DYNAMICS AND THE GODBILLON-VEY CLASS OF C^1 FOLIATIONS

STEVEN HURDER AND RÉMI LANGEVIN

Abstract. Let $\mathcal{F}$ be a codimension–one, C^2-foliation on a manifold M without boundary. In this work we show that if the Godbillon–Vey class $GV(\mathcal{F}) \in H^3(M)$ is non-zero, then $\mathcal{F}$ has a hyperbolic resilient leaf. Our approach is based on methods of C^1-dynamical systems, and does not use the classification theory of C^2-foliations. We first prove that for a codimension–one C^1-foliation with non-trivial Godbillon measure, the set of infinitesimally expanding points $E(\mathcal{F})$ has positive Lebesgue measure. We then prove that if $E(\mathcal{F})$ has positive measure for a C^1-foliation, then $\mathcal{F}$ must have a hyperbolic resilient leaf, and hence its geometric entropy must be positive. The proof of this uses a pseudogroup version of the Pliss Lemma. The first statement then follows, as a C^2-foliation with non-zero Godbillon-Vey class has non-trivial Godbillon measure. These results apply for both the case when M is compact, and when M is an open manifold.

April 6, 2016; revised August 26, 2016.

1. Introduction

Godbillon and Vey introduced the invariant $GV(\mathcal{F}) \in H^3(M;\mathbb{R})$, which is defined for a codimension-one C^2-foliation $\mathcal{F}$ of a manifold M without boundary, in the brief note [26]. While the definition of the Godbillon-Vey class is elementary, understanding its relations to the geometric and dynamical properties of the foliation $\mathcal{F}$ remains an open problem. In the paper [72], Thurston showed that the Godbillon-Vey class can assume a continuous range of values for foliations of closed 3-manifolds, and he also introduced the concept of "helical wobble", which he suggested gives a relation between the value of this class and the Riemannian geometry of the foliation. This geometric relation was made precise in a work by Reinhart and Wood [68]. More recently, Langevin and Walczak in [52, 76, 77] gave further insights into the geometric meaning of the Godbillon-Vey invariant for smooth foliations of closed 3-manifolds, in terms of the conformal geometry of the leaves of the foliation.

The Godbillon-Vey class appears in a surprising variety of contexts, such as the Connes-Moscovici work on the cyclic cohomology of Hopf algebras [13, 15, 14] which interprets the class in non-commutative geometry setting. The works of Leichtnam and Piazza [54] and Moriyoshi and Natsume [58] gave interpretations of the value of the Godbillon-Vey class in terms of the spectral flow of leafwise Dirac operators for smooth foliations.

The problem considered in this work was first posed in papers of Moussu and Pelletier [59] and Sullivan [71], where they conjectured that a codimension-one foliation $\mathcal{F}$ with $GV(\mathcal{F}) \neq 0$ must have leaves of exponential growth. The support for this conjecture at that time was principally a collection of examples, and some developing intuition for the dynamical properties of foliations. The geometry of the helical wobble phenomenon is related to geometric properties of contact flows, such as for the geodesic flow of a compact surface with negative curvature. The weak stable foliations for such flows have all leaves of exponential growth, and often have non-zero Godbillon-Vey classes [72, 62, 68, 41, 27]. Moreover, the work of Thurston in [72] implies that for any positive real number α there exist a C^2-foliation of codimension-one on a compact oriented 3-manifold, whose Godbillon-Vey class is α times the top dimension integral cohomology class. These various results suggest that a geometric interpretation of $GV(\mathcal{F})$ might be related to dynamical invariants such as "entropy", whose values are not limited to a discrete subset of $\mathbb{R}$.

Given a choice of a complete, relatively compact, 1-dimensional transversal $\mathfrak{X} \subset M$ to $\mathcal{F}$, the transverse parallel transport along paths in the leaves defines local homeomorphisms of $\mathfrak{X}$, which yields a 1-dimensional pseudogroup $\mathcal{G}_{\mathcal{F}}$ as recalled in Section 2.2. The study of the properties of foliation pseudogroups has been a central theme of foliation theory since the works of Reeb and Haefliger in the 1950's [66, 67, 28, 29].

The geometric entropy $h(\mathcal{F})$ of a C^1-foliation $\mathcal{F}$ was introduced by Ghys, Langevin and Walczak [24], and can be formulated in terms of the pseudogroup $\mathcal{G}_{\mathcal{F}}$ associated to the foliation. The geometric entropy is a measure of the dynamical complexity of the action of $\mathcal{G}_{\mathcal{F}}$ on $\mathfrak{X}$, and is one of the most important dynamical invariants of C^1-foliations. The Godbillon-Vey class $GV(\mathcal{F})$ vanishes for all the known examples of foliations for which $h(\mathcal{F}) = 0$, and the problem was posed to relate the non-vanishing of the geometric entropy $h(\mathcal{F})$ of a codimension-one C^2-foliation $\mathcal{F}$ with the non-vanishing of its Godbillon-Vey class.

Duminy showed in the unpublished papers [18, 19] that for a C^2-foliation of codimension one, $GV(\mathcal{F}) \neq 0$ implies there are leaves of exponential growth. (See the account of Duminy's results in Cantwell and Conlon [12], and [10, Theorem 13.3.1].) Duminy's proof began by assuming that a C^2-foliation $\mathcal{F}$ has no resilient leaves, or equivalently resilient orbits for $\mathcal{G}_{\mathcal{F}}$ as in Definition 2.3. Then by the Poincaré-Bendixson theory for codimension–one, C^2-foliations [12, 33], Duminy showed that the Godbillon-Vey class of $\mathcal{F}$ must vanish. Thus, if $GV(\mathcal{F}) \neq 0$ then $\mathcal{F}$ must have at least one resilient leaf. If a codimension–one foliation has a resilient leaf, then by an easy argument it follows that $\mathcal{F}$ has an uncountable set of leaves with exponential growth. Duminy's proof is "non-constructive", as it does not show explicitly how a non-trivial value of the Godbillon-Vey class results in resilient leaves for the foliation. One of the points of this present work is to give a direct demonstration of this relation, which we show using techniques of ergodic theory for C^1-foliations.

In the work [24], Theorem 6.1 states that for a codimension-one, C^2-foliation $\mathcal{F}$, if $h(\mathcal{F}) \neq 0$ then $\mathcal{F}$ must have a resilient leaf. Candel and Conlon gave a proof of this result in [9, Theorem 13.5.3] for the special case where the foliation is the suspension of a group action on a circle, but were unable to extend the proof to the general case asserted in [24]. One concludes that for a C^2-foliation $\mathcal{F}$, if the geometric entropy $h(\mathcal{F}) = 0$, then $\mathcal{F}$ has no resilient leaves and thus $GV(\mathcal{F}) = 0$. The proof that $GV(\mathcal{F}) \neq 0$ implies $h(\mathcal{F}) > 0$ given in this work, circumvents these difficulties.

The evolution of vanishing theorems for the Godbillon-Vey invariant, starting with Hermann [37] and progressing up to Duminy's work [18, 19], is discussed in detail in the survey [44]. The technique that is used in all of these works was to use dynamical hypotheses on the foliation to obtain upper bound estimates on the norm of the Godbillon-Vey invariant, defined using smooth forms associated to the foliation. The paper [36] extended these techniques, by making the relation between the value of the Godbillon-Vey invariant and measurable forms explicit. This relationship made it possible to develop more direct relations between the ergodic theory of foliations with the values of their secondary classes in all codimensions. A key idea introduced in [38, 39], was to use techniques from the Oseledets theory of cocycles to study the relation between foliation dynamics and the values of the secondary classes of foliations.

In this paper, we further develop the ergodic theory of C^1-foliations, in order to show that for a C^2-foliation $\mathcal{F}$, the assumption $GV(\mathcal{F}) \neq 0$ implies that the foliation $\mathcal{F}$ has resilient leaves, and thus $h(\mathcal{F}) \neq 0$. An important aspect of our proof, is that the subtle techniques of the Poincaré-Bendixson theory of C^2-foliations are avoided, and the conclusion that there exists resilient leaves follows from straightforward techniques of dynamical systems.

The work of Duminy [18] reformulated the study of the Godbillon-Vey class for C^2-foliations in terms of the *Godbillon measure* $G_{\mathcal{F}}$, which for a C^1-foliation $\mathcal{F}$ of a compact manifold M, is a linear functional defined on the Borel σ-algebra $\mathcal{B}(\mathcal{F})$ formed from the leaf-saturated Borel subsets of M, and by extension this measure is defined on the saturated measurable subsets of M. These ideas are introduced and discussed in the papers [12, 18, 19, 36, 38, 39], and recalled in Section 3 below. Here is our main result, as formulated in these terms:

THEOREM 1.1. *If $\mathcal{F}$ is a codimension–one, C^1-foliation with non-trivial Godbillon measure $G_{\mathcal{F}}$, then $\mathcal{F}$ has a hyperbolic resilient leaf.*

In the course of our proof of this result, resilient orbits of the action of the pseudogroup $\mathcal{G}_{\mathcal{F}}$ are explicitly constructed using a version of the Ping-Pong Lemma, first introduced by Klein in his study of subgroups of Kleinian groups [16], and which is discussed in Section 2.4.

For C^2-foliations, Theorem 4.4 below implies that the Godbillon-Vey class is obtained by evaluating the Godbillon measure on the "Vey class" $[v(\mathcal{F})]$ localized to the hyperbolic set $\mathrm{E}^+(\mathcal{F}) \in \mathcal{B}(\mathcal{F})$ introduced in Definition 4.3. Only the definition of the localized class $[v(\mathcal{F})]|\mathrm{E}^+(\mathcal{F})$ requires that $\mathcal{F}$ be C^2. Thus, for a C^2-foliation $\mathcal{F}$, $GV(\mathcal{F}) \neq 0$ implies that $G_{\mathcal{F}} \neq 0$, and we deduce:

COROLLARY 1.2. *If $\mathcal{F}$ is a codimension–one, C^2-foliation with non-trivial Godbillon-Vey class $GV(\mathcal{F}) \in H^3(M;\mathbb{R})$, then $\mathcal{F}$ has a hyperbolic resilient leaf, and thus the entropy $h(\mathcal{F}) > 0$.*

We next discuss the strategy of the proof of Theorem 1.1. A key idea in dynamical systems of flows is to consider the points for which the dynamics is "infinitesimally exponentially expansive" over long orbit segments, which corresponds to points with positive Lyapunov exponent [2, 5, 61]. The analog for pseudogroup dynamics is to introduce the set of points in the transversal $\mathcal{X}$ for which there are arbitrarily long words in $\mathcal{G}_{\mathcal{F}}$ for which the norm of their transverse derivative matrix is exponentially growing with respect to the word norm on the pseudogroup.

We introduce in Section 4, the $\mathcal{F}$-saturated set $\mathrm{E}^+(\mathcal{F})$ of points in M where the transverse derivative cocycle for $\mathcal{F}$ has positive exponent. A point $x \in \mathrm{E}^+(\mathcal{F}) \cap \mathcal{X}$ if and only if there is a sequence of holonomy maps such that the norms of their derivatives at x grow exponentially fast as a function of "word length" in the foliation pseudogroup, and $\mathrm{E}^+(\mathcal{F})$ is the leaf saturation of this set.

The set $\mathrm{E}^+(\mathcal{F})$ is a fundamental construction for a C^1-foliation. For example, a key step in the proof of the generalized Moussu–Pelletier–Sullivan conjecture in [38] was to show that for a foliation $\mathcal{F}$ with almost all leaves of subexponential growth, the Lebesgue measure $|\mathrm{E}^+(\mathcal{F})| = 0$. Here, we show in Theorem 4.4 that if a measurable, $\mathcal{F}$-saturated subset $B \subset M$ is disjoint from $\mathrm{E}^+(\mathcal{F})$, then the Godbillon measure must vanish on B.

The second step in the proof of Theorem 1.1 is to show that for each point $x \in \mathrm{E}^+(\mathcal{F})$, the holonomy of $\mathcal{F}$ has a uniform exponential estimate along the orbit of x for its transverse expansion along arbitrarily long words in the holonomy pseudogroup. This follows from Proposition 5.3, which is pseudogroup version of what is called the "Pliss Lemma" in the literature for non-uniform dynamics [64, 55, 5]. If $\mathrm{E}^+(\mathcal{F})$ has positive measure, it is then straightforward to construct resilient orbits for the action of $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{X}$, as done in the proof of Proposition 6.4. The proof of Theorem 1.1 then follows by combining Theorem 4.4, Proposition 5.8 and Proposition 6.4.

The proofs of Propositions 5.3 and 5.8 are the most technical aspects of this paper. One important issue that arises in the study of pseudogroup dynamical systems, is that the domain of a holonomy map in the pseudogroup may depend upon the "length" of the leafwise path used to define it, so that composing maps in the pseudogroup often results in a contraction of the domain of definition for the resulting map. This is a key difference between the study of dynamics of a group acting on the circle, and that of a pseudogroup associated to a general codimension–one foliation. One of the key steps in the proof of Proposition 5.8 is to show uniform estimates on the length of the domains of compositions. The proof uses these estimates to produce an abundance of holonomy pseudogroup maps with hyperbolic fixed–points.

We point out one application of Proposition 5.8, which complements the main result of [42].

THEOREM 1.3. *Let $\mathcal{F}$ be a C^1-foliation of codimension-one such that no leaf of $\mathcal{F}$ has a closed loop with hyperbolic transverse holonomy, then the hyperbolic set $\mathrm{E}^+(\mathcal{F})$ is empty.*

Finally, the extension of the methods for closed manifolds to the case of open manifolds requires only a minor modification in the definition of the Godbillon measure, as discussed in Section 7.

For codimension–one foliations, it is elementary that the existence of a resilient leaf implies $h(\mathcal{F}) > 0$. The converse, that $h(\mathcal{F}) > 0$ implies there is a resilient leaf, was proved in [24] for C^2-foliations, and proved in [43] for C^1-foliations. Let "HRL($\mathcal{F}$)" denote the property that $\mathcal{F}$ has a hyperbolic resilient leaf. Let $|E|$ denote the Lebesgue measure of a measurable subset $E \subset M$. The results of this paper are summarized by the following implications:

THEOREM 1.4. *Let $\mathcal{F}$ be a codimension–one, C^1-foliation of a manifold M. Then*

$$g_{\mathcal{F}} \neq 0 \Longrightarrow |\mathrm{E}^+(\mathcal{F})| > 0 \Longrightarrow \text{“HRL}(\mathcal{F})\text{”} \Longleftrightarrow h(\mathcal{F}) > 0 \ . \tag{1}$$

The collaboration of the authors in Spring 1999 leading to this work was made possible by the support of the first author by the Université of Bourgogne, Dijon. This support is gratefully acknowledged.

2. Foliation Basics

In this section, we introduce some standard notions and results of foliation geometry and dynamics. Complete details and further discussions are provided by the texts [8, 9, 25, 34, 75].

We assume that M is a closed oriented smooth Riemannian m-manifold, $\mathcal{F}$ is a C^r-foliation of codimension–1 with oriented normal bundle, for $r \geq 1$, and that the leaves of $\mathcal{F}$ are smoothly immersed submanifolds of dimension $n \geq 2$, where $m = n + 1$. This is sometimes referred to as a $C^{\infty,r}$-foliation, where the holonomy transition maps are C^r, typically for either $r = 1$ or $r = 2$.

2.1. **Regular Foliation Atlas.** A *$C^{\infty,r}$-foliation atlas* on M, for $r \geq 1$, is a finite collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ such that:

(1) $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ is an open covering of M.
(2) $\phi_\alpha : U_\alpha \to (-1,1)^m$ is a $C^{\infty,r}$–coordinate chart; that is, for $(u,w) \in (-1,1)^n \times (-1,1)$, the map $\phi_\alpha^{-1}(u,w)$ is C^∞ in the "leaf" variable u, and together with all the leafwise derivatives with respect to u, it is C^r in the "transverse" variable w.
(3) Each chart ϕ_α is transversally oriented.
(4) Given $x \in U_\alpha \cap U_\beta$ with $\phi_\alpha(x) = (u,w)$, for the change-of-coordinates map $(u',w') = \phi_\beta \circ \phi_\alpha^{-1}(u,w)$, the value of w' is locally constant with respect to u.

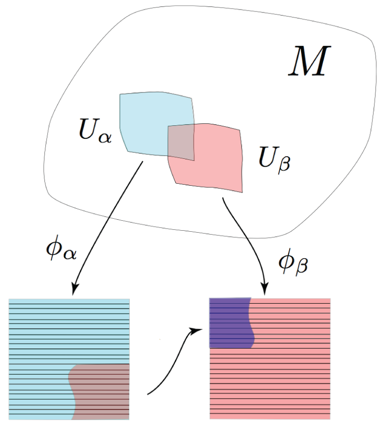

FIGURE 1. Overlapping foliation charts

The collection of sets

$$\mathcal{V}_{\mathcal{F}} \equiv \left\{ V_{\alpha,w} = \phi_\alpha^{-1}(V \times \{w\}) \mid V \subset (-1,1)^n \ , \ w \in (-1,1) \ , \ \alpha \in \mathcal{A} \right\}$$

form a subbasis for the "fine topology" on M. For $x \in M$, let $L_x \subset M$ denote the connected component of this fine topology containing x. Then L_x is path connected, and is called the leaf of $\mathcal{F}$ containing x. Without loss of generality, we can assume that the coordinates are positively oriented, mapping the positive orientation for the normal bundle to $T\mathcal{F}$ to the positive orientation on $(-1,1)$.

Note that each leaf L is a smooth, injectively immersed manifold in M. The Riemannian metric on TM restricts to a smooth metric on each leaf. The path-length metric $d_{\mathcal{F}}$ on a leaf L is defined by

$$d_{\mathcal{F}}(x,y) = \inf \left\{ \|\gamma\| \mid \gamma\colon [0,1] \to L \text{ is } C^1 \ , \ \gamma(0) = x \ , \ \gamma(1) = y \right\} ,$$

where $\|\gamma\|$ denotes the path length of the C^1-curve $\gamma(t)$. If $x, y \in M$ are not on the same leaf, then set $d_{\mathcal{F}}(x,y) = \infty$. It was noted by Plante [63] that for each $x \in M$, the leaf L_x containing the point x, with the induced Riemannian metric from TM is a complete Riemannian manifold with bounded geometry, that depends continuously on x. In particular, bounded geometry implies that for each $x \in M$, there is a leafwise exponential map $\exp_x^{\mathcal{F}} : T_x\mathcal{F} \to L_x$ which is a surjection, and the composition $\iota \circ \exp_x^{\mathcal{F}} : T_x\mathcal{F} \to L_x \subset M$ depends continuously on x in the compact-open topology.

We next recall the notion of a *regular covering*, or what is sometimes called a *nice covering* in the literature (see [9, Chapter 1.2], or [34].) For a regular foliation covering, the intersections of the coverings of leaves by the plaques of the charts have nice metric properties. We first recall a standard fact from Riemannian geometry, as it applies to the leaves of $\mathcal{F}$.

For each $x \in M$ and $r > 0$, let $\overline{B}_{\mathcal{F}}(x,r) = \{y \in L_x \mid d_{\mathcal{F}}(x,y) \leq r\}$ denote the closed ball of radius r in the leaf containing x. The Gauss Lemma implies that there exists $\lambda_x > 0$ such that $\overline{B}_{\mathcal{F}}(x,\lambda_x)$ is a *strongly convex* subset for the metric $d_{\mathcal{F}}$. That is, for any pair of points $y, y' \in \overline{B}_{\mathcal{F}}(x,\lambda_x)$ there is a unique shortest geodesic segment in L_x joining y and y' and it is contained in $\overline{B}_{\mathcal{F}}(x,\lambda_x)$ (cf. [3], [17, Chapter 3, Proposition 4.2]). Then for all $0 < \lambda < \lambda_x$, the disk $\overline{B}_{\mathcal{F}}(x,\lambda)$ is also strongly convex. The compactness of M and the continuous dependence of the Christoffel symbols for a Riemannian metric in the C^2-topology on sections of bundles over M yields:

LEMMA 2.1. *There exists $\lambda_{\mathcal{F}} > 0$ such that for all $x \in M$, $\overline{B}_{\mathcal{F}}(x,\lambda_{\mathcal{F}})$ is strongly convex.*

If $\mathcal{F}$ is defined by a flow without periodic points, so that every leaf is diffeomorphic to $\mathbb{R}$, then the entire leaf is strongly convex, so $\lambda_{\mathcal{F}} > 0$ can be chosen arbitrarily. For a foliation with leaves of dimension $n > 1$, the constant $\lambda_{\mathcal{F}}$ must be less than the injectivity radius for each of the leaves.

Let $d_M\colon M \times M \to [0,\infty)$ denote the path-length metric on M. For $x \in M$ and $\epsilon > 0$, let $B_M(x,\epsilon) = \{y \in M \mid d_M(x,y) < \epsilon\}$ be the open ball of radius ϵ about x, and let $\overline{B}_M(x,\epsilon) = \{y \in M \mid d_M(x,y) \leq \epsilon\}$ denote its closure. Then as above, there exists $\lambda_M > 0$ such that $\overline{B}_M(x,\lambda)$ is a strongly convex ball in M for all $0 < \lambda \leq \lambda_M$.

We use these estimates on the local geometry of M and the leaves of $\mathcal{F}$ to construct a refinement of the given covering of M by foliations charts, which have uniform regularity properties.

Let $\epsilon_{\mathcal{U}} > 0$ be a Lebesgue number for the given covering $\mathcal{U}$ of M.

Then for each $x \in M$, there exists $\alpha_x \in \mathcal{A}$ be such that $x \in B_M(x, \epsilon_{\mathcal{U}}) \subset U_{\alpha_x}$. It follows that for each $x \in M$, there exists $0 < \delta_x \leq \lambda_{\mathcal{F}}$ such that $\overline{B}_{\mathcal{F}}(x, \delta_x) \subset B_M(x, \epsilon_{\mathcal{U}})$.

Let $(u_x, w_x) = \phi_\alpha(x)$, and note that $\phi_\alpha(\overline{B}_{\mathcal{F}}(x, \delta_x)) \subset (-1,1)^n \times \{w_x\}$. Then there exists $\epsilon_x > 0$ so that for each $w \in (w_x - \epsilon_x, w_x + \epsilon_x)$ and $x_w = \phi_\alpha^{-1}(u_x, w)$ we have $\overline{B}_{\mathcal{F}}(x_w, \delta_x) \subset B_M(x, \epsilon_{\mathcal{U}}) \subset U_{\alpha_x}$ is a leafwise convex subset. Define U_x and $\widetilde{U}_x$ to be unions of leafwise strongly convex disks,

$$U_x = \bigcup_{w \in (w_x - \epsilon_x/2, w_x + \epsilon_x/2)} \overline{B}_{\mathcal{F}}(x_w, \delta_x/2) \quad ; \quad \widetilde{U}_x = \bigcup_{w \in (w_x - \epsilon_x, w_x + \epsilon_x)} \overline{B}_{\mathcal{F}}(x_w, \delta_x) \tag{2}$$

so then $U_x \subset \widetilde{U}_x \subset B_M(x, \epsilon_{\mathcal{U}}) \subset U_{\alpha_x}$. The restriction $\phi_{\alpha_x} \colon \widetilde{U}_x \to (-1,1)^{n+1}$ is then a foliation chart, though the image is not onto.

