Dynamics and the Godbillon-Vey Class of C^1 Foliations*

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Abstract

We give a direct proof that a codimension–one, C^2-foliation $\mathcal{F}$ with non-zero Godbillon–Vey class $GV(\mathcal{F}) \in H^3(M)$ has a hyperbolic resilient leaf. Our approach is based on methods of C^1-dynamics, and does not use the classification theory of C^2-foliations. We first prove that for a codimension–one C^1-foliation with non-trivial Godbillon measure, the set of infinitesimally expanding points $E(\mathcal{F})$ has positive Lebesgue measure. We then prove that if $E(\mathcal{F})$ has positive measure for a C^1-foliation $\mathcal{F}$, then $\mathcal{F}$ must have a hyperbolic resilient leaf and hence its geometric entropy must be positive. For a C^2-foliation, $GV(\mathcal{F})$ non-zero implies the Godbillon measure is also non-zero, and the result follows. These results apply for both the case when M is compact, and when M is an open manifold.

Contents

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1 Introduction

In 1971, Godbillon and Vey [18] introduced their invariant $GV(\mathcal{F}) \in H^3(M)$ for a codimension-one C^2-foliation $\mathcal{F}$ of a manifold M. While the definition of the Godbillon-Vey class is elementary, understanding its geometric and dynamical meaning remains an open problem. There have been many results and much progress on this problem, especially for codimension one foliations [15, 31], though far less progress has been made for the case of foliations of codimension greater than one (cf. [28, 32]). The purpose of this paper is to introduce a new technique for the study of the problem, which reveals new insights into the codimension one case, and has prospects for generalizing to the higher codimension case as as well.

Moussu and Pelletier [42] and Sullivan (in [50]) asked whether a foliation $\mathcal{F}$ with $GV(\mathcal{F}) \neq 0$ must have leaves of exponential growth? In 1974, this conjecture was a distant goal, as little was known beyond a collection of examples, and some developing intuition for the dynamical properties of foliations. For example, Thurston's intuitive idea of "helical wobble" [51] is a geometric phenomenon which Reinhart and Wood showed is necessary for $GV(\mathcal{F}) \neq 0$ [47]. The geometry of the helical wobble phenomenon is related to a geometric property of an Anosov flow on a compact 3-manifold, that its transverse 1–form is contact. The weak stable foliations for such flows have all leaves of exponential growth, and non-zero Godbillon-Vey classes [51, 44, 47, 33]. Thurston [51] showed there exist examples of codimension–one foliations on compact 3-manifolds for which the Godbillon-Vey class assumes a continuous range of values, suggesting that a geometric interpretation of $GV(\mathcal{F})$ might involve continuous-valued dynamical information such as "entropy".

In a beautiful work, G. Duminy proved that $GV(\mathcal{F}) \neq 0$ implies there are leaves of exponential growth, and more [10, 11, 8]. Duminy's proof began by assuming that the foliation has no resilient leaves, then used the Poincaré-Bendixson theory for codimension–one, C^2-foliations [8, 23] to deduce that the Godbillon-Vey class must vanish. (See section 3 below for a discussion of this approach.) If a codimension–one foliation has a resilient leaf, then it has an open set of leaves with exponential growth. Hence, $GV(\mathcal{F}) \neq 0$ implies the set of leaves with exponential growth has positive Lebesgue measure. Since the appearance of Duminy's paper, one of the open problems has been to give a direct proof of the existence of resilient leaves assuming $GV(\mathcal{F}) \neq 0$. In this paper, we will provide a direct proof of this fact, using methods of ergodic theory and dynamics. Moreover, our proof of the essential part of the argument requires only that $\mathcal{F}$ is a C^1-foliation, and it suggests an approach to studying the analogous question for codimension $q > 1$.

We introduce in section 2 some basic concepts and terminology of codimension–one foliations [5, 17]. Section 3 briefly discusses the concept of the Godbillon measure introduced by Duminy [10] and its extensions by Heitsch and Hurder [8, 26, 28, 32]. This is a fundamental technique for our current understanding of the dynamical meaning of the Godbillon-Vey invariant (see [31]) and Lemma 3.2 is one of the main tools for proving vanishing theorems. Our main result is formulated in terms of the Godbillon measure:

THEOREM 1.1 *If $\mathcal{F}$ is a codimension–one, C^1-foliation with non-trivial Godbillon measure $g_{\mathcal{F}}$, then $\mathcal{F}$ has a hyperbolic resilient leaf.*

For C^2-foliations, the Godbillon-Vey class is obtained by evaluating the Godbillon measure on the "Vey class". Hence, if $GV(\mathcal{F}) \neq 0$ then $g_{\mathcal{F}} \neq 0$ and we deduce:

COROLLARY 1.2 *If $\mathcal{F}$ is a codimension–one, C^2-foliation with non-trivial Godbillon-Vey class $GV(\mathcal{F}) \in H^3(M; \mathbb{R})$, then $\mathcal{F}$ has a hyperbolic resilient leaf.*

The key idea for the proof of Theorem 1.1 is to introduce the $\mathcal{F}$-saturated set $\mathrm{E}(\mathcal{F})$ of points in M where the transverse Radon-Nikodýn cocycle for $\mathcal{F}$ has positive exponent. (The transverse Radon-Nikodýn cocycle is naturally associated to the holonomy of the C^1-foliation $\mathcal{F}$; see § 2.) A point $x \in \mathrm{E}(\mathcal{F})$ if there is a sequence of holonomy maps whose derivatives at x grow exponentially fast as a function of word length. The set $\mathrm{E}(\mathcal{F})$ is a fundamental invariant of a C^1-foliation. For example, a key step in the proof of the generalized Moussu–Pelletier–Sullivan conjecture in [28] was to show that for a foliation $\mathcal{F}$ with almost all leaves of subexponential growth, the Lebesgue measure $|\mathrm{E}(\mathcal{F})| = 0$.

We then show in Theorem 4.4 that if a measurable, $\mathcal{F}$-saturated subset $B \subset M$ is disjoint from $\mathrm{E}(\mathcal{F})$, then the Godbillon measure must vanish on B. The proof of Theorem 4.4 is of independent interest, as it introduces a new tempering procedure to show that the hypotheses of Lemma 3.2 are satisfied on the complement of $\mathrm{E}(\mathcal{F})$. This tempering procedure combines the idea of the proof that a bounded cocycle is cohomologous to zero, with the tempering methods of [28, 32]. The method works in any codimension.

The second step in the proof is to show that for each point $x \in \mathrm{E}(\mathcal{F})$, the holonomy of $\mathcal{F}$ has a uniform estimate of its transverse expansion. That is, the holonomy is transversally expansive on $\mathrm{E}(\mathcal{F})$, which is proved in Proposition 5.3. If $\mathrm{E}(\mathcal{F})$ has positive measure, it is then a matter of basic dynamics that the holonomy of $\mathcal{F}$ must contain resilient leaves, as proved in Proposition 5.2. The proof of Theorem 1.1 now follows by combining Theorem 4.4, Proposition 5.2 and Proposition 5.3.

The proof of Proposition 5.2 is one of the more technical aspects of this paper, though the techniques used are essentially just calculus. The technical issue is that the domain of a holonomy pseudogroup map may depend upon the "length" of the leafwise path used to define it, so that iterating such maps contracts their domains of definitions. This is a key difference between the study of dynamics of a group acting on the circle, and that of codimension–one foliations. One of the key new ideas of the proof of Proposition 5.2 is how to use the transversally expansive property of $\mathcal{F}$ on $\mathrm{E}(\mathcal{F})$ to get estimates on these domains, and use that to produce an abundance of holonomy pseudogroup maps with hyperbolic fixed–points. This is the hard work in section 5.

The extension of the methods to the case of open manifolds requires only a minor modification in the definition of the Godbillon measure, as discussed in section 6.

The geometric entropy $h(\mathcal{F})$ of a C^1-foliation $\mathcal{F}$ introduced by Ghys, Langevin and Walczak [16] measures the complexity of its dynamics, and is one of the most important dynamical invariants of C^1-foliations. For codimension–one foliations, it is elementary that the existence of a resilient leaf implies $h(\mathcal{F}) > 0$. The converse, that $h(\mathcal{F}) > 0$ implies there is a resilient leaf, was proved in [16] for C^2-foliations, and proved by Hurder [29] for C^1-foliations. Let "HRL($\mathcal{F}$)" denote the property that $\mathcal{F}$ has a hyperbolic resilient leaf. Let $|E|$ denote the Lebesgue measure of a measurable subset $E \subset M$. The results of this paper are the summarized by the following logical implications:

THEOREM 1.3 *Let $\mathcal{F}$ be a codimension–one, C^1-foliation of a manifold M. Then*

$$g_{\mathcal{F}} \neq 0 \Longrightarrow |\mathrm{E}(\mathcal{F})| > 0 \Longrightarrow \text{“HRL}(\mathcal{F})\text{”} \Longleftrightarrow h(\mathcal{F}) > 0$$

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[1]This manuscript is a revised version of a preprint posted on the internet in September 2000. The statements of the results and the ideas for the proofs have not changed, but the authors have added significantly more detail to the proofs of the results in section 5. We have also added several illustrations.

2 Foliation Basics

We assume that M is a closed oriented smooth Riemannian m-manifold, $\mathcal{F}$ is a C^1-foliation of codimension–1 with oriented normal bundle, and that the leaves of $\mathcal{F}$ are smoothly immersed submanifolds. This is sometimes referred to as a $C^{1,\infty}$-foliation. In this section we introduce a number of standard notions of foliation structure theory and dynamics [5, 17, 24].

Regular Foliation Atlas

A *regular foliation atlas* for $\mathcal{F}$ is a finite collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ so that:

1. $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ is a covering of M by $C^{1,\infty}$–coordinate charts $\phi_\alpha : U_\alpha \to (-1,1)^m$
2. Each coordinate chart $\phi_\alpha : U_\alpha \to (-1,1)^m$ admits an extension to a $C^{1,\infty}$–coordinate chart $\widetilde{\phi}_\alpha : \widetilde{U}_\alpha \to (-2,2)^m$ where $\widetilde{U}_\alpha$ is convex in M and contains the closure of the open set U_α
3. For each $z \in (-2,2)$, the preimage $\widetilde{\mathcal{P}}_\alpha(z) = \widetilde{\phi}_\alpha^{-1}((-2,2)^{m-1} \times \{z\}) \subset \widetilde{U}_\alpha$ is the connected component containing $\widetilde{\phi}_\alpha^{-1}(\{0\}\times\{z\})$ of the intersection of the leaf of $\mathcal{F}$ through $\phi_\alpha^{-1}(\{0\}\times\{z\})$ with the set $\widetilde{U}_\alpha$.
4. $\mathcal{P}_\alpha(z)$ and $\widetilde{\mathcal{P}}_\alpha(z)$ are convex subsets for the induced Riemannian metric on leaves.

Note that the convexity hypotheses (2.1.2) and (2.1.4) imply if $U_\alpha \cap U_\beta \neq \emptyset$, then each plaque $\mathcal{P}_\alpha(z)$ intersects exactly one plaque of U_β. The analogous statement holds for pairs $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \neq \emptyset$. The reader interested in the details of the construction of regular coverings and their properties should consult Chapter 1.2 of [5].

The inverse images

$$\mathcal{P}_\alpha(z) = \phi_\alpha^{-1}((-1,1)^{m-1} \times \{z\}) \subset U_\alpha$$

are smoothly embedded discs contained in the leaves of $\mathcal{F}$, called the *plaques* associated to the given foliation atlas. One thinks of the collection of all plaques as "tiling stones" which cover the leaves in a regular fashion. The convexity hypotheses in (2.1.4) implies that an intersection of plaques $\mathcal{P}_{\alpha_1}(z_1) \cap \cdots \cap \mathcal{P}_{\alpha_d}(z_d)$ is either empty, or a convex set.

For each $\alpha \in \mathcal{A}$, the extended chart $\widetilde{\phi}_\alpha$ defines a C^1–embedding

$$t_\alpha = \phi_\alpha^{-1}(\{0\} \times \cdot) : (-2,2) \to \widetilde{U}_\alpha \subset M$$

whose image is denoted by $\widetilde{\mathcal{T}}_\alpha$. We will also assume that these images $\widetilde{\mathcal{T}}_\alpha$ are pairwise disjoint; this can be achieved by a small perturbation of the coordinate charts if necessary. We can also assume that each submanifold $\widetilde{\mathcal{T}}_\alpha$ is everywhere perpendicular to the leaves of $\mathcal{F}$ by adjusting the given Riemannian metric on M in an open tubular neighborhood of each $\widetilde{\mathcal{T}}_\alpha$. We may assume that each $\widetilde{\mathcal{T}}_\alpha$ has diameter at most 1.

Define $\mathcal{T}_\alpha = \phi_\alpha^{-1}(\{0\} \times (-1,1))$. The local coordinate on $\mathcal{T}_\alpha$ is again denoted by $t_\alpha : (-1,1) \to \mathcal{T}_\alpha$. We use this coordinate to identify each transversal $\mathcal{T}_\alpha$ with $(-1,1)$.

We can assume that the coordinates t_α are positively oriented, mapping the positive orientation for the normal bundle to $T\mathcal{F}$ to the positive orientation on $\mathbb{R}$.

The collection of all plaques for the foliation atlas is indexed by the *complete transversal*

$$\mathcal{T} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$$

For a point $x \in \mathcal{T}$, by a mild abuse of language we let $\mathcal{P}_\alpha(x) = \mathcal{P}_\alpha(t_\alpha^{-1}(x))$ denote the plaque containing x.

