COMPACT FOLIATIONS WITH FINITE TRANSVERSE LS CATEGORY

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ABSTRACT. We prove that if F is a foliation of a compact manifold M with all leaves compact submanifolds, and the transverse saturated category of F is finite, then the leaf space M/F is compact Hausdorff. The proof is surprisingly delicate, and is based on some new observations about the geometry of compact foliations. The transverse saturated category of a compact Hausdorff foliation is always finite, so we obtain a new characterization of the compact Hausdorff foliations among the compact foliations as those with finite transverse saturated category.

1. INTRODUCTION

A *compact foliation* is a foliation of a manifold M with all leaves compact submanifolds. For codimension one or two, a compact foliation $\mathcal{F}$ of a compact manifold M defines a fibration of M over its leaf space $M/\mathcal{F}$ which is a Hausdorff space, and has the structure of an orbifold [27, 11, 12, 33, 10].

A compact foliation $\mathcal{F}$ with Hausdorff leaf space is said to be *compact Hausdorff*. Millett [22] and Epstein [12] showed that for a compact Hausdorff foliation $\mathcal{F}$ of a manifold M, the holonomy group of each leaf is finite, a property which characterizes them among the compact foliations. If every leaf has trivial holonomy group, then a compact Hausdorff foliation is a fibration. Otherwise, a compact Hausdorff foliation is a "generalized Seifert fibration", where the leaf space $M/\mathcal{F}$ is a "V-manifold" [29, 17, 22]. In addition, M admits a Riemannian metric so that the foliation is Riemannian.

For codimension three and above, the leaf space $M/\mathcal{F}$ of a compact foliation of a compact manifold need not be a Hausdorff space. This was first shown by an example of Sullivan [30] of a flow on a compact 5-manifold whose orbits are circles, and the lengths of the orbits are not bounded above. Subsequent examples of Epstein and Vogt [13, 35] showed that for any codimension greater than two, there are examples of compact foliations of compact manifolds whose leaf spaces are not Hausdorff, and for which the "bad set" of leaves with infinite holonomy have arbitrary countable hierarchy. Also, Vogt gave a remarkable example of a 1-dimensional, compact C^0-foliation of $\mathbb{R}^3$ with no upper bound on the lengths of the circle leaves in [36]. The results described below apply to the case of compact C^1-foliations of compact manifolds.

A compact foliation whose leaf space is non-Hausdorff has a closed, non-empty saturated subset, the *bad set* $X_1 \subset M$, which is the union of the leaves whose holonomy group is infinite. The image of X_1 in the leaf space $M/\mathcal{F}$ consists of the points which do not have the Hausdorff separation property for the quotient T_1 topology on $M/\mathcal{F}$. That is, for a leaf $L_0 \subset X_1$ with image point $[L_0] \in M/\mathcal{F}$ there exists a leaf $L_1 \subset M$ so that any open neighborhoods in $M/\mathcal{F}$ of $[L_0]$ and $[L_1]$ must have non-trivial intersection. The work by Edwards, Millett and Sullivan [10] established many fundamental properties of the geometry of the leaves of a compact foliation near its bad set, yet there is no general structure theory for compact foliations, comparable to what is understood for compact Hausdorff foliations. The results of §§4, 5 and 6 of this work provide new insights and techniques for the study of these foliations. In particular, we introduce in Definition 5.1 the notion of a *tame point* for the bad set X_1, which is a key idea for this work.

The transverse Lusternik-Schnirelmann (LS) category of foliations was introduced in the 1998 thesis of H. Colman [4, 8]. The key idea is that of a transversally categorical open set. Let $(M, \mathcal{F})$ and

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$(M', \mathcal{F}')$ be foliated manifolds. A homotopy $H: M' \times [0,1] \to M$ is said to be *foliated* if for all $0 \leq t \leq 1$ the map H_t sends each leaf L' of $\mathcal{F}'$ into another leaf L of $\mathcal{F}$. An open subset U of M is *transversely categorical* if there is a foliated homotopy $H: U \times [0,1] \to M$ such that $H_0: U \to M$ is the inclusion, and $H_1: U \to M$ has image in a single leaf of $\mathcal{F}$. Here U is regarded as a foliated manifold with the foliation induced by $\mathcal{F}$ on U. In other words, an open subset U of M is transversely categorical if the inclusion $(U, \mathcal{F}_U) \hookrightarrow (M, \mathcal{F})$ factors through a single leaf, up to foliated homotopy.

DEFINITION 1.1. *The transverse (saturated) category* $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ *of a foliated manifold* $(M, \mathcal{F})$ *is the least number of transversely categorical open saturated sets required to cover* M*. If no such finite covering exists, then* $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) = \infty$*.*

The transverse category $\mathrm{cat}_{\pitchfork}(M, \mathcal{F})$ of a compact Hausdorff foliation $\mathcal{F}$ of a compact manifold M is always finite [8], as every leaf admits a saturated product neighborhood which is transversely categorical. For a non-Hausdorff compact foliation, our main result is that there is no transversely categorical covering of the the bad set.

THEOREM 1.2. *Let* $\mathcal{F}$ *be a compact* C^1*-foliation of a compact manifold* M *with non-empty bad set* X_1*. Then there exists a dense set of tame points* $X_1^t \subset X_1$*. Moreover, for each* $x \in X_1^t$*, there is no transversely categorical saturated open set containing* x*.*

COROLLARY 1.3. *Let* $\mathcal{F}$ *be a compact* C^1*-foliation of a compact manifold* M*. If* M *admits a covering by transversely categorical open saturated sets, then* $\mathcal{F}$ *is compact Hausdorff.*

Recall that a foliation is *geometrically taut* if the manifold M admits a Riemannian metric so that each leaf is an immersed minimal manifold [28, 31, 15]. Rummler proved in [28] that a compact foliation is Hausdorff if and only if it is taut, and thus we can conclude:

COROLLARY 1.4. *A compact* C^1*-foliation of a compact manifold* M *with* $\mathrm{cat}_{\pitchfork}(M, \mathcal{F}) < \infty$ *is geometrically taut.*

The idea of the proof of Theorem 1.2 is as follows. The formal definition of the bad set X_1 in §3 is that it consist of leaves of $\mathcal{F}$ such that every open neighborhood of the leaf contains leaves of arbitrarily large volume. This characterization of the bad set intuitively suggests that it should be a rigid set. That is, any foliated homotopy of an open neighborhood of a point in the bad set should preserve these dynamical properties, hence the open neighborhood cannot be continuously retracted to a single leaf. The proof of this statement is surprisingly delicate, and requires a very precise understanding of the properties of leaves in an open neighborhood of the bad set. A key result is Proposition 5.2, an extension of the Moving Leaf Lemma in [10], which establishes the existence of "tame points".

The overview of the paper is as follows: The first two sections consist of background material, which we recall to establish notations, and also present a variety of technical results required in the later sections. In §2 we give some basic results from foliation theory, and in §3 we recall some basic results about compact foliations, especially the structure theory for the good and the bad sets. In §4 we establish a key homological property for compact leaves under deformation by a homotopy. The techniques introduced in this section are used again in later sections. The most technical results of the paper are contained in §5, where we prove that tame points are dense in the bad set. Finally, in §6 we prove that an open saturated set containing a tame point is not categorical. Theorem 1.2 follows immediately from Propositions 5.2 and 6.3.

2. Foliation preliminaries

We assume that M is a compact smooth Riemannian manifold without boundary of dimension $m = p+q$, that $\mathcal{F}$ is a compact C^1-foliation of codimension-q, and that the leaves of $\mathcal{F}$ are smoothly immersed compact submanifolds, so that $\mathcal{F}$ is more precisely a $C^{1,\infty}$-foliation. For $x \in M$, denote by L_x the leaf of $\mathcal{F}$ containing x.

We recall below some well-known facts about foliations, and introduce some conventions of notation. The books [3, 14, 16] provide excellent basic references; our notation is closest to that used in [3]. Note that the analysis of the bad sets in later sections requires careful estimates on the foliation geometry; not just in each leaf, but also for nearby leaves of a given leaf. This requires a careful description of the local metric geometry of a foliation, as given in this section.

2.1. **Tangential and normal geometry.** Let $T\mathcal{F}$ denote the tangent bundle to $\mathcal{F}$, and let $\Pi\colon Q \to M$ denote its normal bundle, identified with the subbundle $T\mathcal{F}^{\perp} \subset TM$ of vectors orthogonal to $T\mathcal{F}$. The Riemannian metric on TM induces Riemannian metrics on both $T\mathcal{F}$ and Q by fiberwise restriction. For a vector $\vec{v} \in T_xM$, let $\|\vec{v}\|$ denote its length in the Riemannian metric. Then for $\vec{v} \in T_x\mathcal{F}$ the length in the induced leafwise metric is also denoted by $\|\vec{v}\|$.

For $\epsilon > 0$, let $T^{\epsilon}M \subset TM$ denote the disk subbundle of vectors with length less than ϵ, and let $T^{\epsilon}\mathcal{F} \subset T\mathcal{F}$ and $Q^{\epsilon} \subset Q$ be the corresponding ϵ-disk subbundles of $T\mathcal{F}$ and Q, respectively.

Let $d_M\colon M \times M \to [0,\infty)$ be the distance function associated to the Riemannian metric on M. Given $r > 0$ and a set $K \subset M$, let

$$B_M(K, r) = \{y \in M \mid d_M(K, y) < r\} \ . \tag{1}$$

For a leaf $L \subset M$, let $d_L\colon L \times L \to [0,\infty)$ be the distance function on L for the restricted Riemannian metric on L. That is, for $x, x' \in L$ the distance $d_L(x, x')$ is the infimum of the lengths of piece-smooth leafwise paths between x and x'. As L is compact, the manifold L with the metric d_L is a complete metric space, and the distance $d_L(x, x')$ is realized by a leafwise geodesic path from x to x'. We introduce the notation $d_{\mathcal{F}}$ for the collection of leafwise distance functions, where $d_{\mathcal{F}}(x, y) = d_L(x, y)$ if $x, y \in L$, and otherwise $d_{\mathcal{F}}(x, y) = \infty$. Given $r > 0$ and a set $K \subset L$, let

$$B_{\mathcal{F}}(K, r) = \{y \in M \mid d_{\mathcal{F}}(K, y) < r\} \subset L \ . \tag{2}$$

Let $\exp = \exp^M\colon TM \to M$ denote the exponential map for d_M which is well-defined as M is compact. For $x \in M$, we let $\exp^M_x\colon T_xM \to M$ denote the exponential map at x.

For $x \in L$, we let $\exp^{\mathcal{F}}_x\colon T_xL \to L$ denote the exponential map for the leafwise Riemannian metric. Then $\exp^{\mathcal{F}}_x$ maps the ball $B_{T_xL}(0, r)$ of radius r in T_xL onto the set $B_{\mathcal{F}}(x, r)$.

