by Connes, Gromov and Moscovici for the study of the Novikov conjecture for compact manifolds [26, 27], motivated by the work of Gromov and Lawson [50]. These special bundles generate a subgroup $K^0_{af}(M) \subset K^0(M)$ of the Grothendieck group of the manifold M. Almost flat vector bundles are inherently a coarse geometric notion, and in this section we discuss a quantitative measure associated to these bundles, their "non-commutative" isoperimetric functions.

One of the fundamental properties of almost flat bundles is that the the index of the Dirac operator paired with an element of $K^0_{af}(M)$ is a topological invariant [60]. There is also a K^1-version of this property, where a self-adjoint elliptic operator on a manifold yields a projection (the projection onto its positive spectrum) which is then paired with a unitary multiplier, to obtain a generalized Toeplitz operator [9, 33]. The indices of these generalized Toeplitz operators can be explicitly estimated for almost flat unitary maps on M. More generally, for foliations there is a notion of $\mathcal{F}$-almost flat odd K-theory for a foliated manifold $(M, \mathcal{F})$ which has applications to the study of the spectral density function of leafwise elliptic operators.

9.1 Almost flat bundles for foliations

Let $(M, \mathcal{F})$ be a compact foliated measure manifold (or more generally, we must allow for M to be a foliated measure space in the sense of [82], or section 2 [73]) with leaves of dimension m. Assume there is given a leafwise Riemannian metric $\langle \cdot, \cdot \rangle_L$ of bounded geometry, which varies in a bounded measurable way with the (local) transverse parametrization of the leaf. Let ∇^L denote the associated Riemannian connection on the leaf L, and let $\nabla^{\mathcal{F}}$ denote the collection of all the leafwise connections.

A Hermitian vector bundle $\mathbf{E} \to M$ is a *foliated Hermitian flat bundle* if for each foliation chart U_α, there is a trivialization $\Phi_\alpha : \mathbf{E}|U_\alpha \cong \mathbf{C}^N \times \mathbf{D}^m \times \mathcal{T}_\alpha$, such that

- On the overlap of $U_\alpha \cap U_\beta$, the transition function

$$\Phi_\beta^{-1} \circ \Phi_\alpha : \mathbf{C}^N \times \phi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \mathbf{C}^N \times \Phi_\beta(U_\alpha \cap U_\beta)$$

  is a *constant* Hermitian isomorphism when restricted to the "horizontal sets" $\mathbf{D}^m \times \{x\}$

- $\Phi_\beta^{-1} \circ \Phi_\alpha(\vec{v}, x)$ depends measurably on the parameter x for all $\vec{v} \in \mathbf{C}^N$;

Let $\nabla^{\mathbf{E}_L}$ denote the leafwise Hermitian connection for $\mathbf{E}_L$

$U(\mathbf{E})$ denotes the $U(N)$-principal bundle of unitary fiberwise automorphisms of $\mathbf{E}$. Let $C^1_{\mathcal{F}}(U(\mathbf{E}))$ denote the measurable sections whose restrictions to leaves are C^1. In the case where $M = M$ with the one leaf foliation, with $L = M$, we write $C^1(U(\mathbf{E})) = C^1_{\mathcal{F}}(U(\mathbf{E}))$.

Define a C^1-pseudo-norm for $g \in C^1_{\mathcal{F}}(U(\mathbf{E}))$: Let $\{\tilde{h}_1, \ldots, \tilde{h}_N\}$ be a local $\nabla^{\mathbf{E}_L}$-synchronous orthonormal framing about $x \in L$. For example, fix a trivialization $\Phi_\alpha : \mathbf{E}|U_\alpha \cong \mathbf{C}^N \times \mathbf{D}^m \times \mathcal{T}_\alpha$ with $x \in U_\alpha$. Choose an Hermitian framing $\{\vec{v}_1, \ldots, \vec{v}_N\}$ for $\mathbf{C}^N$ for the induced metric on $\mathbf{C}^N$, then set

$$\tilde{h}_\ell(v) = \Phi_\alpha^{-1}(\vec{v}_\ell, \varphi_\alpha(v))$$

The restriction of $\{\tilde{h}_1, \ldots, \tilde{h}_N\}$ to the plaque of L containing x is a local synchronous framing.

For $g \in C^1_{\mathcal{F}}(U(\mathbf{E}))$, let g_L denote the restriction to a leaf L. We use a synchronous framing on L about $x \in L$ to express g_L in matrix form:

$$g_L \cdot \tilde{h}_j \quad = \sum_{1 \leq i \leq N} (g_L)_{ij} \cdot \tilde{h}_i$$

for local C^1-functions g^L_{ij} defined on an open neighborhood in L of x. Then define

$$\|g\|_{(1)} \quad = \quad \sup_{L \subset M} \ \sup_{x \in L} \sqrt{\sum_{1 \leq i,j \leq N} \| \nabla^L (g_L)_{ij}|_x \|^2} \tag{20}$$

A map $g \in C^1_{\mathcal{F}}(U(\mathbf{E}))$ is *admissible* if $\|g\|_{(1)} < \infty$.