Given a subset $\mathcal{Z} \subset U_\alpha$ let $\mathcal{Z}_\mathcal{P}$ denote the union of all plaques in U_α having non-empty intersection with $\mathcal{Z}$. We set $\mathcal{Z}_\mathcal{T} = \mathcal{Z}_\mathcal{P} \cap \mathcal{T}_\alpha$. If $\mathcal{Z}$ is an open subset of U_α, then $\mathcal{Z}_\mathcal{P}$ is open in U_α and $\mathcal{Z}_\mathcal{T}$ is an open subset of $\mathcal{T}_\alpha$.

The Riemannian metric on M induces a Riemannian metric and corresponding distance function $\mathbf{d}_\mathcal{T}$ on each transversal $\widetilde{\mathcal{T}}_\alpha$. For $\alpha \neq \beta$ and $x \in \mathcal{T}_\alpha$, $y \in \mathcal{T}_\beta$ we set $\mathbf{d}_\mathcal{T}(x,y) = \infty$. Given $r > 0$ and $x \in \widetilde{\mathcal{T}}_\alpha$ let $\mathbf{B}_\mathcal{T}(x,r) = \{y \in \widetilde{\mathcal{T}}_\alpha \mid \mathbf{d}_\mathcal{T}(x,y) < r\}$. Given a point $x \in \widetilde{\mathcal{T}}_\alpha$ and $\delta_1, \delta_2 > 0$, we also adopt the notation

$$[x - \delta_1, x + \delta_2] = \{y \in \widetilde{\mathcal{T}}_\alpha \mid \mathbf{d}_\mathcal{T}(x,y) < \delta_2 \text{ if } t_\alpha(y) > t_\alpha(x);\ \mathbf{d}_\mathcal{T}(x,y) < \delta_1 \text{ if } t_\alpha(y) < t_\alpha(x)\}$$

Holonomy Pseudogroup $\mathcal{G}_\mathcal{F}$

A pair of indices (α, β) is *admissible* if $U_\alpha \cap U_\beta \neq \emptyset$. For each admissible pair (α, β) define

$$\mathcal{T}_{\alpha\beta} = \{x \in \mathcal{T}_\alpha \text{ such that } \mathcal{P}_\alpha(x) \cap U_\beta \neq \emptyset\}.$$

Then there is a well-defined transition function $\mathbf{h}_{\beta\alpha} \colon \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha}$, which for $x \in \mathcal{T}_{\alpha\beta}$ is given by

$$\mathbf{h}_{\beta\alpha}(x) = y \text{ where } \mathcal{P}_\alpha(x) \cap \mathcal{P}_\beta(y) \neq \emptyset$$

Note that $\mathbf{h}_{\alpha\alpha} : \mathcal{T}_\alpha \to \mathcal{T}_\alpha$ is the identity map for each $\alpha \in \mathcal{A}$.

The *holonomy pseudogroup* $\mathcal{G}_\mathcal{F}$ associated to the regular foliation atlas for $\mathcal{F}$ is the pseudogroup with object space $\mathcal{T}$, and transformations generated by compositions of the local transformations $\{\mathbf{h}_{\beta\alpha} \mid (\alpha,\beta) \text{ admissible}\}$. (See [21] or [5] for details of the properties of $\mathcal{G}_\mathcal{F}$.)

The $C^{1,\infty}$–hypothesis on the coordinate charts implies that each map $\mathbf{h}_{\beta\alpha}$ is C^1 for the local coordinates,

$$t_\alpha : (-1,1) \to \mathcal{T}_\alpha \text{ and } t_\beta : (-1,1) \to \mathcal{T}_\beta$$

Moreover, the hypothesis (2) on regular foliation charts implies that each $\mathbf{h}_{\beta\alpha}$ admits an extension to a C^1-map $\widetilde{\mathbf{h}}_{\beta\alpha} : \widetilde{\mathcal{T}}_{\alpha\beta} \to \widetilde{\mathcal{T}}_{\alpha\beta}$ defined in a similar fashion. There are only a finite number of admissible pairs, so it is possible to give a uniform estimate on the domains of these extensions.

LEMMA 2.1 *There exists $\epsilon_0 > 0$ so that for every admissible pair (α, β) and $x \in \mathcal{T}_{\alpha\beta}$ then $[x - \epsilon_0, x + \epsilon_0] \subset \widetilde{\mathcal{T}}_{\alpha\beta}$. That is, if $x \in \mathcal{T}_\alpha$ is in the domain of $\mathbf{h}_{\beta\alpha}$ then $[x - \epsilon_0, x + \epsilon_0]$ is in the domain of $\widetilde{\mathbf{h}}_{\beta\alpha}$.* □

For $0 < \delta < \epsilon_0$ we introduce the closed subsets of $\widetilde{\mathcal{T}}$

$$\mathcal{T}[\delta] = \{y \in \widetilde{\mathcal{T}} \mid \exists\, x \in \overline{\mathcal{T}},\ \mathbf{d}_\mathcal{T}(x,y) \leq \delta\}$$

$$\mathcal{T}_{\alpha\beta}[\delta] = \{y \in \widetilde{\mathcal{T}}_{\alpha\beta} \mid \exists\, x \in \overline{\mathcal{T}_{\alpha\beta}},\ \mathbf{d}_\mathcal{T}(x,y) \leq \delta\}$$

Thus, the maps $\mathbf{h}_{\beta\alpha}$ are uniformly C^1 on $\mathcal{T}_{\alpha\beta}[\delta]$ for $\delta < \epsilon_0$.

Composition of elements in $\mathcal{G}_\mathcal{F}$ will be defined via "plaque chains". Given $x, y \in \mathcal{T}$ on the same leaf, a plaque chain of length k between them is a collection of plaques

$$\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$$

where $x_0 = x$, $x_k = y$ and for each $0 \leq i < k$ we have $\mathcal{P}_{\alpha_i}(x_i) \cap \mathcal{P}_{\alpha_{i+1}}(x_{i+1}) \neq \emptyset$. A plaque chain $\mathcal{P}$ also defines an "extended" plaque chain for the charts $\{(\widetilde{U}_\alpha, \widetilde{\phi}_\alpha)\}$,

$$\widetilde{\mathcal{P}} = \{\widetilde{\mathcal{P}}_{\alpha_1}(x_0), \ldots, \widetilde{\mathcal{P}}_{\alpha_k}(x_k)\}$$

We say two plaque chains

$$\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\} \text{ and } \mathcal{Q} = \{\mathcal{P}_{\beta_0}(y_0), \ldots, \mathcal{P}_{\beta_\ell}(y_\ell)\}$$

are *composable* if $x_k = y_0$, hence $\alpha_k = \beta_0$ and $\mathcal{P}_{\alpha_k}(x_k) = \mathcal{P}_{\beta_0}(y_0))$. Their composition is defined by

$$\mathcal{Q} \circ \mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k), \mathcal{P}_{\beta_1}(y_1), \ldots, \mathcal{P}_{\beta_\ell}(y_\ell)\}$$

The holonomy transformation defined by a plaque chain is the local diffeomorphism

$$\mathbf{h}_{\mathcal{P}} = \mathbf{h}_{\alpha_k \alpha_{k-1}} \circ \cdots \circ \mathbf{h}_{\alpha_1 \alpha_0}$$

whose domain $\mathcal{D}_{\mathcal{P}} \subset \mathcal{T}_{\alpha_0}$ contains x_0. Note that $\mathcal{D}_{\mathcal{P}}$ is the largest connected open subset of $\mathcal{T}_{\alpha_0}$ containing x_0 on which $\mathbf{h}_{\alpha_\ell \alpha_{\ell-1}} \circ \cdots \circ \mathbf{h}_{\alpha_1 \alpha_0}$ is defined for all $0 < \ell \leq k$. The dependence of the domain of $\mathbf{h}_{\mathcal{P}}$ on the plaque chain $\mathcal{P}$ is a subtle issue, yet is at the heart of the technical difficulties arising in the study of foliation pseudogroups.

Let $\widetilde{\mathbf{h}}_{\widetilde{\mathcal{P}}}$ be the holonomy associated to the chain $\widetilde{\mathcal{P}}$, with domain $\widetilde{\mathcal{D}}_{\widetilde{\mathcal{P}}} \subset \widetilde{\mathcal{T}}_{\alpha_0}$ the largest maximal open subset containing x_0 on which $\widetilde{\mathbf{h}}_{\alpha_\ell \alpha_{\ell-1}} \circ \cdots \circ \widetilde{\mathbf{h}}_{\alpha_1 \alpha_0}$ is defined for all $1 < \ell \leq k$. By the extension property of a regular atlas, the closure $\overline{\mathcal{D}_{\mathcal{P}}} \subset \widetilde{\mathcal{D}}_{\widetilde{\mathcal{P}}}$ and $\widetilde{\mathbf{h}}_{\widetilde{\mathcal{P}}}$ is an extension of $\mathbf{h}_{\mathcal{P}}$.

Given a plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ and a point $y \in \mathcal{D}_{\mathcal{P}}$, there is a "parallel" plaque chain denoted $\mathcal{P}(y) = \{\mathcal{P}_{\alpha_0}(y), \ldots, \mathcal{P}_{\alpha_k}(y_k)\}$ where $\mathbf{h}_{\mathcal{P}}(y) = y_k$.

Given a plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$, we define its length $\|\mathcal{P}\| = \ell$ where $\mathcal{Q}$ is the shortest plaque chain $\mathcal{Q} = \{\mathcal{P}_{\beta_0}(y_0), \ldots, \mathcal{P}_{\beta_\ell}(y_\ell)$ with $y_0 = x_0$ and $y_\ell = x_k$, and $\mathcal{P}$ and $\mathcal{Q}$ are "homotopic" through plaque chains. The assumption that the plaque chains are homotopic implies that the germs at x_0 of $\mathbf{h}_{\mathcal{P}}$ and $\mathbf{h}_{\mathcal{Q}}$ are equal. Note that the length function $\mathcal{P} \mapsto \|\mathcal{P}\|$ makes $\mathcal{G}_{\mathcal{F}}$ a metric equivalence relation in the sense of [32].

Radon-Nikodýn Cocycle

Given a plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ from $x = x_0$ to $y = x_k$, the derivative map

$$d\mathbf{h}_{\mathcal{P}}(x) : \mathrm{T_x}\widetilde{\mathcal{T}}_{\alpha_1} \longrightarrow \mathrm{T_y}\widetilde{\mathcal{T}}_{\alpha_k}$$

is identified with a real number $\mathbf{h}'_{\mathcal{P}}(x)$ using the norms induced from the Riemannian metric on TM. Given composable plaque chains $\mathcal{P}$ and $\mathcal{Q}$, with $x = x_0, y = x_k = y_0, z = y_\ell$ then by the chain rule

$$\mathbf{h}'_{\mathcal{Q} \circ \mathcal{P}}(x) = \mathbf{h}'_{\mathcal{Q}}(y) \cdot \mathbf{h}'_{\mathcal{P}}(x) \tag{1}$$

The map $d\mathbf{h} \colon \mathcal{G}_{\mathcal{F}} \to \mathbb{R}$ defined by $d\mathbf{h}(\mathcal{P}, y) = \mathbf{h}'_{\mathcal{P}(y)}(y)$ is the Radon-Nikodýn cocycle of the foliation pseudogroup acting on $\widetilde{\mathcal{T}}$.

Resilient Leaves and Ping-Pong Games

A plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ is *closed* if $x_0 = x_k$. A closed plaque chain $\mathcal{P}$ defines a local diffeomorphism $\mathbf{h}_{\mathcal{P}} : \mathcal{D}_{\mathcal{P}} \to \mathcal{T}_\alpha$ with $\mathbf{h}_{\mathcal{P}}(x) = x$, where $x = x_1 \in \mathcal{T}_\alpha$.

A point $y \in \mathcal{D}_{\mathcal{P}}$ is said to be asymptotic by iterates of $\mathbf{h}_{\mathcal{P}}$ to x if $\mathbf{h}^\ell_{\mathcal{P}}(y) \in \mathcal{D}_{\mathcal{P}}$ for all $\ell > 0$ (where $\mathbf{h}^\ell_{\mathcal{P}}$ denotes the composition of $\mathbf{h}_{\mathcal{P}}$ with itself ℓ times), and the iterates $\lim_{\ell \to \infty} \mathbf{h}^\ell_{\mathcal{P}}(y) = x$.

The map $\mathbf{h}_{\mathcal{P}}$ is said to be a contraction at x if there is some $\delta > 0$ so that every $y \in \mathbf{B}_{\mathcal{T}}(x, \delta)$ is asymptotic to x. The map $\mathbf{h}_{\mathcal{P}}$ is said to be a hyperbolic contraction at x if $0 < \mathbf{h}'_{\mathcal{P}}(x) < 1$. In this case, there exists $\epsilon > 0$ and $0 < \lambda < 1$ so that $\mathbf{h}'_{\mathcal{P}}(y) < \lambda$ for all $y \in \mathbf{B}_{\mathcal{T}}(x, \epsilon)$. Hence, every point of $\mathbf{B}_{\mathcal{T}}(x, \epsilon)$ is asymptotic to x, and there exists $0 < \delta < \epsilon$ so that the image of the closed δ–ball about x satisfies

$$\mathbf{h}_{\mathcal{P}}(\overline{\mathbf{B}_{\mathcal{T}}(x, \delta)}) \subset \mathbf{B}_{\mathcal{T}}(x, \delta)$$

DEFINITION 2.2 *We say $x \in \mathcal{T}$ is a hyperbolic resilient point for $\mathcal{G}_{\mathcal{F}}$ if there exists*

1. *a closed plaque chain $\mathcal{P}$ such that $\mathbf{h}_{\mathcal{P}}$ is a hyperbolic contraction at $x = x_0$*
2. *a point $y \in \mathcal{D}_{\mathcal{P}}$ which is asymptotic to x (and $y \neq x$)*
3. *a plaque chain $\mathcal{R}$ from x to y.*

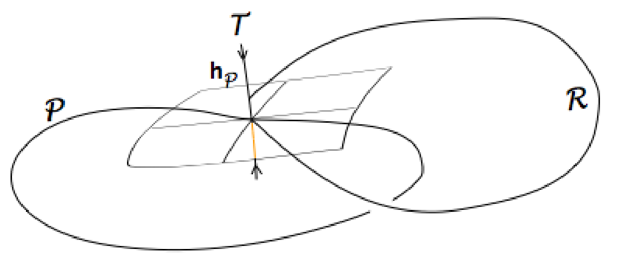

Figure 1: Resilient leaf with contracting holonomy along path $\mathcal{P}$

The "ping-pong lemma" is a key technique for the study of 1-dimensional dynamics. J. Tits used it to prove the existence of free subgroups of non-solvable subgroups of linear groups [52, 9], and it has found many applications since then for studying group actions, especially actions on the circle. A closely related idea is fundamental in the study of the transverse dynamics of codimension one foliations.