We next chose $\epsilon_0 > 0$ so that it satisfies a sequence of conditions, as follows. For each $x \in M$, the differential $D_{\vec{0}} \exp^M_x\colon T_xM \cong T_{\vec{0}}(T_xM) \to T_xM$ is the identity map. It follows that there exists $\epsilon_x > 0$ such that the restriction $\exp^M_x\colon T^{\epsilon_x}_x M \to M$ is a diffeomorphism. As M is compact, there exists $\epsilon_0 > 0$ such that for all $x \in M$, the restriction $\exp^M_x\colon T^{\epsilon_0}_x M \to M$ is a diffeomorphism onto its image. Thus, ϵ_0 is less than the injectivity radius of the Riemannian metric on M. (See [1, 9] for details of the properties of the injectivity radius of the geodesic map.)

We also require that $\epsilon_0 > 0$ be chosen so that for all $x \in M$:

- (ϵ1) The open ball $B_M(x, \epsilon_0)$ is a totally normal neighborhood of x for the metric d_M This means that for any pair of points $y, z \in B_M(x, \epsilon_0)$ there is a unique geodesic contained in $B_M(x, \epsilon_0)$ between y and z. In particular, $B_M(x, \epsilon_0)$ is geodesically convex (See [9, page 72].)
- (ϵ2) The leafwise exponential map $\exp^{\mathcal{F}}_x\colon T^{\epsilon_0}_x\mathcal{F} \to L_x$ is a diffeomorphism onto its image.
- (ϵ3) $B_{\mathcal{F}}(x, \epsilon_0) \subset L_x$ is a totally normal neighborhood of x for the leafwise metric $d_{\mathcal{F}}$.

Let $\exp^Q_x\colon Q_x \to M$ denote the restriction of $\exp^M_x$ to the normal bundle at x. Then for all $x \in M$, $\exp^Q_x\colon Q^{\epsilon_0}_x \to M$ is a diffeomorphism onto its image. We also require that $\epsilon_0 > 0$ satisfy:

- (ϵ4) For all $x \in M$, $\exp^Q_x\colon Q^{\epsilon_0}_x \to M$ is transverse to $\mathcal{F}$, and that the image $\exp^Q_x(Q^{\epsilon_0}_x)$ of the normal disk has angle at least $\pi/4$ with the leaves of the foliation $\mathcal{F}$.

We use the normal exponential map to define a normal product neighborhood of a subset $K \subset L$ for a leaf L. Given $0 < \epsilon \leq \epsilon_0$, let $Q(K, \epsilon) \to K$ denote the restriction of the ϵ-disk bundle $Q^{\epsilon} \to M$ to K. The normal neighborhood $\mathcal{N}(K, \epsilon)$ is the image of the map, $\exp^Q\colon Q(K, \epsilon) \to M$. If $K = \{x\}$ is a point and $0 < \epsilon < \epsilon_0$, then $\mathcal{N}(x, \epsilon)$ is a uniformly transverse normal disk to $\mathcal{F}$.

The restriction of the ambient metric d_M to a leaf L need not coincide (locally) with the leafwise geodesic metric $d_{\mathcal{F}}$ – unless the leaves of $\mathcal{F}$ are totally geodesic submanifolds of M. In any case, the Gauss Lemma implies that the two metrics are locally equivalent. We require that $\epsilon_0 > 0$ satisfy:

(ϵ5) For all $x \in M$, and for all $y, y' \in B_{\mathcal{F}}(x, \epsilon_0)$, then $d_{\mathcal{F}}$ and d_M are related by

$$d_M(y, y')/2 \ \leq \ d_{\mathcal{F}}(y, y') \ \leq \ 2\, d_M(y, y') \ . \tag{3}$$

Let $d\mathrm{vol}$ denote the leafwise volume p-form associated to the Riemannian metric on $T\mathcal{F}$. Given any bounded, Borel subset $A \subset L$ for the leafwise metric, define its leafwise volume by $\mathrm{vol}\,(A) = \int_A \, d\mathrm{vol}\,$.

Let $L \subset M$ be a compact leaf, then there exists $0 < \epsilon_L < \epsilon_0$ such that the normal geodesic map $\exp^Q \colon Q(K, \epsilon_L) \to M$ is a diffeomorphism onto the open neighborhood $\mathcal{N}(K, \epsilon_L)$. We thus obtain a normal projection map along the normal geodesic balls to points in L, which we denote by $\Pi_L \colon \mathcal{N}(K, \epsilon_L) \to L$. Note that the restriction of Π_L to L is the identity map.

Let $\mathcal{F}|\mathcal{N}(K, \epsilon_L)$ denote the restricted foliation whose leaves are the connected components of the leaves of $\mathcal{F}$ intersected with $\mathcal{N}(K, \epsilon_L)$. The tangent bundle to $\mathcal{F}|\mathcal{N}(K, \epsilon_L)$ is just the restriction of $T\mathcal{F}$ to $\mathcal{N}(K, \epsilon_L)$, so for $x' \in \mathcal{N}(K, \epsilon_L)$ and $x = \Pi_L(x')$, the differential of Π_L induces a linear isomorphism $D_{\mathcal{F}}\Pi_L \colon T_{x'}\mathcal{F} \to T_x L$. Then the assumption ($\epsilon$4) on ϵ_0 in Section 2.1, implies that $D_{\mathcal{F}}\Pi_L$ satisfies a Lipschitz estimate for some constant C, which is the identity when restricted to the the leaf tangent bundle TL.

We use this observation in two ways. For $L \subset M$ a compact leaf, assume that $0 < \epsilon_L \leq \epsilon_0$ satisfies, for $x \in L$ and $x' \in \mathcal{N}(K, \epsilon_L)$ such that $x = \Pi_L(x')$:

(ϵ6) for the leafwise Riemannian volume p-form $d\mathrm{vol}_{\mathcal{F}}$

$$(d\mathrm{vol}_{\mathcal{F}}|_{x'})/2 \ \leq \ (D_{\mathcal{F}}\Pi_L)^*(d\mathrm{vol}_{\mathcal{F}}|_x) \ \leq \ 2\,(d\mathrm{vol}_{\mathcal{F}}|_{x'}) \ ; \tag{4}$$

(ϵ7) for the leafwise Riemannian norm $\|\cdot\|_{\mathcal{F}}$ and $\vec{v'} \in T_{x'}\mathcal{F}$,

$$(\|\vec{v'}\|_{\mathcal{F}})/2 \ \leq \ \|D_{\mathcal{F}}\Pi_L(\vec{v'})\|_{\mathcal{F}} \ \leq \ 2\,(\|\vec{v'}\|_{\mathcal{F}}) \ . \tag{5}$$

2.2. **Regular Foliation Atlas.** We next recall some basic properties of foliation charts. A *regular foliation atlas* for $\mathcal{F}$ is a finite collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ so that:

(F1) $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ is a covering of M by $C^{1,\infty}$–coordinate charts $\phi_\alpha : U_\alpha \to (-1,1)^m$ where each U_α is a convex subset with respect to the metric d_M.

(F2) Each coordinate chart $\phi_\alpha : U_\alpha \to (-1,1)^m$ admits an extension to a $C^{1,\infty}$–coordinate chart $\widetilde{\phi}_\alpha : \widetilde{U}_\alpha \to (-2,2)^m$ where $\widetilde{U}_\alpha$ is a convex subset containing the $2\epsilon_0$-neighborhood of U_α, so $B_M(U_\alpha, \epsilon_0) \subset \widetilde{U}_\alpha$. In particular, the closure $\overline{U_\alpha} \subset \widetilde{U}_\alpha$.

(F3) For each $z \in (-2,2)^q$, the preimage $\widetilde{\mathcal{P}}_\alpha(z) = \widetilde{\phi}_\alpha^{-1}((-2,2)^p \times \{z\}) \subset \widetilde{U}_\alpha$ is the connected component containing $\widetilde{\phi}_\alpha^{-1}(\{0\} \times \{z\})$ of the intersection of the leaf of $\mathcal{F}$ through $\phi_\alpha^{-1}(\{0\} \times \{z\})$ with the set $\widetilde{U}_\alpha$.

(F4) $\mathcal{P}_\alpha(z)$ and $\widetilde{\mathcal{P}}_\alpha(z)$ are convex subsets of diameter less than 1 with respect to $d_{\mathcal{F}}$.

The construction of regular coverings is described in chapter 1.2 of [3].

If the tangent bundle $T\mathcal{F}$ and normal bundle $Q = T\mathcal{F}^\perp$ to $\mathcal{F}$ are oriented, then we assume that the charts in the regular covering preserve these orientations.

The inverse images

$$\mathcal{P}_\alpha(z) = \phi_\alpha^{-1}((-1,1)^p \times \{z\}) \subset U_\alpha$$

are smoothly embedded discs contained in the leaves of $\mathcal{F}$, called the *plaques* associated to the given foliation atlas. The convexity hypotheses in (F4) implies that if $U_\alpha \cap U_\beta \neq \emptyset$, then each plaque $\mathcal{P}_\alpha(z)$ intersects at most one plaque of U_β. The analogous statement holds for pairs $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \neq \emptyset$. More generally, an intersection of plaques $\mathcal{P}_{\alpha_1}(z_1) \cap \cdots \cap \mathcal{P}_{\alpha_d}(z_d)$ is either empty, or a convex set.

Recall that a Lebesgue number for the covering $\mathcal{U}$ is a constant $\epsilon > 0$ so that for each $x \in M$ there exists $U \in \mathcal{U}$ with $B_M(x, \epsilon) \subset U$. Every covering of a compact Riemannian manifold (in fact, of a compact metric space) admits a Lebesgue number. We also require that $\epsilon_0 > 0$ satisfy:

($\epsilon 8$) $2\epsilon_0$ is a Lebesgue number for the covering $\{U_\alpha \mid \alpha \in \mathcal{A}\}$ of M by foliation charts.

Then for any $x \in M$, the restriction of $\mathcal{F}$ to $B_M(x, \epsilon_0)$ is a product foliation, and by condition (F1) the leaves of $\mathcal{F} \mid B_M(x, \epsilon_0)$ are convex discs for the metric $d_{\mathcal{F}}$.

For each $\alpha \in \mathcal{A}$, the extended chart $\widetilde{\phi}_\alpha$ defines a C^1–embedding

$$\widetilde{t}_\alpha = \widetilde{\phi}_\alpha^{-1}(\{0\} \times \cdot) : (-2, 2)^q \to \widetilde{U}_\alpha \subset M$$

whose image is denoted by $\widetilde{\mathcal{T}}_\alpha$. We can assume that the images $\widetilde{\mathcal{T}}_\alpha$ are pairwise disjoint. Let t_α denote the restriction of $\widetilde{t}_\alpha$ to $(-1, 1)^q \subset (-2, 2)^q$, and define $\mathcal{T}_\alpha = t_\alpha(-1, 1)^q$. Then the collection of all plaques for the foliation atlas is indexed by the *complete transversal*

$$\mathcal{T} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha \ .$$

For a point $x \in \mathcal{T}$, let $\mathcal{P}_\alpha(x) = \mathcal{P}_\alpha(t_\alpha^{-1}(x))$ denote the plaque containing x.

