Expansive Maps of the Circle

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1 The Theorem

Let Γ be a finitely-generated group, and S^1 the circle of radius 1 with metric $d: S^1 \times S^1 \to [0, \pi]$. We let $\varphi: \Gamma \times S^1 \to S^1$ denote an action of Γ on the circle by homeomorphisms. For notational convenience, given $\gamma \in \Gamma$ we let $\gamma x = \varphi(\gamma)(x)$.

Definition 1.1 *The action $\varphi: \Gamma \times S^1 \to S^1$ is expansive if there exists $\epsilon > 0$ so that for any pair $x \neq y \in S^1$, there exists $\gamma \in \Gamma$ such that $d(\gamma x, \gamma y) > \epsilon$.*

In this paper we answer a question posed by Thomas.B.Ward:

Theorem 1.2 *If $\varphi: \Gamma \times S^1 \to S^1$ is an expansive action, then Γ cannot be an infra-nilpotent group.*

The proof will be given in Section 4, and requires only elementary methods of topological dynamics. We prove two preparatory results in Sections 2 and 3.

After preparing this note, the paper [IT] was discovered which covers the more general situation of "expansive foliations", and in codimension one their result includes Theorem1.2. We note that the proof given here is more self–contained and goes into more detail about the construction of the "ping-pong table" in the topological case, which the reader might still find of interest. Ralf Spatzier has also informed the authors that he has a proof of the main theorem as well.

The last section gives two examples of expansive actions on the circle where Γ is a solvable group with exponential word growth, the first with a unique minimal set consisting of a fixed–point, and the second with an exceptional minimal set of Denjoy type, showing that the conclusion Γ is infra-nilpotent in Theorem 1.2 cannot be strengthened by assuming conditions on the minimal set.

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2 Minimal Actions

Recall that $K \subset S^1$ is a *minimal set* for φ if K is closed, invariant and minimal for these two conditions. Zorn's Lemma implies that every topological action admits at least one minimal set. Recall that every point of a minimal set K has φ–orbit dense in K. If K is infinite, then it contains a limit point of itself, and hence is a perfect set. A perfect minimal set either has no interior, in which case it is a Cantor set, or has interior, so is open and closed and hence must be all of S^1. In this latter case we say the action φ is minimal.

An open, non-empty invariant set $M \subset S^1$ is said to be *locally minimal* if the closure of each orbit in M contains M in its interior. If $M = S^1$ then this reduces to the definition above.

Proposition 2.1 *Suppose that $\varphi: \Gamma \times S^1 \to S^1$ is an expansive action with a locally minimal set M. Then there exists $\alpha, \beta \in \Gamma$ which generate a free sub-semigroup of Γ. In particular, Γ must have exponential word growth and cannot be infra-nilpotent.*

Proof: We will construct a "ping-pong game" [T, delaH] for elements of the action φ and use this to exhibit the elements α and β.

Let $0 < \epsilon$ be the expansive constant for φ. Set $\delta = \epsilon/10$.

Given points $x, y \in S^1$ with $d(x, y) < \pi$ we let $\overline{xy} \subset S^1$ denote the interval (the shortest path in S^1) they determine, and $|x, y|$ denotes the length of this interval.

We begin the proof of the proposition. Since M is invariant and φ is expansive, the diameter of each connected component of M must be at least ϵ. Choose $x_1 \in M$ to be the midpoint of a longest connected interval in M. (In the case where $M = S^1$ select any point.) Let $y_1, z_1 \in M$ be the points with $d(x_1, y_1) = d(x_1, z_1) = \delta/2$ so that $x_1 \in \overline{y_1 z_1} \subset M$. Choose $\gamma_1 \in \Gamma$ with $d(\gamma_1 y_1, \gamma_1 z_1) > \epsilon$. Let J_1 denote the interval $\overline{y_1, z_1}$, and $I_1 = \gamma_1 J_1$.

Now proceed inductively. Assume 6–tuples $\{x_i, y_i, z_i, \gamma_i, J_i, I_i\}$ have been chosen for $1 \leq i < n$ and we select a new 6–tuple $\{x_n, y_n, z_n, \gamma_n, J_n, I_n\}$. Let x_n be the midpoint of I_{n-1} and choose $y_n, z_n \in I_{n-1}$ be distinct points with $d(x_n, y_n) = d(x_n, z_n) = \delta/2^n$. Choose $\gamma_n \in \Gamma$ with $d(\gamma_n y_n, \gamma_n z_n) > \epsilon$. Then set $J_n = \overline{y_n, z_n} \subset I_{n-1}$ and $I_n = \gamma_n J_n$ is a subset of M.