DEFINITION 2.3 *The action of the groupoid $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{T}$ has a "ping-pong game" if there exists $x, y \in \mathcal{T}_\alpha$ with $x \neq y$ and*

1. *a closed plaque chain $\mathcal{P}$ such that $\mathbf{h}_{\mathcal{P}}$ is a contraction at $x = x_0$*
2. *a closed plaque chain $\mathcal{Q}$ such that $\mathbf{h}_{\mathcal{Q}}$ is a contraction at $y = y_0$*
3. *$y \in \mathcal{D}_{\mathcal{P}}$ is asymptotic to x by $\mathbf{h}_{\mathcal{P}}$ and $x \in \mathcal{D}_{\mathcal{Q}}$ is asymptotic to y by $\mathbf{h}_{\mathcal{Q}}$*

We say that the ping-pong game is hyperbolic *if the maps $\mathbf{h}_{\mathcal{P}}$ and $\mathbf{h}_{\mathcal{Q}}$ are hyperbolic contractions.*

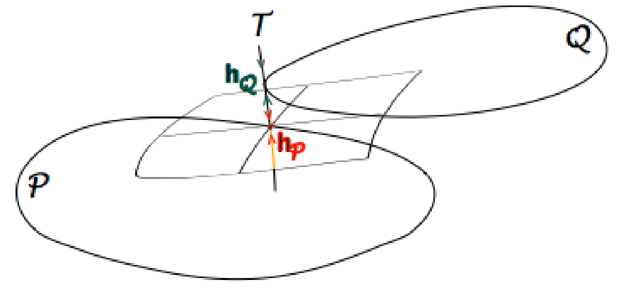

Figure 2: Two paths with contracting holonomy that generate a ping-pong game

The relation between "ping-pong games" and resilient orbits was used already by Ghys, Langevin and Walczak [16], and in the study of 1–dimensional dynamics by the first author in [29, 30] :

PROPOSITION 2.4 *$\mathcal{G}_{\mathcal{F}}$ has a "ping-pong game" if and only if it has a resilient point, and has a "hyperbolic ping-pong game" if and only if it has a hyperbolic resilient point.*

3 The Godbillon Measure

In this section, we recall the basic ideas of the Godbillon measure, and how it is used to estimate the values of the Godbillon-Vey class. The Godbillon functional was introduced by Duminy in [10, 11]; it was extended to Borel measurable sets by Heitsch and Hurder [26]; and techniques for estimating the Godbillon measure were developed in Hurder [28] and Hurder and Katok [32].

Let $\vec{n}$ denote the unit, positively-oriented vector field on M orthogonal to $\mathcal{F}$. Define the 1-form ω on M by setting $\omega(\vec{n}) = 1$, and $\omega(\vec{u}) = 0$ for every vector $\vec{u}$ tangent to $\mathcal{F}$. The assumption that $\mathcal{F}$ is $C^{1,\infty}$ implies $\omega\colon TM \to \mathbb{R}$ is C^1, and for each leaf L of $\mathcal{F}$, the restriction $\omega\colon TM|L \to L$ is C^∞.

The distribution $T\mathcal{F}$ is integrable, so the 2-form $d\omega$ is in the ideal generated by ω – that is, there is some 1-form α with $d\omega = \omega \wedge \alpha$. Define a $C^{0,\infty}$ 1–form $\eta = \iota(\vec{n})d\omega$; then for every vector $\vec{u}$ tangent to $\mathcal{F}$ we have

$$\alpha(\vec{u}) = (\omega \wedge \alpha)(\vec{n}, \vec{u}) = d\omega(\vec{n}, \vec{u}) = \iota(\vec{n})d\omega(\vec{u}) = \eta(\vec{u})$$

Hence, $d\omega = \omega \wedge \eta$. The 1-form η is the "canonical" representative for α which satisfies $\alpha(\vec{n}) = 0$. The form η has an alternate description (see section 3 below) in terms of the gradient of the Radon-Nikodýn derivative along leaves of the "transverse measure" ω.

When $\mathcal{F}$ is a C^2–foliation, then η is a C^1 form, and $d\eta$ is defined as a continuous 2-form. The calculation

$$0 = d(d\omega) = d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta = \omega \wedge \eta \wedge \eta - \omega \wedge d\eta = \omega \wedge d\eta$$

shows that $d\eta$ is a multiple of ω. Consequently, $\eta \wedge d\eta$ is a continuous 3-form ϕ which is closed in the sense of distributions. (That is, given any C^1 form ϕ on M of degree $(n-4)$ with compact support, the integral $\int_M \eta \wedge d\eta \wedge d\phi = 0$.) The cohomology class $GV(\mathcal{F}) \in H^3(M)$ determined by $\eta \wedge d\eta$ is the Godbillon-Vey class. The class $GV(\mathcal{F})$ is an invariant of the diffeomorphism and foliated concordance class of $\mathcal{F}$ (for example, see Chapter 3 of Lawson [38].)

The idea of the Godbillon functional is to separate the roles of the forms η and $d\eta$ in the definition of $GV(\mathcal{F})$, and then study just the contribution from the form η. This requires that we place the 2-form $d\eta$ in a natural domain. When $\mathcal{F}$ is a C^∞ foliation, the idea is easy to describe, so we do that first for clarity. (Alternately, see Chapter 7.1 of [6] for a detailed exposition.) Let $A^*(M,\mathcal{F}) \subset \Omega^*(M)$ denote the ideal in $\Omega^*(M)$ of smooth forms which are a multiple of ω. Since $d\omega = \omega \wedge \eta$, the ideal $A^*(M,\mathcal{F})$ is closed under exterior differentiation, hence forms a graded complex, whose associated cohomology groups are denoted by $H^*(M,\mathcal{F})$. Given a closed form $\zeta \in A^p(M,\mathcal{F})$ we let $[\zeta]_\mathcal{F}$ denotes its class in $H^p(M,\mathcal{F})$. Duminy observed in [10] that the class $[d\eta]_\mathcal{F} \in H^2(M,\mathcal{F})$ is independent of the choice of Riemannian metric on M, hence is an invariant of $\mathcal{F}$, which he called the *Vey class* of $\mathcal{F}$.

The construction of the Vey class can be extended to foliations of lower differentiability by modifying the definition of the ideal $A^*(M,\mathcal{F})$ and taking more care with the definition of the exterior derivative on this ideal. Let $\Omega^*_{(r)}(M)$ denote the graded algebra of C^r–forms on M, for $r \geq 0$, and $\Omega^*_{(r,s)}(M) \subset \Omega^*_{(r)}(M)$ the ideal of forms whose restrictions to leaves of $\mathcal{F}$ are C^s forms, for $r \leq s \leq \infty$. Now let $A^*_{(0,\infty)}(M,\mathcal{F}) \subset \Omega^*_{(0,\infty)}(M)$ denote the ideal in $\Omega^*_{(0,\infty)}(M)$ of forms which are a multiple of ω: a form $\zeta \in A^{p+1}_{(0,\infty)}(M,\mathcal{F})$ if there exists a p–form $\phi \in \Omega^p_{(0,\infty)}(M)$ such that $\zeta = \omega \wedge \phi$. For example, the defining 1-form ω of $\mathcal{F}$ satisfies $\omega \in A^1_{(0,\infty)}(M,\mathcal{F})$, and $d\omega \in A^2_{(0,\infty)}(M,\mathcal{F})$.

Define a linear operator $d_\mathcal{F}\colon A^{p+1}_{(1,\infty)}(M,\mathcal{F}) \to A^{p+2}_{(0,\infty)}(M,\mathcal{F})$, where for $\zeta = \omega \wedge \phi$ with ϕ a C^1-form, we set $d_\mathcal{F}(\zeta) = \omega \wedge d\phi$. Note that if $\zeta = \omega \wedge \phi_1 = \omega \wedge \phi_2$, then $\omega \wedge (\phi_1 - \phi_2) = 0$ hence $\phi_1 - \phi_2 = \omega \wedge \beta$ for some $(p-1)$-form β, and thus

$$\omega \wedge d(\phi_1 - \phi_2) = \omega \wedge d(\omega \wedge \beta) = \omega \wedge d\omega \wedge \beta = 0$$

It follows that $d_\mathcal{F}(\zeta)$ is well-defined. A calculation in local coordinates adapted to the foliation $\mathcal{F}$ shows that $d_\mathcal{F}(\zeta)$ depends only on the leafwise exterior derivative of the restrictions of the form ϕ to the leaves of $\mathcal{F}$, hence the operator $d_\mathcal{F}$ admits a continuous extension to a linear map $d_\mathcal{F}\colon A^{p+1}_{(0,\infty)}(M,\mathcal{F}) \to A^{p+2}_{(0,\infty)}(M,\mathcal{F})$. Note that $d_\mathcal{F}(d_\mathcal{F}\zeta) = 0$ for $\zeta \in A^{p+1}_{(2,\infty)}(M,\mathcal{F})$, hence the extension to $A^{p+1}_{(0,\infty)}(M,\mathcal{F})$ satisfies $d_\mathcal{F}(d_\mathcal{F}\zeta) = 0$.

The operator $d_\mathcal{F}$ is callled the "leafwise differential" in the literature [13, 25, 35].

The operator $d_\mathcal{F}$ can be "twisted" to obtain an extension $d\colon A^{p+1}_{(0,\infty)}(M,\mathcal{F}) \to A^{p+2}_{(0,\infty)}(M,\mathcal{F})$ of the usual exterior differential on smooth forms by setting, for $\zeta = \omega \wedge \phi$,

$$d\zeta = d_\mathcal{F}\zeta + d\omega \wedge \phi = d_\mathcal{F}\zeta - \eta \wedge \zeta$$

Then

$$d(d\zeta) = d(d_\mathcal{F}\zeta - \eta \wedge \zeta) = d_\mathcal{F}(d_\mathcal{F}\zeta - \eta \wedge \zeta) - \eta \wedge (d_\mathcal{F}\zeta - \eta \wedge \zeta) = \eta \wedge d_\mathcal{F}\zeta - \eta \wedge d_\mathcal{F}\zeta = 0$$

so $(A^*_{(0,\infty)}(M,\mathcal{F}), d)$ is a graded exterior differential complex, whose associated cohomology groups are denoted by $H^{p+1}_{(0,\infty)}(M,\mathcal{F})$. Given a closed form $\zeta \in A^p_{(0,\infty)}(M,\mathcal{F})$, let $[\zeta]_\mathcal{F}$ denote its class in $H^p_{(0,\infty)}(M,\mathcal{F})$.

When $\mathcal{F}$ is a C^2–foliation, then $d\eta \in A^2_{(0,\infty)}(M,\mathcal{F})$ and $d(d\eta) = 0$, hence there is a well-defined class $[d\eta]_\mathcal{F} \in H^2_{(0,\infty)}(M,\mathcal{F})$, the *Vey class* of $\mathcal{F}$.

For a general foliation $\mathcal{F}$, the calculation of the cohomology groups $H^{p+1}_{(0,\infty)}(M,\mathcal{F})$ is an intractable problem. Rather, the usefulness of these groups is the existence of pairings with other cohomology theories, and their maps to the usual cohomology groups [26].

Note that the inclusion of the ideal $A^p_{(0,\infty)}(M,\mathcal{F}) \subset \Omega^*_{(0,\infty)}(M)$ induces a map on cohomology $H^*_{(0,\infty)}(M,\mathcal{F}) \to H^*_{(0,\infty)}(M) \cong H^*(M)$.

Given a closed form $\zeta \in: A^{p+1}_{(0,\infty)}(M,\mathcal{F})$, the product $\eta \wedge \zeta \in \Omega^{p+2}_{(0,\infty)}(M)$ is again closed. If $\zeta = d\xi$ for some form $\xi \in A^p_{(0,\infty)}(M,\mathcal{F})$ then $d(-\eta \wedge \xi) = -d\eta \wedge \xi + \eta \wedge d\xi = \eta \wedge \zeta$. Thus, $[\zeta]_{\mathcal{F}}$ determines a well-defined class $g([\zeta]_{\mathcal{F}}) = [\eta \wedge \zeta]_{\mathcal{F}} \in H^{p+2}_{(0,\infty)}(M,\mathcal{F})$. The Godbillon operator is the composition

$$g\colon H^{p+1}(M,\mathcal{F}) \to H^{p+2}(M,\mathcal{F}) \to H^{p+2}(M) \ , \quad g([\zeta]_{\mathcal{F}}) = [\eta \wedge \zeta].$$

The point is that $g([d\eta]_{\mathcal{F}}) = [\eta \wedge d\eta] \in H^3(M)$, or "Godbillon(Vey) = Godbillon-Vey".

We are interested in the values of the cohomology class $[\eta \wedge d\eta] \in H^3(M)$. Since M is assumed to be closed and oriented, Poincaré duality implies that the pairing $H^3(M) \otimes H^{m-3}(M) \to H^m(M) \cong \mathbb{R}$ is non-degenerate, hence the class $[\eta \wedge d\eta]$ is determined by its pairings with classes in $H^{m-3}(M)$.