The Riemannian metric on M induces a Riemannian metric and corresponding distance function $\mathbf{d}_{\mathcal{T}}$ on each extended transversal $\widetilde{\mathcal{T}}_\alpha$. For $\alpha \neq \beta$ and $x \in \mathcal{T}_\alpha$, $y \in \mathcal{T}_\beta$ we set $\mathbf{d}_{\mathcal{T}}(x, y) = \infty$.

Given $x \in \widetilde{\mathcal{T}}_\alpha$ and $r > 0$, let $\mathbf{B}_{\mathcal{T}}(x, r) = \{y \in \widetilde{\mathcal{T}}_\alpha \mid \mathbf{d}_{\mathcal{T}}(x, y) < r\}$.

Given a subset $\mathcal{Z} \subset U_\alpha$ let $\mathcal{Z}_{\mathcal{P}}$ denote the union of all plaques in U_α having non-empty intersection with $\mathcal{Z}$. We set $\mathcal{Z}_{\mathcal{T}} = \mathcal{Z}_{\mathcal{P}} \cap \mathcal{T}_\alpha$. If $\mathcal{Z}$ is an open subset of U_α, then $\mathcal{Z}_{\mathcal{P}}$ is open in U_α and $\mathcal{Z}_{\mathcal{T}}$ is an open subset of $\mathcal{T}_\alpha$.

Given any point $w \in (-1, 1)^p$, we can define a transversal $\mathcal{T}_\alpha(w) = \phi_\alpha^{-1}(\{w\} \times (-1, 1))$. There is a canonical map $\psi_w \colon \mathcal{T}_\alpha(w) \to \mathcal{T}_\alpha(0) = \mathcal{T}_\alpha$ defined by, for $y \in (-1, 1)^q$,

$$\psi_w(\phi_\alpha^{-1}(w \times \{y\}) = \phi_\alpha^{-1}(0 \times \{y\}) \ . \tag{6}$$

The Riemannian metric on M induces also induces a Riemannian metric and distance function on each transversal $\mathcal{T}_\alpha(w)$. By mild abuse of notation we denote all such transverse metrics by $\mathbf{d}_{\mathcal{T}}$. Then by the uniform extension property of the foliation charts, there exists a constant $C_T \geq 1$ so that for all $\alpha \in \mathcal{A}$, $w \in (-1, 1)^p$ and $x, y \in \mathcal{T}_\alpha(w)$,

$$\mathbf{d}_{\mathcal{T}}(x, y)/C_T \ \leq \ \mathbf{d}_{\mathcal{T}}(\psi_w(x), \psi_w(y)) \ \leq \ C_T \, \mathbf{d}_{\mathcal{T}}(x, y) \ . \tag{7}$$

We use the maps (6) to translate points in the coordinate charts U_α to the "center" transversal $\mathcal{T}_\alpha$. The constant C_T is a uniform estimate of the normal distortion introduced by this translation.

We will also consider the normal geodesic ϵ-disk $\mathcal{N}(y, \epsilon)$ at $y = \phi_\alpha^{-1}(w \times \vec{0})$, defined as the image of the map $\exp_y^Q \colon Q_y^\epsilon \to \mathcal{N}(y, \epsilon)$, which for $0 < \epsilon \leq \epsilon_0$ is uniformly transverse to $\mathcal{F}$.

Assume that the image $\mathcal{N}(y, \epsilon) \subset U_\alpha$, then we can project it to the transversal $\mathcal{T}_\alpha$ along the plaques in U_α. Denote this projection by $\Pi_\alpha^{\mathcal{F}} \colon \mathcal{N}(y, \epsilon) \to \mathcal{T}_\alpha$. We also assume that the constant $C_T \geq 1$ is sufficiently large so that for all $y \in M$, for all $0 < \epsilon \leq \epsilon_0$, for all α with $\mathcal{N}(y, \epsilon) \subset U_\alpha$ and for all $z, z' \in \mathcal{N}(y, \epsilon)$ we have

$$\mathbf{d}_M(z, z')/C_T \ \leq \ \mathbf{d}_{\mathcal{T}}(\Pi_\alpha^{\mathcal{F}}(z), \Pi_\alpha^{\mathcal{F}}(z')) \ \leq \ C_T \, \mathbf{d}_M(z, z') \ . \tag{8}$$

2.3. **Transverse holonomy.** The main result of this section is the definition of the module of uniform continuity function for elements of $\mathcal{H}_{\mathcal{F}}^n$, and its application in Lemma 2.1.

We first recall the definition of the holonomy pseudogroup of $\mathcal{F}$. A pair of indices (α, β) is said to be *admissible* if $U_\alpha \cap U_\beta \neq \emptyset$. Let $\mathcal{T}_{\alpha\beta} \subset \mathcal{T}_\alpha$ denote the open set of plaques of U_α which intersect some plaque of U_β. The holonomy transformation $\mathbf{h}_{\alpha\beta}\colon \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha}$ is defined by $y = \mathbf{h}_{\alpha\beta}(x)$ if and only if $\mathcal{P}_\alpha(x) \cap \mathcal{P}_\beta(y) \neq \emptyset$. The finite collection

$$\mathcal{H}^1_{\mathcal{F}} = \{\mathbf{h}_{\alpha\beta}\colon \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha} \mid (\alpha,\beta) \text{ admissible}\} \ . \tag{9}$$

generates the holonomy pseudogroup $\mathcal{H}_{\mathcal{F}}$ of local homeomorphisms of $\mathcal{T}$.

A *plaque chain of length* n, denoted by $\mathcal{P}$, is a collection of plaques

$$\{\mathcal{P}_{\alpha_0}(z_0), \mathcal{P}_{\alpha_1}(z_1), \ldots, \mathcal{P}_{\alpha_n}(z_n)\}$$

satisfying $\mathcal{P}_{\alpha_i}(z_i) \cap \mathcal{P}_{\alpha_{i+1}}(z_{i+1}) \neq \emptyset$ for $0 \leq i < n$. Each pair of indices (α_i, α_{i+1}) is admissible, so determines a holonomy map $\mathbf{h}_{\alpha_i\alpha_{i+1}}$ such that $\mathbf{h}_{\alpha_i\alpha_{i+1}}(z_i) = z_{i+1}$. Let $\mathbf{h}_{\mathcal{P}}$ denote the composition of these maps, so that

$$\mathbf{h}_{\mathcal{P}} = \mathbf{h}_{\alpha_{n-1}\alpha_n} \circ \cdots \circ \mathbf{h}_{\alpha_1\alpha_2} \circ \mathbf{h}_{\alpha_0\alpha_1} \ .$$

Let $\mathcal{H}^n_{\mathcal{F}} = \{\mathbf{h}_{\mathcal{P}} \mid \mathcal{P} \text{ has length at most } n\} \subset \mathcal{H}_{\mathcal{F}}$ denote the collection of maps obtained from the composition of at most n maps in $\mathcal{H}^1_{\mathcal{F}}$.

Each generator $\mathbf{h}_{\alpha\beta}\colon \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha}$ is the restriction of the transition map $\widetilde{\mathbf{h}}_{\alpha\beta}\colon \widetilde{\mathcal{T}}_{\alpha\beta} \to \widetilde{\mathcal{T}}_{\beta\alpha}$ defined by the intersection of the extended charts $\widetilde{U}_\alpha \cap \widetilde{U}_\beta$. The domain $\mathcal{T}_{\alpha\beta} \subset \widetilde{\mathcal{T}}_{\alpha\beta}$ is precompact with $\mathbf{B}_{\widetilde{\mathcal{T}}}(\mathcal{T}_{\alpha\beta}, \epsilon_0) \subset \widetilde{\mathcal{T}}_{\alpha\beta}$, so $\mathbf{h}_{\alpha\beta}$ is a uniformly continuous homeomorphism on its domain. That is, given any $0 < \epsilon < \epsilon_0$, there is a module of continuity $\mu_{\alpha\beta}(\epsilon) > 0$ such that for all $x \in \mathcal{T}_{\alpha\beta}$

$$\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \mu_{\alpha\beta}(\epsilon)) \subset \widetilde{\mathcal{T}}_{\alpha\beta} \quad \text{and} \quad \widetilde{\mathbf{h}}_{\alpha\beta}(\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \mu_{\alpha\beta}(\epsilon))) \subset \mathbf{B}_{\widetilde{\mathcal{T}}}(\mathbf{h}_{\alpha\beta}(x), \epsilon) \ .$$

For the admissible pairs (α, α) we set $\mu_{\alpha\alpha}(\epsilon) = \epsilon$. Given $0 < \epsilon \leq \epsilon_0$, define

$$\mu(\epsilon) = \min\{\mu_{\alpha\beta}(\epsilon) \mid (\alpha,\beta) \text{ admissible}\} \tag{10}$$

so that $0 < \mu(\epsilon) \leq \epsilon$. Then for every admissible pair (α, β) and each $x \in \mathcal{T}_{\alpha\beta}$ the holonomy map $\mathbf{h}_{\alpha\beta}$ admits an extension to a local homeomorphism $\widetilde{\mathbf{h}}_{\alpha\beta}$ defined by the holonomy of $\mathcal{F}$, which satisfies $\widetilde{\mathbf{h}}_{\alpha\beta}(\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \mu(\epsilon))) \subset \mathbf{B}_{\widetilde{\mathcal{T}}}(\mathbf{h}_{\alpha\beta}(x), \epsilon)$.

For an integer $n > 0$ and $0 < \epsilon \leq \epsilon_0$ recursively define $\mu^{(1)}(\epsilon) = \mu(\epsilon)$ and $\mu^{(n)}(\epsilon) = \mu(\mu^{(n-1)}(\epsilon))$, so that $\mu^{(n)}$ denotes the n-fold composition. Then define

$$\mu(n, \epsilon) = \min\{\epsilon, \mu(\epsilon), \mu(\mu(\epsilon)), \ldots, \mu^{(n)}(\epsilon)\} \tag{11}$$

Note that $0 < \mu(\epsilon) \leq \epsilon$ implies $\mu(n, \epsilon) \leq \mu^{(n)}(\epsilon) \leq \epsilon$.