Let x_* be an accumulation point for the set of "midpoints" $\{x_1, x_2, \ldots\}$. Note that since all intervals I_n have length at least ϵ the point x_* lies in the interior of M, and is at least $\epsilon/2$ distance from the boundary of M. By the transitivity of φ there exists $\xi \in \Gamma$ such that $3\delta < d(x_*, \xi x_*) < 4\delta$. Choose $0 < \delta_1 < \delta/2$ such that for the closed interval $W = \{w \in S^1 \mid d(x_*, w) \leq \delta_1\}$, $\xi W \subset \{w \in S^1 \mid d(\xi x_*, w) < \delta_1\}$. That is, both W and its image ξW have diameter less than δ_1. It follows that $W \cap \xi W = \emptyset$.

Choose $0 << p < q$ so that $d(x_*, x_p) < \delta_1/2$ & $\delta/2^p < \delta_1/2$ and $d(x_*, x_q) < \delta_1/2$ & $\delta/2^q < \delta_1/2$. Define

$$\alpha = (\gamma_q \circ \cdots \circ \gamma_p)^{-1} \ ; \quad U_\ell = |y_\ell, z_\ell| \ ; \quad V = \alpha^{-1} U$$

Note that U has diameter $\delta/2^p < \delta_1/2$ and the midpoint $x_p \in U$ satisfies $d(x_*, x_p) < \delta_1/2$ so that $U \subset W$. The midpoint $x_q \in V$ also satisfies $d(x_*, x_q) < \delta_1/2 < \delta$ so that $U \subset V$. Set $\beta = \xi \circ \alpha$, then $\beta V \subset \xi U \subset \xi W \subset V$ and ξW is disjoint from U by the choice of W. It follows that the sub-semigroup of Γ generated by $\{\alpha, \beta\}$ is free. □

3 Finite Minimal Sets

Proposition 3.1 *Let $I \subset S^1$ be a closed interval, assume Γ is a nilpotent group, and suppose that $\varphi: \Gamma \times I \to I$ is an action by orientation–preserving homeomorphisms whose only fixed–points are the endpoints of I. Then φ is not expansive.*

Proof: We assume that the action of φ is expansive, and show this leads to a contradiction. The proof uses induction on the polycyclic length of Γ. Recall that Γ nilpotent implies there is a chain of normal (in Γ) subgroups

$$\Gamma_{d+1} = \{0\} \subset \Gamma_d \subset \cdots \subset \Gamma_1 = \Gamma$$

where each quotient Γ_i/Γ_{i+1} is a rank–one abelian subgroup of the center of Γ/Γ_{i+1}. As Γ is torsion–free, each quotient is isomorphic to $\boldsymbol{Z}$. The integer d is called the polycyclic length of Γ.

Let J denote the interior of I.

First, suppose that Γ has polycyclic length 1, and let $\gamma \in \Gamma$ be a generator. By hypothesis, the action of Γ on J has no fixed points, so if we choose any $x \in J$ then the closed interval $I_1 = \overline{x\gamma x}$ is a fundamental domain for the action of γ. It follows that the action cannot be expansive.

Next, assume Proposition 3.1 is true for all nilpotent groups with polycyclic length less than $d > 1$. Fix a chain $\Gamma_d \subset \cdots \subset \Gamma_1 = \Gamma$ as above, and let $\alpha \in \Gamma_d$ be a generator, which is in the center of Γ.

Let $F \subset I$ be the fixed–point set of α. Note that F is invariant under Γ, and Γ acts without fixed–points on F.