Given a closed form $\zeta \in A^{p+1}_{(0,\infty)}(M)$ and a closed smooth form $\xi \in \Omega^k(M)$, the product $\zeta \wedge \xi \in A^{k+p+1}(M,\mathcal{F})$ is again closed, and if either form is exact in its complex, it is easy to see that $\psi \wedge \xi$ is also exact. Thus, there is a well-defined pairing

$$H^{p+1}_{0,\infty}(M,\mathcal{F}) \otimes H^k(M) \to H^{k+p+1}_{0,\infty}(M,\mathcal{F}) \tag{2}$$

In particular, given a class $u \in H^{m-3}(M)$ we can choose a smooth closed form $\xi \in \Omega^{m-3}(M)$ with $u = [\xi]$, and then form the pairing $[d\eta]_{\mathcal{F}} \cup [\xi] = [d\eta \wedge \xi]_{\mathcal{F}} \in H^{m-1}_{(0,\infty)}(M,\mathcal{F})$.

The Godbillon operator applied to a class in $H^{m-1}_{(0,\infty)}(M,\mathcal{F})$ yields an m-form on M, which can be integrated over the fundamental class to obtain a real number. This composition yields a linear functional denoted by

$$G\colon H^{m-1}_{(0,\infty)}(M,\mathcal{F}) \to \mathbb{R}, \qquad G([\zeta]_{\mathcal{F}}) = \langle [\eta \wedge \zeta], [M] \rangle = \int_M \eta \wedge \zeta \tag{3}$$

Given a class $u \in H^{m-3}(M)$ represented by the smooth closed form $\xi \in \Omega^{m-3}(M)$, observe that

$$G([\eta \wedge d\eta \wedge \xi]_{\mathcal{F}}) = \langle [\eta \wedge d\eta \wedge \zeta], [M] \rangle = \langle [\eta \wedge d\eta] \cup [\zeta], [M] \rangle$$

Poincaré duality for M implies that the values of the pairing $\langle [\eta \wedge d\eta] \cup [\zeta], [M] \rangle$ determine the values of the cohomology class $[\eta \wedge d\eta] \in H^3(M)$. Hence, the values of the Godbillon operator in (3) completely determines the value of the Godbillon-Vey class for $\mathcal{F}$. This elementary, but fundamental observation by Duminy in [10] implies that $GV(\mathcal{F}) = 0$ if G is the trivial functional. The study of the Godbillon-Vey class is thus approached by analyzing the linear functional of (3).

The linear functional G possesses properties that were hinted at in the literature preceding Duminy's work (see the survey [31] for a fuller discussion of the ideas leading up to Duminy's work.) In particular, Duminy showed that G can be restricted to open saturated sets. This observation was systematically generalized by the first author in his work with James Heitsch [26] to show that that G extends to a measure on the measurable saturated subsets of M. Moreover, the values of the measure can be calculated using measurable cocycle data. The extension to measurable data allows the introduction of techniques of ergodic theory. Let us formulate these notions more precisely.

A set $B \subset M$ is $\mathcal{F}$–saturated if for all $x \in B$ the leaf L_x through x is contained in B. Let $\mathcal{B}(\mathcal{F})$ denote the Σ-algebra of Lebesgue measurable $\mathcal{F}$–saturated subsets of M.

THEOREM 3.1 *[10, 26] For each $B \in \mathcal{B}(\mathcal{F})$, there is a well-defined linear functional*

$$G_B \colon H^{m-1}_{(0,\infty)}(M, \mathcal{F}) \to \mathbb{R} \tag{4}$$

defined by $G_B([\zeta]_{\mathcal{F}}) = \int_B \eta \wedge \zeta$. Moreover, the correspondence

$$B \mapsto G_B \in \mathrm{Hom}_{\mathrm{cont}}(H^{m-1}_{(0,\infty)}(M, \mathcal{F}), \mathbb{R})$$

is a countably additive measure on $\mathcal{B}(\mathcal{F})$ which vanishes on sets of Lebesgue measure zero.

G_B is called the *Godbillon measure* of $\mathcal{F}$. We adopt the notation $G_{\mathcal{F}}(B) = G_B$ to emphasize that this is an invariant of $\mathcal{F}$. If $\mathcal{F}$ is C^2, we can also define the *Godbillon-Vey measure* for $\mathcal{F}$,

$$GV_{\mathcal{F}}(B) \colon H^{m-3}(M, \mathcal{F}) \to \mathbb{R} \tag{5}$$

by setting, for $[\zeta] \in H^{m-3}_{(0,\infty)}(M, \mathcal{F})$,

$$GV_{\mathcal{F}}(B)([\zeta]) = \langle G_B([d\eta]), [\zeta] \rangle = \int_B \eta \wedge d\eta \wedge \zeta$$

The linear functional $GV_{\mathcal{F}}(M) \in \mathbf{Hom}(H^{m-3}(M, \mathcal{F}), \mathbb{R})$ is just the Poincaré dual class of $GV(\mathcal{F})$.

Part of the claim of Theorem 3.1 is that the linear functional (4) is independent of the choice of the form ω defining $\mathcal{F}$, which in our construction of ω is determined by the choice of Riemannian metric on M. A remarkable property of the Godbillon measure, proved by Heitsch and Hurder [26], is that ω can be allowed to be transversally measurable. That is, $\eta = \iota(\vec{n}) d_{\mathcal{F}}\omega$ can be defined given a 1-form ω which vanishes on leaves of $\mathcal{F}$ and a tranverse vector field $\vec{n}$ satisfying $\omega(\vec{n}) = 1$, such that ω is leafwise smooth and $\iota(\vec{n}) d_{\mathcal{F}}\omega$ is uniformly bounded on M.

This remark is the idea behind the proof of a more general result, Theorem 4.3 of [28], which we restate and prove in our context:

LEMMA 3.2 *Let $B \in \mathcal{B}(\mathcal{F})$. Suppose there exists a sequence of measurable, leafwise smooth, non-vanishing transverse 1-forms $\{\omega_n \mid n = 1, 2, \ldots\}$ on M for which $\|\iota(\vec{n}) d_{\mathcal{F}}\omega_n\|_B < 1/n$, where $\|\iota(\vec{n}) d_{\mathcal{F}}\omega_n\|_B$ denotes the sup norm over B. Then $G_{\mathcal{F}}(B) = 0$, and hence $GV_{\mathcal{F}}(B) = 0$. If this estimate holds for a conull set B (the measure of $M - B$ is zero), then $GV(\mathcal{F}) = 0$.*

Proof: Let $[\zeta]_{\mathcal{F}} \in H^{m-1}_{(0,\infty)}(M, \mathcal{F})$, then

$$\begin{aligned}
|G_{\mathcal{F}}(B)([\zeta]_{\mathcal{F}})| &= \lim_{n \to \infty} \left| \int_B (\iota(\vec{n}) d_{\mathcal{F}}\omega_n) \wedge \zeta \right| \\
&\leq \lim_{n \to \infty} \int_B \|\iota(\vec{n}) d_{\mathcal{F}}\omega_n\| \, \|\zeta\| \, dvol \\
&\leq \lim_{n \to \infty} n^{-1} \cdot \int_B \|\zeta\| \, dvol \\
&= 0 \qquad \square
\end{aligned}$$

For foliations of differentiability at least C^2, Sacksteder's Theorem [49] implies that if $\mathcal{F}$ has no resilient leaf, then there are no exceptional minimal sets for $\mathcal{F}$. Hence, by the Poincaré-Bendixson theory for C^2–foliations [7, 23], all leaves of $\mathcal{F}$ either lie at finite level, or lie in "arbitrarily thin" open subsets $U \in \mathcal{B}_O(\mathcal{F})$. Thus, for a C^2 foliation without resilient leaves, the problem of showing that the Godbillon-Vey class must vanish can be reduced to showing that the Godbillon measure vanishes on sets consisting of leaves at finite level, and then to show that it vanishes on thin open saturated subsets.

Duminy used a version of Lemma 3.2 for continuous transverse forms on open saturated subsets to show the Godbillon measure vanishes on the sets of leaves at finite level. This part of his proof mimicked the previous results on this problem [54, 27, 41, 40, 53].

The complements of the sets of leaves at finite level are called "thin sets", and in the brilliant note [11], Duminy introduced new techniques to show that Godbillon measure vanishes on "thin sets". Thus, the problem is solved for C^2 foliations. For published details of his arguments, see [5, 8].

In the next two sections, we follow a different line of reasoning to obtain a stronger result about the Godbillon measures, but it is still very interesting to compare the ideas involved. The first part of Duminy's arguments, showing that the Godbillon measure vanishes on leaves of finite level, is very much the same in both approaches. However, Duminy's idea of "thin sets" is still not well-understood as an ergodic theory property of a foliation.

4 $|\mathrm{E}(\mathcal{F})| = 0$ implies $g_{\mathcal{F}} = 0$

In this section, we introduce a measure of the infinitesimal expansion of leaves and the set $\mathrm{E}(\mathcal{F})$. The main theorem is that the Godbillon measure vanishes on the complement of $\mathrm{E}(\mathcal{F})$. Hence, if the Godbillon measure is non zero, since it is non-singular with respect to Lebesgue measure, the set $\mathrm{E}(\mathcal{F})$ must have positive measure. In the next section, we will show that if $\mathrm{E}(\mathcal{F})$ has positive measure, then $\mathcal{F}$ has resilient leaves.

We fix a regular foliation atlas for $\mathcal{F}$, with associated transversal $\mathcal{T}$ and holonomy groupoid $\mathcal{G}_{\mathcal{F}}$.

For $x \in \mathcal{T}$ and each integer $N \geq 0$, define the function

$$\mu_N(x) = \max\{\mathbf{h}'_{\mathcal{P}}(x) \mid x \in \mathcal{D}_{\mathcal{P}} \ \& \ \|\mathcal{P}\| \leq N\} \tag{6}$$

The identity transformation is the holonomy for a plaque chain of length zero, so $\mu_N(x) \geq 1$. For $N > 1$, $\mu_N(x)$ is the maximal transverse infinitesimal expansion of holonomy maps of length at most N. The function μ_N is the maximum of a finite set of continuous functions, so is Borel on $\mathcal{T}$.

Figure 3 below illustrates this definition, where we illustrate a disk of radius N in a given leaf L_x. Several plaque chains starting at x are indicated by the paths traced through their centers on the given leaf. The transverse line segments represent the expansion of a small initial transversal by the holonomy induced by the foliation. Of course, we only care about the infinitesimal expansion of these segments, which is approximated by the lengths of the image segments.

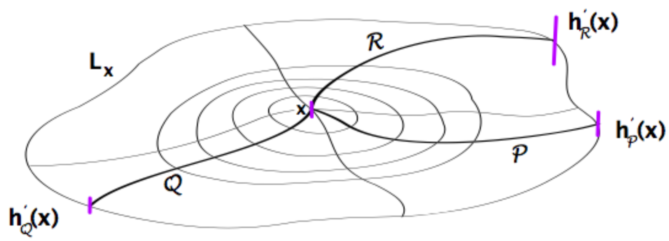

Figure 3: Paths defining plaque chains of length $\leq$ N and their holonomy maps

The following is a basic property of the family of infinitesimal expansion functions μ_N.

LEMMA 4.1 *Let $x \in \mathcal{T}$ and $\mathcal{Q} = \{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$ so that $\mathbf{h}_{\beta\alpha}(x) = y$. Then for all $N > 0$,*

$$\mu_{N-1}(x) \leq \mu_N(y) \cdot \mathbf{h}'_{\beta\alpha}(x) \leq \mu_{N+1}(x) \tag{7}$$

Proof : Let $\mathcal{P}$ be a plaque chain at y with $\|\mathcal{P}\| \leq N$, then $\mathcal{P} \circ \mathcal{Q}$ is a plaque chain at x with $\|\mathcal{P} \circ \mathcal{Q}\| \leq N + 1$, so

$$\mathbf{h}'_{\mathcal{P}}(y) \cdot \mathbf{h}'_{\beta\alpha}(x) = \mathbf{h}'_{\mathcal{P}\circ\mathcal{Q}}(x) \leq \mu_{N+1}(x)$$

As this is true for all plaque chains at y with $\|\mathcal{P}\| \leq N$, we obtain $\mu_N(y) \cdot \mathbf{h}'_{\beta\alpha}(x) \leq \mu_{N+1}(x)$.

Given a plaque chain at x with $\|\mathcal{P}\| \leq N - 1$, the chain $\mathcal{R} = \mathcal{P} \circ \mathcal{Q}^{-1}$ at y has $\|\mathcal{R}\| \leq N$ and

$$\mathbf{h}'_{\mathcal{P}}(x) = \mathbf{h}'_{\mathcal{R}}(y) \cdot \mathbf{h}'_{\beta\alpha}(x) \leq \mu_N(y) \cdot \mathbf{h}'_{\mathcal{Q}}(x)$$

As this is true for all plaque chains at x with $\|\mathcal{P}\| \leq N - 1$, $\mu_{N-1}(x) \cdot \mathbf{h}'_{\beta\alpha}(x) \leq \mu_N(y) \cdot \mathbf{h}'_{\beta\alpha}(x)$. $\square$

Define the *transverse exponent* at x by

$$\lambda_*(x) = \limsup_{N\to\infty} \frac{\ln\{\mu_N(x)\}}{N} \tag{8}$$

The definition states, in words, that given any $0 < \lambda < \lambda_*(x)$ and any $n > 0$, there exists $N > n$ and plaque chain $\mathcal{P}$ with length $\|\mathcal{P}\| \leq N$ so that $\mathbf{h}'_{\mathcal{P}}(x) \geq \exp\{N\lambda\}$.

LEMMA 4.2 *The function λ_* is Borel measurable on $\mathcal{T}$, and constant on the orbits of $\mathcal{G}_{\mathcal{F}}$.*

Proof: For each $N > 0$ the function $\ln\{\mu_N(x)\}/N$ is Borel, so the supremum function is Borel measurable.