Note that for each $x' \in \phi_{\alpha_x}^{-1}(w_x - \epsilon_x, w_x + \epsilon_x)$, the chart ϕ_{α_x} defines a framing of the tangent bundle $T_{x'}L_{x'}$ and this framing depends C^r on the parameter x', so we can then use the Gram-Schmidt process to obtain a C^r-family of orthonormal frames as well. Then using the inverse of the leafwise exponential map and affine rescaling, we obtain foliation charts

$$\begin{aligned} \widetilde{\varphi}_{\alpha_x} \quad &: \quad \widetilde{U}_x \to (-\delta_x, \delta_x)^n \times (w_x - \epsilon_x, w_x + \epsilon_x) \cong (-2,2)^n \times (-2,2) \\ \varphi_{\alpha_x} \quad &: \quad U_x \to (-\delta_x/2, \delta_x/2)^n \times (w_x - \epsilon_x/2, w_x + \epsilon_x/2) \cong (-1,1)^n \times (-1,1) \end{aligned}$$

where φ_{α_x} is the restriction of $\widetilde{\varphi}_{\alpha_x}$. Observe that $\widetilde{\varphi}_{\alpha_x}(x) = (\vec{0}, 0) \in (-1,1)^n \times (-1,1)$ for each x.

The collection of open sets $\{U_x \mid x \in M\}$ forms an open cover of the compact space M, so there exists a finite subcover "centered" at the points $\{x_1, \ldots, x_\nu\} \subset M$. Set

$$\delta_{\mathcal{U}}^{\mathcal{F}} = \min\{\delta_{x_1}/2, \ldots, \delta_{x_\nu}/2\} \ \leq \ \lambda_{\mathcal{F}}/2 \ . \tag{3}$$

This covering by foliation coordinate charts will be fixed and used throughout. To simplify notation, for $1 \leq \alpha \leq \nu$, set $U_\alpha = U_{x_\alpha}$, $\widetilde{U}_\alpha = \widetilde{U}_{x_\alpha}$, $\varphi_\alpha = \varphi_{x_\alpha}$, $\widetilde{\varphi}_\alpha = \widetilde{\varphi}_{x_\alpha}$, and $\mathcal{U} = \{U_1, \ldots, U_\nu\}$.

The resulting collection $\{\varphi_\alpha \colon U_\alpha \to (-1,1)^n \times (-1,1) \mid 1 \leq \alpha \leq \nu\}$ is a *regular covering* of M by foliation charts, in the sense used in [9, Chapter 1.2] or [34].

For each $1 \leq \alpha \leq \nu$, define $\mathcal{T}_\alpha \equiv (-1,1) \cong \{\vec{0}\} \times (-1,1)$ and $\widetilde{\mathcal{T}}_\alpha \equiv (-2,2) \cong \{\vec{0}\} \times (-2,2)$. The extended chart $\widetilde{\varphi}_\alpha$ defines C^r–embeddings

$$\tau_\alpha \colon \mathcal{T}_\alpha \to U_\alpha \quad , \quad \widetilde{\tau}_\alpha \colon \widetilde{\mathcal{T}}_\alpha \to \widetilde{U}_\alpha \ . \tag{4}$$

Let $\mathfrak{X}_\alpha = \tau_\alpha(\mathcal{T}_\alpha)$ and $\widetilde{\mathfrak{X}}_\alpha = \widetilde{\tau}_\alpha(\widetilde{\mathcal{T}}_\alpha)$ denote the images of these maps. For $n \geq 3$, we can assume without loss of generality that the submanifolds $\widetilde{\mathfrak{X}}_\alpha$ and $\widetilde{\mathfrak{X}}_\beta$ are disjoint, for $\alpha \neq \beta$.

Consider $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$ as disjoint spaces for $\alpha \neq \beta$, and similarly for $\widetilde{\mathcal{T}}_\alpha$ and $\widetilde{\mathcal{T}}_\beta$. Introduce the disjoint unions of these spaces, as denoted by

$$\mathcal{T} = \bigcup_{1 \leq \alpha \leq \nu} \mathcal{T}_\alpha \qquad \subset \qquad \widetilde{\mathcal{T}} = \bigcup_{1 \leq \alpha \leq \nu} \widetilde{\mathcal{T}}_\alpha \ , \tag{5}$$

$$\mathfrak{X} = \bigcup_{1 \leq \alpha \leq \nu} \mathfrak{X}_\alpha \qquad \subset \qquad \widetilde{\mathfrak{X}} = \bigcup_{1 \leq \alpha \leq \nu} \widetilde{\mathfrak{X}}_\alpha \ . \tag{6}$$

Note that $\mathfrak{X}$ is a *complete transversal* for $\mathcal{F}$, as the submanifold $\mathfrak{X}$ is transverse to the leaves of $\mathcal{F}$, and every leaf of $\mathcal{F}$ intersects $\mathfrak{X}$. The same is true for $\widetilde{\mathfrak{X}}$.

Let $\tau \colon \mathcal{T} \to \mathfrak{X} \subset M$ denote the map defined by the coordinate chart embeddings τ_α, and similarly define $\widetilde{\tau} \colon \widetilde{\mathcal{T}} \to \widetilde{\mathfrak{X}} \subset M$ using the maps $\widetilde{\tau}_\alpha$.

Let each $\widetilde{\mathcal{T}}_\alpha$ have the metric $\mathbf{d}_{\mathcal{T}}$ induced from the Euclidean metric on $\mathbb{R}$, where $\mathbf{d}_{\mathcal{T}}(x,y) = |x - y|$ for $x, y \in \widetilde{\mathcal{T}}_\alpha$. Extend this to a metric on $\mathcal{T}$ by setting $\mathbf{d}_{\mathcal{T}}(x,y) = \infty$ for $x \in \widetilde{\mathcal{T}}_\alpha$, $y \in \widetilde{\mathcal{T}}_\beta$ with $\alpha \neq \beta$.

Let each $\widetilde{\mathfrak{X}}_\alpha$ have the Riemannian metric induced from the Riemannian metric on M, and let $\mathbf{d}_{\mathfrak{X}}$ denote the resulting path-length metric on $\mathfrak{X}_\alpha$. As before, extend this to a metric on $\mathfrak{X}$ by setting $\mathbf{d}_{\mathfrak{X}}(x,y) = \infty$ for $x \in \widetilde{\mathfrak{X}}_\alpha$, $y \in \widetilde{\mathfrak{X}}_\beta$ with $\alpha \neq \beta$.

Given $r > 0$ and $x \in \widetilde{\mathfrak{X}}_\alpha$ let $\mathbf{B}_{\widetilde{\mathfrak{X}}}(x, r) = \{y \in \widetilde{\mathfrak{X}}_\alpha \mid \mathbf{d}_{\mathfrak{X}}(x, y) < r\}$. Introduce a notation which will be convenient for later work. Given a point $x \in \widetilde{\mathfrak{X}}_\alpha$ and $\delta_1, \delta_2 > 0$, let

$$[x - \delta_1, x + \delta_2] \subset \widetilde{\mathfrak{X}}_\alpha$$

be the connected closed subset bounded below by the point $x - \delta_1$ satisfying by $\mathbf{d}_{\mathfrak{X}}(x, x - \delta_1) = \delta_1$ and $[x - \delta_1, x]$ is an oriented interval in $\mathfrak{X}_\alpha$. The set $[x - \delta_1, x + \delta_2]$ is bounded above by the point $x + \delta_2$ satisfying by $\mathbf{d}_{\mathfrak{X}}(x, x + \delta_2) = \delta_2$ and $[x, x + \delta_1]$ is an oriented interval in $\mathfrak{X}_\alpha$.

For each $1 \leq \alpha \leq \nu$, let $\pi_\alpha \equiv \pi_t \circ \varphi_\alpha \colon U_\alpha \to \mathcal{T}_\alpha$ be the composition of the coordinate map φ_α with the projection $\pi_t \colon \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$. For each $w \in \mathcal{T}_\alpha$, the preimage

$$\mathcal{P}_\alpha(w) = \pi_\alpha^{-1}(w) \subset U_\alpha$$

is called a *plaque* of the chart φ_α. Then the plaques for the foliation atlas are indexed by the set $\mathcal{T}$.

For $x \in U_\alpha$, by a small abuse of notation, we use $\mathcal{P}_\alpha(x)$ to denote the plaque of the chart φ_α containing x. Note that $\mathcal{P}_\alpha(x)$ is the connected component of the intersection of the leaf L_x of $\mathcal{F}$ through x with the set U_α.

The maps $\widetilde{\pi}_\alpha \equiv \pi_t \circ \widetilde{\varphi}_\alpha \colon \widetilde{U}_\alpha \to \widetilde{\mathcal{T}}_\alpha$ are defined analogously, with corresponding plaques $\widetilde{\mathcal{P}}_\alpha(w)$. Again, by an abuse of notation, for $x \in \widetilde{U}_\alpha$ let $\widetilde{\mathcal{P}}_\alpha(x) \subset \widetilde{U}_\alpha$ denote the plaque of the chart $\widetilde{\varphi}_\alpha$ containing x.

Note that each plaque $\mathcal{P}_\alpha(x)$ is strongly convex in the leafwise metric, so if the intersection of two plaques $\{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$ is non-empty, then it is a strongly convex subset. In particular, the intersection $\mathcal{P}_\alpha(x) \cap \mathcal{P}_\beta(y)$ is connected. Thus, each plaque $\mathcal{P}_\alpha(x)$ intersects either zero or one plaque in U_β. The same observations are also true for the extended plaques $\widetilde{\mathcal{P}}_\alpha(x)$.

2.2. **Holonomy Pseudogroup $\mathcal{G}_\mathcal{F}$.** A pair of indices (α, β) is *admissible* if $U_\alpha \cap U_\beta \neq \emptyset$. For each admissible pair (α, β) define

$$\mathcal{T}_{\alpha\beta} = \{x \in \mathcal{T}_\alpha \text{ such that } \mathcal{P}_\alpha(x) \cap U_\beta \neq \emptyset\}, \tag{7}$$

$$\widetilde{\mathcal{T}}_{\alpha\beta} = \{x \in \widetilde{\mathcal{T}}_\alpha \text{ such that } \widetilde{\mathcal{P}}_\alpha(x) \cap \widetilde{U}_\beta \neq \emptyset\}\ . \tag{8}$$

Then there is a well-defined transition function $\mathbf{h}_{\beta,\alpha} \colon \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha}$, which for $x \in \mathcal{T}_{\alpha\beta}$ is given by

$$\mathbf{h}_{\beta,\alpha}(x) = y \text{ where } \mathcal{P}_\alpha(x) \cap \mathcal{P}_\beta(y) \neq \emptyset\ .$$

Note that $\mathbf{h}_{\alpha,\alpha} \colon \mathcal{T}_\alpha \to \mathcal{T}_\alpha$ is the identity map for each $\alpha \in \mathcal{A}$.

The *holonomy pseudogroup* $\mathcal{G}_\mathcal{F}$ associated to the regular foliation atlas for $\mathcal{F}$ is the pseudogroup with object space $\mathcal{T}$, and transformations generated by compositions of the local transformations $\{\mathbf{h}_{\beta,\alpha} \mid (\alpha, \beta) \text{ admissible}\}$. The $C^{\infty,r}$–hypothesis on the coordinate charts implies that each map $\mathbf{h}_{\beta,\alpha}$ is C^r. Moreover, the hypothesis (2) on regular foliation charts implies that each $\mathbf{h}_{\beta\alpha}$ admits an extension to a C^r-map $\widetilde{\mathbf{h}}_{\beta,\alpha} \colon \widetilde{\mathcal{T}}_{\alpha\beta} \to \widetilde{\mathcal{T}}_{\alpha\beta}$ defined in a similar fashion. The number of admissible pairs is finite, so there exists a uniform estimate on the sizes of the domains of these extensions. We note the following consequence of these observations.

LEMMA 2.2. *There exists $\epsilon_0 > 0$ so that for (α, β) admissible and $x \in \mathcal{T}_{\alpha\beta}$, then $[x - \epsilon_0, x + \epsilon_0] \subset \widetilde{\mathcal{T}}_{\alpha\beta}$. That is, if $x \in \mathcal{T}_\alpha$ is in the domain of $\mathbf{h}_{\beta,\alpha}$ then $[x - \epsilon_0, x + \epsilon_0]$ is in the domain of $\widetilde{\mathbf{h}}_{\beta,\alpha}$.*

For $0 < \delta < \epsilon_0$ we introduce the closed subsets of $\widetilde{\mathcal{T}}$

$$\mathcal{T}[\delta] = \{y \in \widetilde{\mathcal{T}} \mid \exists\, x \in \overline{\mathcal{T}},\ \mathbf{d}_\mathcal{T}(x, y) \leq \delta\} \tag{9}$$

$$\mathcal{T}_{\alpha\beta}[\delta] = \{y \in \widetilde{\mathcal{T}}_{\alpha\beta} \mid \exists\, x \in \overline{\mathcal{T}}_{\alpha\beta},\ \mathbf{d}_\mathcal{T}(x, y) \leq \delta\}\ . \tag{10}$$

Thus, the maps $\mathbf{h}_{\beta,\alpha}$ are uniformly C^r on $\mathcal{T}_{\alpha\beta}[\delta]$ for $\delta < \epsilon_0$.

Composition of elements in $\mathcal{G}_\mathcal{F}$ will be defined via "plaque chains". Given $x, y \in \mathcal{T}$ corresponding to points on the same leaf, a *plaque chain $\mathcal{P}$ of length k* between x and y is a collection of plaques

$$\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\},$$

where $x_0 = x$, $x_k = y$ and for each $0 \leq i < k$ we have $\mathcal{P}_{\alpha_i}(x_i) \cap \mathcal{P}_{\alpha_{i+1}}(x_{i+1}) \neq \emptyset$. We write $\|\mathcal{P}\| = k$.

A plaque chain $\mathcal{P}$ also defines an "extended" plaque chain for the charts $\{(\widetilde{U}_\alpha, \widetilde{\phi}_\alpha)\}$,

$$\widetilde{\mathcal{P}} = \{\widetilde{\mathcal{P}}_{\alpha_1}(x_0), \ldots, \widetilde{\mathcal{P}}_{\alpha_k}(x_k)\} \ .$$

We say two plaque chains

$$\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\} \text{ and } \mathcal{Q} = \{\mathcal{P}_{\beta_0}(y_0), \ldots, \mathcal{P}_{\beta_\ell}(y_\ell)\}$$

are *composable* if $x_k = y_0$, hence $\alpha_k = \beta_0$ and $\mathcal{P}_{\alpha_k}(x_k) = \mathcal{P}_{\beta_0}(y_0)$). Their composition is defined by

$$\mathcal{Q} \circ \mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k), \mathcal{P}_{\beta_1}(y_1), \ldots, \mathcal{P}_{\beta_\ell}(y_\ell)\} \ .$$

The holonomy transformation defined by a plaque chain is the local diffeomorphism

$$\mathbf{h}_{\mathcal{P}} = \mathbf{h}_{\alpha_k\alpha_{k-1}} \circ \cdots \circ \mathbf{h}_{\alpha_1\alpha_0}$$

whose domain $\mathcal{D}_{\mathcal{P}} \subset \mathcal{T}_{\alpha_0}$ contains x_0. Note that $\mathcal{D}_{\mathcal{P}}$ is the largest connected open subset of $\mathcal{T}_{\alpha_0}$ containing x_0 on which $\mathbf{h}_{\alpha_\ell\alpha_{\ell-1}} \circ \cdots \circ \mathbf{h}_{\alpha_1\alpha_0}$ is defined for all $0 < \ell \leq k$. The dependence of the domain of $\mathbf{h}_{\mathcal{P}}$ on the plaque chain $\mathcal{P}$ is a subtle issue, yet is at the heart of the technical difficulties arising in the study of foliation pseudogroups.

Let $\widetilde{\mathbf{h}}_{\widetilde{\mathcal{P}}}$ be the holonomy associated to the chain $\widetilde{\mathcal{P}}$, with domain $\widetilde{\mathcal{D}}_{\widetilde{\mathcal{P}}} \subset \widetilde{\mathcal{T}}_{\alpha_0}$ the largest maximal open subset containing x_0 on which $\widetilde{\mathbf{h}}_{\alpha_\ell\alpha_{\ell-1}} \circ \cdots \circ \widetilde{\mathbf{h}}_{\alpha_1\alpha_0}$ is defined for all $1 < \ell \leq k$. By the extension property of a regular atlas, the closure $\overline{\mathcal{D}_{\mathcal{P}}} \subset \widetilde{\mathcal{D}}_{\widetilde{\mathcal{P}}}$ and $\widetilde{\mathbf{h}}_{\widetilde{\mathcal{P}}}$ is an extension of $\mathbf{h}_{\mathcal{P}}$.

Given a plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ and a point $y \in \mathcal{D}_{\mathcal{P}}$, there is a "parallel" plaque chain denoted $\mathcal{P}(y) = \{\mathcal{P}_{\alpha_0}(y), \ldots, \mathcal{P}_{\alpha_k}(y_k)\}$ where $\mathbf{h}_{\mathcal{P}}(y) = y_k$.

For $x \in \mathcal{T}$, let

$$\mathcal{G}_{\mathcal{F}}(x) = \{y = \mathbf{h}_{\mathcal{P}}(x) \in \mathcal{T} \mid \ \mathcal{P} \text{ a plaque chain for which } x \in \mathcal{D}_{\mathcal{P}}\}$$

denote the orbit of x under the action of the pseudogroup. If $L_\xi \subset M$ denotes the leaf containing $\xi \in U_\alpha$ with $\pi_\alpha(\xi) = x \in \mathcal{T}_\alpha$, then $\tau(\mathcal{G}_{\mathcal{F}}(x)) = L_\xi \cap \mathfrak{X}$.

2.3. **The derivative cocycle.** Given a plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ from $x = x_0$ to $y = x_k$, the derivative $\mathbf{h}'_{\mathcal{P}}(x)$ is defined using the identifications $\mathcal{T}_\alpha = (-1, 1)$ for $1 \leq \alpha \leq \nu$. Note that the assumption that the foliation charts are transversally orientation preserving implies that $\mathbf{h}'_{\mathcal{P}}(x) > 0$ for all plaque chains $\mathcal{P}$ and $x \in \mathcal{D}_{\mathcal{P}}$.

Given composable plaque chains $\mathcal{P}$ and $\mathcal{Q}$, with $x = x_0, y = x_k = y_0, z = y_\ell$ the chain rule implies

$$\mathbf{h}'_{\mathcal{Q}\circ\mathcal{P}}(x) = \mathbf{h}'_{\mathcal{Q}}(y) \cdot \mathbf{h}'_{\mathcal{P}}(x) \ . \tag{11}$$

Define the map $D\mathbf{h} \colon \mathcal{G}_{\mathcal{F}} \to \mathbb{R}$ by $D\mathbf{h}(\mathcal{P}, y) = \mathbf{h}'_{\mathcal{P}(y)}(y)$, which is called the *derivative cocycle* for the foliation pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$. The function $\ln\{D\mathbf{h}(\mathcal{P}, y)\} \colon \mathcal{G}_{\mathcal{F}} \to \mathbb{R}$ is the *additive derivative cocycle*, or sometimes the *modular cocycle* for $\mathcal{G}_{\mathcal{F}}$.

2.4. **Resilient Leaves and Ping-Pong Games.** A plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ is *closed* if $x_0 = x_k$. A closed plaque chain $\mathcal{P}$ defines a local diffeomorphism $\mathbf{h}_{\mathcal{P}} \colon \mathcal{D}_{\mathcal{P}} \to \mathcal{T}_{\alpha_0}$ with $\mathbf{h}_{\mathcal{P}}(x) = x$, where $x = x_0 \in \mathcal{T}_{\alpha_0}$.

A point $y \in \mathcal{D}_{\mathcal{P}}$ is said to be *asymptotic* by iterates of $\mathbf{h}_{\mathcal{P}}$ to x, if $\mathbf{h}^\ell_{\mathcal{P}}(y) \in \mathcal{D}_{\mathcal{P}}$ for all $\ell > 0$ (where $\mathbf{h}^\ell_{\mathcal{P}}$ denotes the composition of $\mathbf{h}_{\mathcal{P}}$ with itself ℓ times), and $\lim\limits_{\ell\to\infty} \mathbf{h}^\ell_{\mathcal{P}}(y) = x$.

The map $\mathbf{h}_{\mathcal{P}}$ is said to be a *contraction* at x if there is some $\delta > 0$ so that every $y \in \mathbf{B}_{\mathcal{T}}(x, \delta)$ is asymptotic to x. The map $\mathbf{h}_{\mathcal{P}}$ is said to be a *hyperbolic contraction* at x if $0 < \mathbf{h}'_{\mathcal{P}}(x) < 1$. In this case, there exists $\epsilon > 0$ and $0 < \lambda < 1$ so that $\mathbf{h}'_{\mathcal{P}}(y) < \lambda$ for all $y \in \mathbf{B}_{\mathcal{T}}(x, \epsilon)$. Hence, every point of $\mathbf{B}_{\mathcal{T}}(x, \epsilon)$ is asymptotic to x, and there exists $0 < \delta < \epsilon$ so that the image of the closed δ–ball about x satisfies

$$\mathbf{h}_{\mathcal{P}}(\overline{\mathbf{B}_{\mathcal{T}}(x, \delta)}) \subset \mathbf{B}_{\mathcal{T}}(x, \delta) \ .$$

DEFINITION 2.3. *We say $x \in \mathcal{T}$ is a* hyperbolic resilient point *for $\mathcal{G}_{\mathcal{F}}$ if there exists*

(1) a closed plaque chain $\mathcal{P}$ such that $\mathbf{h}_{\mathcal{P}}$ is a hyperbolic contraction at $x = x_0$
(2) a point $y \in \mathcal{D}_{\mathcal{P}}$ which is asymptotic to x (and $y \neq x$)
(3) a plaque chain $\mathcal{R}$ from x to y.

Figure 2 below illustrates this concept, where the closed plaque chain $\mathcal{P}$ is represented by a path which defines it, and likewise for the plaque chain $\mathcal{R}$ from x to y. Note that the terminal point y is contained in the domain of the contraction $\mathbf{h}_{\mathcal{P}}$ defined by $\mathcal{P}$.