Lemma 3.2 *The interior $U \subset F$ is empty.*

Proof: If not, U is an non–empty, open Γ–invariant set. Let $\{U_1, \ldots, U_n, \ldots\}$ be the set of open connected components of U. The action of φ is expansive, so for $x \neq y \in U_1$ there exists $\gamma \in \Gamma$ so that $d(\gamma x, \gamma y) > \epsilon$. The image of γx is contained in one of the intervals U_i whose length is thus at least ϵ. By reordering if necessary, let $\{U_1, \ldots, U_n\}$ be the subcollection of open intervals such that U_i has length at least ϵ for $1 \leq i \leq n$, and the length of U_i is less than ϵ for $i > n$. Let $\{U_1, \ldots, U_m\}$ for $m \leq n$ be the subcollection of open intervals whose Γ–orbits intersect U_1. That is, for each $1 \leq i \leq m$ there exists $\alpha_i \in \Gamma$ with $\alpha_i U_1 = U_i$, and for $i > m$ we have $\beta U_i \cap U_1 = \emptyset$ for all $\beta \in \Gamma$.

The action of each $\alpha_i: I \to I$ is uniformly continuous, so there exists a constant $\epsilon_1 > 0$ so that for $x, y \in U_1$ with $d(x, y) \leq \epsilon_1$ then $d(\alpha_i x, \alpha_i y) \leq \epsilon$. Without loss we choose $\epsilon_1 < \epsilon$. Thus, for any pair $u, v \in U_i$ with $d(u, v) > \epsilon$ then $d(\alpha_i^{-1} u, \alpha_i^{-1} v) > \epsilon_1$.

Set $M = \overline{U_1}$ be the closure of U_1, and let Γ_M be the subgroup of Γ consisting of elements which leave M invariant. Note that $\Gamma_n \subset \Gamma_M$. We claim that Γ_M acts expansively on M with expansiveness constant ϵ_1. Let $x \neq y \in M$. By hypotheses there exists $\gamma_1 \in \Gamma$ so that $d(\gamma_1 x, \gamma_1 y) > \epsilon$. The image $\gamma_1 U_1$ has length at least ϵ so there exists i with $\gamma_1 U_1 = U_i$. Set $\gamma = \alpha_i^{-1} \gamma_1 \in \Gamma_M$ then $d(\gamma x, \gamma y) > \epsilon_1$.

The subgroup Γ_n acts trivially on M so there is an induced action of the quotient Γ/Γ_n on M. This action has no interior fixed–points and Γ/Γ_n is nilpotent with polycylic length $d - 1$, so by induction the action cannot be expansive, a contradiction. □

We next consider the case where F consists of the endpoints of I. That is, α has no fixed points on the interior $J \subset I$. Choose a point $x \in J$, then the interval $M = \overline{x\alpha x}$ is a fundamental domain for α. Let U denote the interior of M. Let Γ_M denote the subgroup of Γ which leaves M invariant.

Lemma 3.3 *The induced action of Γ_M on M is expansive.*

Proof: Let $x \neq y \in M$ and $\gamma \in \Gamma$ so that $d(\gamma x, \gamma y) > \epsilon$. The image of γM must equal $\alpha^{\ell} M$ for some ℓ, hence $\alpha^{-\ell}\gamma \in \Gamma_M$. Moreover, as γM contains two points separated by at least ϵ, and there are at most finitely many possible images $\alpha^{\ell} M$ with length greater than ϵ, there are at most finitely many such ℓ which arise. As in the case where F had interior, it follows that there is a uniform constant $\epsilon_1 > 0$ so that $d(\gamma x, \gamma y) > \epsilon$ implies $d(\alpha^{-\ell}\gamma x, \alpha^{-\ell}\gamma y) > \epsilon_1$. It follows that the action of Γ_M on M is expansive with constant ϵ_1. □

The group Γ_M is again nilpotent, with polycyclic length at most $d-1$. By induction, we conclude the action on M cannot be expansive, a contradiction.

Finally, we consider the case where F is nowhere–dense and has non-trivial intersection with the interior J of I. We obtain a contradiction using a combination of both of the above arguments.

The complement of F is a countable union of open connected intervals $\{U_1, \ldots, U_n, \ldots\}$, and for each i let M_i denote the closure of U_i in I. Note that α acts without fixed–points on each U_i.