LEMMA 2.1. *Given a plaque chain $\mathcal{P}$ of length n, and $0 < \epsilon \leq \epsilon_0$ set $\delta = \mu(n, \epsilon)$. Then for any x in the domain of $\mathbf{h}_{\mathcal{P}}$, there is an extension to a local homeomorphism $\widetilde{h}_{\mathcal{P}}$ defined by the holonomy of $\mathcal{F}$ whose domain includes the closure of the disk $\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \delta)$ about x in $\widetilde{\mathcal{T}}$, and*

$$\widetilde{h}_{\mathcal{P}}(\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \delta)) \subset \mathbf{B}_{\widetilde{\mathcal{T}}}(\mathbf{h}_{\mathcal{P}}(x), \epsilon) \ . \tag{12}$$

That is, $\mu(n, \epsilon)$ is a module of uniform continuity for all elements of $\mathcal{H}^n_{\mathcal{F}}$.

Proof. For each $0 \leq i < n$, $\mu(n, \epsilon) \leq \mu(i, \epsilon)$ hence there is an extension of

$$\mathbf{h}_i = \mathbf{h}_{\alpha_{i-1}\alpha_i} \circ \cdots \circ \mathbf{h}_{\alpha_0\alpha_1}$$

to $\widetilde{\mathbf{h}}_i$ whose domain includes the disk $\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \delta)$ about x. The image $\mathbf{h}_i(\mathbf{B}_{\widetilde{\mathcal{T}}}(x, \delta)$ is contained in a ball of radius at most $\mu(n-1, \epsilon)$, so that we can continue the extension process to $\mathbf{h}_{i+1}$. □

2.4. **Plaque length and metric geometry.** We make two observations about the metric leafwise geometry of foliations [26]. In particular, the technical result Proposition 2.3 below is a key fact for our proof of the main result of this work.

Let $\gamma\colon [0,1] \to L$ be a leafwise C^1-path. Its leafwise Riemannian length is denoted by $||\gamma||_{\mathcal{F}}$.

The *plaque length* of γ, denoted by $||\gamma||_{\mathcal{P}}$, is the least integer n such that the image of γ is covered by a chain of convex plaques

$$\{\mathcal{P}_{\alpha_0}(z_0), \mathcal{P}_{\alpha_1}(z_1), \ldots, \mathcal{P}_{\alpha_n}(z_n)\}$$

where $\gamma(0) \in \mathcal{P}_{\alpha_0}(z_0)$, $\gamma(1) \in \mathcal{P}_{\alpha_1}(z_1)$, and successive plaques $\mathcal{P}_{\alpha_i}(z_i) \cap \mathcal{P}_{\alpha_{i+1}}(z_{i+1}) \neq \emptyset$.

PROPOSITION 2.2. *For any leafwise C^1-path γ, $||\gamma||_{\mathcal{P}} \leq \lceil(||\gamma||_{\mathcal{F}}/\epsilon_0)\rceil$. Moreover, if γ is leafwise geodesic, then $||\gamma||_{\mathcal{F}} \leq ||\gamma||_{\mathcal{P}} + 1$.*

Proof. Let $N = \lceil(||\gamma||_{\mathcal{F}}/\epsilon_0)\rceil$ be the least integer greater than $||\gamma||_{\mathcal{F}}/\epsilon_0$. Then there exist points $0 = t_0 < t_1 < \cdots < t_N = 1$ such that the restriction of γ to each segment $[t_i, t_{i+1}]$ has length at most ϵ_0. The diameter of the set $\gamma([t_i, t_{i+1}])$ is at most ϵ_0, hence there is some $U_{\alpha_i} \in \mathcal{U}$ with $\gamma([t_i, t_{i+1}]) \subset U_{\alpha_i}$ hence $\gamma([t_i, t_{i+1}]) \subset \mathcal{P}_\alpha(z_i)$ for some z_i. Thus, the image of γ is covered by a chain of convex plaques of length at most N.

Conversely, suppose γ is a leafwise geodesic and $\{\mathcal{P}_{\alpha_0}(z_0), \mathcal{P}_{\alpha_1}(z_1), \ldots, \mathcal{P}_{\alpha_n}(z_n)\}$ is a plaque chain covering the image $\gamma([0,1])$. Each plaque $\mathcal{P}_{\alpha_i}(z_i)$ is a leafwise convex set of diameter at most 1 by the assumption (F4) in Section 2.2, so $||\gamma||_{\mathcal{F}} \leq (n+1) \leq ||\gamma||_{\mathcal{P}} + 1$. □

The extension property (F2) in Section 2.2 implies that for all $\alpha \in \mathcal{A}$ and $z \in (-1,1)^q$, the closure $\overline{\mathcal{P}_\alpha(z)}$ is compact, hence has finite leafwise volume which is uniformly continuous with respect to the parameter z. Hence, there exist constants $0 < C_{min} \leq C_{max}$ such that

$$C_{min} \leq \mathrm{vol}\,(\mathcal{P}_\alpha(z)) \leq C_{max}, \ \forall \alpha \in \mathcal{A}, \ \forall z \in [-1,1]^q \ . \tag{13}$$

We note a consequence of this uniformity which is critical to the proof of the main theorem.

PROPOSITION 2.3. *Let M be a compact manifold. Then there exists a monotone increasing function $v\colon [0,\infty) \to [0,\infty)$ such that if L is a compact leaf, then $\mathrm{vol}\,(L) \leq v(\mathrm{diam}\,(L))$. Conversely, there exists a monotone increasing function $R\colon [0,\infty) \to [0,\infty)$ such that if L is a compact leaf, then $\mathrm{diam}\,(L) \leq R(\mathrm{vol}\,(L))$.*

Proof. The holonomy pseudogroup of $\mathcal{F}$ has a finite set of generators, hence has a *uniform* upper bound on the growth rate of words. This implies that given $r > 0$, there exists a positive integer $e(r)$ such that any subset of a leaf with leaf diameter at most r can be covered by no more that $e(r)$ plaques. Thus, if L is a leaf with diameter at most r, then L has volume at most $v(r) = C_{max} \cdot e(r)$.

Now suppose that L is a compact leaf with diameter $r = \mathrm{diam}\,(L)$. Then, for any pair of points $x, y \in L$, there exists a length minimizing geodesic segment $\gamma\colon [0,1] \to L$ of length $r = d_{\mathcal{F}}(x,y)$, with $\gamma(0) = x$ and $\gamma(1) = y$. Let $B_{\mathcal{F}}(\gamma, \epsilon_0)$ denote the leafwise ϵ_0-tubular neighborhood of the image of γ, defined by (2). Recall that the restricted metric $d_{\mathcal{F}}$ on leaves has uniformly bounded geometry. Then as ϵ_0 is assumed in assumption ($\epsilon 2$) of Section 2.1 to be less than the injectivity radius for the leafwise metric, and γ is a length-minimizing geodesic, there is a constant $V_0 > 0$ so that the leafwise volume $\mathrm{vol}\,(B_{\mathcal{F}}(\gamma, \epsilon_0)) \geq V_0 \cdot r$. Thus, $\mathrm{vol}\,(L) > V_0 \cdot \mathrm{diam}\,(L)$, and then set $R(v) = v/V_0$. □

2.5. **Captured leaves.** The main result is Proposition 2.6, which shows that given a compact leaf L of $\mathcal{F}$, and another compact leaf L' which is sufficiently close to L at some point, where how close depends on $\mathrm{vol}\,(L')$, then L' is "captured" by the holonomy of L. We require some preliminary definitions and observations before giving the proof of this key fact.

Let $L \subset M$ be a compact leaf, and recall that in Section 2.1 the constant $0 < \epsilon_L \leq \epsilon_0$ was defined so that there is a projection map $\Pi_L\colon \mathcal{N}(L, \epsilon_L) \to L$ along the transverse geodesic ϵ_L-disks to L.

Next, recall that Proposition 2.2 shows that for any leafwise C^1-path γ, we have the upper bound $||\gamma||_{\mathcal{P}} \leq \lceil(||\gamma||_{\mathcal{F}}/\epsilon_0)\rceil$ for the number of plaques required to cover γ.

Suppose that L is a compact leaf, and $x \in L$ is a fixed basepoint, then for any $y \in L$ there is a leafwise geodesic $\gamma_{x,y}\colon [0,1] \to L$ from x to y with $\|\gamma\|_{\mathcal{F}} \leq \mathrm{diam}\,(L)$. Thus, $\gamma_{x,y}$ can be covered by at most $\lceil(||\gamma||_{\mathcal{F}}/\epsilon_0)\rceil$ plaques.

For $n > 0$, the number $\mu(n, \epsilon_0) \leq \epsilon_0$ was defined in (11), and the constant $C_T \geq 1$ was introduced in (7) and (8) as a bound on the distortion of the projection maps $\Pi^{\mathcal{F}}_{\alpha_0}\colon \mathcal{N}(x, \epsilon_0) \to \mathcal{T}_{\alpha_0}$. Introduce the function $\Delta(r, \epsilon)$ as given in the following:

DEFINITION 2.4. *For $0 < \epsilon \leq \epsilon_0$ and $r > 0$,*

$$\Delta(r, \epsilon) \equiv \mu(\lceil r/\epsilon_0 \rceil + 2, \epsilon/C_T)/C_T \ . \tag{14}$$

We scale both the domain variable ϵ and the range value of μ by C_T so that we have uniform estimates for pairs of points in any geodesic normal ball in the chart, a fact which will be used in the proofs of Lemma 2.5 and Proposition 2.6.

LEMMA 2.5. *Given $0 < \epsilon \leq \epsilon_0$, and a leafwise C^1-path $\gamma \colon [0,1] \to L$ of length at most r, the transverse holonomy along γ defines a smooth embedding*

$$\mathbf{h}_\gamma \colon \mathcal{N}(\gamma(0), \Delta(r, \epsilon)) \to \mathcal{N}(\gamma(1), \epsilon) \ , \ \mathbf{h}_\gamma(\gamma(0)) = \gamma(1) \ .$$

Proof. For $n = ||\gamma||_{\mathcal{P}}$, let $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(z_0), \mathcal{P}_{\alpha_1}(z_1), \ldots, \mathcal{P}_{\alpha_n}(z_n)\}$ be a covering of γ by a plaque chain, with $z_i \in \mathcal{T}_{\alpha_i}$ for $0 \leq i \leq n$, as in the proof of Proposition 2.2. Set $x = \gamma(0)$ and $y = \gamma(1)$, then $x \in \mathcal{P}_{\alpha_0}(z_0)$ and $y \in \mathcal{P}_{\alpha_n}(z_n)$.