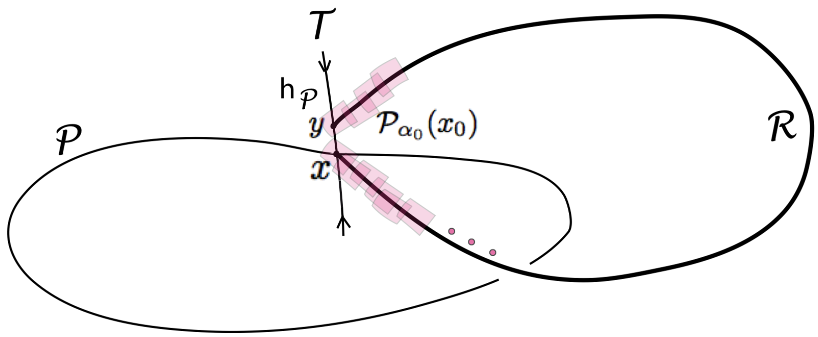

FIGURE 2. Resilient leaf with contracting holonomy along loop $\mathcal{P}$

The "ping-pong lemma" is a key technique for the study of 1-dimensional dynamics, which was used by Klein in his study of subgroups of Kleinian groups [16]. For a pseudogroup, this has the form:

DEFINITION 2.4. *The action of the groupoid $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{T}$ has a* "ping-pong game" *if there exists $x, y \in \mathcal{T}_\alpha$ with $x \neq y$ and*

(1) a closed plaque chain $\mathcal{P}$ such that $\mathbf{h}_{\mathcal{P}}$ is a contraction at $x = x_0$
(2) a closed plaque chain $\mathcal{Q}$ such that $\mathbf{h}_{\mathcal{Q}}$ is a contraction at $y = y_0$
(3) $y \in \mathcal{D}_{\mathcal{P}}$ is asymptotic to x by $\mathbf{h}_{\mathcal{P}}$ and $x \in \mathcal{D}_{\mathcal{Q}}$ is asymptotic to y by $\mathbf{h}_{\mathcal{Q}}$

We say that the ping-pong game is hyperbolic *if the maps $\mathbf{h}_{\mathcal{P}}$ and $\mathbf{h}_{\mathcal{Q}}$ are hyperbolic contractions.*

Figure 3 below illustrates the ping-pong dynamics, where the closed plaque chain $\mathcal{P}$ is represented by a path which defines it, and likewise for the plaque chain $\mathcal{Q}$.

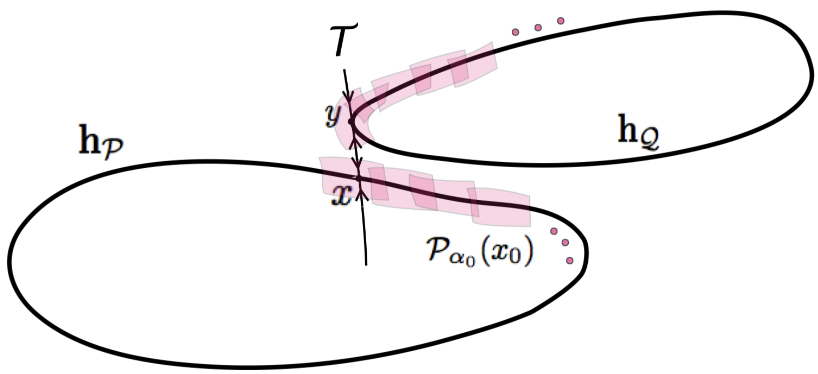

FIGURE 3. Closed paths $\mathcal{P}$ and $\mathcal{Q}$ with contracting holonomy generate a ping-pong game

These two notions are closely related as follows; for example, see [24] for a more detailed discussion.

PROPOSITION 2.5. *$\mathcal{G}_{\mathcal{F}}$ has a "ping-pong game" if and only if it has a resilient point, and has a "hyperbolic ping-pong game" if and only if it has a hyperbolic resilient point.*

3. The Godbillon-Vey invariant

We first recall the definition of the Godbillon-Vey class for C^2-foliations in Section 3.1. We then discuss the Godbillon operator in Section 3.2, as introduced in [18] and [36]. In Section 3.4, we introduce the Godbillon measure, and discuss its calculation using distributional differentials of leafwise forms. This technique was introduced in [36], and is a key point for the estimates of the values of the Godbillon-Vey invariants in terms of dynamical properties of the foliation. In particular, Proposition 3.5 is the key result used to relate the Godbillon-Vey invariant to foliation dynamics. These concepts are also discussed in detail by Candel and Conlon in [10, Chapter 7].

The Godbillon-Vey class is well-defined for C^2-foliations, and the Godbillon measure for C^1-foliations. However, giving these definitions for C^r-foliations adds a layer of notational complexity which obscures the basic ideas of the constructions. Thus, for clarity of the exposition, we assume in the following Section 3.1 that $\mathcal{F}$ is a C^∞-foliation, and leave to the reader the required technical modifications to show the analogous results for C^r-foliations, for $r = 1$ or 2. Alternately, consult the works [18, 36, 38] for further details in these cases.

3.1. **The Godbillon-Vey class.** Assume that M has a Riemannian metric, and that $\mathcal{F}$ is a C^∞-foliation of codimension one. The normal bundle $Q \to M$ to $T\mathcal{F}$ is then identified with the orthogonal space to the tangential distribution $T\mathcal{F}$. We may assume without loss of generality that M is connected, and that both the tangent bundle TM and the normal bundle Q are oriented, as the dynamical properties of foliations that we will be considering are preserved by passing to finite coverings of M. We may thus assume that $T\mathcal{F}$ is defined as the kernel of a non-vanishing 1-form ω on M. Throughout this work, $H^*(M)$ will denote the de Rham cohomology groups of M.

We first recall a basic construction that is used throughout the following discussions. Let $\vec{v}$ be a smooth vector field on M such that $\omega(\vec{v}) = 1$. The integrability of the tangential distribution $T\mathcal{F}$ implies that $d\omega \wedge \omega = 0$. Hence, there exists a 1-form α with $d\omega = \omega \wedge \alpha$. The choice of the 1-form α is not canonical, and so we introduce a procedure for choosing a representative for α.

DEFINITION 3.1. *Let ω be a non-vanishing 1-form on M whose kernel equals $T\mathcal{F}$, and $\vec{v}$ a vector field on M such that $\omega(\vec{v}) = 1$. Define $D^{\vec{v}}\omega = \iota(\vec{v})\, d\omega$.*

For brevity of notation, set $\eta = D^{\vec{v}}\omega$, and note that $\eta(\vec{v}) = 0$. Then for any choice of α such that $d\omega = \omega \wedge \alpha$, let $\vec{u}$ be tangent to $\mathcal{F}$, then we have

$$\eta(\vec{u}) = (\iota(\vec{v})d\omega)(\vec{u}) = d\omega(\vec{v}, \vec{u}) = (\omega \wedge \alpha)(\vec{v}, \vec{u}) = \alpha(\vec{u}) \tag{12}$$

as $\omega(\vec{v}) = 1$ and $\omega(\vec{u}) = 0$ by definition. Thus, for any 1-form α such that $d\omega = \omega \wedge \alpha$ and any leaf L of $\mathcal{F}$, their restrictions satisfy $\alpha|L = \eta|L$. In particular, we have that $d\omega = \omega \wedge \eta$, and calculate

$$0 = d(d\omega) = d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta = \omega \wedge \eta \wedge \eta - \omega \wedge d\eta = -\omega \wedge d\eta \ . \tag{13}$$

We conclude from (13) that the 2-form $d\eta$ is a multiple of ω. Then calculate $d(\eta \wedge d\eta) = d\eta \wedge d\eta = 0$ as $\omega \wedge \omega = 0$, so that $\eta \wedge d\eta$ is a closed 3-form on M.

THEOREM 3.2 (Godbillon and Vey, [26])**.** *The cohomology class $GV(\mathcal{F}) = [\eta \wedge d\eta] \in H^3(M)$ is independent of the choice of the 1-forms ω and η.*

Moreover, the Godbillon-Vey class $GV(\mathcal{F})$ is an invariant of the foliated concordance class of $\mathcal{F}$, as noted for example in Thurston [72] and Lawson [53, Chapter 3].

The definition of the Godbillon-Vey class in Theorem 3.2 reveals very little about the relation of this cohomology class with the dynamics of the foliation $\mathcal{F}$. In the case where the leaves of $\mathcal{F}$ are defined by a smooth fibration $M \to \mathbb{S}^1$, the defining 1-form ω for $\mathcal{F}$ can be chosen to be a closed form, and it is then immediate from the definition that $GV(\mathcal{F}) = 0$. Herman showed in [37] that a foliation defined by the suspension of an action of the abelian group $\mathbb{Z}^2$ on the circle must have $GV(\mathcal{F}) = 0$. The proof used an averaging process to obtain a sequence of defining smooth 1-forms $\{\omega_n \mid n = 1, 2, \ldots\}$ for which the corresponding 1-forms $D^{v_n}\omega_n \to 0$.

3.2. **The Godbillon operator.** We again assume there is given a non-vanishing 1-form ω on M such that $T\mathcal{F}$ equals the kernel of ω, and the Froebenius Theorem implies that $d\omega \wedge \omega = 0$. We define differential graded subalgebras of the de Rham complex $\Omega^*(M)$ of M using this property.

The breakthrough idea of Duminy, which first appeared in his paper with Sergiescu [20], is to separate the roles of the forms η and $d\eta$ in the definition of $GV(\mathcal{F})$, and then study how the contribution from the form η is related to the dynamical properties of $\mathcal{F}$. This is done by introducing the notion of the *Godbillon functional.* First, for $p \geq 1$, introduce the space

$$A^p(M, \mathcal{F}) \equiv \{\xi = \omega \wedge \beta \mid \beta \in \Omega^{p-1}(M)\} \subset \Omega^*(M) \ , \tag{14}$$

which can alternately be defined as the space of p-forms on M which vanish when restricted to each leaf of $\mathcal{F}$. Let $A^*(M, \mathcal{F}) \subset \Omega^*(M)$ denote the sum of these subspaces, which is then a subalgebra with trivial products as $\omega \wedge \omega = 0$. The identity $d\omega = \omega \wedge \eta$ implies that $A^*(M, \mathcal{F})$ is closed under exterior differentiation. More precisely, let $\xi = \omega \wedge \beta \in A^k(M, \mathcal{F})$ for $k \geq 1$, then

$$d\xi = d(\omega \wedge \beta) = d\omega \wedge \beta - \omega \wedge d\beta = (\omega \wedge \eta) \wedge \beta - \omega \wedge d\beta = \omega \wedge (\eta \wedge \beta - d\beta) \in A^{k+1}(M, \mathcal{F}). \tag{15}$$

Thus, $A^*(M, \mathcal{F})$ is a differential graded algebra.

Let $H^*(M, \mathcal{F})$ denote the cohomology of the differential graded complex $\{A^*(M, \mathcal{F}), d\}$. For a closed form $\xi \in A^k(M, \mathcal{F})$, let $[\xi]_{\mathcal{F}} \in H^k(M, \mathcal{F})$ denote its cohomology class.

The inclusion of the ideal $A^*(M, \mathcal{F}) \subset \Omega^*(M)$ induces a map on cohomology $H^*(M, \mathcal{F}) \to H^*(M)$. In general, the induced map need not be injective, and the calculation of the cohomology groups $H^*(M, \mathcal{F})$ is often an intractable problem, as discussed by El Kacimi in [21]. On the other hand, $H^*(M, \mathcal{F})$ is the domain of the Godbillon operator, as we next discuss, which makes it useful.

Let α be any choice of a 1-form satisfying $d\omega = \omega \wedge \alpha$. Then by a calculation analogous to (13), the closed 2-form $d\alpha$ is in the ideal generated by ω, so $d\alpha \in A^2(M, \mathcal{F})$. Duminy observed in [18] (see also [10, Chapter 7],[36]) that the class $[d\alpha]_{\mathcal{F}} \in H^2(M, \mathcal{F})$ is independent of the choices of the 1-forms ω and α, and so is an invariant of $\mathcal{F}$, which he called the *Vey class* of $\mathcal{F}$.

To be precise, we use the form $\eta = D^{\vec{v}}\omega$ to define the Vey class $[d\eta]_{\mathcal{F}} \in H^2(M, \mathcal{F})$. The 2-form $d\eta$ has some properties analogous to those of a symplectic form on M, especially in the geometric interpretation of the Godbillon-Vey invariant as "helical wobble" in [52, 68, 72]. The geometric meaning of the class $[d\eta]_{\mathcal{F}}$ remains obscure, although as noted below, $[d\eta]_{\mathcal{F}} = 0$ implies that $GV(\mathcal{F}) = 0$.

Given a closed form $\xi \in A^p(M, \mathcal{F})$, consider the product $\eta \wedge \xi \in A^{p+1}(M, \mathcal{F})$, and calculate:

$$d(\eta \wedge \xi) = d\eta \wedge \xi = \omega \wedge \eta \wedge \xi = 0 \ , \tag{16}$$

as $\omega \wedge \xi = 0$. Thus, $\eta \wedge \xi$ is a closed form. Moreover, if $\xi = d\beta$ for some form $\beta \in A^p(M, \mathcal{F})$, then $\eta \wedge \beta \in A^{p+1}(M, \mathcal{F})$ and

$$d(-\eta \wedge \beta) = -(d\eta) \wedge \beta + \eta \wedge d\beta = \eta \wedge \xi \ . \tag{17}$$

Thus, given $[\xi]_{\mathcal{F}} \in H^p(M, \mathcal{F})$ we obtain a well-defined class $[\eta \wedge \xi]_{\mathcal{F}} \in H^{p+1}(M, \mathcal{F})$. Multiplication by the 1-form η thus yields a well-defined map

$$\eta \cdot \colon H^p(M, \mathcal{F}) \to H^{p+1}(M, \mathcal{F}) \ . \tag{18}$$

Compose the map (18) with the inclusion induced map $\iota_* \colon H^{p+1}(M, \mathcal{F}) \to H^{p+1}(M)$ to obtain the linear functional

$$g \colon H^p(M, \mathcal{F}) \xrightarrow{\eta \cdot} H^{p+1}(M, \mathcal{F}) \xrightarrow{\iota_*} H^{p+1}(M) \ , \tag{19}$$

which is called the *Godbillon operator*. It was shown above that the 2-form $d\eta$ is a multiple of ω, and is clearly a closed form, so it defines a cohomology class $[d\eta]_{\mathcal{F}} \in H^2(M, \mathcal{F})$. Then we have $g([d\eta]_{\mathcal{F}}) = [\eta \wedge d\eta] = GV(\mathcal{F}) \in H^3(M)$. That is, "Godbillon(Vey) = Godbillon-Vey".

3.3. **The Godbillon functional.** The Godbillon operator takes values in the de Rham cohomology groups $H^*(M)$. For the purposes of showing that this operator vanishes, it is preferable to consider the closely related mappings with values in $\mathbb{R}$, obtained by integrating the image classes of the mappings (19) for $p = m-1$ over the fundamental class of M, where M has dimension m.

If M is a closed 3-manifold with fundamental class $[M]$, then evaluating $GV(\mathcal{F})$ on $[M]$ yields a real number, the *real Godbillon-Vey invariant* of $\mathcal{F}$:

$$\langle GV(\mathcal{F}), [M]\rangle = \int_M \eta \wedge d\eta \ .$$

If M is an open 3-manifold, then $H^3(M) = 0$ so that $GV(\mathcal{F}) = 0$ in this case. However, the class $GV(\mathcal{F})$ need not vanish in the case when M is open and M has dimension $m > 3$. In this case, it is necessary to introduce cohomology with compact supports, in order to obtain a real-valued invariants from the class $GV(\mathcal{F})$.

Let $\Omega^*_c(M) \subset \Omega^*(M)$ denote the differential subalgebra of forms with compact support. The cohomology of this ideal is denoted by $H^*_c(M)$ which is called with the *de Rham cohomology with compact supports* of M. We also consider the differential ideal $A^*_c(M,\mathcal{F}) \subset \Omega^*_c(M)$ consisting of forms in $A^*(M,\mathcal{F})$ with compact support. Its cohomology groups are denoted by $H^*_c(M,\mathcal{F})$, and these groups are called the *foliated cohomology with compact supports*.

Given a closed form $\zeta \in A^p(M,\mathcal{F})$, let $\xi \in \Omega^k_c(M)$ be a closed form with compact support, then the product $\zeta \wedge \xi \in A^{k+p}(M,\mathcal{F})$ is again closed with compact support. If either form ζ or ξ is the boundary of a form with compact support, then $\zeta \wedge \xi$ is also the boundary of a form with compact support. Thus, there is a well-defined pairing

$$H^p(M,\mathcal{F}) \times H^k_c(M) \to H^{k+p}_c(M,\mathcal{F}) \ . \tag{20}$$

In particular, given a class $[\xi] \in H^{m-3}_c(M)$ represented by a smooth closed form $\xi \in \Omega^{m-3}_c(M)$, then the pairing $[d\eta]_{\mathcal{F}} \cup [\xi] = [d\eta \wedge \xi]_{\mathcal{F}} \in H^{m-1}_c(M,\mathcal{F})$ is well-defined.

Recall that the manifold M is assumed to be oriented and connected, so by Poincaré duality the pairing $H^p(M) \otimes H^{m-p}_c(M) \to H^m_c(M) \cong \mathbb{R}$ is non-degenerate for $0 \leq p < m$. In particular, the value of the class $[\eta \wedge d\eta] \in H^3(M)$ is determined by its pairings with classes in $H^{m-3}_c(M)$. This is the idea behind the definition of the Godbillon functional.

The Godbillon operator in (19) applied to a class in $H^{m-1}_c(M,\mathcal{F})$ yields a closed m-form with compact support on M, which can be integrated over the fundamental class to obtain a real number. This composition yields the *Godbillon functional*, denoted by

$$G\colon H^{m-1}_c(M,\mathcal{F}) \to \mathbb{R}, \quad G([\xi]_{\mathcal{F}}) = \langle [\eta \wedge \xi], [M]\rangle = \int_M \eta \wedge \xi \ . \tag{21}$$

Note that we use the notation "g" for the Godbillon operator between cohomology groups, and the notation "G" for the linear functional on the cohomology group $H^{m-1}_c(M,\mathcal{F})$.

With these preliminary preparations, we have the basic result as observed by Duminy in [18]:

PROPOSITION 3.3. *The value of the Godbillon-Vey class $GV(\mathcal{F}) \in H^3(M)$ is determined by the Godbillon functional G in* (21). *In particular, if $G \equiv 0$ then $GV(\mathcal{F}) = 0$.*

Proof. For the case when the dimension $m = 3$ and M is compact, this follows by applying the linear functional G to the class $[d\eta]_{\mathcal{F}} \in H^2(M,\mathcal{F}) = H^2_c(M,\mathcal{F})$. For $m > 3$, then by Poincaré duality, the value of $GV(\mathcal{F}) \in H^3(M)$ is determined by pairing the 3-form $\eta \wedge d\eta$ with closed forms $\xi \in \Omega^{m-3}_c(M)$, followed by integration, to obtain

$$\langle GV(\mathcal{F}) \cup [\xi], [M]\rangle = \int_M (\eta \wedge d\eta) \wedge \xi.$$

Note that $[d\eta \wedge \xi]_{\mathcal{F}} \in H^{m-1}_c(M,\mathcal{F})$, so that $\langle GV(\mathcal{F}) \cup [\xi], [M]\rangle = G([d\eta \wedge \xi]_{\mathcal{F}})$. The claim follows. $\square$

The strategy to proving that $GV(\mathcal{F}) = 0$ is thus to obtain dynamical properties of a foliation which suffice to show that the linear functional G vanishes.

3.4. **The Godbillon measure.** Duminy showed that the integral of the expression $\eta \wedge \xi$ in (21) over a saturated *Borel* subset of M is independent of the choices made to define η, and thus gives a *localized invariant* for $\mathcal{F}$. This observation was systematically generalized in the work [36], to show that the Godbillon functional G extends to a measure on the σ-algebra of *Lebesgue measurable* saturated subsets of M. We show how this observation is used to calculate the Godbillon functional.

A set $B \subset M$ is $\mathcal{F}$–saturated if for all $x \in B$, the leaf L_x through x is contained in B. Let $\mathcal{B}(\mathcal{F})$ denote the σ-algebra of Lebesgue measurable $\mathcal{F}$–saturated subsets of M.

THEOREM 3.4. [18, 36] *For each $B \in \mathcal{B}(\mathcal{F})$, there is a well-defined linear functional*

$$G_{\mathcal{F}}(B)\colon H_c^{m-1}(M,\mathcal{F}) \to \mathbb{R} \quad , \quad G_{\mathcal{F}}(B)([\xi]_{\mathcal{F}}) = \int_B \eta \wedge \xi \tag{22}$$

where $\xi \in A_c^{m-1}(M,\mathcal{F})$ is closed. Moreover, the correspondence

$$B \mapsto G_{\mathcal{F}}(B) \in \mathrm{Hom}_{\mathrm{cont}}(H_c^{m-1}(M,\mathcal{F}),\mathbb{R})$$

is a countably additive measure on the σ-algebra of Borel subsets in $\mathcal{B}(\mathcal{F})$. Note that if B has Lebesgue measure zero, then $G_{\mathcal{F}}(B) = 0$. Thus, $G_{\mathcal{F}}$ extends to the full σ-algebra of Lebesgue measurable saturated subsets of M. This is called the Godbillon measure.

Part of the claim of Theorem 3.4 is that the linear functional (22) is independent of the choice of the smooth 1-form ω defining $\mathcal{F}$. Much more is true, as described below. The key idea, introduced in [36], is to consider representatives for η which belong to the space of leafwise forms on $\mathcal{F}$ which are leafwise smooth, but need only be measurable as functions on M.

Introduce the graded differential algebra $\Omega^*(\mathcal{F})$ consisting of leafwise forms. That is, for $k \geq 0$, the space $\Omega^k(\mathcal{F})$ consists of sections of the dual to the *k-th* exterior power $\Lambda^k(T\mathcal{F})$ of the leafwise tangent bundle $T\mathcal{F}$. Given $\xi \in \Omega^k(\mathcal{F})$, then for each $x \in M$ and k-tuple $(\vec{v}_1,\ldots,\vec{v}_k)$ of vectors in the tangent space $T_x\mathcal{F}$ to the leaf L_x containing x, we obtain a real number $\xi(\vec{v}_1,\ldots,\vec{v}_k) \in \mathbb{R}$. We impose on the sections in $\Omega^k(\mathcal{F})$ the following regularity condition: given $\xi \in \Omega^k(\mathcal{F})$, for each leaf L of $\mathcal{F}$, the restriction $\xi|L$ to L is a smooth form.

