
MR1962152 (2003m:55003)

Colman, Hellen **[Colman Vale, Hellen]** (1-ILCC); **Hurder, Steven** (1-ILCC)

Tangential LS category and cohomology for foliations. (English summary)

Lusternik-Schnirelmann category and related topics (*South Hadley*, *MA*, 2001), 41–64, *Contemp. Math.*, 316, *Amer. Math. Soc.*, *Providence*, *RI*, 2002.
55M30 (55T10 57R30 57R32)

The re-awakening of interest in the Lyusternik-Shnirel′man (LS) category in the past decade provided a springboard for application of the theory to other subjects. One of these new applications (for which the first author was the prime instigator) was to foliations. Let $(M,\mathfrak{F})$ be a foliated manifold and suppose $(U,\mathfrak{F}_U)$ is an open subset with the restricted foliation. Then U is said to be tangentially categorical if the inclusion $(M,\mathfrak{F})\to(U,\mathfrak{F}_U)$ is integrably homotopic to a foliated map $U\to M$ that is constant on each leaf of $\mathfrak{F}_U$. The least number of tangentially categorical open sets required to cover M is the "tangential category" of $(M,\mathfrak{F})$, denoted by $\mathrm{cat}_{\mathfrak{F}}(M)$. (Note that this definition is more aligned with the dynamical definition of category than with the homotopy definition.) This reduces to the ordinary LS category if $\mathfrak{F}$ is the foliation given by taking a single leaf and, just as ordinary category estimates the complexity of a space, the tangential category measures the leafwise topological complexity of $\mathfrak{F}$. Two fundamental lower bounds for $\mathrm{cat}_{\mathfrak{F}}(M)$ were discovered by the first author and E. Macias-Virgós [J. London Math. Soc. (2) **65** (2002), no. 3, 745–756; MR 2003a:57050]; namely, $\mathrm{cat}(L)\le\mathrm{cat}_{\mathfrak{F}}(M)$, where L is any leaf of the foliation and $\mathrm{nil}(H^+_{\mathfrak{F}}(M))\le\mathrm{cat}_{\mathfrak{F}}(M)$, where nil stands for nilpotency degree and $H^+_{\mathfrak{F}}(M)$ denotes the positive degree foliated cohomology. In the present paper, the authors enhance the latter inequality to $\mathrm{nil}(H^+_{\mathfrak{F}}(M))\le\mathrm{nil}(E_1^{*,+})\le\mathrm{cat}_{\mathfrak{F}}(M)$, where $E_1^{*,*}$ is the E_1-term of the foliation spectral sequence. Furthermore, it is also shown that $\mathrm{nil}(E_r^{*,+})\le\mathrm{nil}(E_1^{*,+})$ and methods are developed for finding nontrivial classes. Two sample results that give the flavor of the types of applications possible are the following: (1) If the Godbillon-Vey class $\mathrm{GV}(\mathfrak{F})$ is nonzero, then $\mathrm{cat}_{\mathfrak{F}}(M)\ge n+2$, where n is the codimension of $\mathfrak{F}$; (2) if M is compact and $\mathfrak{F}$ is given by a locally free $\mathbb{R}^m$-action, then $\mathrm{cat}_{\mathfrak{F}}(M)=m+1$. In this last result, equality (rather than an inequality) follows by applying a fundamental result of W. Singhof and E. Vogt [Topology

42 (2003), no. 3, 603–627; MR 2004a:57035] that $\text{cat}_{\mathcal{F}}(M) \leq \dim \mathcal{F} + 1$ (extending the usual category inequality $\text{cat}(M) \leq \dim M$). Finally, let it be noted that the paper is well-written and gives a nice overview of the subject at hand.

{For the entire collection see 2003j:55001}

Reviewed by *John F. Oprea*