The constant C_T was chosen so that for the projection $\Pi^{\mathcal{F}}_{\alpha_0} \colon \mathcal{N}(x, \epsilon_0) \to \mathcal{T}_{\alpha_0}$, and for $x' \in \mathcal{N}(x, \epsilon_0)$, the condition (8) implies that

$$\mathbf{d}_M(x, x')/C_T \ \leq \ \mathbf{d}_{\mathcal{T}}(\Pi^{\mathcal{F}}_{\alpha_0}(x), \Pi^{\mathcal{F}}_{\alpha_0}(x')) \ \leq \ C_T \, \mathbf{d}_M(x, x') \ . \tag{15}$$

Likewise, for $y' \in \mathcal{N}(y, \epsilon_0)$ we have

$$\mathbf{d}_M(y, y')/C_T \ \leq \ \mathbf{d}_{\mathcal{T}}(\Pi^{\mathcal{F}}_{\alpha_n}(y), \Pi^{\mathcal{F}}_{\alpha_n}(y')) \ \leq \ C_T \, \mathbf{d}_M(y, y') \ . \tag{16}$$

Given $x' \in \mathcal{N}(x, \Delta(r, \epsilon))$ then by (15), for $x_0 = \Pi^{\mathcal{F}}_{\alpha_0}(x) \in \mathcal{T}_{\alpha_0}$ and $x'_0 = \Pi^{\mathcal{F}}_{\alpha_0}(x') \in \mathcal{T}_{\alpha_0}$, we have

$$\mathbf{d}_{\mathcal{T}}(x_0, x'_0) \leq C_T \Delta(r, \epsilon) = \mu(\lceil r/\epsilon_0 \rceil + 2, \epsilon/C_T) \ .$$

Then by Lemma 2.1 and the inclusion (12), we have that $d_{\mathcal{T}}(\mathbf{h}_{\mathcal{P}}(x), \mathbf{h}_{\mathcal{P}}(x')) \leq \epsilon/C_T$.

Observe that $\mathbf{h}_{\mathcal{P}}(x) = \Pi^{\mathcal{F}}_{\alpha_n}(\gamma(1))$ and so we set $\mathbf{h}_\gamma(x') = (\Pi^{\mathcal{F}}_{\alpha_n})^{-1}(\mathbf{h}_{\mathcal{P}}(x'))$. Then formula (16) implies that $d_M(\mathbf{h}_\gamma(x), \mathbf{h}_\gamma(x')) \leq \epsilon$, and so $\mathbf{h}_\gamma(x') \in \mathcal{N}(y, \epsilon)$ as was to be shown. □

We apply Lemma 2.5 to obtain the following "captured leaf" property.

PROPOSITION 2.6. *Let L be a compact leaf of $\mathcal{F}$, with the constant $0 < \epsilon_L \leq \epsilon_0$ as introduced in Section 2.1. Given $\Lambda > 0$, there exists $0 < \delta_\Lambda \leq \epsilon_L$ so that if L' is a compact leaf with volume $\mathrm{vol}\,(L') \leq \Lambda$ and $L' \cap \mathcal{N}(L, \delta_\Lambda) \neq \emptyset$, then $L' \subset \mathcal{N}(L, \epsilon_L)$.*

Proof. Let $R(\Lambda)$ be the constant introduced in Proposition 2.3, so that $\mathrm{vol}\,(L') \leq \Lambda$ implies that $\mathrm{diam}\,(L') \leq R(\Lambda)$, and set $\delta_\Lambda = \Delta(R(\Lambda), \epsilon_L/2)$. Note that $\mu(n, \epsilon) \leq \epsilon$ implies $\delta_\Lambda \leq \epsilon_L/2 \leq \epsilon_0/2$.

Recall that by condition (ϵ1) in Section 2.1, given any two points $y, z \in B_M(x, \epsilon_0)$, there is a unique geodesic for the metric d_M between y and z.

Given that $L' \cap \mathcal{N}(L, \delta_\Lambda) \neq \emptyset$, there exists $x \in L$ such that there exists $y' \in L' \cap \mathcal{N}(x, \delta_\Lambda)$. Let $K_{y'} \subset L' \cap B_M(x, \epsilon_0)$ be the connected component containing y'. Then there exists a point $y \in K_{y'}$ which minimizes $d_M(x, y)$, and by the choice of y' we have $d_M(x, y) \leq \epsilon_L/2 \leq \epsilon_0/2$. This implies that y is an interior point for $K_{y'}$ hence the geodesic from x to y is contained in $B_M(x, \epsilon_0)$ and intersects $K_{y'}$ orthogonally. Thus, $x \in \mathcal{N}(y, \delta_\Lambda)$.

Let $z \in L'$, then there is a leafwise geodesic path $\gamma_{y,z} \colon [0,1] \to L'$ with $\|\gamma_{y,z}\|_{\mathcal{F}} \leq R(\Lambda)$, $y = \gamma_{y,z}(0)$ and $z = \gamma_{y,z}(1)$. Then by Lemma 2.5, the holonomy map $\mathbf{h}_{\gamma_{y,z}} \colon \mathcal{N}(y, \delta_\Lambda) \to \mathcal{N}(z, \epsilon_L)$ is well-defined.

As $x \in L \cap \mathcal{N}(y, \delta_\Lambda)$ then $x' = \mathbf{h}_{\gamma_{y,z}}(x) \in L_x \cap \mathcal{N}(z, \epsilon_\Lambda)$. Thus, $z \in B_M(x', \epsilon_\Lambda) \subset \mathcal{N}(L, \epsilon_L)$.

It follows that $L' \subset \mathcal{N}(L, \epsilon_L)$, as was to be shown. □

Proposition 2.6 has the following useful consequence.

COROLLARY 2.7. *Let L_0 be a compact leaf of $\mathcal{F}$. Given $\Lambda > 0$, there exists $0 < \delta_\Lambda < \epsilon_{L_0}$ so that if L_1 is a compact leaf with volume $\mathrm{vol}\,(L_1) < \Lambda$ and $L_1 \cap \mathcal{N}(L_0, \delta_\Lambda) \neq \emptyset$, then $L_1 \subset \mathcal{N}(L_0, \epsilon_{L_0})$. Moreover, the projection map $\Pi_{L_0} \colon \mathcal{N}(L_0, \epsilon_{L_0}) \to L_0$ restricted to L_1 is a covering map onto L_0. Furthermore, if the tangent bundle $T\mathcal{F}$ is orientable, then $\mathrm{vol}\,(L_1) \leq 2\, d_* \,\mathrm{vol}\,(L_0)$ where d_* is the homological degree of the covering map $\Pi_{L_0} \colon L_1 \to L_0$.*

Proof. Let δ_Λ be as defined in Proposition 2.6, then $L_1 \subset \mathcal{N}(L_0, \epsilon_{L_0})$ follows.

By the assumption (ϵ4) in Section 2.1, the leaves of $\mathcal{F}$ are uniformly transverse to the fibers of $\Pi_{L_0} \colon \mathcal{N}(L_0, \epsilon_{L_0}) \to L_0$, so the restriction to L_1 is a covering map. As L_0 and L_1 are compact, the map $\Pi_{L_0} \colon L_1 \to L_0$ is onto. Assume that the tangent bundle $T\mathcal{F}$ is oriented, then we can choose a positively-oriented Riemannian volume form on the leaves of $\mathcal{F}$, whose restriction to a leaf L is denoted by ω_L. We have that $\mathrm{vol}\,(L) = \int_L \omega_L$, so the closed p-form $\mathrm{vol}\,(L)^{-1} \cdot \omega_L$ on L is dual to the fundamental class $[L]$. Thus, the homological degree d_* of $\Pi_{L_0} \colon L_1 \to L_0$ is given by

$$d_* = \int_{L_1} \Pi_{L_0}^*(\mathrm{vol}\,(L_0)^{-1} \cdot \omega_{L_0}) = \mathrm{vol}\,(L_0)^{-1} \cdot \int_{L_1} \Pi_{L_0}^*(\omega_{L_0}) \ . \tag{17}$$

Condition (ϵ6) of Section 2.1 gives that for $x' \in L_1$ and $x = \Pi_{L_0}(x') \in L_0$ we have

$$1/2 \cdot \omega_{L_1}|_{x'} \ \leq \ \Pi_{L_0}^*(\omega_{L_0})|_x \ \leq \ 2 \cdot \omega_{L_1}|_{x'} \ , \tag{18}$$

and thus

$$1/2 \cdot \mathrm{vol}\,(L_1) = 1/2 \cdot \int_{L_1} \omega_{L_1} \leq \int_{L_1} \Pi_{L_0}^*(\omega_{L_0}) \leq 2 \cdot \int_{L_1} \omega_{L_1} = 2 \cdot \mathrm{vol}\,(L_1) \ . \tag{19}$$

By (17) we have $d_* \cdot \mathrm{vol}\,(L_0) = \int_{L_1} \Pi_{L_0}^*(\omega_{L_0})$ and thus $\mathrm{vol}\,(L_1) \leq 2\, d_* \cdot \mathrm{vol}\,(L_0)$. □

3. Properties of compact foliations

In this section, $\mathcal{F}$ is assumed to be a compact foliation of a manifold M without boundary. The geometry of compact foliations has been studied by Epstein [11, 12], Millett [22], Vogt [33, 34, 35] and Edwards, Millett and Sullivan [10]. We recall some of their results.

3.1. **The good and the bad sets.**

Let $\mathrm{vol}\,(L)$ denote the volume of a leaf L with respect to the Riemannian metric induced from M. Define the volume function on M by setting $v(x) = \mathrm{vol}\,(L_x)$. Clearly, the function $v(x)$ is constant along leaves of $\mathcal{F}$, but need not be continuous on M.

The *bad set* X_1 of $\mathcal{F}$ consists of the points $y \in M$ where the function $v(x)$ is not bounded in any open neighborhood of y. By its definition, the bad set X_1 is saturated. Note also that

$$X_1 = \bigcup_{n=1}^{\infty} X_1 \cap \mathrm{vol}^{-1}(0, n] \ .$$

The leaves in the intersection $X_1 \cap \mathrm{vol}^{-1}(0, n]$ have volume at most n, while $v(x)$ is not locally bounded in any open neighborhood of $y \in X_1$, therefore each set $X_1 \cap \mathrm{vol}^{-1}(0, n]$ has no interior. By the Baire category theorem, X_1 has no interior.

The complement $G = M \setminus X_1$ is called the *good set.* The holonomy of every leaf $L \subset G$ is finite, thus by the Reeb Stability Theorem, L has an open saturated neighborhood consisting of leaves with finite holonomy. Hence, G is an open set, X_1 is closed, and the leaf space $G/\mathcal{F}$ is Hausdorff.

Inside the good set is the open dense saturated subset $G_e \subset G$ consisting of leaves without holonomy. Its complement $G_h = G \setminus G_e$ consists of leaves with non-trivial finite holonomy.