For $\xi \in \Omega^k(\mathcal{F})$, define $D_{\mathcal{F}}(\xi) \in \Omega^{k+1}(\mathcal{F})$ is as follows. For each leaf L of $\mathcal{F}$, the restriction $\xi|L$ is a smooth k-form on L, so there is a well-defined exterior differential $d(\xi|L)$. The collection of leafwise forms $\{d(\xi|L) \in \Omega^{k+1}(L) \mid L \subset M\}$ defines $D_{\mathcal{F}}(\xi) \in \Omega^{k+1}(\mathcal{F})$. Thus, there is a well-defined leafwise exterior differential operator,

$$D_{\mathcal{F}}\colon \Omega^k(\mathcal{F}) \to \Omega^{k+1}(\mathcal{F}) \quad , \quad D_{\mathcal{F}}(\xi)|L = d(\xi|L) \quad \text{for each leaf } L \subset M \ . \tag{23}$$

The cohomology of $\{\Omega^*(\mathcal{F}), D_{\mathcal{F}}\}$ is called the *foliated cohomology* of $\mathcal{F}$.

A key observation in the definition of the exterior differential in (23) is that it does not require any regularity for the transverse behavior of the leafwise forms. Thus, one can consider the subcomplex $\Omega^*_\infty(\mathcal{F}) \subset \Omega^*(\mathcal{F})$ of smooth leafwise forms, and the differential $D_{\mathcal{F}}$ restricted to $\Omega^*_\infty(\mathcal{F})$ yields a differential graded subalgebra. Its smooth foliated cohomology groups $H^*_\infty(\mathcal{F})$ were used, for example, by Heitsch in [35] to study the deformation theory of foliations. We can also consider the the differential graded subalgebra $\Omega^*_c(\mathcal{F}) \subset \Omega^*(\mathcal{F})$ consisting of the continuous leafwise forms, whose cohomology spaces $H^*_c(\mathcal{F})$ were studied by El Kacimi-Alaoui in [21].

Finally, one can also consider the differential graded subalgebra $\Omega^k_{me}(\mathcal{F}) \subset \Omega^*(\mathcal{F})$ of measurable (or bounded measurable) sections of the dual to the *k-th* exterior power of the leafwise tangent bundle $T\mathcal{F}$. A form $\xi \in \Omega^k_{me}(\mathcal{F})$ is required to be smooth when restricted to leaves of $\mathcal{F}$, but is only required to be a Borel measurable function on M. We also demand that for $\xi \in \Omega^k_{me}(\mathcal{F})$, its leafwise differential $D_{\mathcal{F}}\xi \in \Omega^{k+1}_{me}(\mathcal{F})$. The cohomology $H^*_{me}(\mathcal{F})$ of the complex $\Omega^*_{me}(\mathcal{F})$ is called the *measurable leafwise cohomology* of $\mathcal{F}$. These groups were used by Zimmer in [79, 80] to study the rigidity theory for measurable group actions.

A function $f\colon M \to \mathbb{R}$ is said to be *transversally measurable* if it is a measurable function, and for each leaf L of $\mathcal{F}$, the restriction $f|L$ is smooth and the leafwise derivatives of f are measurable functions as well. Such a function f is the typical element in $\Omega^0_{me}(\mathcal{F})$. Given a function $f \in \Omega^0_{me}(\mathcal{F})$ and a form $\xi \in \Omega^k_c(\mathcal{F})$, then the product $f \cdot \xi \in \Omega^k_{me}(\mathcal{F})$.

We next introduce norms on the spaces $\Omega^k_{me}(\mathcal{F})$. For each $x \in M$, the Riemannian metric on T_xM defines a norm on T_xM, which restricts to a norm on the leafwise tangent space $T_x\mathcal{F}$. The norm on the space $T_x\mathcal{F}$ induces a dual norm on the cotangent bundle $T^*_x\mathcal{F}$, and also induces norms on each exterior vector space $\Lambda^k T_x\mathcal{F}$ and on its dual $\Omega^k(T_x\mathcal{F})$, for all $k > 1$. We denote this norm by $\|\cdot\|_x$ in each of these cases. For a function $f \in \Omega^0_{me}(\mathcal{F})$, let $\|f\|_x = |f(x)|$. Given a subset $B \subset M$, and a leafwise form $\xi \in \Omega^k_{me}(\mathcal{F})$ for $k \geq 0$, define the sup-norm over B by

$$\|\xi\|_B = \sup_{x \in B} \ \|\xi\|_x \ .$$

Next, given a smooth function $f\colon M \to \mathbb{R}$, set $\omega_f = \exp(f)\cdot\omega$. Let $\vec{v}$ be a vector field such that $\omega(\vec{v}) = 1$, then set $\vec{v}_f = \exp(-f)\cdot\vec{v}$ so that $\omega_f(\vec{v}_f) = 1$. Then by Definition 3.1,

$$D^{\vec{v}_f}\omega_f = \exp(-f)\cdot\iota(\vec{v})\, d\{\exp(f)\cdot\omega\} = \iota(\vec{v})\,(df\wedge\omega + d\omega) \ .$$

Thus, for $\zeta \in A^p(M,\mathcal{F})$ we evaluate

$$(D^{\vec{v}_f}\omega_f)\wedge\zeta = -df\wedge\zeta + D^{\vec{v}}\omega\wedge\zeta = -D_{\mathcal{F}}f\wedge\zeta + D^{\vec{v}}\omega\wedge\zeta \ . \tag{24}$$

By the Leafwise Stokes' Theorem [36, Proposition 2.6], given a closed form with compact support $\zeta \in A^{m-1}_c(M,\mathcal{F})$ and $B \in \mathcal{B}(\mathcal{F})$, then

$$\int_B (D^{\vec{v}_f}\omega_f)\wedge\zeta = \int_B D^{\vec{v}}\omega\wedge\zeta \ . \tag{25}$$

Observe that for ζ a closed form, the leafwise coboundary term $D_{\mathcal{F}}f\wedge\zeta$ in (24) depends only on the leafwise derivatives of f. This observation is the idea behind the proof of [36, Theorem 2.7] which shows that if $f \in \Omega^0_{me}(\mathcal{F})$ satisfies $\|D_{\mathcal{F}}f\|_B < \infty$, and $\zeta \in A^{m-1}_c(M,\mathcal{F})$ is a closed form with compact support, then

$$G_{\mathcal{F}}(B)([\zeta]_{\mathcal{F}}) \ = \ \int_B -D_{\mathcal{F}}f\wedge\zeta + D^{\vec{v}}\omega\wedge\zeta \ , \tag{26}$$

where $D_{\mathcal{F}}f$ is defined by (23). The formula (26) is the motivation for introducing the following terminology, where given $f \in \Omega^0_{me}(\mathcal{F})$, set $\omega_f = \exp(f)\cdot\omega$ and $\vec{v}_f = \exp(-f)\cdot\vec{v}$, then define

$$\overline{D}^{\vec{v}_f}\omega_f = -D_{\mathcal{F}}f + D^{\vec{v}}\omega \ . \tag{27}$$

If the function f is smooth on M and $\zeta \in A^{m-1}_c(M,\mathcal{F})$, then $(\overline{D}^{\vec{v}_f}\omega_f)\wedge\zeta = (D^{\vec{v}_f}\omega_f)\wedge\zeta$, where the latter term is defined in the sense of Definition 3.1. Thus, the definition (27) can be viewed as the extension of the Definition 3.1 in the sense of distributions to the measurable complex $\Omega^*_{me}(\mathcal{F})$.

We now recall a fundamental result, Theorem 4.3 of [38], which is a broad generalization of the ideas in the seminal work by Herman [37].

PROPOSITION 3.5. *Let $B \in \mathcal{B}(\mathcal{F})$. Suppose there exists a sequence of transversally measurable functions $\{f_n \mid n = 1, 2, \ldots\}$ on M so that the 1-forms $\{\omega_n = \exp(f_n)\cdot\omega \mid n = 1, 2, \ldots\}$ on M satisfy $\|\overline{D}^{\vec{v}_n}\omega_n\|_B < 1/n$ where $\vec{v}_n = \exp(-f_n)\cdot\vec{v}$. Then $G_{\mathcal{F}}(B) = 0$.*

Proof. For each $n \geq 1$, set

$$\eta_n = \overline{D}^{\vec{v}_n}\omega_n = -D_{\mathcal{F}}f_n + D^{\vec{v}}\omega \ . \tag{28}$$

For $[\zeta]_{\mathcal{F}} \in H^{m-1}_c(M,\mathcal{F})$ and each $n \geq 1$, then by (26) we have

$$G_{\mathcal{F}}(B)([\zeta]_{\mathcal{F}}) = \int_B \eta_n\wedge\zeta \ . \tag{29}$$

Estimate the norms of the integrals in (29):

$$\begin{aligned} |G_{\mathcal{F}}(B)([\zeta]_{\mathcal{F}})| \ &= \ \lim_{n\to\infty} \left|\int_B \eta_n\wedge\zeta\right| \\ &\leq \ \lim_{n\to\infty} \int_B \|\eta_n\|_B \ \|\zeta\|_B \ dvol \\ &\leq \ \lim_{n\to\infty} (1/n)\cdot\int_B \|\zeta\|_B \ dvol = 0 \ . \end{aligned}$$

As this holds for all $[\zeta]_{\mathcal{F}} \in H^{m-1}_c(M,\mathcal{F})$, the claim follows. □

We note two important aspects of the proof of Proposition 3.5. First, the n-form $\eta_n \wedge \zeta$ in the integrand of (29) depends only on the restrictions $\eta_n|L$ for leaves L of $\mathcal{F}$. Thus, the pairing $\eta_n \wedge \zeta$ is well-defined when $\mathcal{F}$ is a $C^{\infty,1}$-foliation. Also, the convergence of the integral in (29) as $n \to \infty$ uses the Lebesgue dominated convergence theorem, and can be applied assuming only that the form $\zeta \in A_c^{m-1}(M, \mathcal{F})$ is continuous. In particular, for a $C^{\infty,2}$-foliation the form $d\eta$ is continuous, so the calculation above applies to multiples of this form as required for the proof of Proposition 3.3.

Proposition 3.5 gives an effective method for showing that the Godbillon-Vey class vanishes on a set $B \in \mathcal{B}(\mathcal{F})$, provided that one can construct a sequence of 1-forms $\{\omega_n = \exp(f_n)\cdot\omega \mid n = 1, 2, \ldots\}$ on M satisfying the hypotheses of Proposition 3.5. In hindsight, one can see that an analogous estimate was used in the previous works [20, 37, 56, 57, 74, 78] to show that $GV(\mathcal{F}) = 0$ for C^2-foliations of codimension one, for foliations with various types of dynamical properties.

For a C^2-foliation $\mathcal{F}$, Sacksteder's Theorem [70] implies that if $\mathcal{F}$ has no resilient leaf, then there are no exceptional minimal sets for $\mathcal{F}$. Hence, by the Poincaré-Bendixson theory, all leaves of $\mathcal{F}$ either lie at finite level, or lie in "arbitrarily thin" open subsets $U \in \mathcal{B}(\mathcal{F})$. In his works [18, 19], Duminy used a result analogous to Proposition 3.5 to show that $G_{\mathcal{F}}(B) = 0$, where B is a union of leaves at finite level. Thus, for a C^2-foliation with no resilient leaves, the Godbillon measure vanishes on the union of the leaves of finite level, and also vanishes on any Borel set in their complement. Thus, $GV(\mathcal{F}) = 0$ for a C^2-foliation of codimension-one with no resilient leaves. See [9, 12] for a published version of this proof. In the next two sections, we follow a different, more direct approach to obtain this conclusion. From the assumption $G_{\mathcal{F}} \neq 0$, we conclude that the holonomy pseudogroup of a $C^{\infty,1}$-foliation $\mathcal{F}$ must contain resilient orbits. Thus for a C^2-foliation $\mathcal{F}$ with $GV(\mathcal{F}) \neq 0$, we have that $G_{\mathcal{F}} \neq 0$ and hence $\mathcal{F}$ must contain resilient leaves.

4. Asymptotically expansive holonomy

In this section, we assume there is given a codimension-one foliation $\mathcal{F}$ on a compact manifold M, so that M admits a finite regular $C^{\infty,1}$-foliation atlas $\{\varphi_\alpha \colon U_\alpha \to (-1,1)^n \times (-1,1) \mid 1 \leq \alpha \leq \nu\}$ which is a *regular covering* of M by foliation charts, as in Section 2.1, with associated transversal $\mathcal{T}$, and associated holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ as in Section 2.2. Then $\mathcal{G}_{\mathcal{F}}$ is generated by a finite collection of local C^1-diffeomorphisms defined on open subsets of $\mathcal{T}$. Recall that the charts in the foliation atlas are assumed to be transversally oriented, so for each plaque chain $\mathcal{P}$, the derivative $\mathbf{h}'_{\mathcal{P}}(x) > 0$ for all $x \in \mathcal{D}_{\mathcal{P}}$ in its domain.

The main result of this section, Theorem 4.4, implies that the Godbillon measure $G_{\mathcal{F}}$ is supported on the set $\mathrm{E}^+(\mathcal{F})$ introduced in Section 4.2. That is, for any $B \in \mathcal{B}(\mathcal{F})$, we have $G_{\mathcal{F}}(B) = G_{\mathcal{F}}(B \cap \mathrm{E}^+(\mathcal{F}))$. Hence, $G_{\mathcal{F}} \neq 0$ implies the set $\mathrm{E}^+(\mathcal{F})$ must have positive Lebesgue measure..

4.1. **The transverse expansion exponent function.** We introduce the notion of *asymptotically expansive* holonomy for a leaf of $\mathcal{F}$. For all $x \in \mathcal{T}$, set $\mu_0(x) = 1$. Then for each integer $n \geq 1$, define the *maximal n-expansion function*

$$\mu_n(x) = \sup\{\, \mathbf{h}'_{\mathcal{P}}(x) \mid x \in \mathcal{D}_{\mathcal{P}} \;\&\; \|\mathcal{P}\| \leq n\} \ . \tag{30}$$

The function $x \mapsto \mu_n(x)$ is the maximum of a finite set of continuous functions, so is a Borel function on $\mathcal{T}$, and $\mu_n(x) \geq 1$ as the identity transformation is the holonomy for a plaque chain of length 1.

LEMMA 4.1. *Let $x \in \mathcal{T}$, and let $\mathcal{Q} = \{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$ be a plaque chain of length 1. For the holonomy map $\mathbf{h}_{\beta,\alpha}$ of this length-one plaque-chain, we have $\mathbf{h}_{\beta,\alpha}(x) = y$. Then for all $n > 0$,*

$$\mu_{n-1}(x) \leq \mu_n(y) \cdot \mathbf{h}'_{\beta,\alpha}(x) \leq \mu_{n+1}(x) \ . \tag{31}$$

Proof. Let $\mathcal{P}$ be a plaque chain at y with $\|\mathcal{P}\| \leq n$, then $\mathcal{P} \circ \mathcal{Q}$ is a plaque chain at x with $\|\mathcal{P} \circ \mathcal{Q}\| \leq n+1$, so

$$\mathbf{h}'_{\mathcal{P}}(y) \cdot \mathbf{h}'_{\beta,\alpha}(x) = \mathbf{h}'_{\mathcal{P}\circ\mathcal{Q}}(x) \leq \mu_{n+1}(x) \ .$$

As this is true for all plaque chains at y with $\|\mathcal{P}\| \leq n$, we obtain $\mu_n(y) \cdot \mathbf{h}'_{\beta,\alpha}(x) \leq \mu_{n+1}(x)$.

Given a plaque chain $\mathcal{P}$ at x with $\|\mathcal{P}\| \leq n-1$, the chain $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}^{-1}$ at y has $\|\mathcal{R}\| \leq n$ and

$$\mathbf{h}'_{\mathcal{P}}(x) = \mathbf{h}'_{\mathcal{R}}(y) \cdot \mathbf{h}'_{\beta,\alpha}(x) \leq \mu_n(y) \cdot \mathbf{h}'_{\beta,\alpha}(x) \ . \tag{32}$$

As (32) holds for all plaque chains at x with $\|\mathcal{P}\| \leq n-1$, we have $\mu_{n-1}(x) \leq \mu_n(y) \cdot \mathbf{h}'_{\beta,\alpha}(x)$. $\square$

Define $\lambda_n(x) = \ln(\mu_n(x))$, so that $\lambda_n(x) = \sup\{\ln(\mathbf{h}'_{\mathcal{P}}(x)) \mid x \in \mathcal{D}_{\mathcal{P}} \ \& \ \|\mathcal{P}\| \leq n\}$.

Then the *transverse expansion exponent* at $x \in \mathcal{T}$ is defined by

$$\lambda_*(x) = \limsup_{n\to\infty} \frac{\lambda_n(x)}{n} \ . \tag{33}$$

LEMMA 4.2. *The transverse expansion exponent function λ_* is Borel measurable on $\mathcal{T}$, and constant on the orbits of $\mathcal{G}_{\mathcal{F}}$.*

Proof. For each $n \geq 1$, the function $\lambda_n(x)/n$ is Borel, so the supremum function in (33) is also Borel.

Let $x \in \mathcal{T}$, and let $\mathcal{Q} = \{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$ be a plaque chain, then the estimate (31) implies that,

$$\frac{\ln(\mu_{n+1}(x))}{n+1} \geq \frac{\ln(\mu_n(y)\cdot \mathbf{h}'_{\beta,\alpha}(x))}{n} \cdot \frac{n}{n+1} = \left\{\frac{\ln(\mu_n(y))}{n} + \frac{\ln(\mathbf{h}'_{\beta,\alpha}(x))}{n}\right\} \cdot \frac{n}{n+1} \tag{34}$$

so that

$$\lambda_*(x) = \limsup_{n\to\infty} \left\{\frac{\ln(\mu_{n+1}(x))}{n+1}\right\} \geq \limsup_{n\to\infty} \left\{\frac{\ln(\mu_n(y))}{n}\right\} = \lambda_*(y) \ . \tag{35}$$

The converse inequality follows similarly.

Thus, $\lambda_*(x) = \lambda_*(y)$ if there is a plaque chain $\mathcal{Q} = \{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$. The pseudogroup $\mathcal{G}_{\mathcal{F}}$ is generated by the holonomy defined by plaque chains of length 1, so that for each point $y \in \mathcal{G}_{\mathcal{F}}(x)$, there is a finite plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ with $x_0 = x$ and $x_k = y$. Then $\lambda_*(x_\ell) = \lambda_*(x_{\ell+1})$ for each $0 \leq \ell < k$, from which it follows that $\lambda_*(x) = \lambda_*(y)$. $\square$

4.2. **The expansion decomposition.** The transverse expansion exponent function λ_* is defined on the space $\mathcal{T}$. We use the conclusion of Lemma 4.2 to lift the function λ_* from $\mathcal{T}$ to M, and then use this lifted function to define a Borel saturated decomposition of M.

Let $\mathfrak{X} \subset M$ be the transversal to $\mathcal{F}$ as defined in (6). For each $\xi \in \mathfrak{X}$ there exists $1 \leq \alpha \leq \nu$ such that $\xi \in \mathfrak{X}_\alpha$, and $x_\alpha \in \mathcal{T}_\alpha$ with $\tau_\alpha = x$. Define the function $\overline{\lambda}_*$ on $\mathfrak{X}$ by setting $\overline{\lambda}_*(\xi) = \lambda_*(x_\alpha)$. Extend the function $\overline{\lambda}_*$ to a function on M, where for $\xi \in M$ choose and index $1 \leq \alpha \leq \nu$ such that $\xi \in U_\alpha$, then set $\overline{\lambda}_*(\xi) = \lambda_*(\pi_\alpha(\xi))$. The value $\lambda_*(x)$ is independent of the choice of open set with $\xi \in U_\alpha$ by Lemma 4.2. Moreover, the function $\overline{\lambda}_*$ is then constant on leaves, as the function λ_* is constant on the orbits of $\mathcal{G}_{\mathcal{F}}$. By abuse of notation, we denote by $\overline{\lambda}_*(L)$ this constant value, so that $\overline{\lambda}_*(L) = \overline{\lambda}_*(\xi)$ for some $\xi \in L$.

DEFINITION 4.3. *Define the $\mathcal{G}_{\mathcal{F}}$-saturated Borel subsets of $\mathcal{T}$*

$$\begin{aligned} \mathrm{E}^+(\mathcal{T}) &= \{x \in \mathcal{T} \mid \lambda_*(x) > 0\} \\ \mathrm{E}^+_a(\mathcal{T}) &= \{x \in \mathcal{T} \mid \lambda_*(x) > a\}, \text{ for } a \geq 0 \\ \mathrm{S}(\mathcal{T}) &= \mathcal{T} - \mathrm{E}^+(\mathcal{T}) \ , \end{aligned}$$

and the $\mathcal{F}$-saturated Borel subsets of M

$$\begin{aligned} \mathrm{E}^+(\mathcal{F}) &= \{\xi \in M \mid \overline{\lambda}_*(\xi) > 0\} \\ \mathrm{E}^+_a(\mathcal{F}) &= \{\xi \in M \mid \overline{\lambda}_*(\xi) > a\}, \text{ for } a \geq 0 \\ \mathrm{S}(\mathcal{F}) &= M - \mathrm{E}^+(\mathcal{F}) \ . \end{aligned}$$

A point $x \in \mathrm{E}^+(\mathcal{F})$ is said to be *infinitesimally expansive.* The set $\mathrm{E}^+(\mathcal{F})$ is called the *hyperbolic set* for $\mathcal{F}$, and is the analog for codimension-one foliations of the hyperbolic set for diffeomorphisms in Pesin theory [2, 61]. The set $\mathrm{S}(\mathcal{F})$ consists of the leaves of $\mathcal{F}$ for which the transverse infinitesimal holonomy has "slow growth". Both sets $\mathrm{E}^+(\mathcal{F})$ and $\mathrm{S}(\mathcal{F})$ are fundamental for the study of the dynamics of the foliation $\mathcal{F}$.

Note that for $x \in \mathcal{T}$, if there is an holonomy map $\mathbf{h}_{\mathcal{P}}$ with $x \in \mathcal{D}_{\mathcal{P}}$, $\mathbf{h}_{\mathcal{P}}(x) = x$ and $\mathbf{h}'_{\mathcal{P}}(x) = \lambda > 1$, then $x \in \mathrm{E}^+(\mathcal{T})$. If $\mathcal{P}$ is a plaque-chain of length k, then $x \in \mathrm{E}^+_a(\mathcal{T})$ for any $0 < a < \ln(\lambda)/k$. The plaque chain $\mathcal{P}$ determines a closed loop $\gamma_{\mathcal{P}}$ based at x in the leaf L_x, and the transverse holonomy along $\gamma_{\mathcal{P}}$ is linearly expanding in some open neighborhood of x. Such transversally hyperbolic elements of the leaf holonomy have a fundamental role in the study of foliation dynamics, in particular in the works by Sacksteder [70], by Bonatti, Langevin and Moussu [4], and the works [40, 42]. However, given $x \in \mathrm{E}^+(\mathcal{T})$ there may not be a holonomy map $\mathbf{h}_{\mathcal{P}}$ with $\mathbf{h}_{\mathcal{P}}(x) = x$ and $\mathbf{h}'_{\mathcal{P}}(x) = \lambda > 1$. What is always true, is that there exists a sequence of holonomy maps whose lengths tend to infinity, each of which has infinitesimally expansive holonomy at x. We make this statement precise.