Let $x \in \mathcal{T}$ and $\mathcal{Q} = \{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$ then by the estimate (7),

$$\frac{\ln\{\mu_{N+1}(x)\}}{N+1} \geq \frac{\ln\{\mu_N(y) \cdot \mathbf{h}'_{\beta\alpha}(x)\}}{N} \cdot \frac{N}{N+1} = \left\{ \frac{\ln\{\mu_N(y)\}}{N} + \frac{\ln\{\mathbf{h}'_{\beta\alpha}(x)\}}{N} \right\} \cdot \frac{N}{N+1}$$

so that

$$\lambda_*(x) = \limsup_{N\to\infty} \left\{ \frac{\ln\{\mu_{N+1}(x)\}}{N+1} \right\} \geq \limsup_{N\to\infty} \left\{ \frac{\ln\{\mu_N(y)\}}{N} \right\} = \lambda_*(y)$$

The converse inequality follows similarly.

Thus, the function $y \mapsto \lambda_*(y)$ is constant when x and y are joined by a plaque chain of length 2. The pseudogroup $\mathcal{G}_{\mathcal{F}}$ is generated by the holonomy of plaque chains of length 2, and as each point y in the intersection of L_x with the transversal $\mathcal{T}$ is in the orbit of x under the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$, $y \mapsto \lambda_*(y)$ is constant on all such points as claimed. $\square$

Given a leaf L, set $\lambda_*(L) = \lambda_*(x)$ for some $x \in L \cap \mathcal{T}$. The following is a fundamental concept.

DEFINITION 4.3 $\mathrm{E}(\mathcal{F}) = \{z \in M \mid \lambda_*(x) > 0\}$. *For $a \geq 0$,* $\mathrm{E}_a(\mathcal{F}) = \{x \in M \mid \lambda_*(x) > a\}$.

By Lemma 4.2, $\mathrm{E}(\mathcal{F})$ and $\mathrm{E}_a(\mathcal{F})$ for $a \geq 0$ are saturated subsets of M.

A point $x \in \mathrm{E}(\mathcal{F})$ is said to be *infinitesimally expansive*. If there is an holonomy map $\mathbf{h}_{\mathcal{P}}$ with $x \in \mathcal{D}_{\mathcal{P}}$ and $\mathbf{h}_{\mathcal{P}}(x) = x$, $\mathbf{h}'_{\mathcal{P}}(x) \neq 1$, then $x \in \mathrm{E}(\mathcal{F})$. This case is classical, and the study of the dynamics of such local hyperbolic elements originated in the work of Poincaré [46], and also appeared in fundamental way in the works of Sacksteder [49], Bonatti, Langevin and Moussu [2] and many other authors. However, a point $x \in \mathrm{E}(\mathcal{F})$ may not be expanded by a just single element of holonomy – it may happen there is a sequence of holonomy elements whose length tends to infinity which realizes the condition $\lambda_*(x) > 0$ in definition (8), but which are not the powers of single hyperbolic element of holonomy as happens in the classical case. In fact, the sequence of holonomy elements need not even be associated to an infinite word of which each finite path is a truncation. This makes the analysis of the set $\mathrm{E}(\mathcal{F})$ much more delicate.

To understand the properties of the sets $\mathrm{E}_a(\mathcal{F})$ better, consider a point $x \in \mathcal{T} \cap \mathrm{E}_a(\mathcal{F})$ and λ with $a < \lambda < \lambda_*(x)$. Then for all $n > 0$, there exists $N > n$ and plaque chain $\mathcal{P}$ with length $\|\mathcal{P}\| \leq N$ so that $\mathbf{h}'_{\mathcal{P}}(x) \geq \exp\{N\lambda\}$. By the continuity of the derivative function on $\mathcal{T}$, there exists an open interval $(x - \epsilon, x + \epsilon) \subset \mathcal{T}$ so that

$$\mathbf{h}'_{\mathcal{P}}(y) \geq \exp\{N\lambda/2\}\ , \qquad \text{for all} \quad x - \epsilon \leq y \leq x + \epsilon$$

Thus, by the Mean Value Theorem, $\mathbf{h}_{\mathcal{P}}$ expands the interval $(x-\epsilon, x+\epsilon)$ by an exponential amount. Of course, $\epsilon > 0$ depends upon the choices of N, λ and x, and in particular ϵ may become exponentially small as $N \to \infty$. It is a strong condition to have this expansiveness at a fixed point x for all $N \to \infty$, and this is what makes the set $\mathrm{E}(\mathcal{F})$ a fundamental part of the study of foliation dynamics, exactly in analog with the role of the Pesin set in smooth dynamics.

Here is the main result of this section:

THEOREM 4.4 *For any set $B \in \mathcal{B}(\mathcal{F})$, the Godbillon measure $G_{\mathcal{F}}(B) = G_{\mathcal{F}}(B \cap \mathrm{E}(\mathcal{F}))$. Hence, if $\mathrm{E}(\mathcal{F})$ has Lebesgue measure zero, then $G_{\mathcal{F}}(B) = 0$ for all $B \in \mathcal{B}(\mathcal{F})$. That is,*

$$|\mathrm{E}(\mathcal{F})| = 0 \Longrightarrow G_{\mathcal{F}} = 0$$

Proof: It suffices to show that $G_{\mathcal{F}}(M - \mathrm{E}(\mathcal{F})) = 0$, as for any $B \in \mathcal{B}(\mathcal{F})$, we have

$$G_{\mathcal{F}}(B \cap (M - \mathrm{E}(\mathcal{F}))) \leq G_{\mathcal{F}}(M - \mathrm{E}(\mathcal{F})) = 0, \text{ so}$$

$$G_{\mathcal{F}}(B) = G_{\mathcal{F}}(B \cap \mathrm{E}(\mathcal{F})) + G_{\mathcal{F}}(B \cap (M - \mathrm{E}(\mathcal{F}))) = G_{\mathcal{F}}(B \cap \mathrm{E}(\mathcal{F}))$$

Set $B = M - \mathrm{E}(\mathcal{F})$. By Lemma 3.2, it will suffice to construct a sequence of measurable, leafwise smooth, non-vanishing transverse 1-forms $\{\omega_n \mid n = 1, 2, \ldots\}$ for which $\|d_{\mathcal{F}}\omega_n\|_B < 1/n$, where the norm is the sup norm over B. The construction of the forms $\{\omega_n\}$ follows the method of [28]. The first, and crucial step, is to construct an ϵ–tempered cocycle over the foliations which is cohomologous to the Radon-Nykodyn additive cocycle. This cocycle is then used to produce the 1-forms $\{\omega_n\}$, using a procedure adapted from the methods of [3] and [34, 36]. We define local 1–forms $\{\omega_\epsilon^\alpha \mid n = 1, 2, \ldots\}$ on the coordinate charts U_α, then use a partition of unity to extend these to a 1-form ω_ϵ on all of M.

For $x \in \mathcal{T} \cap B$, $\lambda_*(x) = 0$ implies that for all $\epsilon > 0$, there exists N_ϵ so that $N \geq N_\epsilon$ implies $\ln\{\mu_N(x)\} \leq N\epsilon/2$ and hence $\mu_N(x) \leq \exp\{N\epsilon/2\}$. Define a Borel function f_ϵ on $\mathcal{T} \cap B$ by

$$f_\epsilon(x) = \sum_{N=0}^{\infty} \exp\{-N\epsilon\} \cdot \mu_N(x)$$

which converges by the above estimates. Note that while $f_\epsilon(x)$ is finite for each $x \in \mathcal{T} \cap B$, there need not be an upper bound for its values on $\mathcal{T} \cap B$.

For $x \in \mathcal{T}$ but $x \notin B$, set $f_\epsilon(x) = 1$. We then obtain a measurable function f_ϵ defined on all of $\mathcal{T}$.

Let dx denote the Riemannian volume form on $\mathcal{T}$, so that $\mathbf{h}_{\mathcal{Q}}^*(dx) = \mathbf{h}_{\mathcal{Q}}'\, dx$. On the transversal $\mathcal{T}_\alpha$ set $\omega_\epsilon^\alpha = f_\epsilon\, dx$. Extend each form ω_ϵ^α from $\mathcal{T}_\alpha$ to U_α via the projection $U_\alpha \to \mathcal{T}_\alpha$ along plaques.

Choose a partition of unity $\{\rho_\alpha \mid \alpha \in \mathcal{A}\}$ subordinate to the cover $\{U_\alpha \mid \alpha \in \mathcal{A}\}$. Note that the form ω_ϵ^α on U_α is constant along the plaques, so on U_α its leafwise differential $d_{\mathcal{F}}\omega_\epsilon^\alpha = 0$.

Define the 1-form ω_ϵ on M by specifying its restriction to each open set U_α

$$\omega_\epsilon | U_\alpha = \sum_{U_\beta \cap U_\alpha \neq \emptyset} \rho_\beta\, \mathbf{h}_{\beta\alpha}^*(\omega_\epsilon^\beta) \tag{9}$$

We will calculate the Godbillon measure of B starting with the 1-form ω_ϵ. We make a preliminary estimate.

LEMMA 4.5 *For all* α, $x \in \mathcal{T}_\alpha \cap B$ *and* $\mathcal{Q} = \{\mathcal{P}_\alpha(x), \mathcal{P}_\beta(y)\}$,

$$\exp\{-\epsilon\} \cdot \omega_\epsilon^\alpha \leq \mathbf{h}_{\beta\alpha}^*(\omega_\epsilon^\beta) \leq \exp\{\epsilon\} \cdot \omega_\epsilon^\alpha \tag{10}$$

Proof: Calculate for $x \in \mathcal{T}_\alpha$ using the estimate (7),

$$\begin{aligned}
\mathbf{h}_{\beta\alpha}^*(\omega_\epsilon^\beta)(x) &= f_\epsilon(\mathbf{h}_{\beta\alpha}(x))\ \mathbf{h}'_{\beta\alpha}(x)\ dx \\
&= \left\{ \sum_{N=0}^{\infty} \exp\{-N\epsilon\} \cdot \mu_N(y)\ \mathbf{h}'_{\beta\alpha}(x) \right\} dx \\
&\leq \sum_{N=0}^{\infty} \exp\{-N\epsilon\} \cdot \mu_{N+1}(x) \\
&\leq \exp\{\epsilon\} \cdot \sum_{N=1}^{\infty} \exp\{-N\epsilon\} \cdot \mu_N(x) \\
&< \exp\{\epsilon\} \cdot \omega_\epsilon^\alpha(x)
\end{aligned} \tag{11}$$

Similarly, we have

$$\begin{aligned}
\mathbf{h}_{\beta\alpha}^*(\omega_\epsilon^\beta)(x) &\geq \sum_{N=1}^{\infty} \exp\{-N\epsilon\} \cdot \mu_{N-1}(x) \\
&\geq \exp\{-\epsilon\} \cdot \sum_{N=0}^{\infty} \exp\{-N\epsilon\} \cdot \mu_N(x) \\
&> \exp\{-\epsilon\} \cdot \omega_\epsilon^\alpha(x) \qquad \square
\end{aligned} \tag{12}$$

Recall that $\vec{n}$ denotes the unit, positively-oriented vector field on M orthogonal to $\mathcal{F}$. The Godbillon measure can then be calculated using the form $\eta_\epsilon = \iota(\vec{n}) d\omega_\epsilon / \omega_\epsilon(\vec{n})$. We estimate the norm $\|\eta_\epsilon\|$. First, consider the restriction of η_ϵ when restricted to U_α. For α fixed, we set $\phi_\beta(x) = f_\epsilon(\mathbf{h}_{\beta\alpha}(x)) \cdot \mathbf{h}'_{\beta\alpha}(x)$. Note that $\phi_\alpha(x) = f_\epsilon(x)$, and by Lemma 4.1,

$$\exp\{-\epsilon\} f_\epsilon(x) \leq \phi_\beta(x) \leq \exp\{\epsilon\} f_\epsilon(x)$$

Now estimate, using that $0 = d_{\mathcal{F}}(1) = d_{\mathcal{F}}(\sum \rho_\beta) = \sum d_{\mathcal{F}} \rho_\beta$:

$$\begin{aligned}
\|\eta_\epsilon | U_\alpha\| &= \left\| \frac{\iota(\vec{n}) d\omega_\epsilon}{\iota(\vec{n})\omega_\epsilon} \right\| \\
&= \left\| \frac{\sum_{U_\beta \cap U_\alpha \neq \emptyset} (d_{\mathcal{F}}\rho_\beta)\ f_\epsilon(\mathbf{h}_{\beta\alpha}(x)) \cdot \mathbf{h}'_{\beta\alpha}(x)}{\sum_{U_\beta \cap U_\alpha \neq \emptyset}\ \rho_\beta\ f_\epsilon(\mathbf{h}_{\beta\alpha}(x)) \cdot \mathbf{h}'_{\beta\alpha}(x)} \right\| \\
&= \left\| \frac{\sum_{U_\beta \cap U_\alpha \neq \emptyset} (d_{\mathcal{F}}\rho_\beta)\ (\phi_\beta(x) - \phi_\alpha(x))}{\sum_{U_\beta \cap U_\alpha \neq \emptyset}\ \rho_\beta\ \phi_\beta(x)} \right\| \\
&\leq \frac{\sum_{U_\beta \cap U_\alpha \neq \emptyset} \|d_{\mathcal{F}}\rho_\beta\|\ \|\phi_\beta(x) - f_\epsilon(x)\|}{\exp\{-\epsilon\} \cdot f_\epsilon(x)} \\
&\leq \{\exp\{2\epsilon\} - 1\} \cdot \sum_{U_\beta \cap U_\alpha \neq \emptyset} \|d_{\mathcal{F}}\rho_\beta\|
\end{aligned}$$

The right hand side tends to 0 as $\epsilon \to 0$. For each $n > 0$ we choose $\epsilon_n > 0$ so that $\|\eta_{\epsilon_n}\| \leq 1/n$ and set $\omega_n = \omega_{\epsilon_n}$. $\square$

COROLLARY 4.6 *If* $GV(\mathcal{F}) \neq 0$*, then* $\mathrm{E}(\mathcal{F})$ *has positive Lebesgue measure.*

5 $|\mathrm{E}(\mathcal{F})| > 0$ implies $h(\mathcal{F}) > 0$

The purpose of this section is to prove:

THEOREM 5.1 *Let $\mathcal{F}$ be a C^1 foliation of a compact manifold. If $\mathrm{E}(\mathcal{F})$ has positive Lebesgue measure, then $\mathcal{F}$ has a hyperbolic resilient leaf, and hence $h(\mathcal{F}) > 0$.*

Proof: The assumption that $|\mathrm{E}(\mathcal{F})| > 0$ is used in two ways in the proof. First, the set $\mathrm{E}(\mathcal{F})$ is an increasing union of the sets $\mathrm{E}_a(\mathcal{F})$ for $a > 0$, so $|\mathrm{E}(\mathcal{F})| > 0$ implies $\mathrm{E}_a(\mathcal{F})$ has positive Lebesgue measure for some $a > 0$. Secondly, almost every point of a measurable set is a point of positive Lebesgue density, so $|\mathrm{E}_a(\mathcal{F})| > 0$ implies that $\mathrm{E}_a(\mathcal{F})$ has a "pre-perfect" subset of points with expansion greater than a. Following these two observations, we construct an infinite sequence of hyperbolic fixed-points arbitrarily close to the support of $\mathrm{E}_a(\mathcal{F})$ with domains of uniform length. These domains have to overlap since $\mathcal{T}$ is bounded, and this will produce the ingredients for the ping-pong game, unless a resilient leaf is encountered first. We formulate these two steps as independent propositions, as they are of independent interest for the study of foliation dynamics.