3.2. **The Epstein filtration.** The restriction of the volume function $v(x)$ to X_1 again need not be locally bounded, and the construction of the bad set can be iterated to obtain the *Epstein filtration*:

$$M = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_\alpha \supset \cdots .$$

The definition of the sets X_α proceeds inductively: Let $\alpha > 1$ be a countable ordinal, and assume that X_β has been defined for $\beta < \alpha$. If α is a successor ordinal, let $\alpha = \gamma + 1$ and define X_α to be the closed saturated set of $y \in X_\gamma$ where the function $v(x)$ is not bounded in any relatively open neighborhood of $y \in X_\gamma$ in X_γ.

If α is a limit ordinal, then define $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$.

For $\beta < \alpha$, the set X_α is nowhere dense in X_β. Note that since each set $M \setminus X_\alpha$ is open, the filtration is at most countable. The *filtration length* of $\mathcal{F}$ is the ordinal α such that $X_\alpha \neq \emptyset$ and $X_{\alpha+1} = \emptyset$.

Vogt [35] showed that for any finite ordinal α, there is a compact foliation of a compact manifold with filtration length α. He also remarked that given any countable ordinal α, the construction can be modified to produce a foliation with filtration length α. Such examples show that the bad set X_1 and the subspaces X_α need not be finite unions or intersections of submanifolds; they may have pathological topological structure, especially when the filtration length is an infinite ordinal.

3.3. **Regular points.** A point $x \in X_1$ is called a *regular point* if the restricted holonomy of $\mathcal{F}|X_1$ is trivial at x. Equivalently, the regular points are the points of continuity for the restricted volume function $v|X_1 \colon X_1 \to \mathbb{R}$. If $X_1 \neq \emptyset$, then the regular points form an open and dense subset of $X_1 \setminus X_2$. We recall a key result of Edwards, Millett, and Sullivan (see §5 of [10].)

PROPOSITION 3.1 (Moving Leaf). *Let $\mathcal{F}$ be a compact foliation of an oriented manifold M with orientable normal bundle. Suppose that X_1 is compact and non-empty. Let $x \in X_1$ be a regular point. Then there exists a leaf $L \subset G_e$ and a smooth isotopy $h \colon L \times [0,1) \to G_e$ such that:*

- *For all $0 \leq t < 1$, $h_t \colon L \to L_t \subset M$ is a diffeomorphism onto its image L_t*
- *L_x is in the closure of the leaves $\bigcup_{t>1-\delta} L_t$ for any $\delta > 0$*
- $\limsup_{t\to 1} \operatorname{vol}(L_t) = \infty$

While the "moving leaf" L_t limits on X_1, the moving leaf cannot accumulate on a single compact leaf of X_1. This follows because a compact leaf L admits a relative homology dual cycle, which for $\epsilon > 0$ sufficiently small and $x \in L$, is represented by the transverse disk $\mathbf{B}_{\mathcal{T}}(x, \epsilon)$. This disk intersects L precisely in the point x, hence the relative homology class $[\mathbf{B}_{\mathcal{T}}(x,\epsilon), \partial\mathbf{B}_{\mathcal{T}}(x,\epsilon)]$ is Poincaré dual to the fundamental class $[L]$. Assuming that $\{L_t\}$ limits on L, for $t < 1$ sufficiently close to 1, each $L_t \subset \mathcal{N}(L,\epsilon)$ and so the intersection number $[L_t \cap \mathbf{B}_{\mathcal{T}}(x,\epsilon)] = [L_t] \cap [\mathbf{B}_{\mathcal{T}}(x,\epsilon), \partial\mathbf{B}_{\mathcal{T}}(x,\epsilon)]$ is constant. Thus the leaves $\{L_t\}$ have bounded volume as $t \to 1$, which is a contradiction.

It is precisely this "non-localized limit behavior" for leaves with unbounded volumes approaching the bad set which makes the study of compact foliations with non-empty bad sets so interesting, and difficult. There seem to be no results in the literature describing how these paths of leaves must behave in the limit.

3.4. **Structure of the good set.** Epstein [12] and Millett [22] showed that for a compact foliation $\mathcal{F}$ of a manifold V, then

$$v(x) \text{ is locally bounded} \Leftrightarrow V/\mathcal{F} \text{ is Hausdorff} \Leftrightarrow \text{ the holonomy of every leaf is finite } .$$

By definition, the leaf volume function is locally bounded on the good set G, hence the restriction of $\mathcal{F}$ to G is compact Hausdorff, and all leaves of $\mathcal{F}|G$ have finite holonomy group. Epstein and Millett showed there is a much more precise structure theorem for the foliation $\mathcal{F}$ in an open neighborhood of a leaf of the good set:

PROPOSITION 3.2. *Let V denote an open connected component of the good set G, and $V_e = V \cap G_e$ the set of leaves with no holonomy. There exists a "generic leaf" $L_0 \subset V_e$, such that for each $x \in V$ with leaf L_x containing x,*

(1) there is a finite subgroup H_x of the orthogonal group $\mathbf{O}(\mathbf{q})$ and a free action α_x of H_x on L_0
(2) there exists a diffeomorphism of the twisted product

$$\phi_x\colon L_0 \times_{H_x} \mathbb{D}^q \to V_x \tag{20}$$

onto an open saturated neighborhood V_x of L_x (where $\mathbb{D}^q$ denotes the unit disk in $\mathbb{R}^q$)
(3) the diffeomorphism ϕ_x is leaf preserving, where $L_0 \times_{H_x} \mathbb{D}^q$ is foliated by the images of $L_0 \times \{w\}$ for $w \in \mathbb{D}^q$ under the quotient map $\mathcal{Q}\colon L_0 \times \mathbb{D}^q \to L_0 \times_{H_x} \mathbb{D}^q$
(4) ϕ_x maps $L_0/H_x \cong L_0 \times_{H_x} \{0\}$ diffeomorphically to L_x

In particular, if $x \in V_e$ then H_x is trivial, and ϕ_x is a product structure for a neighborhood of L_x.

The open set V_x is called a standard neighborhood of L_x, and the 4–tuple $(V_x, \phi_x, H_x, \alpha_x)$ is called a *standard local model* for $\mathcal{F}$. Note that, by definition, $V_x \subset G$ hence $V_x \cap X_1 = \emptyset$.

The Hausdorff space $G/\mathcal{F}$ is a Satake manifold; that is, for each point $b \in G/\mathcal{F}$ and $\pi(x) = b$ the leaf L_x has an open foliated neighborhood V_x as above, and $\phi_x\colon L_0 \times_{H_x} \mathbb{D}^q \to V_x$ induces a coordinate map $\varphi_b\colon \mathbb{D}^q/H_x \to W_b$, where $W_b = \pi(V_x)$. The open sets $W_b \subset G/\mathcal{F}$ are called basic open sets for $G/\mathcal{F}$. Note also that π is a closed map [12, 22].

4. Properties of foliated homotopies

In this section, we study some of the geometric and topological properties of a foliated homotopy of a compact leaf. These results play an essential role in our proof of Proposition 6.3, and hence of Theorem 1.2. The main result of this section yields an upper bound on both the volumes of the compact leaves and the topological degrees of the covering maps which arise in a homotopy of a compact leaf. Note that the results of this section apply for all C^1-foliations of a manifold M. We first recall a "stability" result from the work [18].

THEOREM 4.1. [18, Corollary 1.4] *Let $\mathcal{F}$ be a C^1 foliation of a compact manifold M. Let L be a compact leaf, and $H\colon L \times [0,1] \to M$ be a foliated homotopy for which H_0 is the inclusion map. Then for all $0 \leq t \leq 1$, the image $H_t(L)$ is contained in a compact leaf L_t of $\mathcal{F}$, and moreover, the map $H_t\colon L \to L_t$ is surjective.*

The following technical result is at the heart of the proof of Theorem 1.2.

PROPOSITION 4.2. *Let $\mathcal{F}$ be a C^1 foliation of a manifold M, and let L be a compact leaf. Suppose that $H\colon L \times [0,1] \to M$ is a foliated homotopy for which H_0 is the inclusion map, and let L_t denote the compact leaf containing $H_t(L)$. Assume that both the tangent bundle $T\mathcal{F}$ and the normal bundle Q to $\mathcal{F}$ are oriented. Then there exists $d_* > 0$, depending on H and L, such that*

$$1 \leq \deg(H_t\colon L_0 \to L_t) \leq d_* \ . \tag{21}$$

Moreover, there exists an integer $k \geq 0$ such that

$$vol(L_t) \leq 4^k d_* \cdot \mathrm{vol}\,(L) \ , \quad \text{for all } 0 \leq t \leq 1 \ . \tag{22}$$

Proof. Let $0 \leq t \leq 1$, then there exists $0 < \epsilon_t = \epsilon_{L_t} \leq \epsilon_0$ such that we have a normal ϵ_t-bundle projection map $\Pi_{L_t}\colon \mathcal{N}(L_t, \epsilon_t) \to L_t$. The subset $\mathcal{N}(L_t, \epsilon_t) \subset M$ is open, and H is uniformly continuous, so there exists $\delta_t > 0$ such that $H_s(L) \subset \mathcal{N}(L_t, \epsilon_t)$ for all $t - \delta_t < s < t + \delta_t$. For such s, the map $H_s\colon L_0 \to L_s$ is onto, so the leaf $L_s \subset \mathcal{N}(L_t, \epsilon_t)$, hence the restriction $\Pi_{L_t}\colon L_s \to L_t$ is a covering map.

The maps $H_s, H_t\colon L_0 \to \mathcal{N}(L_t, \epsilon_t)$ are homotopic in $\mathcal{N}(L_t, \epsilon_t)$, hence for their induced maps on fundamental classes, their degrees satisfy

$$\deg(H_t\colon L_0 \to L_t) = \deg(\Pi_{L_t}\colon L_s \to L_t) \cdot \deg(H_s\colon L_0 \to L_s) \ . \tag{23}$$

The homological degree of a covering map equals its covering degree, thus the covering degree of $\Pi_{L_t}: L_s \to L_t$ divides the homological degree of $\deg(H_t: L_0 \to L_t)$.