Consider a point $x \in \mathrm{E}^+_a(\mathcal{T})$ for $a > 0$, and choose λ with $a < \lambda < \lambda_*(x)$. Then for all $N > 0$, there exists $n \geq N$ such that $\lambda_n(x) \geq n\lambda$. By the definition of $\lambda_n(x)$, this means there exists a plaque chain $\mathcal{P}$ with length $\|\mathcal{P}\| \leq n$ starting at x such that $\mathbf{h}'_{\mathcal{P}}(x) \geq \exp\{n\lambda\}$. By the continuity of the derivative function on $\mathcal{T}$, there exists $\epsilon_n > 0$ such that on the open interval $(x - \epsilon_n, x + \epsilon_n) \subset \mathcal{T}$,

$$\mathbf{h}'_{\mathcal{P}}(y) \geq \exp\{n\lambda/2\} \text{ for all } x - \epsilon_n \leq y \leq x + \epsilon_n \ .$$

By the Mean Value Theorem, $\mathbf{h}'_{\mathcal{P}}$ is expanding on the interval $(x - \epsilon_n, x + \epsilon_n)$ by a factor at least $\exp\{n\lambda/2\}$. Thus, the assumption $\lambda_*(x) > \lambda > 0$ and the definition in (33) implies that we can choose a sequence of plaque chains $\mathcal{P}_\ell$ with lengths $\|\mathcal{P}_\ell\| = n_\ell$ starting at x such that n_ℓ is strictly increasing, and so tends to infinity, and the corresponding holonomy maps satisfy

$$\mathbf{h}'_{\mathcal{P}_\ell}(y) \geq \exp\{n_\ell\lambda/2\} \text{ for all } x - \epsilon_{n_\ell} \leq y \leq x + \epsilon_{n_\ell} \ . \tag{36}$$

The constant $\epsilon_{n_\ell} > 0$ in (36) depends upon ℓ, λ and x, and is exponentially decreasing as $\ell \to \infty$.

It is a strong condition to have a sequence of holonomy maps as in (36) for elements of the holonomy pseudogroup at points x, whose plaque lengths tend to infinity. This is what gives the set $\mathrm{E}^+(\mathcal{F})$ a fundamental role in the study of foliation dynamics, exactly in analog with the role of the Pesin set in smooth dynamics [2, 51, 61, 69]. The works [46, 47] give further study of the relation between the hyperbolic set $\mathrm{E}^+(\mathcal{F})$ and the dynamics of the foliation. In contrast, for the *slow set* $\mathrm{S}(\mathcal{F})$, the dynamics of $\mathcal{F}$ on $\mathrm{S}(\mathcal{F})$ has "less complexity", as discussed in [47].

4.3. **A vanishing criterion.** For an arbitrary saturated Borel set $B \in \mathcal{B}(\mathcal{F})$, we have

$$G_{\mathcal{F}}(B) = G_{\mathcal{F}}(B \cap \mathrm{E}^+(\mathcal{F})) + G_{\mathcal{F}}(B \cap \mathrm{S}(\mathcal{F})) \ . \tag{37}$$

We use the criteria of Proposition 3.5 to show that $G_{\mathcal{F}}(\mathrm{S}(\mathcal{F})) = 0$, so that $G_{\mathcal{F}} \neq 0$ implies the set $\mathrm{E}^+(\mathcal{F})$ must have positive Lebesgue measure.

The strategy is to construct a sequence of transversally measurable, non-vanishing transverse 1-forms $\{\omega_n \mid n = 1, 2, \ldots\}$ on M for which $\|D^{\vec{v_n}}\omega_n\|_{\mathrm{S}(\mathcal{F})} < 1/n$, where the leafwise 1-form $D^{\vec{v_n}}\omega_n$ is defined as in (28) The construction of the forms $\{\omega_n\}$ follows the method introduced in [38]. The first, and crucial step, is to construct an ϵ–tempered cocycle over the pseudogroup $\mathcal{G}_{\mathcal{F}}$ which is cohomologous to the additive derivative cocycle, using a procedure adapted from [39]. This tempered cocycle is then used to produce the sequence of defining 1-forms ω_n, using the methods of [7] and [48, 50]. These are the ingredients used in the proof of the following result.

THEOREM 4.4. *The Godbillon measure $G_{\mathcal{F}}(\mathrm{S}(\mathcal{F})) = 0$. Hence, by* (37) *for any set $B \in \mathcal{B}(\mathcal{F})$, the Godbillon measure $G_{\mathcal{F}}(B) = G_{\mathcal{F}}(B \cap \mathrm{E}^+(\mathcal{F}))$. In particular, if $\mathrm{E}^+(\mathcal{F})$ has Lebesgue measure zero, then $G_{\mathcal{F}}(B) = 0$ for all $B \in \mathcal{B}(\mathcal{F})$.*

Proof. The first step in the proof is to use the properties of the holonomy action for points in the slow set $\mathrm{S}(\mathcal{T})$ to construct the forms $\{\omega_n\}$ as mentioned above.

Fix $\epsilon > 0$. For $x \in \mathrm{S}(\mathcal{T})$, by the definition of $\lambda_*(x) = 0$ in (33), there exists $N_{\epsilon,x}$ such that $n \geq N_{\epsilon,x}$ implies $\ln\{\mu_n(x)\} \leq n\epsilon/2$, and hence the maximal n-expansion $\mu_n(x) \leq \exp\{n\epsilon/2\}$.

We define the coboundary g_ϵ function next. Set $g_\epsilon(x) = 1$ for $x \in \mathcal{T}$ but $x \notin \mathrm{S}(\mathcal{T})$. For $x \in \mathrm{S}(\mathcal{T})$ set

$$g_\epsilon(x) = \sum_{n=0}^{\infty} \exp\{-n\epsilon\} \cdot \mu_n(x). \tag{38}$$

For x in the slow set $\mathrm{S}(\mathcal{T})$, the sum in (38) converges as the function $\exp\{-n\epsilon\} \cdot \mu_n(x)$ decays exponentially fast as $n \to \infty$. Note that while $g_\epsilon(x)$ is finite for each $x \in \mathcal{T}$, there need not be an upper bound for its values on $\mathrm{S}(\mathcal{T})$. Also, g_ϵ is a Borel measurable function defined on all of $\mathcal{T}$.

The definition of the function g_ϵ in (38) is analogous to the definition of the Lyapunov metric in Pesin theory. Its role is to give a "change of gauge" with respect to which the expansion rates of the dynamical system is "normalized" for the action of $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{T}$, as made precise by Lemma 4.5 below.

Let $x \in \mathcal{T}$, and let $\mathcal{Q} = \{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$ be a plaque chain of length 1. Let $\mathbf{h}_{\beta,\alpha}$ denote the holonomy map of the plaque-chain $\mathcal{Q}$, with $\mathbf{h}_{\beta,\alpha}(x) = y$. The following result estimates the value of g_ϵ under a change of coordinates, for charts such that $\mathcal{P}_\alpha(x) \cap \mathcal{P}_\beta(y) \neq \emptyset$. Let $S_\alpha = \pi_\alpha(\mathrm{S}(\mathcal{F}) \cap U_\alpha) \subset \mathcal{T}_\alpha$.

LEMMA 4.5. *For $x \in S_\alpha$ and $\mathcal{Q} = \{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$,*

$$\exp\{-\epsilon\} \cdot g_\epsilon^\alpha(x) \leq g_\epsilon^\beta(y) \cdot \mathbf{h}'_{\beta,\alpha}(x) \leq \exp\{\epsilon\} \cdot g_\epsilon^\alpha(x) \ . \tag{39}$$

Proof. Use the estimate (31), noting that $\mathbf{h}_{\beta,\alpha}(x) = y$, to obtain:

$$\begin{aligned} g_\epsilon^\beta(y) \cdot \mathbf{h}'_{\beta,\alpha}(x) &= \left\{\sum_{n=0}^{\infty} \exp\{-n\epsilon\} \cdot \mu_n(y)\right\} \mathbf{h}'_{\beta,\alpha}(x) \\ &\leq \sum_{n=0}^{\infty} \exp\{-n\epsilon\} \cdot \mu_{n+1}(x) \\ &< \exp\{\epsilon\} \cdot \left\{\sum_{n=1}^{\infty} \exp\{-n\epsilon\} \cdot \mu_n(x) + \mu_0(x)\right\} \\ &= \exp\{\epsilon\} \cdot g_\epsilon^\alpha(x) \ . \end{aligned} \tag{40}$$

Similarly, we have

$$\begin{aligned} g_\epsilon^\beta(y) \cdot \mathbf{h}'_{\beta,\alpha}(x) &= \left\{\sum_{n=0}^{\infty} \exp\{-n\epsilon\} \cdot \mu_n(y)\right\} \mathbf{h}'_{\beta,\alpha}(x) \\ &\geq \sum_{n=1}^{\infty} \exp\{-n\epsilon\} \cdot \mu_{n-1}(x) + \mu_0(x) \cdot \mathbf{h}'_{\beta,\alpha}(x) \\ &\geq \exp\{-\epsilon\} \cdot \ g_\epsilon^\alpha(x) \ . \end{aligned} \tag{41}$$

This completes the proof of Lemma 4.5. □

Introduce the notation, for $x \in U_\alpha \cap U_\beta$,

$$k_{\epsilon,\beta,\alpha}(x) = g_\epsilon^\beta \circ \mathbf{h}_{\beta,\alpha} \circ \pi_\alpha(x) \cdot \mathbf{h}'_{\beta,\alpha} \circ \pi_\alpha(x) \ . \tag{42}$$

Note that $k_{\epsilon,\alpha\alpha} = g_\epsilon^\alpha \circ \pi_\alpha$. Then in this notation, the estimate (39) implies that

$$\exp(-\epsilon) \cdot k_{\epsilon,\alpha\alpha}(x) \leq k_{\epsilon,\beta,\alpha}(x) \leq \exp(\epsilon) \cdot k_{\epsilon,\alpha\alpha}(x) \ . \tag{43}$$

We next use the given covering $\{U_\beta \mid 1 \leq \beta \leq \nu\}$ of M to construct a smooth 1-form ω on M which defines $\mathcal{F}$. Recall that $\mathcal{T}_\beta \equiv (-1,1)$, and let dx_β denote the coordinate 1-form on $\mathcal{T}_\beta$. Use the projection $\pi_\beta \colon U_\beta \to \mathcal{T}_\beta$ along plaques to pull-back the form dx_β to the closed 1-form $\omega_\beta = \pi_\beta^*(dx_\beta)$. Choose a partition of unity $\{\rho_\beta \mid 1 \leq \beta \leq \nu\}$ subordinate to the cover $\{U_\beta \mid 1 \leq \beta \leq \nu\}$. Then for each $1 \leq \beta \leq \nu$, the 1-form $\rho_\beta \cdot \omega_\beta$ has support contained in U_β, and set $\omega = \displaystyle\sum_{1 \leq \beta \leq \nu} \rho_\beta \cdot \omega_\beta$.

Next, for $\epsilon > 0$ we construct a measurable 1-form ω_ϵ on M. Recall that the function g_ϵ on $\mathcal{T}$ was defined by (38). For each $1 \leq \beta \leq \nu$, introduce the notation $g_\epsilon^\beta = g_\epsilon | \mathcal{T}_\beta$, and define the 1-form $\phi_\beta^\epsilon = g_\epsilon^\beta \ dx_\beta$ on $\mathcal{T}_\beta$. Then define $\omega_\epsilon^\beta = \pi_\beta^*(\phi_\beta^\epsilon) = \left(g_\epsilon^\beta \circ \pi_\beta\right) \cdot \omega_\beta$ which is a transversally measurable leafwise 1-form on U_β. Recall that $\omega_\beta = \pi_\beta^*(dx_\beta)$, so that $d\omega_\beta = 0$, and hence $D_{\mathcal{F}}\omega_\epsilon^\beta = 0$ on U_β. Finally, define the 1-form $\omega_\epsilon = \sum \rho_\beta \cdot \omega_\epsilon^\beta$ on M.

For each $1 \leq \alpha \leq \nu$, consider the 1-forms ω and ω_ϵ restricted to the chart U_α:

$$\omega | U_\alpha = \sum_{U_\beta \cap U_\alpha \neq \emptyset} \rho_\beta|_{U_\alpha} \cdot \omega_\beta|_{U_\alpha} \tag{44}$$

$$\omega_\epsilon | U_\alpha = \sum_{U_\beta \cap U_\alpha \neq \emptyset} \rho_\beta|_{U_\alpha} \cdot \omega_\epsilon^\beta|_{U_\alpha} = \sum_{U_\beta \cap U_\alpha \neq \emptyset} \rho_\beta|_{U_\alpha} \cdot \left(g_\epsilon^\beta \circ \pi_\beta\right) \cdot \omega_\beta|_{U_\alpha} \ . \tag{45}$$

We express the terms on the right-hand-sides of (44) and (45) in terms of the 1-form ω_α. For β with $U_\alpha \cap U_\beta \neq \emptyset$, let $\mathcal{Q} = \{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$ be the plaque chain with holonomy map $\mathbf{h}_{\beta,\alpha}$. Then $\mathbf{h}^*_{\beta,\alpha}(dx_\beta) = \mathbf{h}'_{\beta,\alpha} \cdot dx_\alpha$ and so by the identity $\pi_\beta = \mathbf{h}_{\beta,\alpha} \circ \pi_\alpha$ on $U_\alpha \cap U_\beta$ we have

$$\begin{aligned} \omega_\beta|_{U_\alpha \cap U_\beta} &= \pi^*_\beta(dx_\beta)|_{U_\alpha \cap U_\beta} = \pi^*_\alpha \circ \mathbf{h}^*_{\beta,\alpha}(dx_\beta)|_{U_\alpha \cap U_\beta} = \pi^*_\alpha(\mathbf{h}'_{\beta,\alpha} \cdot dx_\alpha)|_{U_\alpha \cap U_\beta} \\ &= (\mathbf{h}'_{\beta,\alpha} \circ \pi_\alpha) \cdot \omega_\alpha|_{U_\alpha \cap U_\beta} \ . \end{aligned} \tag{46}$$

Using the identity $\mathbf{h}^*_{\beta,\alpha}(\phi^\epsilon_\beta) = \left(g^\beta_\epsilon \circ \mathbf{h}_{\beta,\alpha}\right) \cdot \mathbf{h}'_{\beta,\alpha} \cdot dx_\alpha$, we obtain the corresponding expression

$$\begin{aligned} \omega^\beta_\epsilon|_{U_\alpha \cap U_\beta} &= \pi^*_\beta(dx^\epsilon_\beta)|_{U_\alpha \cap U_\beta} = \pi^*_\alpha \circ \mathbf{h}^*_{\beta,\alpha}(dx^\epsilon_\beta)|_{U_\alpha \cap U_\beta} = \pi^*_\alpha(g^\beta_\epsilon \circ \mathbf{h}_{\beta,\alpha} \cdot \mathbf{h}'_{\beta,\alpha} \cdot dx_\alpha)|_{U_\alpha \cap U_\beta} \\ &= (g^\beta_\epsilon \circ \mathbf{h}_{\beta,\alpha} \circ \pi_\alpha) \cdot (\mathbf{h}'_{\beta,\alpha} \circ \pi_\alpha) \cdot \omega_\alpha|_{U_\alpha \cap U_\beta} \ . \end{aligned} \tag{47}$$

Thus, for $x \in U_\alpha \cap U_\beta$, we have $\omega^\beta_\epsilon|_x = k_{\epsilon,\beta,\alpha}(x) \cdot \omega_\alpha|_x$. For each $1 \leq \alpha \leq \nu$, on U_α define:

$$\Phi_\alpha = \sum_{U_\beta \cap U_\alpha \neq \emptyset} \rho_\beta|_{U_\alpha} \cdot \mathbf{h}'_{\beta,\alpha} \circ \pi_\alpha \quad , \quad \Phi^\epsilon_\alpha = \sum_{U_\beta \cap U_\alpha \neq \emptyset} \rho_\beta|_{U_\alpha} \cdot k_{\epsilon,\beta,\alpha} \ . \tag{48}$$

We then have

$$\omega|_{U_\alpha} = \Phi_\alpha \cdot \omega_\alpha|_{U_\alpha} \quad , \quad \omega_\epsilon|_{U_\alpha} = \Phi^\epsilon_\alpha \cdot \omega_\alpha|_{U_\alpha} \ . \tag{49}$$

We return to the proof that $G_{\mathcal{F}}(\mathrm{S}(\mathcal{F})) = 0$. Let $\vec{v}$ be a vector field on M such that $\omega(\vec{v}) = 1$. Then set $\eta = D^{\vec{v}}\omega = \iota(\vec{v})d\omega$, as in Definition 3.1 Define a function $f_\epsilon \in \Omega^0_{me}(\mathcal{F})$ by $\omega_\epsilon = \exp(f_\epsilon) \cdot \omega$, and set $\vec{v}_\epsilon = \exp(-f_\epsilon) \cdot \vec{v}$ so that $\omega_\epsilon(\vec{v}_\epsilon) = 1$. Then as in (27), define

$$\eta_\epsilon = \overline{D}^{\vec{v}_\epsilon}\omega_\epsilon = -D_{\mathcal{F}} f_\epsilon + D^{\vec{v}}\omega = -D_{\mathcal{F}} f_\epsilon + \eta \ , \tag{50}$$

where the 1-form $D_{\mathcal{F}} f_\epsilon$ is defined by (23). Then by (26), the Godbillon measure $G_{\mathcal{F}}(B)$ can be calculated using the 1-form η_ϵ restricted to B.

Next, estimate the norm $\|\eta_\epsilon\|_x$ for $x \in U_\alpha$ using the expression (50). We first calculate

$$\eta|_{U_\alpha} = \iota(\vec{v})d(\omega|_{U_\alpha}) = \iota(\vec{v})d(\Phi_\alpha \cdot \omega_\alpha|_{U_\alpha}) = \iota(\vec{v})d(\exp\{\ln(\Phi_\alpha)\} \cdot \omega_\alpha|_{U_\alpha}) = -D_{\mathcal{F}} \ln(\Phi_\alpha|_{U_\alpha}) \ . \tag{51}$$

Then by (49), we have $\exp(f_\epsilon)|_{U_\alpha} \cdot \Phi_\alpha = \Phi^\epsilon_\alpha$, so $f_\epsilon|_{U_\alpha} = \ln\,(\Phi^\epsilon_\alpha) - \ln\,(\Phi_\alpha)$, and calculate

$$\eta_\epsilon|_{U_\alpha} = -D_{\mathcal{F}} f_\epsilon|_{U_\alpha} + \eta|_{U_\alpha} = (\{D_{\mathcal{F}} \ln\,(\Phi_\alpha) - D_{\mathcal{F}} \ln\,(\Phi^\epsilon_\alpha)\} - \{D_{\mathcal{F}} \ln\,(\Phi_\alpha)\} = -D_{\mathcal{F}} \ln\,(\Phi^\epsilon_\alpha) \ . \tag{52}$$

Note that $D_{\mathcal{F}} k_{\epsilon,\beta,\alpha} = 0$, as each function $k_{\epsilon,\beta,\alpha}$ is constant along the plaques in $U_\alpha \cap U_\beta$, so its leafwise differential is zero. Use this observation and the definition (48) to obtain:

$$\|\eta_\epsilon|_x\| = \|D_{\mathcal{F}} \ln\,(\Phi^\epsilon_\alpha)\|_x = (\Phi^\epsilon_\alpha(x))^{-1} \cdot \left\| \sum_{U_\beta \cap U_\alpha \neq \emptyset} D_{\mathcal{F}} \rho_\beta|_{U_\alpha} \cdot k_{\epsilon,\beta,\alpha} \right\|_x \ . \tag{53}$$

The leafwise differential of the constant function is zero, so we have the identity

$$0 = D_{\mathcal{F}} 1 = D_{\mathcal{F}}(\sum \rho_\beta) = \sum D_{\mathcal{F}} \rho_\beta$$

which implies that

$$\sum_{U_\beta \cap U_\alpha \neq \emptyset} D_{\mathcal{F}} \rho_\beta \cdot k_{\epsilon,\alpha\alpha} = k_{\epsilon,\alpha\alpha} \cdot \sum_{U_\beta \cap U_\alpha \neq \emptyset} D_{\mathcal{F}} \rho_\beta = 0 \ . \tag{54}$$

Then continuing from (53), and using the identities (54) and (43), for $x \in U_\alpha$ we have:

$$\begin{aligned} \|\eta_\epsilon|_x\| &= (\Phi^\epsilon_\alpha(x))^{-1} \cdot \left\| \sum_{U_\beta \cap U_\alpha \neq \emptyset} D_{\mathcal{F}} \rho_\beta \cdot \{k_{\epsilon,\beta,\alpha} - k_{\epsilon,\alpha\alpha}\} \right\|_x \\ &\leq (\Phi^\epsilon_\alpha(x))^{-1} \cdot \sum_{U_\beta \cap U_\alpha \neq \emptyset} \|D_{\mathcal{F}} \rho_\beta\|_x \cdot |k_{\epsilon,\beta,\alpha}(x) - k_{\epsilon,\alpha\alpha}(x)| \\ &\leq (\Phi^\epsilon_\alpha(x))^{-1} \cdot \sup_{x \in U_\alpha} \|D_{\mathcal{F}} \rho_\beta\|_x \cdot \sum_{U_\beta \cap U_\alpha \neq \emptyset} |k_{\epsilon,\beta,\alpha}(x) - k_{\epsilon,\alpha\alpha}(x)| \\ &\leq (\Phi^\epsilon_\alpha(x))^{-1} \cdot \sup_{x \in U_\alpha} \|D_{\mathcal{F}} \rho_\beta\|_x \cdot \sum_{U_\beta \cap U_\alpha \neq \emptyset} (\exp(\epsilon) - 1) \cdot k_{\epsilon,\alpha\alpha}(x) \ . \end{aligned} \tag{55}$$

It remains to estimate $(\Phi^\epsilon_\alpha(x))^{-1}$ in (55). Use (48) and the estimates (43) to obtain for $x \in U_\alpha$ that

$$(56)\quad \Phi^\epsilon_\alpha(x) = \sum_{U_\beta \cap U_\alpha \neq \emptyset} \rho_\beta(x) \cdot k_{\epsilon,\beta,\alpha}(x) \geq \sum_{U_\beta \cap U_\alpha \neq \emptyset} \rho_\beta(x) \cdot \exp(-\epsilon) \cdot k_{\epsilon,\alpha\alpha}(x) = \exp(-\epsilon) \cdot k_{\epsilon,\alpha\alpha}(x) \ .$$

Thus, we obtain the estimate

$$(\Phi^\epsilon_\alpha(x))^{-1} \leq \exp(\epsilon) \cdot k_{\epsilon,\alpha\alpha}(x)^{-1} \ . \tag{57}$$

Then combining (55) and (57), and noting that the number of indices β for which $U_\beta \cap U_\alpha \neq \emptyset$ is bounded by the cardinality ν of the covering, we obtain

$$\|\eta_\epsilon|_x\| \leq \sup_{x \in U_\alpha} \|D_{\mathcal{F}}\rho_\beta)\|_x \cdot \nu \cdot \exp(\epsilon) \cdot (\exp(\epsilon) - 1) \tag{58}$$

Note that the right hand side in (58) tends to 0 as $\epsilon \to 0$, so that for each $n > 0$, we can choose $\epsilon_n > 0$ such that $\|\eta_{\epsilon_n}\| \leq 1/n$. Then set $\omega_n = \omega_{\epsilon_n}$, and the claim of the Theorem 4.4 follows from Proposition 3.5. □

5. Uniform hyperbolic expansion

In this section, we assume that $\mathcal{F}$ is a C^1-foliation with non-empty hyperbolic set $\mathrm{E}^+(\mathcal{F})$, and show that there exists a hyperbolic fixed-point for the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$. The proof uses a pseudogroup version of the *Pliss Lemma*, which is fundamental in the study of non-uniformly hyperbolic dynamics (see [1] or [5, Lemma 11.5], or the original article by Pliss [64].)