DEFINITION 9.1 (cf. Definition 5.1 [60]) *An* almost flat odd cocycle *for $(M, \mathcal{F})$ consists of the data $g^{af} = \{(g_i, N_i) \mid i \geq 1\}$ such that for each $i \geq 1$:*

- *$\mathbf{E}_i \to M$ is a foliated Hermitian flat bundle of dimension N_i*
- *$g_i \in C^1_{\mathcal{F}}(U(\mathbf{E}_i))$ is an admissible map with $||g_i||_{(1)} \leq 1/i$*
- *The stabilized vector bundles $\mathbf{E}_i \oplus \mathbf{C}^\infty$ are all isomorphic to a common Hermitian vector bundle $\mathbf{E}_\infty \to M$*
- *there is a continuous family of admissible maps $g_t \in C^1_{\mathcal{F}}(U(\mathbf{E}_\infty))$ for $i \leq t \leq i+1$ interpolating between the stabilized sections g_i and g_{i+1}.*

We say that two almost flat odd cycles $\{g^{af}\}$ and $\{h^{af}\}$ are *equivalent* if there exists admissible maps $H_i(t) \in C^1_{\mathcal{F}}(U(\mathbf{E}_\infty))$ interpolating between g_i and h_i for all $i \geq 0$.

PROPOSITION 9.2 *The set of equivalence classes of almost flat odd cocycles for $(M, \mathcal{F})$ forms a group, $K^1_{af}(M, \mathcal{F})$, called the* almost flat odd K-theory *of $\mathcal{F}$.*

Suppose that M is a compact Riemannian manifold with fundamental group $\Lambda = \pi_1(M, y_0)$, and $\tilde{M}_\Gamma \to M$ is the covering associated to a surjection $\rho : \Lambda \to \Gamma$, with the covering group Γ acting on the left on $\tilde{M}_\Gamma$. There is a version of the suspension construction for an action of Γ on a measure space X: Let X denote a standard, second countable Borel measure space, with $\tilde{\mu}$ a Borel probability measure on X. Consider a Borel action $\varphi : \Gamma \times X \to X$ which preserves $\tilde{\mu}$. The product of the deck action on $\tilde{M}_\Gamma$ with the φ-action on X defines an action of Γ on $\tilde{M}_\Gamma \times X$. Form the quotient measure space,

$$M_\varphi = \Gamma \setminus (\tilde{M}_\Gamma \times X).$$

The product foliation on $\tilde{M}_\Gamma \times X$, with typical leaf $\tilde{L} = \tilde{M}_\Gamma \times \{x\}$ for $x \in X$, descends to a measurable foliation denoted by $\mathcal{F}_\varphi$ on M_φ. The measure $\tilde{\mu}$ descends to a holonomy-invariant transverse measure μ for $\mathcal{F}_\varphi$.

Exactly as before, let $K_\varphi \subset \Lambda$ denote the subgroup of elements which act trivially on X under φ, and let $\Gamma_\varphi = \Lambda / K_\varphi$ denote the quotient group. The *global holonomy group* of $\mathcal{F}_\varphi$ is the isomorphic image

$$\Gamma_\varphi \overset{\varphi}{\cong} \mathcal{H}_{\mathcal{F}_\varphi} \subset \mathrm{Aut}(X).$$

The projection onto the first factor map, $\tilde{M}_\Gamma \times X \to \tilde{M}_\Gamma$, descends to a map $\pi : M_\varphi \to M$, and π restricted to the leaves $\mathcal{F}_\varphi$ is a covering map. The Riemannian metric on TM lifts via π^* to a leafwise metric on $T\mathcal{F}_\varphi$. The foliated spaces $(M_\varphi, \mathcal{F}_\varphi)$ are prototypical.

There is a natural construction of Borel measure space $(X_\Gamma, \tilde{\mu}_\Gamma)$ associated to a group Γ, equipped with a measure preserving ergodic action φ of Γ. Endow the two-point space $\mathbf{Z_2} = \{0, 1\}$ with the "$\frac{1}{2} - \frac{1}{2}$" probability measure, and set

$$X_\Gamma = \prod_{\gamma \in \Gamma} (\mathbf{Z_2})_\gamma$$

equipped with the product topology from the factors and product measure $\tilde{\mu}_\Gamma = \prod_{\gamma \in \Gamma} \mu_\gamma$. A typical element of X_Γ is a string $x = \{a_\gamma\} = \{a_\gamma \mid a_\gamma \in \mathbf{Z_2} \text{ for } \gamma \in \Gamma\}$. Let $\varphi : \Gamma \times X_\Gamma \to X_\Gamma$ be the "shift" action of Γ on X_Γ, defined by $\varphi(\delta, \{a_\gamma\}) = \{a_{\delta\gamma}\}$. The shift action is continuous, transitive, measure-preserving, ergodic and free for $\tilde{\mu}_\Gamma$-a.e. $x \in X_\Gamma$.