The collection of open intervals $\{\mathcal{I}_t = (t-\delta_t, t+\delta_t) \mid 0 \leq t \leq 1\}$ is an open covering of $[0,1]$, so there exists a finite set $\{0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = 1\}$ so that the collection $\{\mathcal{I}_{t_i} \mid 0 \leq i \leq k\}$ is a finite covering of $[0,1]$. Choose a sequence $\{0 < s_1 < \cdots < s_{k-1} < 1\}$ such that

$$t_{\ell-1} < s_\ell < t_\ell \quad , \quad t_\ell - \delta_{t_\ell} < s_\ell < t_{\ell-1} + \delta_{t_{\ell-1}}$$

and hence $s_\ell \in \mathcal{I}_{t_{\ell-1}} \cap \mathcal{I}_{t_\ell}$. Thus for the choices of the constants δ_t for each $0 < \ell < k$, we have the inclusions $L_{s_\ell} \subset \mathcal{N}(L_{t_{\ell-1}}, \epsilon_{t_{\ell-1}}) \cap \mathcal{N}(L_{t_\ell}, \epsilon_{t_\ell})$. Thus, there are finite covering maps

$$\Pi_{L_{t_{\ell-1}}}: L_{s_\ell} \to L_{t_{\ell-1}} \ , \ \Pi_{L_{t_\ell}}: L_{s_\ell} \to L_{t_\ell} \quad , \text{ for each } 1 \leq \ell < k-1 \ . \tag{24}$$

The collection of maps (24) is called a *geometric correspondence* from L_0 to L_1. We have shown:

LEMMA 4.3. *Let $\mathcal{F}$ be a C^1 foliation of M, L a compact leaf of $\mathcal{F}$, and $H: L \times [0,1] \to M$ a foliated homotopy for which H_0 is the inclusion map. Then there exists a geometric correspondence from $L_0 = L$ to $L_1 = H_1(L)$.*

Introduce the following integer constants associated to a correspondence (24), for $0 < \ell < k$:

$$a_\ell \ = \ \deg(\Pi_{L_{t_{\ell-1}}}: L_{s_\ell} \to L_{t_{\ell-1}}) \tag{25}$$

$$b_\ell \ = \ \deg(\Pi_{L_{t_\ell}}: L_{s_\ell} \to L_{t_\ell}) \ . \tag{26}$$

Note that a_ℓ and b_ℓ are equal to the covering degrees of the covering maps in (25) and (26), and that $a_1 = 1$ as the leaf L_{s_1} must be a diffeomorphic covering of L_0. Then the choice of each $\epsilon_t \leq \epsilon_0$ we can apply the Condition (ϵ6) of Section 2.1 and the estimate (19) in the proof of Corollary 2.7, to obtain for $1 \leq \ell < k$,

$$\begin{array}{lclcl} a_\ell/2 \cdot \mathrm{vol}\,(L_{t_{\ell-1}}) & \leq & \mathrm{vol}\,(L_{s_\ell}) & \leq & 2a_\ell \cdot \mathrm{vol}\,(L_{t_{\ell-1}}) \\ b_\ell/2 \cdot \mathrm{vol}\,(L_{t_\ell}) & \leq & \mathrm{vol}\,(L_{s_\ell}) & \leq & 2b_\ell \cdot \mathrm{vol}\,(L_{t_\ell}) \end{array}$$

Combine these sequences of upper and lower estimates to obtain the estimate:

$$4^{-k}\, \frac{b_1 \cdots b_{k-1}}{a_1 \cdots a_{k-1}} \cdot \mathrm{vol}\,(L_0) \ \leq \ \mathrm{vol}\,(L_1) \ \leq \ 4^k\, \frac{a_1 \cdots a_{k-1}}{b_1 \cdots b_{k-1}} \cdot \mathrm{vol}\,(L_0) \ . \tag{27}$$

Set $d_* = \dfrac{a_1 \ldots a_{k-1}}{b_1 \ldots b_{k-1}}$ and we obtain the estimate (22) for $t = 1$. The uniform bound (21) follows from the argument above, considering only the homological degrees of the covering maps and ignoring the volume estimates. A minor modification of the above arguments also yields these estimates for the values $0 < t < 1$. This completes the proof of Proposition 4.2. □

5. Tame points in the bad set

In this section, we introduce the concept of a "tame point" in the bad set X_1, which is a point $x \in X_1$ that can be approached by a path in the good set. The main result of this section proves the existence of tame points, using a more careful analysis of the ideas of the Moving Leaf Proposition 3.1. Tame points are used in section 6 for studying the deformations of the bad set under foliated homotopy.

Recall that the "good set" $G \subset M$ is the union of leaves whose holonomy group is finite, and its complement is the bad set $X_1 \subset M$ which is the union of the points $y \in M$ where the leaf volume function $v(x)$ is not bounded in any open neighborhood of y. Then there is an open dense saturated subset $G_e \subset G$ consisting of leaves without holonomy.

The bad set X_1 is closed, saturated and has no interior. A point $x_1 \in X_1$ is said to be *regular* if the restriction to X_1 of leaf volume function $v: X_1 \to \mathbb{R}^+$ is continuous at x_1. Equivalently, $x_1 \in X_1$ is a regular point if the holonomy of the restriction of $\mathcal{F}$ to X_1 is trivial in some relatively open neighborhood of $x_1 \in X_1$.

DEFINITION 5.1. *A regular point $x_1 \in X_1$ is* tame *if there exists $\epsilon > 0$ and a transverse C^1-path*

$$\gamma\colon [0,1] \to (\mathcal{N}(x_1,\epsilon) \cap G_e) \cup \{x_1\} \tag{28}$$

with $\gamma(t) \in G_e$ for $0 \leq t < 1$, $\gamma(1) = x_1$ and such that $v(\gamma(t)))$ tends uniformly to infinity as $t \to 1$.

Let $X_1^t \subset X_1$ denote the subset of tame points.

Since the restricted path $\gamma\colon [0,1) \to G_e$ lies in the set of leaves without holonomy, it follows that there is a foliated isotopy $\Gamma\colon L_{\gamma(0)} \times [0,1) \to G_e$ such that $\Gamma_t(\gamma(0)) = \gamma(t)$. Thus, a tame point x is directly approachable by a family of moving leaves whose volumes tend uniformly to infinity.

In the examples constructed by Sullivan [30], it is easy to see that every regular point is a tame point. In general, though, Edwards, Millet, and Sullivan specifically point out that their proof of the Moving Leaf Proposition 3.1 in [10] does not claim that a regular point is a tame point. The problem is due to the possibility that the complement of the bad set need not be locally connected in a neighborhood of a point in the bad set. In their proof, the moving leaf is defined by a curve that follows an end ω of the good set out to infinity, passing through points where the volume is tending to infinity along the way. This end ω of the good set is contained in arbitrarily small ϵ-neighborhoods of the bad set, but they do not control the behavior of the end. Thus, the existence of a tame point is asserting the existence of a "tame end" of the good set on which the volume function is unbounded, and which is defined by open neighborhoods of some point in the bad set.

PROPOSITION 5.2. *Let $\mathcal{F}$ be a compact, C^1-foliation of a manifold M, and assume that the tangent bundle $T\mathcal{F}$ and the normal bundle Q to $\mathcal{F}$ are oriented. Then the set of tame points X_1^t is dense in X_1.*

The proof of this result involves several technical steps, so we first give an overview of the strategy of the proof. Let $x_1 \in X_1$ be a regular point, and L_1 the leaf through x_1. We use a key result in the proof of the Moving Leaf Lemma to obtain an open neighborhood U of x_1 in its transversal space, on which the volume function is unbounded. We then choose a regular point $x_* \in U \cap X_1$ which is sufficiently close to x_1, so that the leaf L_* through x_* is a diffeomorphic covering of the leaf L_1. Moreover, the point x_* is approachable by a path in the good set. Then we argue by contradiction, that if the leaf volume function does not tend uniformly to infinity along this path, then each leaf through a point in the set $U \cap G$ is also a covering space of L_1 with uniformly bounded covering degree, from which we conclude that the volume function is bounded on the leaves through points in U, contrary to choice. It follows that x_* is a tame point which is arbitrarily close to x_1. The precise proofs of these claims requires that we first establish some technical properties of the foliation $\mathcal{F}$ in a normal neighborhood of L_1.

5.1. **Technical preliminaries.** The leaf L_1 is compact, hence has finitely-generated fundamental group. Thus, we can choose a finite generating set $\{[\tau_1], \ldots, [\tau_k]\}$ for $\pi_1(L_1, x_1)$, where $[\tau_i]$ is represented by a smooth closed path $\tau_i\colon [0,1] \to L_1$ with basepoint x_1. Let $\|\tau_i\|$ denote the path length of τ_i. Then set

$$D_{L_1} = 2 \max \{\operatorname{diam}(L_1), \|\tau_1\|, \ldots, \|\tau_k\|\} \ . \tag{29}$$

Recall that in Section 2.1, given a compact leaf L the constant $0 < \epsilon_L \leq \epsilon_0$ was defined so that there is a projection map $\Pi_L\colon \mathcal{N}(L, \epsilon_L) \to L$ along the transverse geodesic ϵ_L-disks to L. Set $\epsilon_1 = \epsilon_{L_1}$ so that the normal projection map $\Pi_{L_1}\colon \mathcal{N}(L_1, \epsilon_1) \to L_1$ is well-defined. Then set $\epsilon_2 = \Delta(D_{L_1}, \epsilon_1)$ where $\Delta(D_{L_1}, \epsilon_1)$ is defined in Definition 2.4. Then by Lemma 2.5, for any path $\sigma\colon [0,1] \to L_1$ with $\sigma(0) = x_1$ and $\|\sigma\| \leq D_{L_1}$ the transverse holonomy maps are defined for all $0 \leq t \leq 1$,

$$\mathbf{h}_\sigma\colon \mathcal{N}(x_1, \epsilon_2) \to \mathcal{N}(\sigma(t), \epsilon_1) \ . \tag{30}$$

In particular, the holonomy map $\mathbf{h}_i$ along each closed path τ_i is defined on the transverse disk $\mathcal{N}(x_1, \epsilon_2)$. That is, the transverse holonomy along τ_i is represented by a local homeomorphism into

$$\mathbf{h}_i\colon \mathcal{N}(x_1, \epsilon_2) \to \mathcal{N}(x_1, \epsilon_1) \ . \tag{31}$$

The assumption that $x_1 \in X_1$ is a regular point implies that the germinal holonomy at x_1 of the restricted foliation $\mathcal{F}|X_1$ is trivial. Thus we can choose $0 < 2\delta \leq \epsilon_2$ sufficiently small, so that each holonomy map $\mathbf{h}_i$ restricted to $X_1 \cap \mathcal{N}(x_1, 2\delta)$ is the identity map. It follows that the holonomy of $\mathcal{F}$ restricted to the closure

$$Z_1 = \overline{X_1 \cap \mathcal{N}(x_1, \delta)} = X_1 \cap \overline{\mathcal{N}(x_1, \delta)} \subset X_1 \cap \mathcal{N}(x_1, 2\delta) \tag{32}$$

is trivial. Hence, every point in Z_1 is a regular point of the bad set. It follows that the saturation $Z_{\mathcal{F}}$ of Z_1 is a fibration over the closed set Z_1, and that the leaf volume function $v(y)$ is uniformly continuous and hence bounded on the compact set Z_1. Thus, we may assume that δ is sufficiently small so that $Z_{\mathcal{F}} \subset \mathcal{N}(L_1, \epsilon_1)$. That is, for each $z \in Z_1$ the leaf $L_z \subset \mathcal{N}(L_1, \epsilon_1)$.