The goal is to construct hyperbolic contractions in the holonomy pseudogroup. The length of the path defining the holonomy element is not important, but rather it is important to obtain uniform estimates on the size of the domain of the hyperbolic element thus obtained, estimates which are independent of the length of the path. This is a key technical point for the application of the constructions of this section in the next Section 6, where we construct sufficiently many contractions so that they result in the existence of a resilient orbit for the action of the holonomy pseudogroup.

We note that the existence of a hyperbolic contraction can also be deduced using the foliation geodesic flow methods introduced in [42], though that method does not yield estimates on the size of the domain of the hyperbolic element in the foliation pseudogroup.

5.1. **Uniform hyperbolicity and the Pliss Lemma.** We fix a regular covering on M as in Section 2.1, with transversals $\mathfrak{X}$ and $\widetilde{\mathfrak{X}}$ as in (6), and let $\mathcal{G}_{\mathcal{F}}$ denote the resulting pseudogroup acting on the spaces $\mathcal{T}$ and $\widetilde{\mathcal{T}}$ as in (5). Recall that by Lemma 2.2, there exists $\epsilon_0 > 0$ so that for every admissible pair (α, β) and $x \in \mathcal{T}_{\alpha\beta}$ then $[x - \epsilon_0, x + \epsilon_0] \subset \widetilde{\mathcal{T}}_{\alpha\beta}$. Recall that the space $\mathcal{T}_{\alpha\beta}$ was defined in (7), and $\widetilde{\mathcal{T}}_{\alpha\beta}$ was defined in (8).

DEFINITION 5.1. *Given* $0 < \epsilon_1 \leq \epsilon_0$*, a constant* $0 < \delta_0 \leq \epsilon_1$ *is said to be a* logarithmic modulus of continuity *for* $\mathcal{G}_{\mathcal{F}}$ *with respect to* ϵ_1*, if for* $y, z \in \mathcal{T}_{\alpha\beta}[\delta_0]$ *with* $\mathbf{d}_{\mathcal{T}}(y, z) \leq \delta_0$*, then*

$$\left|\log\{\widetilde{\mathbf{h}}'_{\beta,\alpha}(y)\} - \log\{\widetilde{\mathbf{h}}'_{\beta,\alpha}(z)\}\right| \leq \epsilon_1 \ . \tag{59}$$

LEMMA 5.2. *Given* $0 < \epsilon_1 \leq \epsilon_0$*, there exists a constant* $0 < \delta_0 \leq \epsilon_1$ *which is a logarithmic modulus of continuity for* $\mathcal{G}_{\mathcal{F}}$ *with respect to* ϵ_1*.*

Proof. By the choice of $0 < \epsilon_1 \leq \epsilon_0$, for each admissible pair $\{\alpha, \beta\}$, the logarithmic derivative $\log\{\widetilde{\mathbf{h}}'_{\beta,\alpha}(y)\}$ is continuous on the compact subset $\mathcal{T}_{\alpha\beta}[\epsilon_1] \subset \widetilde{\mathcal{T}}_{\alpha\beta}$. Thus, there exists $\delta_0(\alpha, \beta) > 0$ such that (59) holds for this choice of $\{\alpha, \beta\}$. Define $\delta_0 = \min\{\delta_0(\alpha, \beta) \mid \{\alpha, \beta\} \text{ admissible}\}$. As the number of admissible pairs is finite, we have $\delta_0 > 0$. □

The next result shows that if $\mathrm{E}^+(\mathcal{F})$ is non-empty, then there are words in $\mathcal{G}_{\mathcal{F}}$ of arbitrarily long length, along which the holonomy is "uniformly expansive". That is, there exists a constant $\lambda_* > 0$ such that for such a word $\mathbf{h}_n$ defined by a plaque chain $\mathcal{P}$ of length n, then $\mathbf{h}'_n(y) \geq \exp\{n\lambda_*)$ for all $y \in \mathcal{D}_{\mathcal{P}}$. The proof is technical, but also notable as it develops a version for pseudogroup actions of the Pliss Lemma, which is used in the study of the dynamics of partially hyperbolic diffeomorphisms, as for example in [5, 55, 64].

Note that Definition 4.3 implies that the set $\mathrm{E}^+(\mathcal{F})$ is an increasing union of the sets $\mathrm{E}^+_a(\mathcal{F})$ for $a > 0$, and thus given $\xi \in \mathrm{E}^+(\mathcal{F})$, there exist $a > 0$ such that $\xi \in \mathrm{E}^+_a(\mathcal{F})$.

We introduce a convenient notation for working with the set $\mathrm{E}^+_a(\mathcal{F})$. For each $1 \leq \alpha \leq \nu$, let

$$\begin{aligned} \mathrm{E}^+_a(\mathcal{F}) \cap \mathcal{T}_\alpha &= \pi_\alpha(\mathrm{E}^+_a(\mathcal{F}) \cap U_\alpha) \subset \mathcal{T}_\alpha \\ \mathrm{E}^+_a(\mathcal{F}) \cap \mathcal{T} &= (\mathrm{E}^+_a(\mathcal{F}) \cap \mathcal{T}_1) \cup \cdots \cup (\mathrm{E}^+_a(\mathcal{F}) \cap \mathcal{T}_\nu). \end{aligned}$$

Recall that the transversals $\mathfrak{X}_\alpha$ and their images $\mathcal{T}_\alpha$ in the coordinates U_α were defined in (4).

PROPOSITION 5.3. *Let $x \in \mathrm{E}^+_a(\mathcal{F}) \cap \mathcal{T}$ for $a > 0$, let $0 < \epsilon_1 < \min\{\epsilon_0, a/100\}$, and let δ_0 be the logarithmic modulus of continuity for $\mathcal{G}_\mathcal{F}$ with respect to ϵ_1, as chosen in Lemma 5.2.*

Then for each integer $n > 0$, there exist a point $y_n \in \mathcal{G}_\mathcal{F}(x)$, a closed interval $I^x_n \subset \widetilde{\mathcal{T}}_\alpha$ containing x in its interior, and a holonomy map $\mathbf{h}^x_n \colon I^x_n \to J^x_n$ such that for $y_n = \mathbf{h}^x_n(x)$, $J^x_n = [y_n - \delta_0/2, y_n + \delta_0/2] \subset \widetilde{\mathcal{T}}$ and $I^x_n = (\mathbf{h}^x_n)^{-1}(J^x_n)$, we have

$$(\mathbf{h}^x_n)'(z) > \exp\{na/2\} \text{ for all } z \in I^x_n \ . \tag{60}$$

It follows that $|I^x_n| < \delta_0 \exp\{-na/2\}$. This is illustrated in Figure 4.

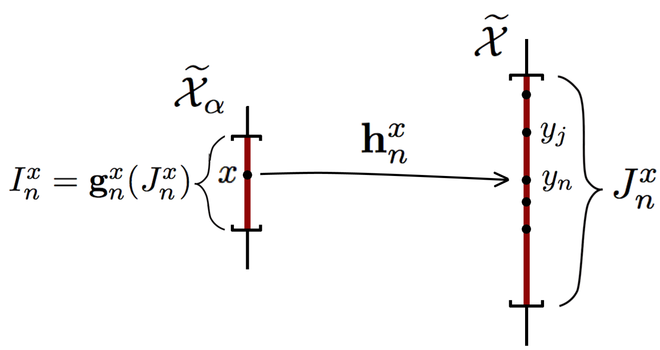

FIGURE 4. Expanding holonomy map $\mathbf{h}^x_n$

Proof. Fix a choice of $0 < \epsilon_1 < \min\{\epsilon_0, a/100\}$, and then choose a logarithmic modulus of continuity $\delta_0 > 0$ as in Lemma 5.2.

The set $\mathcal{T}_{\alpha\beta}[\epsilon_1]$, as defined in (10) for $\delta = \epsilon_1$, is compact, so there exists $C_0 > 0$ so that for all (α, β) admissible and $y \in \mathcal{T}_{\alpha\beta}[\epsilon_1]$, we have $1/C_0 \leq \widetilde{\mathbf{h}}'_{\beta,\alpha}(y) \leq C_0$.

From the definition of $\lambda_*(x)$ as a lim sup in (33), the assumption that $\lambda_*(x) > a$ implies that for each integer $n > 0$, we can choose a plaque chain of length $\ell_n \geq n$, given by $\mathcal{P}_n = \{\mathcal{P}_{\alpha_0}(z_0), \ldots, \mathcal{P}_{\alpha_{\ell_n}}(z_{\ell_n})\}$ with $z_0 = x$, such that $\log\{\mathbf{h}'_{\mathcal{P}_n}(z_0)\} > \ell_n \cdot a$. Fix n and the choice of the plaque chain $\mathcal{P}_n$ as above.

For each $1 \leq j \leq \ell_n$ let $\mathbf{h}_{\alpha_j,\alpha_{j-1}}$ be the holonomy transformation defined by $\{\mathcal{P}_{\alpha_{j-1}}, \mathcal{P}_{\alpha_j}\}$, and so $\mathbf{h}^{-1}_{\alpha_j,\alpha_{j-1}} = \mathbf{h}_{\alpha_{j-1},\alpha_j}$. Introduce the notation $\widehat{\mathbf{h}}_0 = Id$, and for $1 \leq j \leq \ell_n$ let

$$\widehat{\mathbf{h}}_j = \mathbf{h}_{\alpha_j,\alpha_{j-1}} \circ \cdots \circ \mathbf{h}_{\alpha_1,\alpha_0} \tag{61}$$

denote the partial composition of generators. Note that $z_j = \widehat{\mathbf{h}}_j(z_0)$ and $z_0 = x$, and that we have the relations $\widehat{\mathbf{h}}_{j+1} = \mathbf{h}_{\alpha_{j+1},\alpha_j} \circ \widehat{\mathbf{h}}_j$ and $z_{j+1} = \mathbf{h}_{\alpha_{j+1},\alpha_j}(z_j)$ for $0 \leq j < \ell_n$. For each $1 \leq j \leq \ell_n$, set

$$\lambda_j = \log\{\widehat{\mathbf{h}}'_{\alpha_{j-1},\alpha_j}(z_j)\} = -\log\{\widehat{\mathbf{h}}'_{\alpha_j,\alpha_{j-1}}(z_{j-1})\} \ . \tag{62}$$

In particular, $\log\{\widehat{\mathbf{h}}'_{\ell_n}(x)\} = -(\lambda_1 + \cdots + \lambda_{\ell_n})$. Note that if $\lambda_j < 0$ then the map $\widehat{\mathbf{h}}_{\alpha_{j-1},\alpha_j}$ is an infinitesimal contraction at z_j, and $\widehat{\mathbf{h}}_{\alpha_j,\alpha_{j-1}}$ is an infinitesimal expansion at z_{j-1}.

The following algebraic definition and lemma provide the key to the analysis of the hyperbolic expansion properties of the partial compositions of the maps $\widehat{\mathbf{h}}_j$.

DEFINITION 5.4. *Let $\{\lambda_1, \ldots, \lambda_m\}$ be given, and $\vartheta > 0$. An index $1 \leq j \leq m$ is said to be* ϑ-regular *if the following sequence of partial sum estimates hold:*

$$\begin{aligned} \lambda_j + \ \vartheta &< 0 \\ \lambda_{j-1} + \lambda_j + 2\vartheta &< 0 \\ &\vdots \\ \lambda_1 + \cdots + \lambda_j + j\vartheta &< 0\ . \end{aligned} \tag{63}$$

Condition (63) is a weaker hypothesis than assuming the uniform estimates $\lambda_i < -\vartheta$ for all $1 \leq i \leq j$, but is sufficient for our purposes. The next result shows that ϑ-regular indices always exist.

LEMMA 5.5. *Assume there are given real numbers $\{\lambda_1, \ldots, \lambda_m\}$ such that*

$$\lambda_1 + \cdots + \lambda_m \leq -a\, m\ . \tag{64}$$

Then for any $0 < \epsilon_1 < a$, there exists an ϵ_1-regular index q_m, for some $1 \leq q_m \leq m$, which satisfies

$$\lambda_1 + \cdots + \lambda_{q_m}\ \leq\ (-a + \epsilon_1)\, m\ . \tag{65}$$

Proof. The existence of the index q_m satisfying this property is shown by contradiction. We introduce the concept of an ϵ_1-*irregular* index, for which the ϵ_1-*regular* condition fails, and show by contradiction that not all indices can be ϵ_1-irregular.

We say that an index $k \leq m$ is ϵ_1-*irregular* if

$$\lambda_k + \cdots + \lambda_m + (m - k + 1)\epsilon_1 \geq 0\ . \tag{66}$$

If there is no irregular index, then observe that $q_m = m$ is an ϵ_1-regular index. Otherwise, suppose that there exists some index k which is ϵ_1-irregular. The inequality (64) states that the index $k = 1$ is not ϵ_1-irregular. Let $j_m \leq m$ be the least ϵ_1-irregular index, so that

$$\lambda_{j_m} + \cdots + \lambda_m + (m - j_n + 1)\epsilon_1 \geq 0 \tag{67}$$

By (64), $j_m = 1$ is is not ϵ_1-irregular, so we have $2 \leq j_m \leq m$.

Set $q_m = j_m - 1$, then we claim that q_m is an ϵ_1-regular index. If not, then at least one of the inequalities in (63) must fail to hold. That is, there is some $i \leq q_m$ with

$$\lambda_i + \cdots + \lambda_{q_m} + (q_m - i + 1)\epsilon_1 \geq 0\ . \tag{68}$$

Add the inequalities (67) and (68), and noting that $q_m = j_m - 1$, we obtain that i is also an ϵ_1-irregular index. As $i < j_m$, this is contrary to the choice of j_m. Hence, q_m is an ϵ_1-regular index.

It remains to show that the estimate (65) holds. As $j_m = q_m + 1$ is irregular, subtract (66) for $k = j_m$ from (64) to obtain

$$\lambda_1 + \cdots + \lambda_{q_m} \leq -am + (m - q_m)\epsilon_1 \leq (-a + \epsilon_1)\, m$$

as claimed. □

We return to considering the maps $\widehat{\mathbf{h}}_j$ defined by (61), and the exponents λ_j defined by (62). The following result then follows directly from Lemma 5.5 and the definitions.

COROLLARY 5.6. *Assume that there is given $a > 0$ with $x \in \mathrm{E}_a^+(\mathcal{F}) \cap \mathcal{T}$, a choice of integer $n > 0$, and plaque-chain $\mathcal{P}_n = \{\mathcal{P}_{\alpha_0}(z_0), \ldots, \mathcal{P}_{\alpha_{\ell_n}}(z_{\ell_n})\}$ with $\ell_n \geq n$, such that $\log\{\mathbf{h}'_{\mathcal{P}_n}(z_0)\} \geq \ell_n \cdot a$ and $z_0 = x$. Given $0 < \epsilon_1 < a$, by Lemma 5.5 there exists an ϵ_1-regular index q_n, for some $1 \leq q_n \leq \ell_n$ chosen as in Lemma 5.5, such that for the map $\widehat{\mathbf{h}}_{q_n}$ defined by (61),*

$$\log\{\widehat{\mathbf{h}}'_{q_n}(x)\} \geq (a - \epsilon_1)\, \ell_n \geq (a - \epsilon_1)\, n\ . \tag{69}$$

The estimate (69) can be interpreted as stating that "most" of the infinitesimal expansion of the map $\widehat{\mathbf{h}}_{\ell_n}$ at z_0 is achieved by the action of the partial composition $\widehat{\mathbf{h}}_{q_n}$.

Recall that we have a fixed choice of $0 < \epsilon_1 < \min\{\epsilon_0, a/100\}$, as given in the statement of Proposition 5.3, and $\delta_0 > 0$ is chosen so that the uniform continuity estimate (59) in Lemma 5.2 is satisfied.

Then let $1 \leq q_n \leq \ell_n$ be the ϵ_1-regular index defined in Lemma 5.5 which satisfies (65). We next use the ϵ_1-regular condition to obtain uniform estimates on the domains for which the inverses $\widehat{\mathbf{h}}_j^{-1}$ are contracting, for $1 \leq j \leq q_n$.

Recall that $\widetilde{\mathbf{h}}_{\beta,\alpha}$ denotes the continuous extension of the map $\mathbf{h}_{\beta,\alpha}$ to the domain $\widetilde{\mathcal{T}}_{\alpha\beta}$. Introduce extensions $\mathbf{h}_n^x$ of $\widehat{\mathbf{h}}_{q_n}$ and $\mathbf{g}_n^x$ of its inverse $\widehat{\mathbf{h}}_{q_n}^{-1}$, which are defined by

$$\mathbf{h}_n^x = \widetilde{\mathbf{h}}_{\alpha_{q_n},\alpha_{q_n-1}} \circ \cdots \circ \widetilde{\mathbf{h}}_{\alpha_1,\alpha_0} \tag{70}$$

$$\mathbf{g}_n^x = (\mathbf{h}_n^x)^{-1} = \widetilde{\mathbf{h}}_{\alpha_0,\alpha_1} \circ \cdots \circ \widetilde{\mathbf{h}}_{\alpha_{q_n-1},\alpha_{q_n}} \ . \tag{71}$$

Set $y_n = \mathbf{h}_n^x(x) = z_{\ell_n}$, then by the estimate (69) we have

$$\log\{(\mathbf{g}_n^x)'(y_n)\} \ = \ \lambda_1 + \cdots + \lambda_{q_n} \ \leq \ (-a + \epsilon_1)\, \ell_n < 0 \ . \tag{72}$$

We next show that $\mathbf{g}_n^x$ is uniformly contracting on an interval with uniform length about y_n.

LEMMA 5.7. *Set $\delta_0' = \delta_0/8$. Then the interval $J_n^x = [y_n - 4\delta_0', y_n + 4\delta_0']$ is in the domain of $\mathbf{g}_n^x$, and for all $y \in J_n^x$,*

$$\exp\{(-a - 2\epsilon_1)\, \ell_n\} \leq (\mathbf{g}_n^x)'(y) \leq \exp\{(-a + 2\epsilon_1)\, \ell_n\} \ . \tag{73}$$

Hence, for $I_n^x = \mathbf{g}_n^x(J_n^x)$,

$$|I_n^x| \leq \delta_0 \exp\{(-a + 2\epsilon_1)\, \ell_n\} < \exp\{(-a/2)\, \ell_n\} \ . \tag{74}$$

Proof. By the choice of δ_0', the uniform continuity estimate (59) implies that for all $y \in J_n^x$

$$\left| \log\{\widetilde{\mathbf{h}}'_{\alpha_{q_n-1},\alpha_{q_n}}(y)\} - \log\{\widetilde{\mathbf{h}}'_{\alpha_{q_n-1},\alpha_{q_n}}(y_n)\} \right| \leq \epsilon_1 \ .$$

Thus, by the definition of λ_{q_n} we have that, for all $y \in J_n^x$,

$$\exp\{\lambda_{q_n} - \epsilon_1\} \leq \widetilde{\mathbf{h}}'_{\alpha_{q_n-1},\alpha_{q_n}}(y) \leq \exp\{\lambda_{q_n} + \epsilon_1\} \ .$$

The assumption that q_n is ϵ_1-regular implies $\lambda_{q_n} + \epsilon_1 < 0$, hence $\exp\{\lambda_{q_n} + \epsilon_1\} < 1$. Thus, for all $y \in J_n^x$ we have

$$\mathbf{d}_{\mathcal{T}}(\widetilde{\mathbf{h}}_{\alpha_{q_n-1},\alpha_{q_n}}(y_n), \widetilde{\mathbf{h}}_{\alpha_{q_n-1},\alpha_{q_n}}(y)) \leq 4\delta_0' \exp\{\lambda_{q_n} + \epsilon_1\} < 4\delta_0' \ . \tag{75}$$

Now proceed by downward induction. For $0 < j \leq q_n$ set

$$\mathbf{g}_{n,j}^x = \widetilde{\mathbf{h}}_{\alpha_{j-1},\alpha_j} \circ \cdots \circ \widetilde{\mathbf{h}}_{\alpha_{q_n-1},\alpha_{q_n}} \ , \ \ J_{n,j}^x = \mathbf{g}_{n,j}^x(J_n^x) \ , \ \ y_{n,j} = \mathbf{g}_{n,j}^x(y_n) = z_{j-1} \ .$$

Assume that for $1 < j \leq q_n$, we are given that for all $y \in J_{n,j}^x$ the estimates

$$\exp\{\lambda_j + \cdots + \lambda_{q_n} - (q_n - j + 1)\, \epsilon_1\} \leq (\mathbf{g}_{n,j}^x)'(y) \leq \exp\{\lambda_j + \cdots + \lambda_{q_n} + (q_n - j + 1)\, \epsilon_1\} \ , \tag{76}$$

$$\mathbf{d}_{\mathcal{T}}(y, y_{n,j}) \leq 4\delta_0' \ . \tag{77}$$

The choice of δ_0 and the hypothesis (77) imply that for $y \in J_{n,j}^x$,

$$\left| \log\{\widetilde{\mathbf{h}}'_{\alpha_{j-2},\alpha_{j-1}}(y)\} - \log\{\widetilde{\mathbf{h}}'_{\alpha_{j-2},\alpha_{j-1}}(y_{n,j})\} \right| \leq \epsilon_1 \ .$$

Recall that $z_{j-1} = y_{n,j}$, and that $\lambda_{j-1} = \log\{\widetilde{\mathbf{h}}'_{\alpha_{j-2},\alpha_{j-1}}(y_{n,j})\}$ by (62), so for all $y \in J_{n,j}^x$ we have for the inverse map $\widetilde{\mathbf{h}}_{\alpha_{j-2},\alpha_{j-1}} = \widetilde{\mathbf{h}}^{-1}_{\alpha_{j-1},\alpha_{j-2}}$ that

$$\exp\{\lambda_{j-1} - \epsilon_1\} \leq \widetilde{\mathbf{h}}'_{\alpha_{j-2},\alpha_{j-1}}(y) \leq \exp\{\lambda_{j-1} + \epsilon_1\} \ . \tag{78}$$

Then by the chain rule, the estimates (78) and the inductive hypothesis (76) yield the estimates

$$\exp\{\lambda_{j-1} + \cdots + \lambda_{q_n} - (q_n - j + 2)\, \epsilon_1\} \leq (\mathbf{g}_{n,j-1}^x)'(y) \leq \exp\{\lambda_{j-1} + \cdots + \lambda_{q_n} + (q_n - j + 2)\, \epsilon_1\}. \tag{79}$$

Now the assumption that q_n is ϵ_1-regular implies $\lambda_{j-1} + \cdots + \lambda_{q_n} + (q_n - j + 2)\, \epsilon_1 < 0$ hence $\exp\{\lambda_{j-1} + \cdots + \lambda_{q_n} + (q_n - j + 2)\, \epsilon_1\} < 1$.