For each quotient group $\Lambda \to \Gamma$, introduce the foliated measure space $M_\Gamma = \Gamma \setminus (\tilde{M}_\Gamma \times X_\Gamma)$ with foliation $\mathcal{F}_\Gamma$, transverse invariant measure μ_Γ and μ_Γ-typical leaf isometric to $\tilde{M}_\Gamma$.

DEFINITION 9.3 *A* Γ-almost flat odd cocycle *for* M *consists of the data* $g^{af} = \{(g_i, \mathbf{E}_i, N_i) \mid 0 \leq i\}$ *which satisfy:*

- $\mathbf{E}_0 \to M$ *is the product bundle with fibers of dimension* N_0
- $\mathbf{E}_i \to M$ *is an Hermitian flat bundle of dimension* N_i *associated to a holonomy homomorphism* $\Lambda \xrightarrow{\rho} \Gamma \xrightarrow{\alpha} U(N_i)$
- $g_i \in C^1(U(\mathbf{E}_i))$ *is an admissible map with* $||g_i||_{(1)} \leq 1/i$
- *There is an Hermitian vector bundle* $\mathbf{E}_\infty$ *so that* $\mathbf{E}_i \oplus \mathbf{C}^\infty \cong \mathbf{E}_\infty$ *for all* i
- *For each* $i \geq 0$*, there is an admissible map* $g_t \in C^1_{\mathcal{F}_\Gamma}(U(\pi^!\mathbf{E}_\infty))$ *for* $i \leq t \leq i+1$ *interpolating between the stabilized sections* $\pi^* g_i$ *and* $\pi^* g_{i+1}$*, where* $\pi : M_\Gamma \to M$*.*

Let $[g^{af}] \in K^1(M)$ denote the class of the map $g_0 : M \to U(N_0)$.

DEFINITION 9.4 *For a quotient group* $\rho : \Lambda \to \Gamma$*, the* Γ-almost flat odd K-theory *of* M *is the subgroup* $K^1_{\Gamma af}(M) \subset K^1(M)$ *of elements* $[g^{af}]$*, where* g^{af} *is a* Γ*-almost flat odd cocycle for* M*.*

DEFINITION 9.5 *The* almost flat odd K-theory *of* M *is the group* $K^1_{af}(M) = K^1_{\Lambda af}(M)$ *associated to the fundamental group* Λ *of* M*.*

Almost flat odd K-theory is functorial:

PROPOSITION 9.6 *Let* $\Lambda \xrightarrow{\rho} \Gamma \xrightarrow{q} \Gamma'$ *be a composition of submersions. Then there is a natural map*

$$q^! : K^1_{\Gamma' af}(M) \to K^1_{\Gamma af}(M).$$

In particular, for all $\rho : \Lambda \to \Gamma$*, there is a map* $\rho^! : K^1_{\Gamma af}(M) \to K^1_{af}(M)$*.*

PROPOSITION 9.7 ([69]) *There is a natural map*

$$\pi^! : K^*_{\Gamma af}(M) \to K^*_{af}(M_\Gamma, \mathcal{F}_\Gamma). \tag{21}$$

which is injective on rational K-theory.

One of the most fundamental questions about almost flat K-theory is to establish a precise relation with the corona space. Here is the precise problem:

PROBLEM 9.8 *Let* $\Gamma \cong \pi_1(B, b_0)$ *where* B *is a compact simplicial space with contractable universal covering. Construct a natural, non-trivial boundary map*

$$\partial_{af} \colon K^0(\partial_h \Gamma) \longrightarrow K^1_{af}(B\Gamma)$$

9.2 Non-commutative isoperimetric functions

In this section, we introduce a numerical measure of how efficiently an almost flat bundle can be realized on a leaf or covering. To obtain a small gradient ϵ, one must use bundles of increasingly large dimensions. In the standard examples (see §§9.3 & 9.4 below), the dimension required is related to the degree of a covering on which the bundle data can be sufficiently smoothed. Hence the bundle dimension corresponds to a "volume measure" in K-theory, and it is natural to study the function relating the smoothness ϵ with the "volume" required to achieve this smoothness. This is the reasoning behind our definition of "non-commutative isoperimetric functions".