The action of φ is expansive, so for $x \neq y \in U_1$ there exists $\gamma_1 \in \Gamma$ so that $d(\gamma_1 x, \gamma_1 y) > \epsilon$. The image of $\gamma_1 x$ must lie in the complement of F, hence in one of the intervals U_i whose length is at least ϵ. By reordering if necessary, let $\{U_1, \ldots, U_n\}$ be the subcollection of open intervals such that U_i has length at least ϵ for $1 \leq i \leq n$, and the length of U_i is less than ϵ for $i > n$. Let $\{U_1, \ldots, U_m\}$ for $m \leq n$ be the subcollection of open intervals whose Γ–orbits intersect U_1. That is, for each $1 \leq i \leq m$ there exists $\alpha_i \in \Gamma$ with $\alpha_i U_1 = U_i$, and for $i > m$ we have $\beta U_i \cap U_1 = \emptyset$ for all $\beta \in \Gamma$.

The action of each $\alpha_i \colon I \to I$ is uniformly continuous, so there exists a constant $\epsilon_1 > 0$ so that for $x, y \in U_1$ with $d(x,y) \leq \epsilon_1$ then $d(\alpha_i x, \alpha_i y) \leq \epsilon$. Without loss we choose $\epsilon_1 < \epsilon$. Thus, for any pair $u, v \in U_i$ with $d(u,v) > \epsilon$ then $d(\alpha_i^{-1} u, \alpha_i^{-1} v) > \epsilon_1$.

Set $M = \overline{U_1}$ be the closure of U_1, and let Γ_M be the subgroup of Γ consisting of elements which leave M invariant. Note that $\Gamma_n \subset \Gamma_M$. We claim that Γ_M acts expansively on M with expansiveness constant ϵ_1. Let $x \neq y \in M$. By hypotheses there exists $\gamma_1 \in \Gamma$ so that $d(\gamma_1 x, \gamma_1 y) > \epsilon$. The image $\gamma_1 U_1$ has length at least ϵ so there exists i with $\gamma_1 U_1 = U_i$. Set $\gamma = \alpha_i^{-1}\gamma_1 \in \Gamma_M$ then $d(\gamma x, \gamma y) > \epsilon_1$.

Now, Γ_M is nilpotent with polycyclic length at most d, and the action of Γ_n on M has no interior fixed–points. By the previous case above, the action cannot be expansive, a contradiction.

This concludes the proof of Proposition 3.1. □

4 Proof of Theorem

Suppose that $\varphi: \Gamma \times S^1 \to S^1$ is an expansive action and Γ is infra–nilpotent. We show this leads to a contradiction, proving Theorem 1.2.

If φ admits a minimal set K with interior, and hence $K = S^1$, then we are done by Proposition 2.1. We can thus assume that every minimal set K for the action φ is nowhere–dense.

Lemma 4.1 *Let $I \subset S^1$ be a closed invariant subset for α, and $\Gamma_0 \subset \Gamma$ be a subgroup of finite index. Then Γ acts expansively on I if and only if Γ_0 acts expansively on I.*

Proof: Let $\{\gamma_1, \ldots, \gamma_n\}$ be elements of Γ so that $\Gamma = \gamma_1\Gamma_0 \cup \cdots \cup \gamma_n\Gamma_0$. As I is compact, there exists $\epsilon_0 > 0$ so that for any points $x, y \in I$ with $d(x, y) > \epsilon$ then $d(\gamma_i^{-1}x, \gamma_i^{-1}y) > \epsilon_0$ for all $1 \leq i \leq n$.

Assume that Γ acts expansively on I. Then given $x \neq y \in I$, there exists $\gamma \in \Gamma$ for which $d(\gamma x, \gamma y) > \epsilon$. Then for some $1 \leq i \leq n$ we have $\gamma_i^{-1}\gamma \in \Gamma_0$, and $d(\gamma_i^{-1}\gamma x, \gamma_i^{-1}\gamma y) > \epsilon_0$, thus Γ_0 acts expansively with constant ϵ_0.

Conversely, if Γ_0 acts expansively, then obviously the same holds for $\Gamma \supset \Gamma_0$. □

By definition, Γ has a subgroup Γ_0 of finite index which is nilpotent. Moreoever, by passing to a subgroup of index two if necessary, we can assume Γ_0 acts on S^1 via orientation–preserving homeomorphisms. By Lemma 4.1 the action of Γ_0 on S^1 is again expansive. Hence, by changing notation, we can assume that Γ is nilpotent and the action of φ is expansive and orientation–preserving.