PROPOSITION 5.2 *Given $a > 0$, there exists $\delta > 0$ so that for all $\eta > 0$, $0 < \lambda < 1$, and $x \in \mathrm{E}_a(\mathcal{F}) \cap \mathcal{T}$, there exists $x^* \in \mathcal{T}$ such that $(x^* - 2\delta, x^* + 2\delta) \subset \widetilde{\mathcal{T}}$ and a holonomy map $\mathbf{k}_*$ with*

1. *$(x^* - 2\delta, x^* + 2\delta) \subset \widetilde{\mathcal{T}}$ is in the domain of $\mathbf{k}_*$, hence also the closed interval $\mathcal{J}_x = [x^* - \delta, x^* + \delta]$ is in the domain of $\mathbf{k}_*$*
2. *$\mathbf{k}_*(x^*) = x^*$ and $\mathcal{I}_x = \mathbf{k}_*(\mathcal{J}_x) \subset \mathcal{J}_x$*
3. *$0 < \mathbf{k}'_*(y) < \lambda$ for all $y \in \mathcal{J}_x$*
4. *there exists a holonomy map $\mathbf{h}_{\mathcal{P}}$ defined on an open neighborhood of $\mathcal{I}_x$ so that $x \in \mathbf{h}_{\mathcal{P}}(\mathcal{I}_x)$ and $|\mathbf{h}_{\mathcal{P}}(\mathcal{I}_x)| < \eta$, hence $\mathbf{d}_{\mathcal{T}}(x, h_{\mathcal{P}}(x^*)) < \eta$, where $\mathbf{d}_{\mathcal{T}}$ is the distance on the tranversal $\mathcal{T}$.*

Thus, the leaf L_{x^*} through x^* has a closed loop whose holonomy is a uniform hyperbolic contraction, and L_{x^*} approaches within distance η of the initially given point x.

Proof of Proposition 5.2: Let $\eta > 0$, $0 < \lambda < 1$, and $x \in \mathrm{E}_a(\mathcal{F}) \cap \mathcal{T}$ be given, so $\lambda_*(x) > a$.

Recall that by Lemma 2.1, there exists $\epsilon_0 > 0$ so that for every admissible pair (α, β) and $x \in \mathcal{T}_{\alpha\beta}$ then $[x - \epsilon_0, x + \epsilon_0] \subset \widetilde{\mathcal{T}}_{\alpha\beta}$. That is, if $x \in \mathcal{T}_\alpha$ is in the domain of $\mathbf{h}_{\beta\alpha}$ then $[x - \epsilon_0, x + \epsilon_0]$ is in the domain of $\widetilde{\mathbf{h}}_{\beta\alpha}$.

Choose $0 < \epsilon_0 < a/100$ so that the conclusions of Lemma 2.1 are valid. Then, by uniform continuity, there exists $\delta > 0$ with $0 < 8\delta < \epsilon_0$ so that if $y, z \in \mathcal{T}_{\alpha\beta}[8\delta]$ and $\mathbf{d}_{\mathcal{T}}(y, z) \leq 8\delta$, then

$$\left| \log\{\widetilde{\mathbf{h}}'_{\beta\alpha}(y)\} - \log\{\widetilde{\mathbf{h}}'_{\beta\alpha}(z)\} \right| < \epsilon_0 \tag{13}$$

Choose $\mathcal{M} > 0$ satisfying $1/\mathcal{M} \leq \widetilde{\mathbf{h}}'_{\beta\alpha}(y) \leq \mathcal{M}$ for all admissible (α, β) and $y \in \mathcal{T}_{\alpha\beta}[8\delta]$.

The key to the proof of Proposition 5.2 is the following technical proposition, which shows that the pseudogroup $\mathcal{G}_{\mathcal{F}}$ expands an exponentially small neighborhood of x in $\mathcal{T}$ to a 2δ-wide interval.

PROPOSITION 5.3 *For each integer $n > 0$, there exists a closed interval $I_n^x \subset \mathcal{T}_\alpha$ containing x in its interior, and holonomy map $\mathbf{h}_n^x\colon I_n^x \to J_n^x$ such that*

1. $x_n = \mathbf{h}_n^x(x) \in I_n^x$
2. $J_n^x = [x_n - 4\delta, x_n + 4\delta] \subset \widetilde{\mathcal{T}}$
3. $(\mathbf{h}_n^x)'(y) > \exp\{na/2\} \quad \forall y \in I_n^x$
4. $|I_n^x| < 8\delta \exp\{-na/2\}$.

Proof: Choose a plaque chain $\{\mathcal{P}_n\}$ starting at x such that $\ell_n = \|\mathcal{P}_n\| \geq n$ and $\log\{\mathbf{h}'_{\mathcal{P}_n}(x)\}/\ell_n > a$. The maps $\mathbf{h}_{\mathcal{P}_n}$ are the initial candidates for the h_n^x of the proposition, but there is a serious technical problem to overcome. We are given that $\mathbf{h}'_{\mathcal{P}_n}(x) > \exp\{\ell_n a\}$ and would like to conclude this holds in an interval about x. To show this, we write the map $\mathbf{h}_{\mathcal{P}_n}$ as a composition of adjacent holonomy elements, and then apply the uniform estimate (13). The technical point is that this requires that the domains of the maps appearing in the compositions of pseudogroup elements stay bounded above in length so that they remain in the domain of the succeeding maps in the composition.

Another technical problem is that $\mathbf{h}'_{\mathcal{P}_n}(x)$ is the product of the derivatives of a sequence of factors in a composition, and it might happen that some of these factors expand much faster than "average", and others contract. The rapid expansion could inflate the images so fast that they no longer satisfy the uniform continuity hypotheses, so we loose control over the estimates. The solution is to remove any sections of the plaque chain which are "unnecessarily contracting", or at least to make sure these contractions take place at the end of the composition, and then to omit them. The solution to this problem given below is reminiscent of techniques used in Pesin Theory. (The original works of Pesin [43] and its applications in Ruelle [48] and Katok [37] are the best place to start exploring this topic, though the more recent book by Pollicott [45] gives a gentler introduction, and there is also now a textbook on the subject by Barreira and Pesin [1].)

Fix an index $n > 0$, and let $\mathcal{P}_n = \{\mathcal{P}_{\alpha_0}(z_0), \ldots, \mathcal{P}_{\alpha_{\ell_n}}(z_{\ell_n})\}$. Here, $\alpha_0 = \alpha$ and $z_0 = x$. For each $1 \leq j \leq \ell_n$ let $\mathbf{h}_{\alpha_j,\alpha_{j-1}}$ be the holonomy transformation defined by $\{\mathcal{P}_{\alpha_{j-1}}, \mathcal{P}_{\alpha_j}\}$, and so $\mathbf{h}^{-1}_{\alpha_j,\alpha_{j-1}} = \mathbf{h}_{\alpha_{j-1},\alpha_j}$. Introduce the notation $\mathbf{H}_0 = Id$, and for $1 \leq j \leq \ell_n$

$$\mathbf{H}_j = \mathbf{h}_{\alpha_j,\alpha_{j-1}} \circ \cdots \circ \mathbf{h}_{\alpha_1,\alpha_0} \tag{14}$$

Note that $z_j = \mathbf{H}_j(x)$, and that we have the recursion relation $\mathbf{H}_{j+1} = \mathbf{h}_{\alpha_{j+1},\alpha_j} \circ \mathbf{H}_j$ for $1 \leq j < \ell_n$. For each $1 \leq j \leq \ell_n$ set $\lambda_j = \log\{\mathbf{H}'_{\alpha_{j-1},\alpha_j}(z_j)\} = -\log\{\mathbf{H}'_{\alpha_j,\alpha_{j-1}}(z_{j-1})\}$. Then

$$\log\{\mathbf{H}'_{\ell_n}(x)\} = -(\lambda_1 + \cdots + \lambda_{\ell_n})$$

An index $1 \leq j \leq \ell_n$ is said to be ϵ_0-*regular* if all of the partial sum estimates hold:

$$\begin{array}{rcl} \lambda_j + \ \ \epsilon_0 & < & 0 \\ \lambda_{j-1} + \lambda_j + 2\epsilon_0 & < & 0 \\ & \vdots & \\ \lambda_1 + \cdots + \lambda_j + j\epsilon_0 & < & 0 \end{array} \tag{15}$$

The conditions (15) imply that for fixed $j > 0$ and each $i < j$, the composition $\mathbf{h}_{\alpha_i,\alpha_{i+1}} \circ \cdots \circ \mathbf{h}_{\alpha_{j-1},\alpha_j}$ is a uniformly strong linear contraction on a prescribed domain so that we have control of the estimates on its derivatives under composition. (This will be discussed in detail below.)

LEMMA 5.4 *There exists an ϵ_0-regular value $j = \xi_n$ where $1 \leq \xi_n \leq \ell_n$ which satisfies*

$$\lambda_1 + \cdots + \lambda_{\xi_n} \ \leq \ (-a + \epsilon_0)\, \ell_n \tag{16}$$

Proof: An index $k \leq \ell_n$ is ϵ_0-*irregular* if

$$\lambda_k + \cdots + \lambda_{\ell_n} + (\ell_n - k + 1)\epsilon_0 \geq 0 \tag{17}$$

If there is no irregular index, then $\xi_n = \ell_n$ is an ϵ_0-regular index. Otherwise, suppose that there exists some index k which is ϵ_0-irregular. We are given that

$$\lambda_1 + \cdots + \lambda_{\ell_n} \leq -a\ell_n \leq -\epsilon_0\, \ell_n \tag{18}$$

so the index $j = 1$ is not ϵ_0-irregular, hence $k > 1$. Let $j_0 \geq 2$ be the least ϵ_0-irregular index.

Set $\xi_n = j_0 - 1$. We claim that ξ_n is an ϵ_0-regular index. If not, then there is some $i \leq \xi_n$ with

$$\lambda_i + \cdots + \lambda_{\xi_n} + (\xi_n - i + 1)\epsilon_0 \geq 0 \tag{19}$$

Add the estimates (19) to (17) to obtain that $i < j_0$ is also an ϵ_0-irregular index, contrary to the choice of j_0. As $\epsilon_0 + 1 = j_0$ is irregular, subtract (17) from (18) to obtain

$$\lambda_1 + \cdots + \lambda_{\xi_n} \leq -a\ell_n + (\ell_n - j_0)\epsilon_0 \leq (-a + \epsilon_0)\, \ell_n$$

which yields the inequality (16). □

The inequality (16) means that almost all of the "infinitesimal expansion" of the map $\mathbf{H}_{\ell_n}$ at x is achieved by the action of $\mathbf{H}_{\xi_n}$.