We first consider the case of unitaries over a compact space $B\Gamma$. Fix $u \in K^1(M)$. Realize the classifying space of the discrete group Γ with a simplicial space $B\Gamma$ endowed with a compatible Riemannian metric on the simplices. For each $\epsilon > 0$, let $D_{\Gamma,u}(\epsilon)$ denote the minimum dimension of a Hermitian flat bundle $\mathbf{E}_\epsilon \to B\Gamma$ so that u is represented by a fiberwise unitary $g_\epsilon \in C^1(U(\mathbf{E}_\epsilon))$ with $||u_\epsilon||_{(1)} \leq \epsilon$. If no such bundle exists, set $D_{\Gamma,u}(\epsilon) = \infty$.

DEFINITION 9.9 (Non-commutative Γ-isoperimetric function) *For $\epsilon > 0$, set*

$$\mathcal{I}_{\Gamma,u}(\epsilon) = \frac{1}{D_{\Gamma,u}(\epsilon)} \tag{22}$$

Introduce an equivalence relation on positive functions, where $f \sim g$ if there exists a constant $a > 0$ such that

$$g(\frac{\epsilon}{a}) \leq f(\epsilon) \leq g(a\epsilon) \qquad \text{for all } \epsilon > 0.$$

The following result states that $\mathcal{I}_{\Gamma,u}(\epsilon)$ is a coarse invariant.

PROPOSITION 9.10 ([69]) *Let $\Gamma \cong \pi_1(B, b_0)$ where B is a compact simplicial space with contractable universal covering. Then for each $u \in K^1(B\Gamma)$, the class of $\mathcal{I}_{\Gamma,u}(\epsilon)$ is a topological invariant.*

There is a similar definition of the non-commutative isoperimetric function for foliated manifolds. Let $(M, \mathcal{F})$ be a compact foliated measure space. Fix $u \in K^1_{af}(M, \mathcal{F})$, represented by a fiberwise Hermitian automorphism $g \in C^1_{\mathcal{F}}(\mathbf{E})$. For each $\epsilon > 0$, let $D_{\mathcal{F},u}(\epsilon)$ denote the minimum dimension of a foliated Hermitian flat bundle $\mathbf{E}_\epsilon \to M$ so that u is represented by a fiberwise unitary $g_\epsilon \in C^1_{\mathcal{F}}(U(\mathbf{E}_\epsilon))$ with $||g_\epsilon||_{(1)} < \epsilon$. If no such bundle exists, set $D_{\mathcal{F},u}(\epsilon) = \infty$.

DEFINITION 9.11 *The non-commutative foliated isoperimetric function is defined by*

$$\mathcal{I}_{\mathcal{F},u}(\epsilon) = \frac{1}{D_{\mathcal{F},u}(\epsilon)} \tag{23}$$

The same ideas as used in the proof of Proposition 9.10 also establish:

PROPOSITION 9.12 *Let $(M, \mathcal{F})$ be a C^1-foliation of a* compact *manifold M. Then for each $u \in K^1_{af}(M, \mathcal{F})$, the class of $\mathcal{I}_{\mathcal{F},u}(\epsilon)$ is a leafwise homotopy invariant.*

9.3 Profinite bundles

The idea of almost flat bundles is best illustrated by considering the special case of bundles over residually finite manifolds, which leads to the concept of *profinite K^1-cocycles* for $K_{af}(M)$.

For each $N > 0$, $U(N) \subset M(N, \mathbf{C}) \cong \mathbf{C}^{N^2}$ denotes the group of $N \times N$-unitary matrices considered as a subspace of the vector space of all matrices. Let $U(\infty)$ denote the stabilized super-group with the weak limit topology. The C^1-semi-norm of a C^1-function $g : \tilde{M} \to U(N)$ is defined as the supremum of the norms of the covariant derivatives of its matrix entries,

$$||g||_{(1)} = \sup_{x \in \tilde{M}} \sup_{1 \leq k, \ell \leq N} ||\nabla g_{k\ell}||_x \tag{24}$$

DEFINITION 9.13 (Profinite K^1-Γ-cocycles) *Let $\rho : \Lambda \to \Gamma$ be a submersion. A profinite Γ-cocycle for M consists of the data $g^{pf} = \{(g_i, \Gamma_i, N_i) \mid 0 \leq i\}$ which satisfy:*

- *Γ_i is a finite quotient group of Γ, with $\Gamma_0 = \Gamma$*
- *$\pi_i : \tilde{M}_i \to M$ is the covering of M associated to the surjection $\Lambda \to \Gamma \to \Gamma_i$*
- *$g_i : \tilde{M}_i \to U(N_i)$ is a C^1 mapping with $||g_i||_{(1)} < 1/i$*
- *For each $i \geq 0$, $|\Gamma_i| \cdot [g_i] = [g_0 \circ \pi_i] \in K^1(\tilde{M}_i)$.*

Let $[g^{pf}] \in K^1(M)$ denote the homotopy class of the stabilized map $g_0 \to U(N_0) \subset U(\infty)$.