By the Mean Value Theorem, this yields the distance bound $\mathbf{d}_{\mathcal{T}}(y_{n,j-1}, y) \leq 4\delta_0'$, which is the hypothesis (77) for $j-1$. This completes the inductive step. Thus, we may take $j = 1$ in inequality (76) and combined with the inequality (65), for all $y \in J_n^x$ we have that

$$(80) \quad (\mathbf{g}_n^x)'(y) \leq \exp\{\lambda_1 + \cdots + \lambda_{q_n} + q_n\,\epsilon_1\} \leq \exp\{-a\,\ell_n + (\ell_n + q_n)\,\epsilon_1\} \leq \exp\{(-a + 2\epsilon_1)\,\ell_n\}\ .$$

Set $I_n^x = \mathbf{g}_n^x(J_n^x)$, then the estimate (74) follows by the Mean Value Theorem. □

Since $a - 2\epsilon_1 > a/2$ and $\ell_n \geq n$, this completes the proof of Proposition 5.3. □

5.2. **Hyperbolic fixed-points.** We show the existence of hyperbolic fixed-points for $\mathcal{G}_{\mathcal{F}}$ contained in the closure of $\mathrm{E}^+(\mathcal{F}) \cap \mathcal{T}$ in $\widetilde{\mathcal{T}}$, with uniform estimates on the lengths of their domains of contraction.

PROPOSITION 5.8. *Let $x \in \mathrm{E}_a^+(\mathcal{F}) \cap \mathcal{T}$ for $a > 0$, let $0 < \epsilon_1 < \min\{\epsilon_0, a/100\}$, and let δ_0 be chosen as in Lemma 5.2, and set $\delta_0' = \delta_0/8$. Given $0 < \delta_1 < \delta_0'$ and $0 < \mu < 1$, then there exists holonomy maps $\phi_1, \psi_1 \in \mathcal{G}_{\mathcal{F}}$, points $u_1, v_1 \in \overline{\mathcal{T}}$ such that $\mathbf{d}_{\mathcal{T}}(x, v_1) < \delta_1$, such that we have:*

(1) $\Phi_1 = \phi_1 \circ \psi_1$ has fixed point $\Phi_1(u_1) = u_1$;
(2) $\mathcal{J}_1 \equiv [u_1 - \delta_0', u_1 + \delta_0']$ is contained in the domain of Φ_1;
(3) $\Phi'(y) < \mu$ for all $y \in \mathcal{J}_1$;
(4) $\Psi_1 = \psi_1 \circ \phi_1$ has fixed point $\Psi_1(v_1) = v_1$;
(5) $\mathcal{K}_1 \equiv \psi_1(\mathcal{J}_1) \subset (x - \delta_1, x + \delta_1)$.

Proof. The idea of the proof is to consider a sequence of maps as given by Proposition 5.3, for $n \geq 1$, and consider a subsequence of these for which the sequence of points $\{y_n = \mathbf{h}_n^x(x) = z_{\ell_n} \mid n \geq 1\}$ cluster at a limit point. We then use the estimates (74) on the sizes of the domains to show that the appropriate compositions of these maps are defined, and have a hyperbolic fixed point. The details of this argument follow.

Set $\delta_* = \min\{1, \delta_0'/4, \delta_1/4\}$. Then by Proposition 5.3, for each integer $n > 0$, we can choose a map $\mathbf{h}_n^x \colon I_n^x \to J_n^x$ as in (70), which satisfies condition (60). Label the resulting sequence of points $y_n = \mathbf{h}_n^x(x) \in \mathcal{T}$, and the inverse maps $\mathbf{g}_n^x = (\mathbf{h}_n^x)^{-1}$. Let p_n denote the length of the plaque chain defining $\mathbf{h}_n^x$, then p_n equals the ϵ_1-regular index $1 \leq q_n \leq \ell_n$ chosen as in the proof of Corollary 5.6.

Recall that $\mathcal{T}$ has compact closure in $\widetilde{\mathcal{T}}$, so there exists an accumulation point $y_* \in \overline{\mathcal{T}} \subset \widetilde{\mathcal{T}}$ for the set $\{y_n \mid n > 0\} \subset \mathcal{T}$. We can assume that $\mathbf{d}_{\mathcal{T}}(y_*, y_n) < \delta_*/4$ for all $n > 0$, first by passing to a subsequence $\{y_{n_i}\}$ which converges to y_* and satisfies this metric estimate, and then reindexing the sequence.

Let $J_n^x = [y_n - 4\delta_0', y_n + 4\delta_0']$, and set $J_* = [y_* - 3\delta_0', y_* + 3\delta_0']$. Then for all $n > 0$, we have $y_n \in (y_* - \delta_0', y_* + \delta_0') \subset J_* \subset J_n^x$. In particular, $y_1 \in J_* \subset J_1^x$ is an interior point of J_*, so $x = \mathbf{g}_1^x(y_1)$ is an interior point of $\mathbf{g}_1^x(J_*)$.

Also recall from Proposition 5.3, that $I_n^x = \mathbf{g}_n^x(J_n^x)$ with $x \in I_n^x$ for all n, and the interval I_n^x has length $|I_n^x| < \delta_0 \exp\{-na/2\} = 8\delta_0' \exp\{-na/2\}$. Hence, for n sufficiently large, the interval I_n^x is contained in the interior of $\mathbf{g}_1^x(J_*)$. Without loss of generality, we again pass to a subsequence and reindex the sequence, so that we have $I_n^x \subset \mathbf{g}_1^x(J_*)$ and $\ell_{n+1} > \ell_n$ for all $n > 0$. We then have the inclusions

$$(81) \qquad \mathbf{g}_n^x(J_*) \subset \mathbf{g}_n^x(J_n^x) = I_n^x \subset \mathbf{g}_1^x(J_*)\ .$$

Thus, for each $n > 0$ the composition $\mathbf{h}_1^x \circ \mathbf{g}_n^x \colon J_* \to \mathbf{h}_1^x \circ \mathbf{g}_1^x(J_*) \subset J_*$ is defined. (See Figure 5.)

Recall that p_1 denotes the length of the plaque-chain which defines $\mathbf{h}_1^x$, and C_0 is the Lipschitz constant defined in the proof of Proposition 5.3. Let N_0 be chosen so that for $n \geq N_0$ we have

$$(82) \qquad C_0^{p_1}\ \exp\{-a\,n/2\} \quad < \quad \min\{\mu, 1/2\}$$

$$(83) \qquad \delta_0'\ \exp\{-a\,n/2\} \quad < \quad \delta_1/2\ .$$

With the above notations, we then have:

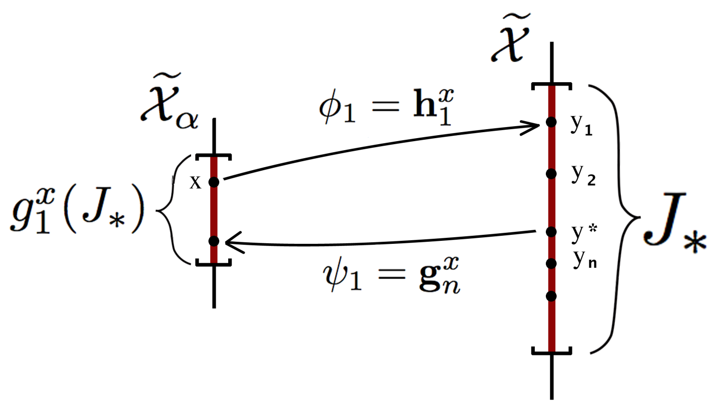

FIGURE 5. The contracting holonomy map $\mathbf{h}_1^x \circ \mathbf{g}_n^x$

LEMMA 5.9. *Fix $n \geq N_0$, then the map $\mathbf{h}_1^x \circ \mathbf{g}_n^x$ is a hyperbolic contraction on J_* with fixed-point $v_* \in J_*$ satisfying $\mathbf{d}_{\mathcal{T}}(v_*, y_n) \leq \delta_1/2$ and $(\mathbf{h}_1^x \circ \mathbf{g}_n^x)'(v_*) < \mu$.*

Proof. By the choice of C_0 we have $(\mathbf{h}_1^x)'(y) \leq C_0^{p_1}$ for all y in its domain. Recall that $\mathbf{g}_n^x$ is the inverse of $\mathbf{h}_n^x$ which is defined by a plaque-chain of length $\ell_n \geq n$, so the same holds for $\mathbf{g}_n^x$. The derivative of $\mathbf{g}_n^x$ satisfies the estimates (73) by Lemma 5.7, so we have

$$\exp\{(-a - 2\epsilon_1)\,\ell_n\} \leq (\mathbf{g}_n^x)'(y) \leq \exp\{(-a + 2\epsilon_1)\,n\}\ . \tag{84}$$

Thus by (82), for all $y \in J_*$ the composition $\mathbf{h}_1^x \circ \mathbf{g}_n^x$ satisfies

$$(\mathbf{h}_1^x \circ \mathbf{g}_n^x)'(y) \leq C_0^{p_1}\ \exp\{(-a + 2\epsilon_1)\,n\} < C_0^{p_1}\ \exp\{-a\,n/2\} < \min\{\mu, 1/2\} \tag{85}$$

where we use that the choice of $\epsilon_1 < a/100$ implies that $(-a + 2\epsilon_1) < -a/2$. Thus, $\mathbf{h}_1^x \circ \mathbf{g}_n^x$ is a hyperbolic contraction on J_* and it follows that $\mathbf{h}_1^x \circ \mathbf{g}_n^x$ has a unique fixed-point $v_* \in J_*$. Define a sequence of points $w_\ell = (\mathbf{h}_1^x \circ \mathbf{g}_n^x)^\ell(y_n) \in J_*$ for $\ell \geq 0$, then $v_* = \lim\limits_{\ell \to \infty}\ w_\ell$.

Observe that $\mathbf{h}_1^x \circ \mathbf{g}_n^x(y_n) = \mathbf{h}_1^x(x) = y_1$, and recall that $\mathbf{d}_{\mathcal{T}}(y_*, y_n) < \delta_*/4$ for all n, hence, $\mathbf{d}_{\mathcal{T}}(y_1, y_n) < \delta_*/2$. Since $w_0 = y_n$ and $w_1 = y_1$, the estimate (85) implies that

$$\mathbf{d}_{\mathcal{T}}(w_\ell, w_{\ell+1}) < 2^{-\ell} \cdot \mathbf{d}_{\mathcal{T}}(w_0, w_1) < 2^{-\ell} \cdot \delta_*/2\ .$$

Summing these estimates for $\ell \geq 1$, we obtain that $\mathbf{d}_{\mathcal{T}}(w_0, v_*) = \mathbf{d}_{\mathcal{T}}(y_n, v_*) \leq \delta_*$ so that

$$\mathbf{d}_{\mathcal{T}}(y_*, v_*) \leq \mathbf{d}_{\mathcal{T}}(y_*, y_n) + \mathbf{d}_{\mathcal{T}}(y_n, v_*) < 2\delta_* \leq \delta_1/2\ . \tag{86}$$

Then by (85) we have $(\mathbf{h}_1^x \circ \mathbf{g}_n^x)'(v_*) \leq \mu$, as was to be shown. □

The conclusions of Lemma 5.9 essentially yield the proof of Proposition 5.8, except that it remains to make a change of notation so the results are in the form stated in the proposition, and check that conditions (1) to (5) of Proposition 5.8.1 are satisfied. This change of notation is done so that the conclusions are in a standard format, which will be invoked recursively in the following Section 6 to prove there exists "ping-pong" dynamics in the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$.

Choose $n \geq N_0$ so that the hypotheses of Lemma 5.9 are satisfied, then define $\phi_1 = \mathbf{h}_1^x$ and $\psi_1 = \mathbf{g}_n^x$ so that $\Phi_1 = \phi_1 \circ \psi_1 = \mathbf{h}_1^x \circ \mathbf{g}_n^x$, and recall that $g_1^x(J_*)$

$$J_* = [y_* - 3\delta_0', y_* + 3\delta_0'] \subset J_n^x = [y_n - 4\delta_0', y_n + 4\delta_0'] \tag{87}$$

for δ_0' and y_* as defined above. Set $u_1 = v_*$ and $v_1 = \mathbf{g}_n^x(v_*)$.

We check that conditions (5.8.1) and (5.8.4) of Proposition 5.8 are satisfied:

$$\Phi_1(u_1) = \phi_1 \circ \psi_1(u_1) = \mathbf{h}_1^x \circ \mathbf{g}_n^x(v_*) = v_* = u_1\ ,$$

$$\Psi_1(v_1) = \psi_1 \circ \phi_1(v_1) = \mathbf{g}_n^x \circ \mathbf{h}_1^x(\mathbf{g}_n^x(v_*)) = \mathbf{g}_n^x(v_*) = v_1\ .$$

Next, for $\mathcal{J}_1 = [u_1 - \delta_0', u_1 + \delta_0'] = [v_* - \delta_0', v_* + \delta_0']$ as defined in (5.8.2), by the estimate (86) we have $\mathbf{d}_{\mathcal{T}}(y_*, v_*) < 2\delta_* \leq \delta_0'/2$ from which it follows that $\mathcal{J}_1 \subset J_*$. Then condition (5.8.3) follows from (85) since $u_1 = v_* \in J_*$.

Finally, to show condition (5.8.5) of Proposition 5.8 is satisfied, recall that $\psi_1(y_n) = \mathbf{g}_1^x(y_n) = x$, that $\mathbf{d}_{\mathcal{T}}(y_n, v_*) < \delta_* \leq 1$ by the proof of Lemma 5.9, and that $\delta_* = \min\{1, \delta_0'/4, \delta_1/4\}$. Also, the estimate (73) combined with (83) and the choice of $\delta_0' \leq 1$ in Definition 5.1 yields that, for all $y \in J_*$

$$(\mathbf{g}_n^x)'(y) \leq \exp\{(-a + 2\epsilon_1)\,\ell_n\} < \delta_0' \cdot \exp\{-a\,n/2\} < \delta_1/2 \ . \tag{88}$$

Thus, by the Mean Value Theorem and the estimate $\mathbf{d}_{\mathcal{T}}(y_n, v_*) \leq \delta_* \leq 1$, we have that

$$\mathbf{d}_{\mathcal{T}}(x, v_1) = \mathbf{d}_{\mathcal{T}}(\mathbf{g}_n^x(y_n), \mathbf{g}_n^x(v_*)) \leq \delta_1/2 \cdot \mathbf{d}_{\mathcal{T}}(y_n, v_*) \leq \delta_1/2 \ .$$

For any $y \in \mathcal{J}_1 = [v_* - \delta_0', v_* + \delta_0']$ we also have that

$$\mathbf{d}_{\mathcal{T}}(\mathbf{g}_n^x(y), v_1) = \mathbf{d}_{\mathcal{T}}(\mathbf{g}_n^x(y), \mathbf{g}_n^x(v_*)) \leq \delta_1/2 \cdot \mathbf{d}_{\mathcal{T}}(y, v_*) \leq \delta_0'\delta_1/2 < \delta_1/2 \ .$$

Thus,

$$\mathbf{d}_{\mathcal{T}}(\mathbf{g}_n^x(y), x) \leq \mathbf{d}_{\mathcal{T}}(\mathbf{g}_n^x(y), v_1) + \mathbf{d}_{\mathcal{T}}(x, v_1) < \delta_1 \ ,$$

so that $\mathcal{K}_1 = \psi_1(\mathcal{J}_1) \subset [x - \delta_1, x + \delta_1]$, as was to shown.

This completes the proof of Proposition 5.8. □

6. Hyperbolic sets with positive measure

The main result of this section is:

THEOREM 6.1. *Let $\mathcal{F}$ be a C^1-foliation of codimension-one of a compact manifold M for which $\mathrm{E}^+(\mathcal{F})$ has positive Lebesgue measure. Then $\mathcal{F}$ has a hyperbolic resilient leaf, and hence the geometric entropy $h(\mathcal{F}) > 0$.*

The assumption that the Lebesgue measure $|\mathrm{E}^+(\mathcal{F})| > 0$ is used in two ways. First, the set $\mathrm{E}^+(\mathcal{F})$ is an increasing union of the sets $\mathrm{E}_a^+(\mathcal{F})$ for $a > 0$, so $|\mathrm{E}^+(\mathcal{F})| > 0$ implies $|\mathrm{E}_a^+(\mathcal{F})| > 0$ for some $a > 0$. For each $x \in \mathrm{E}_a^+(\mathcal{F})$, we obtain from Proposition 5.8 uniform hyperbolic contractions with fixed-points arbitrarily close to the given $x \in E$, and with prescribed bounds on their domains.

Secondly, almost every point of a measurable set is a point of positive Lebesgue density, hence $|\mathrm{E}_a^+(\mathcal{F})| > 0$ implies that $\mathrm{E}_a^+(\mathcal{F})$ has a "pre-perfect" subset of points with expansion greater than a. This observation enables us to construct an infinite sequence of hyperbolic fixed-points arbitrarily close to the support of $\mathrm{E}_a^+(\mathcal{F})$, whose domains have to eventually overlap since the closure $\overline{\mathcal{T}}$ is compact. This yields the existence of a resilient orbit for $\mathcal{G}_{\mathcal{F}}$, hence a ping-pong game dynamics as defined in Section 2.4, which implies that $h(\mathcal{F}) > 0$.

DEFINITION 6.2. *A set $\mathcal{E}$ is said to be* pre-perfect *if it is non-empty, and its closure $\overline{\mathcal{E}}$ is a perfect set. Equivalently, $\mathcal{E}$ is pre-perfect if it is not empty, and no point is isolated.*

The following observation is a standard property of sets with positive Lebesgue measure.

LEMMA 6.3. *If $X \subset \mathbb{R}^q$ has positive Lebesgue measure, then there is a pre-perfect subset $\mathcal{E} \subset X$.*

Proof. Let $\mathcal{E} \subset X$ be the set of points with Lebesgue density 1. Recall that this means that for each $x \in X$ and each $\delta > 0$, the Lebesgue measure $|B_X(x,\delta) \cap X| > 0$, and $\displaystyle\lim_{\delta \to 0} \frac{|B_X(x,\delta) \cap X|}{|B_X(x,\delta)|} = 1$.

It is a standard fact of Lebesgue measure theory that $|\mathcal{E}| = |X|$, so that $|X| > 0$ implies that $\mathcal{E} \neq \emptyset$. Moreover, if $x \in \mathcal{E}$ is isolated in $\mathcal{E}$, then x is a point with Lebesgue density 0, thus each $x \in \mathcal{E}$ cannot be isolated. It follows that $\mathcal{E}$ is pre-perfect. □

Theorem 6.1 now follows from Lemma 6.3 and the following result:

PROPOSITION 6.4. *Let $a > 0$, and suppose there exists a pre-perfect subset $\mathcal{E} \subset \mathrm{E}_a^+(\mathcal{F})$, then $\mathcal{F}$ has a resilient leaf contained in the closure $\overline{\mathrm{E}_a^+(\mathcal{F})}$.*

Proof. Let $a > 0$ and let $\mathcal{E} \subset \mathrm{E}_a^+(\mathcal{F})$ be a pre-perfect set. The saturation of a pre-perfect set under the action of the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ is pre-perfect, so we can assume that $\mathcal{E}$ is saturated. We assume that $\mathcal{F}$ does not have a resilient leaf in $\overline{\mathrm{E}_a^+(\mathcal{F})}$, and show this leads to a contradiction.

We follow the notation introduced in the proof of Proposition 5.8, which will be invoked repeatedly, and the resulting maps and constants will be labeled according to the stage of the induction. Choose $0 < \epsilon_1 < \min\{\epsilon_0, a/100\}$, and let δ_0 be chosen as in Definition 5.1.

Fix a choice of $0 < \mu < 1$, and choose $0 < \delta_1 < \delta_0$ and $x_1 \in \mathcal{E} \cap \mathcal{T}_\alpha$. Then by Proposition 5.8, there exists holonomy maps $\phi_1, \psi_1 \in \mathcal{G}_{\mathcal{F}}$ and points $u_1 \in \overline{\mathcal{T}}$ and $v_1 = \psi_1(u_1)$, such that $\mathbf{d}_{\mathcal{T}}(x_1, v_1) < \delta_1$ and which are fixed-points for the maps Φ_1, Ψ_1 respectively. Moreover, we have the sets

(1) $\mathcal{J}_1 \equiv [u_1 - \delta_0, u_1 + \delta_0]$
(2) $\mathcal{I}_1 \equiv \Phi_1(\mathcal{J}_1) \subset (u_1 - \delta_0, u_1 + \delta_0)$
(3) $\mathcal{K}_1 \equiv \psi_1(\mathcal{J}_1) \subset (x_1 - \delta_1, x_1 + \delta_1)$

whose properties were given in Proposition 5.8. In particular, $\Phi_1 \colon \mathcal{J}_1 \to \mathcal{I}_1 \subset \mathcal{J}_1$ is a hyperbolic contraction with fixed-point u_1. In particular, note that $\bigcap\limits_{\ell > 0} \Phi_1^\ell(\mathcal{J}_1) = \{u_1\}$.