Recall that $\widetilde{\mathbf{h}}_{\alpha,\beta}$ denotes the continuous extension of the map $\mathbf{h}_{\alpha,\beta}$. Define extensions of $\mathbf{H}_{\ell_n}$ and its inverse by

$$\mathbf{h}_n^x \ = \ \widetilde{\mathbf{h}}_{\alpha_{\xi_n},\alpha_{\xi_n - 1}} \circ \cdots \circ \widetilde{\mathbf{h}}_{\alpha_1,\alpha_0} \tag{20}$$

$$\mathbf{g}_n^x \ = \ \widetilde{\mathbf{h}}_{\alpha_0,\alpha_1} \circ \cdots \circ \widetilde{\mathbf{h}}_{\alpha_{\xi_n - 1},\alpha_{\xi_n}} \tag{21}$$

Set $x_n = \mathbf{h}_n^x(x)$ and $J_n^x = [x_n - 4\delta, x_n + 4\delta]$. Then by the inequality (16) we have

$$\log\{(\mathbf{g}_n^x)'(x_n)\} \ = \ \lambda_1 + \cdots \lambda_{\xi_n} \ \leq \ (-a + \epsilon_0)\, \ell_n$$

LEMMA 5.5 *J_n^x is in the domain of $\mathbf{g}_n^x$ and for all $y \in J_n^x$,*

$$\exp\{(-a - 2\epsilon_0)\, \ell_n\} \leq (\mathbf{g}_n^x)'(y) \leq \exp\{(-a + 2\epsilon_0)\, \ell_n\} \tag{22}$$

Hence, for $I_n^x = \mathbf{g}_n^x(J_n^x)$,

$$|I_n^x| \leq 8\delta \exp\{(-a + 2\epsilon_0)\ \ell_n\} \tag{23}$$

Proof: By the choice of δ, the uniform continuity estimate (13) implies that for all $y \in J_n^x$

$$\left| \log\{\widetilde{\mathbf{h}}'_{\alpha_{\xi_n - 1},\alpha_{\xi_n}}(y)\} - \log\{\widetilde{\mathbf{h}}'_{\alpha_{\xi_n - 1},\alpha_{\xi_n}}(x_n)\} \right| \leq \epsilon_0$$

Thus, by the definition of λ_{ξ_n} we have for all $y \in J_n^x$

$$\exp\{\lambda_{\xi_n} - \epsilon_0\} \leq \widetilde{\mathbf{h}}'_{\alpha_{\xi_n-1},\alpha_{\xi_n}}(y) \leq \exp\{\lambda_{\xi_n} + \epsilon_0\}$$

hence for all $y \in J_n^x$

$$4\delta \exp\{\lambda_{\xi_n} - \epsilon_0\} \leq \mathbf{d}_{\mathcal{T}}(\widetilde{\mathbf{h}}_{\alpha_{\xi_n-1},\alpha_{\xi_n}}(x_n), \widetilde{\mathbf{h}}_{\alpha_{\xi_n-1},\alpha_{\xi_n}}(y)) \leq 4\delta \exp\{\lambda_{\xi_n} + \epsilon_0\} \tag{24}$$

Now the assumption that ξ_n is ϵ_0-regular implies $\lambda_{\xi_n} - \epsilon_0 < \lambda_{\xi_n} + \epsilon_0 < 0$, hence $4\delta \exp\{\epsilon_0 + \lambda_{\xi_n}\} < 4\delta$. Now proceed by downward induction. For $0 < j \leq \xi_n$ set

$$\mathbf{g}_{n,j}^x = \widetilde{\mathbf{h}}_{\alpha_{j-1},\alpha_j} \circ \cdots \circ \widetilde{\mathbf{h}}_{\alpha_{\xi_n-1},\alpha_{\xi_n}} \ , \ \ J_{n,j}^x = \mathbf{g}_{n,j}^x(J_n^x) \ , \ \ x_{n,j} = \mathbf{g}_{n,j}^x(x)$$

Assume that for $1 < j \leq \xi_n$ we are given for all $y \in J_{n,j}^x$ the two estimates

$$\exp\{\lambda_j + \cdots + \lambda_{\xi_n} - (\xi_n - j + 1)\,\epsilon_0\} \leq (\mathbf{g}_{n,j}^x)'(y) \leq \exp\{\lambda_j + \cdots + \lambda_{\xi_n} + (\xi_n - j + 1)\,\epsilon_0\} \tag{25}$$

$$\mathbf{d}_{\mathcal{T}}(x_{n,j}, y) \leq 4\delta \tag{26}$$

For $y \in J_{n,j}^x$ the uniform continuity of the maps $\widetilde{\mathbf{h}}'_{\alpha,\beta}$, the choice of δ and the hypothesis (26) imply

$$\left|\log\{\widetilde{\mathbf{h}}'_{\alpha_{j-2},\alpha_{j-1}}(y)\} - \log\{\widetilde{\mathbf{h}}'_{\alpha_{j-2},\alpha_{j-1}}(x_{n,j})\}\right| \leq \epsilon_0$$

Thus, by the definition of $\lambda_{j-1} = \log\{\widetilde{\mathbf{h}}'_{\alpha_{j-2},\alpha_{j-1}}(x_{n,j})$ we have for all $y \in J_{n,j}^x$

$$\exp\{\lambda_{j-1} - \epsilon_0\} \leq \widetilde{\mathbf{h}}'_{\alpha_{j-2},\alpha_{j-1}}(y) \leq \exp\{\lambda_{j-1} + \epsilon_0\} \tag{27}$$

Then by the chain rule, the estimates (27) and the inductive hypothesis (25) yield the estimates

$$\exp\{\lambda_{j-1} + \cdots + \lambda_{\xi_n} - (\xi_n - j + 2)\,\epsilon_0\} \leq (\mathbf{g}_{n,j-1}^x)'(y) \leq \exp\{\lambda_{j-1} + \cdots + \lambda_{\xi_n} + (\xi_n - j + 2)\,\epsilon_0\}. \tag{28}$$

Now the assumption that ξ_n is ϵ_0-regular implies $\lambda_{j-1} + \cdots + \lambda_{\xi_n} + (\xi_n - j + 2)\,\epsilon_0 < 0$ hence $2\delta \exp\{\lambda_{j-1} + \cdots + \lambda_{\xi_n} + (\xi_n - j + 2)\,\epsilon_0\} < 2\delta$. By the Mean Value Theorem, this yields the distance bound (26) for $j-1$.

Take $j = 1$ in inequality (25) then combined with the inequality (16), we obtain for all $y \in J_{n,j}^x$

$$(\mathbf{g}_n^x)'(y) \leq \exp\{\lambda_1 + \cdots + \lambda_{\xi_n} + \xi_n\,\epsilon_0\} \leq \exp\{-a\,\ell_n + (\ell_n + \xi_n)\,\epsilon_0\} \leq \exp\{(-a + 2\epsilon_0)\,\ell_n\}$$

The estimate (23) then follows by the Mean Value Theorem. □

Since $a - 2\epsilon_0 > a/2$ and $\ell_n \geq n$, this completes the proof of Proposition 5.3. □

We now complete the proof of Proposition 5.2. Recall that $0 < \lambda < 1$ and $\eta > 0$ are given, we have fixed the choices of $\delta > 0$ and $0 < \epsilon_0 < a/100$. We need to exhibit the fixed-point x^* and the maps $\mathbf{k}_*$ and $\mathbf{h}_{\mathcal{P}}$ satisfying the conclusions of 5.2.

By Proposition 5.3, for each integer $n > 0$ we can choose a map $\mathbf{h}_n^x \colon I_n^x \to J_n^x$, where $x_n = \mathbf{h}_n^x(x)$, and $J_n^x = [x_n - 4\delta, x_n + 4\delta]$ with inverse map $\mathbf{g}_n^x = (\mathbf{h}_n^x)^{-1} \colon J_n^x \to I_n^x$.

The set $\{x_n \mid n = 1, 2, \ldots\} \subset \mathcal{T}$ has an accumulation point $x_* \in \overline{\mathcal{T}} \subset \widetilde{\mathcal{T}}$, and passing to a subsequence if necessary, we can assume $x_n \to x_*$ and $\mathbf{d}_{\mathcal{T}}(x_*, x_n) < \min\{\eta/2, \delta/2\}$ for all $n > 0$.

Define $J_* = [x_* - 3\delta, x_* + 3\delta]$.

For all $n > 0$, $x_n \in (x_* - \delta, x_* + \delta) \subset J_*$. Moreover, if $y \in J_*$ then

$$\mathbf{d}_{\mathcal{T}}(y, x_n) \leq \mathbf{d}_{\mathcal{T}}(y, x_*) + \mathbf{d}_{\mathcal{T}}(x_*, x_n) \leq 2\delta + \delta/2 < 3\delta$$

hence $J_* \subset J_n^x$. In particular, $x_1 \in (x_* - \delta, x_* + \delta) \subset J_* \subset J_1^x$ is an interior point of J_* so $x = g_1^x(x_1)$ is an interior point of $g_1^x(J_*)$.

Note that $x \in I_n^x$ for all n, and the interval $I_n^x = \mathbf{g}_n^x(J_n^x)$ has length $|I_n^x| < 8\delta \exp\{-na/2\}$, so for $n \gg 0$ the interval I_n^x is in the interior of $g_1^x(J_*)$. Without loss of generality, we pass to a subsequence of the index set $\{\ell_n \mid n = 1, 2, \ldots\}$ to obtain $I_n^x \subset g_1^x(J_*)$ and $\ell_n > \ell_{n-1}$ for all $n > 0$.

Combine these inclusions to obtain

$$\mathbf{g}_n^x(J_*) \subset \mathbf{g}_n^x(J_n^x) = I_n^x \subset \mathbf{g}_1^x(J_*) \tag{29}$$

Thus $\mathbf{h}_1^x \circ \mathbf{g}_n^x : J_* \to \mathbf{h}_1^x \circ \mathbf{g}_1^x(J_*) = J_*$. As J_* is compact, $\mathbf{h}_1^x \circ \mathbf{g}_n^x$ has a fixed point in J_*.

LEMMA 5.6 $\mathbf{h}_1^x \circ \mathbf{g}_n^x$ *is a hyperbolic contraction on* J_* *for* $n \gg 0$. *Thus, there exists a unique fixed-point* $y_n \in J_*$ *for* $\mathbf{h}_1^x \circ \mathbf{g}_n^x$.

Proof: By the choice of $\mathcal{M}$ we have $(\mathbf{h}_1^x)'(y) \leq \mathcal{M}^{\xi_1}$ while $\mathbf{g}_n^x$ satisfies the estimates (22), so for all $y \in J_*$ the composition $\mathbf{h}_1^x \circ \mathbf{g}_n^x$ satisfies

$$(\mathbf{h}_1^x \circ \mathbf{g}_n^x)'(y) \leq \mathcal{M}^{\xi_1}\ \exp\{(-a + 2\epsilon_0)\,\ell_n\} \tag{30}$$

and $\mathcal{M}^{\xi_1}\ \exp\{(-a + 2\epsilon_0)\,\ell_n\} < 1$ for n sufficiently large. □

Now choose n sufficiently large so that

$$\mathcal{M}^{\xi_1}\ \exp\{-a\,\ell_n/2\} \quad < \quad \lambda/12 \tag{31}$$

$$\exp\{-a\,\ell_n/2\} \quad < \quad \eta/2\delta \tag{32}$$

Then set $\mathbf{k}_* = \mathbf{h}_1^x \circ \mathbf{g}_n^x$ and $\mathbf{h}_{\mathcal{P}} = \mathbf{g}_1^x$. Set $x^* = y_n$ and $\mathcal{J}_x = [y_n - \delta, y_n + \delta]$. See Figure 4.

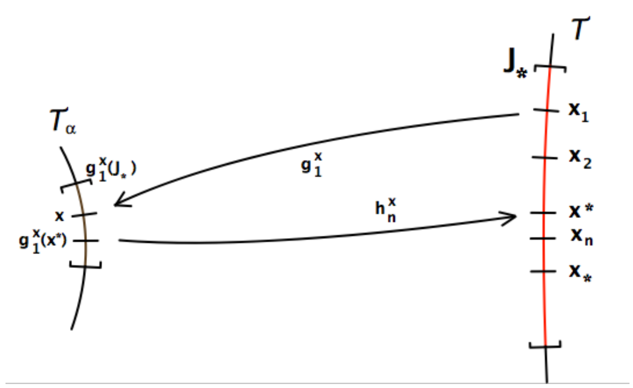

Figure 4: Construction of the contracting holonomy map $\mathbf{k}_*$

We next check that the conditions (5.2.1) to (5.2.4) are satisfied.

Note that $x_n, y_n \in J_*$ implies $\mathbf{d}_{\mathcal{T}}(x_n, y_n) < 6\delta$, so by the Mean Value Theorem and the estimates (22) we have

$$\mathbf{d}_{\mathcal{T}}(x, \mathbf{g}_n^x(y_n)) = \mathbf{d}_{\mathcal{T}}(\mathbf{g}_n^x(x_n), \mathbf{g}_n^x(y_n)) < 6\delta \ \exp\{-a\,\ell_n/2\} \tag{33}$$

Now by the definition of $\mathcal{M}$ and the hypothesis (31) the images under $\mathbf{h}_1^x$ satisfy

$$\mathbf{d}_{\mathcal{T}}(x_1, y_n) = \mathbf{d}_{\mathcal{T}}(\mathbf{h}_1^x(x), \mathbf{h}_1^x(\mathbf{g}_n^x(y_n))) < \mathcal{M}^{\xi_n}\, 6\delta \ \exp\{-a\,\ell_n/2\} < \delta\,\lambda/2 < \delta/2$$

Thus, $\mathbf{d}_{\mathcal{T}}(x_*, y_n) \leq \mathbf{d}_{\mathcal{T}}(x_*, x_1) + \mathbf{d}_{\mathcal{T}}(x_1, y_n) < \delta/2 + \delta/2 = \delta$ and hence the open interval

$$(y_n - 2\delta, y_n + 2\delta) \subset (x_* - 3\delta, x_* + 3\delta) \subset J_*$$

lies in the domain of $\mathbf{k}_* = \mathbf{h}_1^x \circ \mathbf{g}_n^x$. This proves (5.2.1).

The condition (5.2.2) states that $\mathbf{k}_*(y_n) = y_n$ and $\mathbf{k}_*([y_n - \delta, y_n + \delta]) \subset [y_n - \delta, y_n + \delta]$ which are both true by choice of the map $\mathbf{k}_*$.