Let $K^1_{\Gamma pf}(M) \subset K^1(M)$ denote the subset of classes represented by profinite K^1-Γ-cocycles. When $\Gamma = \Lambda = \pi_1(M, x_0)$, then we simply write $K^1_{pf}(M)$.

PROPOSITION 9.14 ([69]) *$K^1_{\Gamma pf}(M)$ is a subgroup of $K^1_{\Gamma af}(M)$.*

9.4 Calculations of isoperimetric functions

The universal covering of the m-torus $\mathbf{T}^m$ is $\mathbf{R}^m$, and identify the deck action of the fundamental group $\Lambda \cong \mathbf{Z}^m$ on $\mathbf{R}^m$ with the translation action of the subgroup $\mathbf{Z}^m \subset \mathbf{R}^m$. A Riemannian metric on $\mathbf{T}^m$ lifts to a $\mathbf{Z}^m$-periodic Riemannian metric on $\mathbf{R}^m$, and an Hermitian vector bundle $\mathbf{E}^0_{\mathbf{T}^m} \to \mathbf{T}^m$ lifts to a $\mathbf{Z}^m$-periodic bundle $\mathbf{E}^0 \to \mathbf{R}^m$.

Given a C^1-map $g\colon M \to U(p)$, set

$$\|[g_0]\|_{(1)} = \inf\left\{\| g \|_{(1)} \mid g\colon M \to U(p) \text{ and } g \sim g_0\right\}$$

PROPOSITION 9.15 *Let $0 \neq u \in K^1(\mathbf{T}^m)$ be represented by $g_0 : \mathbf{T}^m \to U((m+1)/2)$.*

$$D_{\mathbf{Z}^m,u}(\|[g_0]\|_{(1)}/\ell) \leq \frac{(m+1)\cdot \ell^m}{2} \tag{25}$$

and hence $\mathcal{I}_{\mathbf{Z}^m,u}(\epsilon) \sim \epsilon^m$ for ϵ small.

Proof: We follow the re-scaling method of of Gromov and Lawson [50]: For each integer $i > 0$ let $\pi_i : \tilde{M}_i \to \mathbf{T}^m$ denote the covering corresponding to the subgroup $\Lambda_i = i\cdot \mathbf{Z}^m \subset \mathbf{Z}^m$ with index $[\Lambda : \Lambda_i] = i^m$. Let $\Phi_i : \tilde{M}_i \cong \mathbf{T}^m$ be the canonical diffeomorphism which decreases distances by the factor $1/i$ and define a unitary $g_i = g_0 \circ \Phi_i : \tilde{M}_i \to U$. Thus, each map g_i is topologically the same as the map g_0 but is considered as a map on the covering $\tilde{M}_i$ which is a metric re-scaling of the base torus. The sequence $g^{pf} = \{(g_i, \Lambda_i, N_i) \mid 0 \leq i\}$ for $N_i = (m+1)/2$ clearly satisfies the conditions of Definition 9.13. □

REMARK 9.16 The motivation for calling $\mathcal{I}_{\Gamma,u}(\epsilon)$ an "isoperimetric function" appears in the above derivation of the estimate (25). Recall the usual isoperimetric constant for a complete Riemannian manifold X (cf. Theorem 1, [99]):

$$h(X) = \inf_{U\subset X}\ \inf_{f\in C^1_c(U)} \frac{\int_U \| \nabla f \|\, dvol}{\int_U | f |\, dvol} \tag{26}$$

For a typical test function f which satisfies $|f| \leq 1$, the isoperimetric constant is dominated by the ratio of the supremum of $\| \nabla f \|$ on U to the mass of U. Observe that for a class $u \in K^1_{\Gamma pf}(M)$, the function $\mathcal{I}_{\Gamma,u}(\epsilon)$ is dominated by the ratio of the supremum of $\| \nabla g \|$, for $g : \tilde{M}_i \to U(\infty)$ in the class of u, to the number $|\Gamma_i|$ which is proportional to the mass of $\tilde{M}_i$. Thus, the function $\mathcal{I}_{\Gamma,u}(\epsilon)$ measures how "efficiently" the K-theory class u can be realized on the open manifold $\tilde{M}_\Gamma$ in terms of volume.