Suppose now that every minimal set of φ is finite; let K be one such. As K is invariant, there is a subgroup of finite index $\Gamma_0 \subset \Gamma$ such that Γ_0 fixes the points of K. By Lemma 4.1 the action of Γ_0 is again expansive, with expansiveness constant ϵ_0.

Note that every minimal set for the action of Γ_0 must also be finite, and therefore consists of fixed–points. Let $F \subset S^1$ denote the closed set of fixed–points for the action of Γ_0, then every minimal set for Γ_0 is contained in F.

The complement of F consists of a countable union of open connected intervals, which we denote by $\{J_1, \ldots, J_n, \ldots\}$ and let for each n let I_n denote the closure of J_n in S^1.

Now observe that F is fixed implies that each interval I_n is also invariant for Γ_0, hence the action of φ on I_n must be expansive. That is, we obtain an expansive action of a nilpotent group Γ_0 on a closed interval $I = I_n$, which contradicts the conclusion of Proposition 3.1.

The final case to consider is when φ admits a minimal set K which is not finite and not dense, hence K is a nowhere–dense perfect set. Such minimal sets are called "exceptional" in the foliation literature [CC, G].

The group Γ is nilpotent, hence is amenable, so its action on K admits an invariant probability measure μ. We consider μ as an invariant measure for the action of Γ on S^1. The measure μ defines an invariant "length coordinate" on S^1, for which there is an induced action $\phi_\mu: \Gamma \times S^1 \to$ by rotations [S, CC, G]. Moreover, μ also defines a continuous map $h_\mu: S^1 \to S^1$ which is a semi-conjugacy between the actions φ and ϕ_μ.

Let $J \subset S^1$ be a maximal connected open interval in the complement of K, and I its closure. Define Γ_I to be the subgroup of elements which leave I invariant. Define Γ_μ to the kernel of ϕ_μ.

Lemma 4.2 $\Gamma_I = \Gamma_\mu$. *Thus, each element of* Γ_I *leaves invariant every maximal connected open interval in the complement of* K.

Proof: By construction, h_μ is locally constant on the complement of K, hence maps I to a point $\theta \in S^1$. If $\gamma \in \Gamma_I$ then $\phi_\mu(\gamma)$ fixes θ. As each image of ϕ_μ consists of rotations of S^1, $\phi_\mu(\gamma)$ must be the identity. Hence $\gamma \in \Gamma_\mu$. Conversely, $\gamma \in \Gamma_\mu$ fixes every point of S^1 hence leaves $I = h_\mu(\theta)$ invariant, so $\gamma \in \Gamma_I$. □

The action of φ is expansive, so for $x \neq y \in I$ with $d(x, y) < \delta$ there exists $\gamma \in \Gamma$ so that $d(\gamma x, \gamma y) > \epsilon$. Let $I_1 = \gamma I$ which is a closed maximal connected interval whose interior is disjoint from K. Note that by Lemma 4.2, $\Gamma_I = \Gamma_{I_1}$. We change notation and set $I = I_1$.

Let $\{I_1, \ldots, I_n\}$ be the collection of closed intervals such that there exists $\alpha_i \in \Gamma$ with $I_i = \alpha_i I$ and I_i has length at least ϵ. The set is non-empty, and finite as the intervals $I_i \cap I_j = \emptyset$ for $i \neq j$. Let $\{\alpha_1, \ldots, \alpha_n\}$ be elements of Γ with $\alpha_i I = I_i$. The action of each $\alpha_i \colon I \to I_i$ is uniformly continuous, so there exists a constant $\epsilon_1 > 0$ so that for $x, y \in I$ with $d(x, y) \leq \epsilon_1$ then $d(\alpha_i x, \alpha_i y) \leq \epsilon$. Without loss we choose $\epsilon_1 < \epsilon$. Thus, for any pair $u, v \in I_i$ with $d(u, v) > \epsilon$ then $d(\alpha_i^{-1} u, \alpha_i^{-1} v) > \epsilon_1$.

Lemma 4.3 Γ_I *acts expansively on* I *with expansiveness constant* ϵ_1.