If the orbit of u_1 under $\mathcal{G}_{\mathcal{F}}$ intersects $\mathcal{J}_1$ in a point other than u_1, then by definition, u_1 is a hyperbolic resilient point, which by assumption does not exist. Therefore, the $\mathcal{G}_{\mathcal{F}}$-orbit of u_1 intersects the interval $\mathcal{J}_1$ exactly in the interior point u_1, and intersects $\mathcal{K}_1$ exactly in the interior point v_1.

Note that $x_1 \in \mathcal{K}_1 \cap \mathcal{E}$ so there exists $x_2 \in (\mathcal{K}_1 - \{x_1, v_1\}) \cap \mathcal{E}$ as $\mathcal{E}$ is pre-perfect. Choose $0 < \delta_2 < \delta_1$ so that $(x_2 - \delta_2, x_2 + \delta_2) \subset (\mathcal{K}_1 - \{x_1, v_1\})$. The $\mathcal{G}_{\mathcal{F}}$-orbit of v_1 intersects $\mathcal{K}_1$ only in the point v_1, thus the interval $(x_2 - \delta_2, x_2 + \delta_2)$ is disjoint from the orbit of v_1. We then repeat the construction in the proof of Proposition 5.8, to obtain holonomy maps $\phi_2, \psi_2 \in \mathcal{G}_{\mathcal{F}}$ and points $u_2 \in \overline{\mathcal{T}}$ and $v_2 = \psi_2(u_2)$, such that $\mathbf{d}_{\mathcal{T}}(x_2, v_2) < \delta_2$ and which are fixed-points for the maps Φ_2, Ψ_2 respectively. Again, define the sets

(1) $\mathcal{J}_2 \equiv [u_2 - \delta_0, u_2 + \delta_0]$
(2) $\mathcal{I}_2 \equiv \Phi_2(\mathcal{J}_2) \subset (u_2 - \delta_0, u_2 + \delta_0)$
(3) $\mathcal{K}_2 \equiv \psi_2(\mathcal{J}_2) \subset [x_2 - \delta_2, x_2 + \delta_2]$.

We then repeat this construction recursively. Let $\{u_1, u_2, \ldots\} \subset \overline{\mathcal{T}}$ be the resulting centers of contraction for the hyperbolic maps $\{\Phi_i \mid i > 0\}$. As $\overline{\mathcal{T}}$ is compact, there exists an accumulation point $u_* \in \overline{\mathcal{T}}$. In particular, there exists distinct indices $i_1, i_2 > 0$ such that $\mathbf{d}_{\mathcal{T}}(u_*, u_{i_1}) < \delta_0/10$ and $\mathbf{d}_{\mathcal{T}}(u_*, u_{i_2}) < \delta_0/10$ and thus $\mathbf{d}_{\mathcal{T}}(u_{i_1}, u_{i_2}) < \delta_0/5$.

Recall that the intervals $\mathcal{J}_{i_1} = [u_{i_1} - \delta_0, u_{i_1} + \delta_0]$ and $\mathcal{J}_{i_2} = [u_{i_2} - \delta_0, u_{i_2} + \delta_0]$ have uniform width, and moreover $\{u_{i_1}, u_{i_2}\} \subset \mathcal{J}_{i_1} \cap \mathcal{J}_{i_2}$. As u_{i_1} and u_{i_2} are disjoint fixed-points of hyperbolic attractors, we can choose integers $m_1, m_2 > 0$ so that $\Phi_{i_1}^{m_1}(\mathcal{J}_{i_1}) \cap \Phi_{i_2}^{m_2}(\mathcal{J}_{i_2}) = \emptyset$ and $\Phi_{i_j}^{m_j}(\mathcal{J}_{i_j}) \subset \mathcal{J} = \mathcal{J}_{i_1} \cap \mathcal{J}_{i_2}$ for $j = 1, 2$. Then the action of the contracting maps $\mathbf{H} = \Phi_{i_1}^{m_1}$ and $\mathbf{G} = \Phi_{i_2}^{m_2}$ on $\mathcal{J}$ define a "ping-pong game" as in Definition 2.4.

Now let $x = u_{i_1}$, $y = \mathbf{G}(x) \neq x$, then $\mathbf{H}^\ell(y) \to x$ as $\ell \to \infty$, so that the orbit of x under the action $\mathcal{G}_{\mathcal{F}}$ is resilient, contrary to assumption.

Hence, if there exists a pre-perfect set $\mathcal{E} \subset \mathrm{E}_a^+(\mathcal{F})$ for $a > 0$, then there exists a resilient leaf. □

7. Open manifolds

In this section, we extend the methods above from compact manifolds to open manifolds, using the techniques of [38, Section 5].

THEOREM 7.1. *Let $\mathcal{F}$ be a codimension-one C^2-foliation of an open complete manifold M. If the Godbillon-Vey class $GV(\mathcal{F}) \in H^3(M; \mathbb{R})$ is non-zero, the $\mathcal{F}$ has a hyperbolic resilient leaf.*

Proof. The class $GV(\mathcal{F}) \in H^3(M; \mathbb{R})$ is determined by its pairing with the compactly supported cohomology group $H_c^{m-3}(M; \mathbb{R})$, so $GV(\mathcal{F}) \neq 0$ implies there exists a closed $m - 3$ form ξ with compact support on M such that $\langle GV(\mathcal{F}), [\xi] \rangle \neq 0$. Let $|\xi| \subset M$ denote the support of ξ, which is a compact set. As the support $|\xi|$ is compact, there is a finite open cover of $|\xi|$ by a regular foliation atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ for $\mathcal{F}$ on M (as in Section 2 above). Let M_0 denote the union of the sets $\{U_\alpha \mid \alpha \in \mathcal{A}\}$, then the closure $\overline{M_0}$ is a compact subset of M and $|\xi| \subset M_0$. Thus we have $GV(\mathcal{F}|M_0) \neq 0$. If M_0 is not connected, we can choose a connected component $M_1 \subset M_0$ for which $GV(\mathcal{F}|M_1) \neq 0$. Thus, we may assume that M_0 is connected.

The proof of Theorem 4.4 used only the properties of the pseudogroup generated by a regular foliation atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ – the compactness of M was not used except in the construction of this atlas. The definition and properties of the Godbillon measure also apply to open manifolds, as was discussed in [38, Section 5]. Hence, by the same proof we obtain that the set $\mathrm{E}^+(\mathcal{F}|M_0)$ has positive measure.

The proofs of Propositions 5.8 and 6.4 use only the assumption that the pseudogroup $\mathcal{G}_\mathcal{F}$ is compactly generated, as defined by Haefliger [32], and do not require the compactness of M, hence apply directly to show that $\mathcal{GF}|M_0$ has a hyperbolic resilient point if $\mathrm{E}(\mathcal{F}|M_0)$ has positive measure. Thus, $\mathcal{F}|M_0$ must have a resilient leaf, and so also must $\mathcal{F}$. □

Here is an application of Theorem 7.1. Let $\mathbf{B}\Gamma_1^{(2)}$ denote the Haefliger classifying space of codimension–one C^2-foliations [30, 31]. There is a universal Godbillon-Vey class $GV \in H^3(\mathbf{B}\Gamma_1^{(2)}; \mathbb{R})$ such that for every codimension–one C^2-foliation $\mathcal{F}$ of a manifold M, there is a classifying map $h_\mathcal{F} \colon M \to \mathbf{B}\Gamma_1^{(2)}$ such that $h_\mathcal{F}^* GV = GV(\mathcal{F})$ (see [6, 53].) The first two integral homotopy groups $\pi_1(\mathbf{B}\Gamma_1^{(2)}) = 0 = \pi_2(\mathbf{B}\Gamma_1^{(2)})$, while Thurston showed in [72] that the Godbillon-Vey class defines a *surjection* $GV \colon \pi_3(\mathbf{B}\Gamma_1^{(2)}) \to \mathbb{R}$. It follows from Thurston's work in [73], that for a closed oriented 3-manifold M and any $a > 0$, there exists a codimension–one foliation $\mathcal{F}_a$ on M such that $\langle GV(\mathcal{F}_a), [M]\rangle = a$. Each such foliation $\mathcal{F}_a$ for $a \neq 0$ must then have resilient leaves.

More generally, given any finite CW complex X, a continuous map $h \colon X \to \mathbf{B}\Gamma_1^{(2)}$ defines a foliated microbundle over X, whose total space M is an open manifold with a codimension–one foliation $\mathcal{F}_h$ such that $h^*GV = GV(\mathcal{F}_h)$. This is discussed in detail by Haefliger [31], who introduced the technique. (See also Lawson [53].) Thus, using homotopy methods to construct the map h so that $h^*GV \neq 0$, one can construct many examples of open foliated manifolds with non-trivial Godbillon-Vey classes. Theorem 7.1 implies that all such examples have resilient leaves.

References

[1] J. Alves, C. Bonatti, and M. Viana, *SRB measures for partially hyperbolic systems whose central direction is mostly expanding*, **Invent. Math.**, 140:351–398, 2000.

[2] L. Barreira and Ya.B. Pesin, **Introduction to smooth ergodic theory**, Graduate Studies in Mathematics, Vol. 148, American Mathematical Society, Providence, RI, 2013.

[3] R. Bishop and R. Crittenden, **Geometry of manifolds**, (reprint of the 1964 original), AMS Chelsea Publishing, Providence, RI, 2001.

[4] C. Bonatti, R. Langevin, and R. Moussu, *Feuilletages de CP(n): de l'holonomie hyperbolique pour les minimaux exceptionnels*, **Inst. Hautes tudes Sci. Publ. Math.**, 75:123–134, 1992.

[5] C. Bonatti, L. Díaz, M. Viana, **Dynamics beyond uniform hyperbolicity**, Encyclopaedia of Mathematical Sciences, Vol. 102, Springer-Verlag, Berlin, 2005.

[6] R. Bott and A. Haefliger, *On characteristic classes of Γ-foliations*, **Bull. Amer. Math. Soc.**, 78:1039–1044, 1972.

[7] R. Bott, *On some formulas for the characteristic classes of group actions*, In **Differential topology, foliations and Gelfand-Fuks cohomology (Proc. Sympos., Pontifícia Univ. Católica, Rio de Janeiro, 1976)**, Lect. Notes in Math. Vol. 652, Springer–Verlag, Berlin, 1978:25–61.

[8] C. Camacho and A. Lins Neto, **Geometric Theory of Foliations**, Translated from the Portuguese by Sue E. Goodman, Progress in Mathematics, Birkhäuser Boston, MA, 1985.

[9] A. Candel and L. Conlon, **Foliations I**, Amer. Math. Soc., Providence, RI, 2000.

[10] A. Candel and L. Conlon, **Foliations II**, Amer. Math. Soc., Providence, RI, 2003.

[11] J. Cantwell and L. Conlon, *Poincaré-Bendixson theory for leaves of codimension one*, **Transactions Amer. Math. Soc.**, 265:181–209, 1981.

[12] J. Cantwell and L. Conlon, *The dynamics of open, foliated manifolds and a vanishing theorem for the Godbillon-Vey class*, **Advances in Math.**, 53:1–27, 1984.

[13] A. Connes, *Cyclic cohomology and the transverse fundamental class of a foliation*, In **Geometric methods in operator algebras (Kyoto, 1983)**, Pitman Res. Notes Math. Ser. Vol. 123, 1986:52–144.

[14] A. Connes, **Noncommutative geometry**, Academic Press Inc., 1994.

[15] A. Connes and H. Moscovici, *Hopf algebras, cyclic cohomology and the transverse index theorem*, **Comm. Math. Phys.**, 198:199–246, 1998.

[16] P. de la Harpe, **Topics in geometric group theory**, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000.

[17] M. do Carmo, **Riemannian geometry**, Translated from the second Portuguese edition by Francis Flaherty, Birkhäuser Boston Inc., Boston, 1992.

[18] G. Duminy, *L'invariant de Godbillon–Vey d'un feuilletage se localise dans les feuilles ressort*, unpublished preprint, Université de Lille, I, 1982.

[19] G. Duminy, *Sur les cycles feuilletés de codimension un*, unpublished manuscript, Université de Lille, I, 1982.

[20] G. Duminy and V. Sergiescu, *Sur la nullité de l'invariant de Godbillon-Vey*, **C.R. Acad. Sci. Paris**, 292:821–824, 1981.

[21] A. El Kacimi-Alaoui, *Sur la cohomologie feuilletée*, **Compositio Mathematica**, 49:195–215, 1983.

[22] D. Fried, *Entropy and twisted cohomology*, **Topology**, 25:455–470, 1986.

[23] É. Ghys, *Sur l'invariant de Godbillon-Vey*, In **Séminaire Bourbaki, Vol. 1988/89**, Asterisque, 177-178, Société Mathématique de France, 1989, 155–181.

[24] É. Ghys, R. Langevin, and P. Walczak, *Entropie géométrique des feuilletages*, **Acta Math.**, 160:105–142, 1988.

[25] C. Godbillon, **Feuilletages: Etudes géométriques I, II**, Publ. IRMA Strasbourg (1985-86); Progress in Math., Vol. 98, Birkhäuser, Boston, Mass., 1991.

[26] C. Godbillon and J. Vey, *Un invariant des feuilletages de codimension 1*, **C.R. Acad. Sci. Paris**, 273:92–95, 1971.

[27] J. Gomes and R. Ruggiero, *Rigidity of surfaces whose geodesic flows preserve smooth foliations of codimension 1*, **Proc. Amer. Math. Soc.**, 135:507–515 (electronic), 2007.

[28] A. Haefliger, *Structures feulletées et cohomologie à valeur dans un faisceau de groupoïdes*, **Comment. Math. Helv.**, 32:248–329, 1958.

[29] A. Haefliger, *Variétés feulletées*, **Ann. Scuola Norm. Sup. Pisa**, 16:367–397, 1962.

[30] A. Haefliger, *Feuilletages sur les variétés ouvertes*, **Topology**, 9:183–194, 1970.

[31] A. Haefliger, *Homotopy and integrability*, In **Manifolds–Amsterdam 1970 (Proc. Nuffic Summer School)**, Lect. Notes in Math. Vol. 197, Springer–Verlag, Berlin, 1971:133–163.

[32] A. Haefliger, *Foliations and compactly generated pseudogroups* In **Foliations: geometry and dynamics (Warsaw, 2000)**, World Sci. Publ., River Edge, NJ, 2002:275–295.

[33] G. Hector, *Architecture des feuilletages de classe C^2*, **Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982)**, Astérisque, 107, Société Mathématique de France 1983, 243–258.

[34] G. Hector and U. Hirsch, **Introduction to the Geometry of Foliations, Parts A,B**, Vieweg, Braunschweig, 1981.

[35] J. Heitsch, *A cohomology for foliated manifolds*, **Comment. Math. Helv.**, 50:197–218 ,1975.

[36] J. Heitsch and S. Hurder, *Secondary classes, Weil measures and the geometry of foliations*, **Jour. Differential Geom.**, 20:291–309, 1984.

[37] M.R. Herman, *The Godbillon-Vey invariant of foliations by planes of T^3*, In **Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976)**, Lect. Notes in Math. Vol. 597, Springer–Verlag, Berlin, 1977, 294–307.

[38] S. Hurder, *The Godbillon measure of amenable foliations*, **Jour. Differential Geom.**, 23:347–365, 1986.

[39] S. Hurder and A. Katok, *Ergodic Theory and Weil measures for foliations*, **Ann. of Math. (2)**, 126:221–275, 1987.

[40] S. Hurder, *Ergodic theory of foliations and a theorem of Sacksteder*, In **Dynamical Systems: Proceedings, University of Maryland 1986-87**, Lect. Notes in Math. Vol. 1342, Springer-Verlag, 1988: 291–328.

[41] S. Hurder and A. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows*, **Publ. Math. Inst. Hautes Etudes Sci.**, 72:5–61, 1991.

[42] S. Hurder, *Exceptional minimal sets of $C^{1+\alpha}$ actions on the circle*, **Ergodic Theory Dynam. Systems**, 11:455-467, 1991.

[43] S. Hurder, *Entropy and Dynamics of C^1–Foliations*, *preprint*, August 2000. Available at http://www.math.uic.edu/∼hurder/publications.

[44] S. Hurder, *Dynamics and the Godbillon-Vey class: a History and Survey*, In **Foliations: Geometry and Dynamics (Warsaw, 2000)**, World Scientific Publishing Co. Inc., River Edge, N.J., 2002:29–60.

[45] S. Hurder and R. Langevin, *Dynamics and the Godbillon-Vey Class of C^1 Foliations*, *preprint*, March 2004. Available at http://www.math.uic.edu/ hurder/papers/59manuscript.pdf

[46] S. Hurder, *Classifying foliations*, **Foliations, Geometry and Topology. Paul Schweitzer Festschrift**, (eds. Nicoalu Saldanha et al), Contemp Math. Vol. 498, American Math. Soc., Providence, RI, 2009, pages 1–61.

[47] S. Hurder, *Lectures on Foliation Dynamics: Barcelona 2010*, **Foliations: Dynamics, Geometry and Topology**, Advanced Courses in Mathematics CRM Barcelona, 2014; arXiv:1104.4852.

[48] F.W. Kamber and Ph. Tondeur, *Algèbres de Weil semi-simpliciales*, **C.R. Acad. Sci. Paris**, 276:A1177–A1179, 1973.

[49] F.W. Kamber and Ph. Tondeur, **Foliated bundles and characteristic classes**, Lect. Notes in Math. Vol. 493, Springer-Verlag, Berlin, 1975.

[50] F.W. Kamber and Ph. Tondeur, *Semisimplicial Weil algebras and characteristic classes for foliated bundles in Čech cohomology*, In **Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 1, Stanford Univ., Stanford, Calif., 1973)**, Amer. Math. Soc., Providence, R.I., 1975:283–294.

[51] A. Katok, *Lyapunov exponents, entropy and periodic orbits for diffeomorphisms*, **Inst. Hautes tudes Sci. Publ. Math.**, 51:137–173, 1980.

[52] R. Langevin and P. Walczak, *Conformal geometry of foliations*, **Geom. Dedicata**, 132:135–178, 2008.

[53] H.B. Lawson, Jr., **The Quantitative Theory of Foliations**, NSF Regional Conf. Board Math. Sci., Vol. 27, 1977.

[54] E. Leichtnam and P. Piazza, *Étale groupoids, eta invariants and index theory*, **J. Reine Angew. Math.**, 587:169–233, 2005.

[55] R. Mañé, **Ergodic theory and differentiable dynamics**, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 8, Springer-Verlag, Berlin, 1987.

[56] T. Mizutani, S. Morita and T. Tsuboi, *The Godbillon-Vey classes of codimension one foliations which are almost without holonomy*, **Ann. of Math. (2)**, 113:515–527, 1981.

[57] S. Morita and T. Tsuboi, *The Godbillon-Vey class of codimension one foliations without holonomy*, **Topology**, 19:43–49, 2001.

[58] H. Moriyoshi and T. Natsume, *The Godbillon-Vey cyclic cocycle and longitudinal Dirac operators*, **Pacific J. Math.**, 172:483–539, 1996.

[59] R. Moussu and F. Pelletier, *Sur le théorème de Poincaré–Bendixson*, **Ann. Inst. Fourier (Grenoble)**, 14:131–148, 1974.

[60] T. Nishimori, *SRH-decompositions of codimension one foliations and the Godbillon-Vey classes*, **Tôhoku Math. Jour.**, 32:9–34, 1980.

[61] Ja.B. Pesin, *Characteristic Ljapunov exponents, and smooth ergodic theory*, **Uspehi Mat. Nauk**, 32:55–112, 1977.

[62] J. Plante and W. Thurston, *Anosov flows and the fundamental group*, **Topology**, 11:147–150, 1972.

[63] J. Plante, *Foliations with measure-preserving holonomy*, **Ann. of Math.**, 102:327–361, 1975.

[64] V.A. Pliss, *On a conjecture of Smale*, **Differencial′nye Uravnenija**, 8:268–282, 1972.

[65] H. Poincaré, *Mémoires sur les courbes définies par une équation différentielle*, **J. Math. Pure et Appl.** (Série 3), 7:375–422,1881.

[66] G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, **Actualité Sci. Indust. 1183**, Hermann, Paris (1952).

[67] G. Reeb, *Sur les structures feuilletées de codimension un et sur un théorème de M. A. Denjoy*, **Ann. Inst. Fourier, Grenoble**, 11:185–200, 1961.

[68] B.L. Reinhart and J. Wood, *A metric formula for the Godbillon-Vey invariant for foliations*, **Proc. Amer. Math. Soc.**, 38:427–430, 1973.

[69] D. Ruelle, *Ergodic theory of differentiable dynamical systems*, **Inst. Hautes Études Sci. Publ. Math.**, 50:27–58, 1979.

[70] R. Sacksteder, *Foliations and pseudogroups*, **Amer. J. Math.**, 87:79–102, 1965.

[71] P.A. Schweitzer, *Some problems in foliation theory and related areas*, In **Differential topology, foliations and Gelfand-Fuks cohomology (Proc. Sympos., Pontifícia Univ. Católica, Rio de Janeiro, 1976)**, Lect. Notes in Math. Vol. 652, Springer–Verlag, Berlin, 1978:240–252.

[72] W.P. Thurston, *Noncobordant foliations of* $\mathbb{S}^3$, **Bull. Amer. Math. Soc.** 78:511–514, 1972.

[73] W.P. Thurston, *Existence of codimension-one foliations* **Ann. of Math. (2)** 104: 249–268, 1976.

[74] N. Tsuchiya, *The Nishimori decompositions of codimension-one foliations and the Godbillon-Vey classes*, **Tôhoku Math. J. (2)**, 34:343–365, 1982.

[75] P. Walczak, **Dynamics of foliations, groups and pseudogroups**, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)], Vol. 64. Birkhäuser Verlag, Basel, 2004.

[76] P. Walczak, *Orthogonal total foliations: Godbillon-Vey forms via local conformal invariants*, in Foliations, geometry, and topology, Contemp. Math. Vol. 498, Amer. Math. Soc., Providence, RI, 2009, pages 155–160.

[77] P. Walczak, *Tautness and the Godbillon-Vey class of foliations*, in Foliations 2012, World Scientific, Singapore, 2013, pages 205–213.

[78] G. Wallet, *Nullité de l'invariant de Godbillon-Vey d'un tore*, **C. R. Acad. Sci. Paris Sér. A-B**, 283:A821–A823, 1976.

[79] R.J. Zimmer, *On the cohomology of ergodic actions of semisimple Lie groups and discrete subgroups*, **Amer. J. Math.**, 103:937–951, 1981.

[80] R.J. Zimmer, *Infinitesimal rigidity for smooth actions of discrete subgroups of Lie groups*, **J. Differential Geom.**, 31:301–322, 1990.

Steven Hurder, Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045, USA

E-mail address: `hurder@uic.edu`

Rémi Langevin, Université de Bourgogne - Franche Comté, Institut de Mathématiques de Bourgogne UMR CNRS 5584, 21078 Dijon, France

E-mail address: `Remi.Langevin@u-bourgogne.fr`