The above example $M = \mathbf{T}^m$ is a special case of a general class of manifolds for which one can derive an estimate for $\mathcal{I}_{\Lambda,u}(\epsilon)$. Recall the definition of an a compactly enlargeable manifold due to Gromov and Lawson ([50]). A Riemannian manifold is *enlargeable* of dimension m if for every $\epsilon > 0$, there exists a covering (possibly infinite) $\tilde{M}_\epsilon \to M$ and a degree one map $f_\epsilon : \tilde{M}_\epsilon \to S^m$ which is constant at infinity and has $\| \nabla f_\epsilon \| < \epsilon$. The manifold M is *compactly enlargeable* if for each $\epsilon > 0$, there exists a finite covering $\tilde{M}_\epsilon$ with these properties. There are many examples of compactly enlargeable manifolds:

THEOREM 9.17 (Theorems 5.3, 5.4, [79]) *The following are compactly enlargeable:*

1. *A compact Riemannian manifold which admits a globally expanding self-map.*
2. *A compact arithmetic manifold with constant non-positive sectional curvatures.*
3. *The product of compactly enlargeable manifolds.*
4. *The connected sum of any compact manifold with a compactly enlargeable manifold.*
5. *Any manifold which admits a map of non-zero degree onto an enlargeable manifold.*

Recall that a map f from a metric space (X, d_X) to a metric space (Y, d_Y) is called *globally expanding* if for any two points $x_1, x_2 \in X$ with $x_1 \neq x_2$, one has $d_Y(f(x_1), f(x_2)) > d_X(x_1, x_2)$. John Franks proved that the fundamental group $\Lambda = \pi_1(M, x_0)$ of a compact Riemannian manifold M which admits a *globally expanding self-map* has polynomial growth, hence by the celebrated theorem of Gromov, Λ must contain a nilpotent subgroup of finite index. Thus, by Shub's criteria the map f is topologically conjugate to an expanding infra-nil-endomorphism of M. See the Introduction and section 1 of the paper of Gromov, [46], for a discussion and references concerning globally expanding self-maps.

The *covering degree function* of a compactly enlargeable Riemannian manifold M is defined for all $\epsilon > 0$:

$$\begin{aligned} CD_M(\epsilon) \;\; = \;\; \inf\{[\Lambda : \Lambda_i] \mid \Lambda_i = \pi_1(\tilde{M}_i) \; & \text{ and there exists a degree one map} \\ & f_\epsilon : \tilde{M}_i \to S^m \;\text{ with }\; \| \nabla f_\epsilon \| < \epsilon\}. \end{aligned}$$

The proof of Proposition 9.14 yields the estimate:

LEMMA 9.18 *Let M be a compactly enlargeable, odd dimensional Riemannian manifold with fundamental group Λ, and $u = [\iota \circ f] \in K^1(M)$ the K-theory class of $\iota: S^m \to U((m+1)/2)$ composed with a degree-one map $f: M \to S^m$. Then there exists a constant $C(M) > 0$ so that*

$$D_{\Lambda,u}(\epsilon) \leq C(M) \cdot CD_M(\epsilon)$$

In particular, this implies that the reciprocal function $\mathcal{I}_{\Lambda,u}(\epsilon) > 0$ when $\epsilon > 0$. □

10 Coarse invariance of the leafwise spectrum

Our final topic considers relations between the spectrum of operators and coarse geometry. The results we describe just open the door to a whole other area where the ideas of coarse geometry are being applied (cf. [12, 13, 65, 69, 88]. Recall the spectrum of a symmetric elliptic differential operator on a complete open Riemannian manifold is a closed subset of the real line, but there are few other *a priori* restrictions on its nature. One knows from the Weyl test formula that the topological nature of the spectrum is determined by the behavior of the operator on test functions whose supports tend to infinity. This suggests searching for properties of the spectrum of an elliptic operator $\mathcal{D}$ on a complete Riemannian manifold L of bounded geometry, which depend only on the coarse geometry of L. The idea is to prove that the measure theory and geometry at infinity for L control aspects of the local spectrum of $\mathcal{D}$. For example, the Poisson kernel formula for constructing harmonic functions on the Poincaré disk is an example of this phenomenon, an one speculates that similar properties might be found for more general open manifolds. We will give two cases where a coarse geometric property of L implies properties of the local spectrum of an elliptic differential operator on L. Both results are proved using index theory techniques.

10.1 Coronas and leafwise spectrum

Roe observed (Proposition 5.21 [88]) that the existence of a gap in the spectrum of a geometric operator $\mathcal{D}$ on a complete open manifold L implies the exotic index of $\mathcal{D}$ vanishes. This "gap" property is an important source of relations between coarse geometry and index theory, via the Lichnerowicz formalism [49, 50, 88, 89].

Fix a foliated manifold M with foliation $\mathcal{F}$ having leaves of dimension p, and a leafwise-smooth Riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{F}}$ on $T\mathcal{F}$ such that each leaf L is a complete manifold with bounded geometry. We assume the leafwise metrics, with the C^2-topology, vary continuously with the transverse parameter. Recall that ∇^L is the associated Riemannian connection on the leaf L with $\nabla^{\mathcal{F}}$ denoting the family of leafwise connections.