Proof: Let $x \neq y \in I$. By hypotheses there exists $\gamma_1 \in \Gamma$ so that $d(\gamma_1 x, \gamma_1 y) > \epsilon$. The image $\gamma_1 I$ has length at least ϵ so there exists i with $\gamma_1 I = I_i$. Set $\gamma = \alpha^{-1}\gamma_1 \in \Gamma_I$ then $d(\gamma x, \gamma y) > \epsilon_1$. □

Let $F \subset I$ be the closed set of fixed–points for the action of Γ_I. If F consists or more than the endpoints of I, then we consider the restriction of the action of Γ_I to the closure of each maximal connected component F in $I \setminus F$, so are reduced to the case where F consists of the endpoints of I. Then as G_I is nilpotent, Proposition 3.1 implies the action cannot be expansive, a contradiction.

This completes the proof of Theorem 1.2. □

5 Two Examples

We give two examples of expansive C^1–actions, one of an action of the "ax +b" group on the circle with one fixed–point, and the second example where the "ax + b" group is embedded into the gaps of a Denjoy example to produce an expansive action with a Cantor type minimal set.

Example 5.1 *An expansive action of a solvable group on S^1 with minimal set a point.*

Define maps of the real line by $f(x) = 2x$ and $g(x) = x + 1$. Embed the real line into the circle $S^1 = [-\pi, \pi]/\{-\pi \sim \pi\}$ using the inverse tangent map $\theta = h(x) = 2\arctan(x)$. Set $\alpha = h \circ f \circ h^{-1}$ and $\beta = h \circ g \circ h^{-1}$, and let Γ be the subgroup of $\mathbf{Homeo(S^1)}$ they generate.

The action of f, g on $\boldsymbol{R}$ has every orbit dense, hence the action of Γ on S^1 admits a unique fixed–point π, and every other orbit is dense.

Given $x \neq y \in S^1$, at least one of these cannot be the fixed–point π. Hence, there exists ℓ such that $g^\ell(h^{-1}(x))$ and $g^\ell(h^{-1}(y))$ lie on opposite sides of the origin in $\boldsymbol{R}$. Applying a suitable power $k > 0$ of f we can ensure that their images $f^k \circ g^\ell(h^{-1}(x))$ and $f^k \circ g^\ell(h^{-1}(y))$ span an interval containing either [0,1] or [-1,0] in $\boldsymbol{R}$, hence $h \circ f^k \circ g^\ell(h^{-1}(x))$ and $h \circ f^k \circ g^\ell(h^{-1}(y))$ are ϵ separated in S^1 for $\epsilon = 2\arctan(1) = \pi/2$. □

Example 5.2 *An expansive action of a solvable group on S^1 with Cantor minimal set.*

Let $\gamma\S^1 \to S^1$ be a C^1-diffeomorphism with an invariant exceptional minimal set K. The complement of K consists of a disjoint union of open intervals $\{U_1, U_2, \ldots\}$ and γ acts transitively on the set of intervals. Thus, we can index the open sets by $\boldsymbol{Z}$ where $U_\ell = \gamma^\ell U_0$.

Choose a diffeomorphism $h\colon \boldsymbol{R} \to U_0$.

Define $\alpha \in \mathbf{Homeo(S^1)}$ with fixed–point set K, and on U_ℓ we define

$$\alpha | U_\ell = \gamma^\ell \circ h \circ f \circ h^{-1} \circ \gamma^{-\ell}$$

Define $\beta \in \mathbf{Homeo(S^1)}$ with fixed–point set K, and on U_ℓ we define

$$\beta | U_\ell = \gamma^\ell \circ h \circ g \circ h^{-1} \circ \gamma^{-\ell}$$

Let $\Gamma \subset \mathbf{Homeo(S^1)}$ be the subgroup generated by $\{\alpha, \beta\}$. Clearly, K is the unique minimal set for the action of Γ, and every point in the complement of K has dense orbit in S^1.

It is also easy to see that the action of Γ is expansive. There are two cases to consider. If $x \neq y \in S^1$, and both points lie in the same connected component of the complement of K, then a suitable power $\gamma^\ell(x) \in U_0$ so we can proceed as above. If $x \neq y$ do not lie in the closure of the same connected component of the complement of K, then there exists some U_ℓ contained in the interval $\overline{xy}$, thus $\gamma^{-\ell}(x)$ and $\gamma^{-\ell}(y)$ contain U_0 in the interval they determine. □

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