The condition (5.2.3) that $\mathbf{k}_*'(y) < \lambda$ for $y \in \mathcal{J}_x$ now follows from estimates (30) and (31)

$$\mathbf{k}_*'(y) = (\mathbf{h}_1^x \circ \mathbf{g}_n^x)'(y) \leq \mathcal{M}^{\xi_1} \ \exp\{(-a + 2\epsilon_0)\,\ell_n\} < \mathcal{M}^{\xi_1} \ \exp\{-a\,\ell_n/2\} < \lambda/12 < 1$$

Finally, $\mathbf{h}_{\mathcal{P}}(\mathcal{I}_x) = \mathbf{g}_1^x(\mathbf{h}_1^x \circ \mathbf{g}_n^x(\mathcal{J}_x)) = \mathbf{g}_n^x(\mathcal{J}_x))$ so by inequalities (22) and (32) we obtain

$$|\mathbf{h}_{\mathcal{P}}(\mathcal{I}_x)| < \exp\{-a\,\ell_n/2\}\, 2\,\delta < \eta$$

This shows (5.2.4) and completes the proof of Proposition 5.2. $\square$

The second part of the proof of Theorem 5.1 is to show that the uniformly hyperbolic contractions constructed in Proposition 5.2 force the existence of a resilient orbit. By our previous remarks, $|\mathrm{E}(\mathcal{F})| > 0$ implies there exists some $a > 0$ such that $|\mathrm{E}_a(\mathcal{F})| > 0$, and hence there is a subset $\mathcal{E} \subset \mathrm{E}_a(\mathcal{F})$ with positive Lebesgue measure for which every point $y \in \mathcal{E}$ has positive Lebesgue density. What we need to show - that there exists resilient leaves – is actually much weaker than $|\mathrm{E}_a(\mathcal{F})| > 0$, so we formulate and prove Proposition 5.8 in this greater generality.

Say that a set $\mathcal{E}$ is *pre-perfect* if it is non-empty and its closure $\overline{\mathcal{E}}$ is a perfect set. Equivalently, $\mathcal{E}$ is pre-perfect if it is not empty, and no point is isolated.

LEMMA 5.7 *If $X \subset \mathbb{R}^q$ has positive Lebesgue measure, then there is a pre-perfect subset $\mathcal{E} \subset X$.*

Proof: Let $\overline{X}$ denote the closure of X. The closure can be written as a disjoint union of sets $\overline{X} = Y \cup Z$, where Z is perfect and Y is countable. Then

$$0 < |X| = |X \cap \overline{X}| = |X \cap Y| + |X \cap Z| = |X \cap Z|$$

as Y is a countable set. Thus $X \cap Z$ is not empty. We set $\mathcal{E} = Z \cap X$, and then note that Z closed implies $\overline{\mathcal{E}} = \overline{X \cap Z} = \overline{X} \cap Z = Z$. $\square$

PROPOSITION 5.8 *Let $a > 0$. If there exists a pre-perfect subset $\mathcal{E} \subset \mathrm{E}_a(\mathcal{F})$, then $\mathcal{F}$ has a resilient leaf.*

Proof of Proposition 5.8: Let $a > 0$ and let $\mathcal{E} \subset \mathrm{E}_a(\mathcal{F})$ be a pre-perfect set. The saturation of a pre-perfect set under the action of the holonomy pseudogroup Γ is pre-perfect, so we can assume that $\mathcal{E}$ is saturated. Let $\delta > 0$ be the constant chosen in the proof of Proposition 5.2. We assume that $\mathcal{F}$ does not have a resilient leaf, and show this leads to a contradiction.

Choose $x \in \mathcal{E}$, then following the proof of Proposition 5.2 with a slight modification of notation, let $\mathcal{J}_1 = [x_* - \delta, x_* + \delta]$, and let $\mathbf{k}_1 = \mathbf{k}_* = \mathbf{h}_1^x \circ \mathbf{g}_n^x : \mathcal{J}_1 \to \mathcal{J}_1$ be a uniform hyperbolic contraction with fixed-point $z_1 \in \mathcal{J}_1$. Let $\mathcal{I}_1$ denote the image $\mathbf{k}_1(\mathcal{J}_1)$, and for the holonomy map $\mathbf{h}_{\mathcal{P}}$ set $\mathcal{K}_1 = \mathbf{h}_{\mathcal{P}}(\mathcal{I}_1)$ which by hypothesis (5.2.3) can be chosen so that $|\mathcal{K}_1| < \eta$ for any prescribed value of $0 < \eta < 1$. Note that Proposition 5.2 gives that $x_1 \in \mathcal{K}_1$ is an interior point.

If the orbit of z_1 under $\mathcal{G}_{\mathcal{F}}$ intersects $\mathcal{K}_1$ (or $\mathcal{J}_1$) in a point other than z_1, then by definition, z_1 is a hyperbolic resilient point, which by assumption does not exist. Therefore, the leaf L_{z_1} through z_1 intersects the interval $\mathcal{K}_1$ exactly in one interior point, which we denote by w_1. Note this implies the leaf L_{z_1} is proper.

We next use induction to show the existence of an infinite sequence of uniform contractions with domains of uniform width. Assume that we are given, for $1 \leq i \leq n$,

- distinct points $\{x_1, x_2, \ldots, x_n\} \subset \mathcal{E}$
- distinct points $\{z_1, z_2, \ldots, z_n\}$
- intervals $\mathcal{J}_i = [z_i - \delta, z_i + \delta]$
- uniformly hyperbolic contractions $\mathbf{k}_i : \mathcal{J}_i \to \mathcal{J}_i$ with $\mathbf{k}_i(z_i) = z_i$

which satisfy the conditions

- the closed intervals $\mathcal{K}_i = \mathbf{h}_{\mathcal{P},i}(\mathcal{I}_i)$ where $\mathcal{I}_i = \mathbf{k}_i(\mathcal{J}_i)$ form a proper descending nested chain,

$$\mathcal{K}_1 \supset \mathrm{int}(\mathcal{K}_1) \supset \mathcal{K}_2 \supset \mathrm{int}(\mathcal{K}_2) \supset \cdots \supset \mathcal{K}_n$$

- the orbit of z_ℓ is disjoint from $\mathcal{K}_i$ for $\ell < i$
- the orbit of z_i intersects $\mathcal{K}_i$ exactly in one interior point, denoted by w_i
- $x_i \in \mathrm{int}(\mathcal{K}_i)$ for $i \leq i \leq n$

By assumption, the orbit of z_n under the holonomy pseudogroup intersects $\mathcal{K}_n$ in exactly one point, while the orbits of z_k for $k < n$ are disjoint. Moreover, $\mathrm{int}(\mathcal{K}_n)$ is an open neighborhood of $x_n \in \mathcal{E}$ so by the hypothesis that $\mathcal{E}$ is pre-perfect, there exists $x_{n+1} \in \mathcal{E} \cap (\mathcal{I}_n - \{x_n, w_n\})$. Let $\eta > 0$ be chosen so that $[x_{n+1} - \eta, x_{n+1} + \eta] \subset \mathcal{I}_n - \{x_n, w_n\}$. Then by Proposition 5.2 we can find $z_{n+1} \in \widetilde{\mathcal{T}}$ and a uniformly hyperbolic contraction $\mathbf{k}_{n+1} : \mathcal{J}_{n+1} \to \mathcal{I}_{n+1} \subset \mathcal{J}_{n+1}$ with $\mathbf{k}_i(z_{n+1}) = z_{n+1}$ where $\mathcal{J}_{n+1} = [z_{n+1} - \delta, z_{n+1} + \delta]$. Moreover, there is a holonomy map $\mathbf{h}_{\mathcal{P},n+1} : \mathcal{I}_{n+1} \to \widetilde{\mathcal{T}}$ whose image contains x_{n+1} in its interior, and has length $|\mathbf{h}_{\mathcal{P},n+1}(\mathcal{I}_{n+1})| < \eta$.

Set $\mathcal{K}_{n+1} = \mathbf{h}_{\mathcal{P},n+1}(\mathcal{I}_{n+1})$, so that $\mathcal{K}_{n+1} \subset (x_{n+1} - \eta, x_{n+1} + \eta) \subset \mathrm{int}(\mathcal{K}_n)$. By the induction hypotheses and the choice of η, this implies that the orbit of z_k is disjoint from $\mathcal{K}_{n+1}$ for $k \leq n$.

Moreover, the intersection of the orbit of z_{n+1} with $\mathcal{K}_{n+1}$ contains the point $w_{n+1} = \mathbf{h}_{\mathcal{P},n+1}(z_{n+1})$. If the orbit of z_{n+1} intersects $\mathcal{K}_{n+1}$ in a second point, then this implies the orbit of z_{n+1} also intersects the domain of the contraction $\mathbf{k}_{n+1} \colon \mathcal{J}_{n+1} \to \mathcal{J}_{n+1}$ in a point distinct from z_{n+1}, which contradicts the assumption that there are no resilient orbits.

This completes the inductive step, so by inductive recursion, we obtain a sequence of points $\{z_1, z_2, \ldots\} \subset \mathcal{T}[8\delta]$ and maps as above. Let $z_* \in \mathcal{T}[8\delta]$ denote an accumulation point for this sequence. Then there exists indices $m, n \gg 0$ so that $\mathbf{d}_{\mathcal{T}}(z_m, z_*) < \delta/100$ and $\mathbf{d}_{\mathcal{T}}(z_n, z_*) < \delta/100$. In the proof of Proposition 5.2 the uniform contractions $\mathbf{k}_i \colon \mathcal{J}_i \to \mathcal{J}_i$ were actually chosen with exponents less than $\lambda/12$, hence $\mathbf{k}_m \colon \mathcal{J}_m \to \mathcal{J}_*$ and $\mathbf{k}_n \colon \mathcal{J}_m \to \mathcal{J}_*$. Thus, $\mathbf{k}_m$ and $\mathbf{k}_n$ define a "ping-pong game" as in Definition 2.3, which by Proposition 2.4 implies there is a resilient orbit. This is contrary to assumption.

Hence, if there exists a pre-perfect set $\mathcal{E} \subset \mathrm{E}_a(\mathcal{F})$ for $a > 0$ then there exist a resilient leaf. $\square$

6 Open manifolds

Let $\mathbf{B}\Gamma_1^{(2)}$ denote the Haefliger classifying space of codimension–one C^2-foliations [19, 20]. There is a universal Godbillon-Vey class $GV \in H^3(\mathbf{B}\Gamma_1^{(2)})$ such that for every codimension–one C^2-foliation $\mathcal{F}$ of a manifold M, there is a classifying map $h_{\mathcal{F}}: M \to \mathbf{B}\Gamma_1^{(2)}$ such that $h_{\mathcal{F}}^* GV = GV(\mathcal{F})$ (see [4, 38].) It is known that the first two integral homotopy groups $\pi_1(\mathbf{B}\Gamma_1^{(2)}) = 0 = \pi_2(\mathbf{B}\Gamma_1^{(2)})$. Thus, a consequence of Thurston's construction of codimension–one foliations on a compact 3-manifold $(M^3, \mathcal{F}_a)$ with $\langle GV(\mathcal{F}_a, [M]\rangle = a$ for any $a > 0$ is that evaluation of the universal Godbillon-Vey class defines a surjection of the integral homotopy group, $GV: \pi_3(\mathbf{B}\Gamma_1^{(2)}) \to \mathbb{R}$.

One of the points of the introduction of the classifying space $\mathbf{B}\Gamma_1^{(2)}$ is that given any finite CW complex $\mathbf{X}$, a map $h: \mathbf{X} \to \mathbf{B}\Gamma_1^{(2)}$ defines a foliated microbundle over X, whose total space M is an open manifold with a codimension–one foliation $\mathcal{F}_h$ such that $h^* GV = GV(\mathcal{F}_h)$. This is discussed in detail by Haefliger [20], who introduced the technique. (A similar approach to classifying foliations via foliated microbundles was also introduced by Milnor [39].) Thus, using homotopy methods to construct the map h so that $h^* GV \neq 0$, one can construct many examples of open foliated manifolds with non-trivial Godbillon-Vey classes. In this section, we comment on how to apply the methods of this paper in this open manifold case, to conclude that every codimension-one foliation with non-trivial Godbillon-Vey class must have a hyperbolic resilient leaf.

Let M be an open m-manifold with a codimension–one C^2-foliation $\mathcal{F}$ such that $GV(\mathcal{F}) \neq 0$. The class $GV(\mathcal{F})$ is then determined by its pairing with the compactly supported cohomology group $H_c^{m-3}(M)$, so there exists a closed $m-3$ form ξ with compact support on M such that $\langle GV(\mathcal{F}), [\xi]\rangle \neq 0$. Let $|\xi| \subset M$ denote the closed support of ξ. Then there exists an open subset $M_0 \subset M$ such that the closure $\overline{M_0}$ is a compact subset of M and $|\xi| \subset M_0$. Choose a Riemannian metric on TM whose restriction to M_0 will then have bounded geometry. That is, there is a positive lower bound on the injectivity radius in M for points in M_0. This is all that is required to construct a finite open cover of $|\xi|$ by a regular foliation atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ for $\mathcal{F}$ on M (as in section 2 above) contained in M_0 (see Chapter 1.2 of [5].)

Let $\mathcal{V} \subset M_0$ given by the union of the open sets in this open cover of $|\xi|$, then note that $|\xi| \subset \mathcal{V}$ hence $GV(\mathcal{F}|\mathcal{V}) \neq 0$. The proof of Theorem 4.4 used only the properties of the pseudogroup generated by the regular foliation atlas $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ – the compactness of M was not used except in the construction of this atlas. The definition and properties of the Godbillon measure also applies to open manifolds, as was discussed in section 5 of [28]. Hence, by the same proof we obtain that the set $\mathrm{E}(\mathcal{F}|\mathcal{V})$ has positive measure.

The proofs of Propositions 5.2 and 5.8 were formulated strictly in terms of the groupoid $\mathcal{G}_{\mathcal{F}}$ and also do not use the compactess of M, hence apply directly to show that $\mathcal{G}\mathcal{F}|\mathcal{V}$ has a hyperbolic resilient point if $\mathrm{E}(\mathcal{F}|\mathcal{V})$ has positive measure. Thus, $\mathcal{F}|\mathcal{V}$ must have a resilient leaf.

These remarks complete the proof of Theorem 1.1 for open manifolds.

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