Let $\mathcal{S} \to M$ be the Clifford bundle of spinors associated to the Clifford algebra bundle $\mathcal{C}(T\mathcal{F})$, and for each leaf $L \subset M$, let $\mathcal{S}_L \to L$ denote the restricted bundle. Then $\not{\mathcal{D}}_L : C_c^\infty(\mathcal{S}_L) \to C_c^\infty(\mathcal{S}_L)$ denotes the corresponding leafwise Dirac operator. Given a smooth Hermitian vector bundle $\mathbf{E}^0 \to M$, we can introduce the leafwise geometric operators

$$\mathcal{D}_L = \not{\mathcal{D}}_L \otimes \nabla^{\mathbf{E}^0_L} \tag{27}$$

defined on the (leafwise) compactly supported sections $C_c^\infty(\mathbf{E}_L)$ of the bundle $\mathbf{E} = \mathcal{S} \otimes \mathbf{E}^0$ restricted to the leaves of $\mathcal{F}$.

DEFINITION 10.1 *A foliation geometric operator $\mathcal{D}$ for $(M, \mathcal{F})$ is a collection of leafwise geometric operators $\{\mathcal{D}_L \mid L \subset M\}$ defined as in* (27) *for some leafwise Riemannian metric for $\mathcal{F}$ and some Hermitian vector bundle $\mathbf{E}^0$ as above.*

DEFINITION 10.2 *We say that the spectrum of $\mathcal{D}_{\mathcal{F}}$ has a* uniform gap *about $\lambda \in \mathbf{R}$ if there exists $\delta > 0$ such that, for each $x \in M$, the intersection $\sigma(\mathcal{D}_x) \cap (\lambda - \delta, \lambda + \delta)$ is empty for all $x \in M$.*

Roe's observations about exotic indices and spectral gaps carry over to the case of foliations:

THEOREM 10.3 ([67]) *Let $\mathcal{D}_{\mathcal{F}}$ be a leafwise geometric operator for $\mathcal{F}$ with coefficients in an Hermitian bundle $\mathbf{E} \to M$.*

1. *Suppose that $\mathcal{D}_{\mathcal{F}}$ has uniform gap about 0. Then for any self-adjoint grading ϵ for $\mathcal{D}_{\mathcal{F}}$ and class $[u] \in K^\ell(\partial_h \mathcal{F})$ the exotic index $\rho[u](Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)) \in K^{\ell+1}(M)$ is trivial.*

2. *Suppose there exists $\lambda \in \mathbf{R}$ such that $\mathcal{D}_{\mathcal{F}}$ has a* uniform gap *about λ. Then for any and class $[u] \in K^\ell(\partial_h \mathcal{F})$, the self-adjoint exotic index $\rho[u](Ind(\mathcal{D}_{\mathcal{F}})) \in K^\ell(M)$ is trivial.*

The point is that the exotic index $\rho[u](Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)) \in K^{\ell+1}(M)$ is calculated in terms of a pairing between the Chern character of the symbol of the leafwise operator $\mathcal{D}_{\mathcal{F}}$ and the transgression of the Chern character of the boundary K-theory class. The existence of the hypothesized boundary K-theory class is strictly a coarse geometric property of $\mathcal{F}$.

Theorem 10.3 has also applications to the existence of metrics of positive scalar curvature (cf. Rosenberg [89]; Zimmer [103]; section 6C of Roe [88]):

COROLLARY 10.4 *Let $\mathcal{F}$ be a C^∞-foliation with even dimensional leaves of a compact manifold M, and assume the tangential distribution $T\mathcal{F}$ admits a spin structure. If there exists a Riemannian metric on $T\mathcal{F}$ so that each leaf of $\mathcal{F}$ has positive scalar curvature, then the exotic index $\rho[u](Ind(\mathcal{D}_{\mathcal{F}}, \epsilon)) = 0$ of the leafwise Dirac operator for any class $[u] \in K^\ell(\partial_h \mathcal{F})$. For a foliation with odd dimensional leaves, the corresponding statement holds for the odd exotic index classes.*

10.2 Spectral density and isoperimetric functions

The "Vafa-Witten method" (section III, [96] & section 3, [4]) can be combined with the foliation index theorem for leafwise Toeplitz operators ([22, 32, 33]) to obtain topological obstructions to the existence of a gap in the spectrum of a geometric operator $\sigma(\mathcal{D})$ on an open manifold. What is more, the method yields *estimates* on the spectral density function for the operator $\mathcal{D}$ in terms of the non-commutative isoperimetric function $\mathcal{I}_{\Gamma,u}(\epsilon)$ and the index pairing between the K-theory class $[u]$ and that of the symbol of the operator. We state the typical result for the case where $L = \tilde{M}$ is a covering of a compact manifold M:

THEOREM 10.5 *Let M be a compact orientable odd-dimensional Riemannian manifold with fundamental group $\Lambda = \pi_1(M, y_0)$. For a quotient group $\rho : \Lambda \to \Gamma$, let $\pi : \tilde{M}_\Gamma \to M$ be the associated normal covering. Fix an element of odd K-theory $u \in K^1(B\Gamma)$.*

Given a first-order, symmetric, geometric operator $\mathcal{D}_M$ acting on the sections of a Hermitian vector bundle $\mathbf{E}_M \to M$, let $\mathcal{D}_\Gamma : C^1_c(\mathbf{E}_\Gamma) \to C^1_c(\mathbf{E}_\Gamma)$ denote the lifted operator acting on the compactly supported sections of the lifted Hermitian bundle $\mathbf{E}_\Gamma = \pi^!(\mathbf{E}_M) \to \tilde{M}_\Gamma$.

Finally, let $\tilde{\mathcal{D}}$ be a Γ-invariant, relatively compact perturbation of $\mathcal{D}_\Gamma$.

Then there exists a constant $\kappa(\tilde{\mathcal{D}}) > 0$, which depends on the Riemannian geometry of M and the perturbation $\tilde{\mathcal{D}}$, so that for all $\lambda \in \mathbf{R}$ and all $\epsilon > 0$,

$$Tr_\Gamma \left(\{\chi_{[\lambda,\lambda+\epsilon)}(\tilde{\mathcal{D}})\} \right) \geq \frac{1}{4} \cdot |\langle ch^*(B\rho^* u), ch_*[\mathcal{D}_M]\rangle| \cdot \mathcal{I}_{\Gamma,u}(\epsilon/4\kappa(\tilde{\mathcal{D}})) \qquad (28)$$

where Tr_Γ is the Γ-trace of Atiyah [3], $\chi_{[\lambda,\lambda+\epsilon]}(\tilde{\mathcal{D}})$ is the spectral projection associated to the characteristic function $\chi_{[\lambda,\lambda+\epsilon]}$, and the pairing in (28) is the (integral) odd Toeplitz index of the compression of the unitary multiplier for $B\rho^* u$ with the positive projection of $\mathcal{D}_M$.

In particular, if $\langle ch^*(B\rho^* u, ch_*[\mathcal{D}_M]\rangle \neq 0$ for some $u \in K^1(M)$, and $\mathcal{I}_{\Gamma,u}(\epsilon) > 0$ when $\epsilon > 0$, then the spectrum $\sigma(\tilde{\mathcal{D}}) = \mathbf{R}$.

The number $Tr_\Gamma\left(\{\chi_{[\lambda,\lambda+\epsilon)}(\tilde{\mathcal{D}})\}\right)$ is the "average spectral density" for the operator $\tilde{\mathcal{D}}$ in the interval $[\lambda, \lambda+\epsilon)$. If the spectrum of $\tilde{\mathcal{D}}$ is isolated in this interval, then $Tr_\Gamma\left(\{\chi_{[\lambda,\lambda+\epsilon)}(\tilde{\mathcal{D}})\}\right)$ is the integral over a fundamental domain in $\tilde{M}_\Gamma$ of the Γ-periodic function $\sum_n \parallel f_n \parallel^2$, where $\{f_n\}$ is an orthogonal basis for the eigensections of $\tilde{\mathcal{D}}$ in $[\lambda, \lambda+\epsilon)$. The result is a type of dimension: for a compact manifold, this integral will be the dimension of the sum of the eigenspaces in this interval. More generally, it is an average density of the eigenspaces in the interval $[\lambda, \lambda+\epsilon)$, which makes sense whether the spectrum is isolated or not.

A fundamental point of Theorem 10.5 is that the function class of the right-hand-side of (28) is a coarse geometric invariant of the symbol of the operator $\mathcal{D}_M$ and the K-theory class u, so that when the index pairing is non-trivial we obtain a topologically determined lower bound on the Γ-spectral density function for the Γ-periodic lift $\mathcal{D}_\Gamma$. For example, when $\Gamma \cong \mathbf{Z}^n$ for n odd and u is the top odd dimensional K-theory generator, then $\mathcal{I}_{\Gamma,u}(\epsilon) \sim \epsilon^n$ for ϵ small.

This result can be considered as parallel to the results of R. Brooks [13, 14] and Sunada [91, 92] on the spectrum of the Laplacian on open manifolds, which are based on the relation between the Cheeger isoperimetric constant for $\tilde{M}_\Gamma$ and the spectrum of the Laplacian.

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