Next consider the properties of the the normal projection $\Pi_{L_1} \colon \mathcal{N}(L_1, \epsilon_1) \to L_1$ when restricted to leaves in $\mathcal{N}(L_1, \epsilon_1)$. The restriction $\pi^z \equiv \Pi|L_z \colon L_z \to L_1$ is a covering projection, which is a diffeomorphism as $\mathcal{F}|Z_1$ has no holonomy, and by the assumption that $\epsilon_1 \leq \epsilon_0$, the estimate (5) implies the map π^z is a quasi-isometry with expansion constant bounded by 2.

Note that $\mathcal{N}(x_1, 2\delta)$ is contained in the the normal transversal $\mathcal{N}(x_1, \epsilon_1)$, so by definition for $z \in Z_1$ we have $\pi^z(z) = x_1$, and thus given a path $\sigma \colon [0,1] \to L_1$ with $\sigma(0) = x_1$, there is a lift $\sigma^z \colon [0,1] \to L_z$ with $\sigma^z(0) = z$ and $\pi^z \circ \sigma^z(t) = \sigma(t)$ for all $0 \leq t \leq 1$. In particular, the closed loop τ_i lifts via π^z to a closed loop $\tau_i^z \colon [0,1] \to L_z$ with endpoints z. The homotopy classes of the lifts, $\{[\tau_1^z], \ldots, [\tau_k^z]\}$, yield a generating set for $\pi_1(L_z, z)$, which have a uniform bound $\|\tau_i^z\| \leq D_{L_1}$ on their path lengths.

For an arbitrary point $y_0 \in \mathcal{N}(x_1, \delta)$ and path $\sigma \colon [0,1] \to L_1$ with $\sigma(0) = x_1$ and path length $\|\sigma\| \leq D_{L_1}$, by the choice of δ the transverse holonomy map in (30) is defined at y_0 hence there is a lift of the path σ to a path $\sigma^y \colon [0,1] \to L_y \cap \mathcal{N}(L_1, \epsilon_1)$ with $\sigma^y(0) = y_0$ and $\pi^y \circ \sigma^y(t) = \sigma(t)$ for all $0 \leq t \leq 1$. This lifting property need not hold for paths longer than D_{L_1}, as there may be leaves of $\mathcal{F}$ which intersect the normal neighborhood $\mathcal{N}(L_1, \delta)$ but are not contained in $\mathcal{N}(L_1, \epsilon_1)$.

We observe a technical point about the distances in the submanifold $\mathcal{N}(x_1, \epsilon_1) \subset M$. The inclusion $\mathcal{N}(x_1, \epsilon_1) \subset M$ induces a Riemannian metric on $\mathcal{N}(x_1, \epsilon_1)$ which then defines a path-length distance function on this subspace. Unless $\mathcal{N}(x_1, \epsilon_1)$ is a totally geodesic submanifold of M, the induced distance function on $\mathcal{N}(x_1, \epsilon_1)$ need not agree with the restricted path-length metric from M. For $y \in \mathcal{N}(x_1, \epsilon_1)$ and $0 < \lambda \leq \epsilon_1$, let $B_T(y, \lambda) \subset \mathcal{N}(x_1, \epsilon_1)$ denote the open ball of radius λ about y for the induced Riemannian metric on $\mathcal{N}(x_1, \epsilon_1)$.

Now consider an arbitrary point $y \in \mathcal{N}(x_1, \delta)$ and assume that $L_y \subset \mathcal{N}(L_1, \delta)$, so that L_y is in the domain of the projection $\Pi_{L_1} \colon \mathcal{N}(L_1, \epsilon_1) \to L_1$. Given a path $\sigma \colon [0,1] \to L_y$ with $\sigma(0) = y$, then Lemma 2.5 implies that there exists $0 < \lambda < \delta$, which depends on the length $\|\sigma\|$, so that for $y' \in B_T(y, \lambda)$ there is a path $\sigma^{y'} \colon [0,1] \to L_{y'}$ satisfying $\Pi_{L_y}(\sigma(t)) = \Pi_{L_y}(\sigma^{y'}(t))$ for $0 \leq t \leq 1$. We call the path $\sigma^{y'}$ a lifting of σ^y from L_y to $L_{y'}$.

5.2. **Proof of Proposition 5.2.** We first recall a key fact from the proof in [10] of the Moving Leaf Proposition 3.1, whose proof was in turn based on ideas of Montgomery [24] and Newman [25]. (In particular, Figure 3 on page 23 of [10] and the arguments following it are pertinent.)

LEMMA 5.3. *For $\delta > 0$ sufficiently small, there is an open connected component U of $\mathcal{N}(x_1, \delta) \setminus Z_1$ on which the volume function $v(y)$ is unbounded on the open neighborhood $U \cap \mathcal{N}(x_1, \delta/2)$.*

Next, fix a choice of regular point $x_1 \in X_1$ and sufficiently small constant $\delta > 0$ as above so that (32) holds, then choose a point $y_0 \in U \cap \mathcal{N}(x_1, \delta/2)$. Let $x_* \in Z_1$ be a closest point to y_0 for the induced metric on $\mathcal{N}(x_1, \delta)$. That is, consider the sequence of closed balls $\overline{B_T(y_0, \lambda)} \subset \mathcal{N}(x_1, \delta) \setminus Z_1$ for $\lambda > 0$, expanding until there is a first contact with the frontier of U, then x_* is contained in this intersection. Let $\delta_0 \leq \delta/2$ denote the radius of first contact, hence δ_0 equals the distance from y_0 to x_* in the induced path-length metric on $\mathcal{N}(x_1, \epsilon_1)$. Then $B_T(y_0, \delta_0) \subset U$ and $x_* \in \overline{B_T(y_0, \delta_0)} \cap Z_1$. (This is illustrated in Figure 1 below.) Let $L_* = L_{x_*}$ denote the leaf containing x_*.

We claim that x_* is a tame point. As $\delta > 0$ was chosen to be arbitrarily small, and the regular points are dense in the bad set, the proof of Proposition 5.2 then follows from this claim.

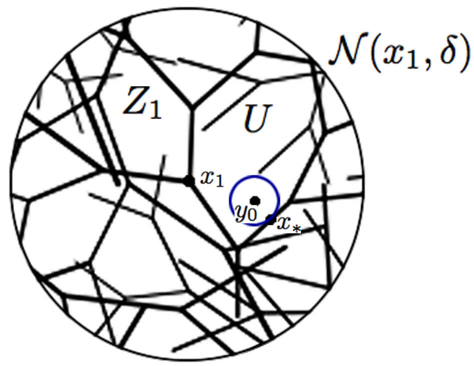


FIGURE 1. A tame point in the bad set

By the choice of $B_T(y_0, \delta_0) \subset U$, there is a path $\gamma\colon [0,1] \to \mathcal{N}(x_1, \delta)$ such that $\gamma(0) = y_0$, $\gamma(1) = x_*$ and $\gamma[0,1) \subset B_T(y_0, \delta_0)$. The complement of X_1 is the good set, hence the image $\gamma[0,1) \subset G$. The set of leaves with holonomy G_h in G is a union of submanifolds with codimension at least 2 by Proposition 3.2. Thus, by a small C^1-perturbation of the path γ in U, we can assume that its image is disjoint from the set G_h. That is, $\gamma(t) \in G_e$ for all $0 \le t < 1$, and $\gamma(1) \in L_*$. Let L_t denote the leaf containing $\gamma(t)$.

We claim that the volumes of the leaves L_t tend uniformly to infinity. Assume not, so there exists a constant $V_{max} > 0$ and a sequence $0 < t_1 < \cdots < t_n \cdots \to 1$ such that $x_n = \gamma(t_n) \to x_*$ and the volumes of the leaves $L_n = L_{x_n}$ are bounded above by V_{max}. We show this yields a contradiction to our assumptions. What we show in the following is that if there exists a leaf L_y for $y \in U \cap G_e$ sufficiently close to L_* with prescribed bounded volume, then using Proposition 2.6 and Corollary 2.7, we show this implies that all leaves intersecting U have bounded volume, which yields the contradiction.

PROPOSITION 5.4. *For $V_{max} > 0$, there is an $\epsilon_* > 0$ so that if there exists $y \in U \cap G_e$ such that $d(y, x_*) < \epsilon_*$ and* $\mathrm{vol}\,(L_y) \le V_{max}$*, then for all $y' \in U$, the leaf $L_{y'}$ containing y' has the volume bound* $\mathrm{vol}\,(L_{y'}) \le 2\,V_{max}$.

Proof. By Proposition 2.3, there is a function $R\colon [0,\infty) \to [0,\infty)$ such that if $L \subset M$ satisfies $\mathrm{vol}\,(L) \le V_{max}$ then $\mathrm{diam}\,(L) \le D_* \equiv R(V_{max})$.

Recall that δ was chosen so that $2\delta \le \epsilon_2$ where $\epsilon_2 = \Delta(D_{L_1}, \epsilon_1)$ was defined after (29), and so that (32) holds, hence $Z_{\mathcal{F}} \subset \mathcal{N}(L_1, \epsilon_1)$. Thus by the choice $x_* \in \overline{B_T(y_0, \delta_0)} \cap Z_1$, we have that $L_* = L_{x_*} \subset \mathcal{N}(L_1, \epsilon_1)$. Let $\pi^* = \pi^{x_*}\colon L_* \to L_1$ denote the normal projection, whose restriction to L_* is a covering map, which is a diffeomorphism as $x_* \in Z_1$.

We next choose $y \in U \cap G_e$ which is sufficiently close to L_* so that $L_y \subset \mathcal{N}(L_1, \epsilon_1)$ and the holonomy maps of L_* based at x_* are defined on L_y. This will imply that L_y is a finite covering of L_*.

For each $1 \le i \le k$, let $\tau_i^*\colon [0,1] \to L_*$ be the lift of τ_i with basepoint x_*. By the definition of D_{L_1} in (29) and the estimate (5), each lifted path has bounded length $\|\tau_i^*\| \le D_{L_1}$ and their homotopy classes $\{[\tau_1^*], \ldots, [\tau_k^*]\}$ form a generating set for $\pi_1(L_*, x_*)$. Denote the holonomy along τ_i^* by $\mathbf{h}_i^*$.

As $L_* \subset \mathcal{N}(L_1, \epsilon_1)$, there exists $0 < \epsilon_3 \le \epsilon_2$ be such that $\mathcal{N}(L_*, \epsilon_3) \subset \mathcal{N}(L_1, \epsilon_1)$.

Set $\epsilon_* = \Delta(D_{L_1}, \epsilon_3)$.

By assumption, there exists $y \in U \cap G_e \cap B_T(x_*, \epsilon_*)$ with $\mathrm{vol}\,(L_y) \le V_{max}$, and by the choice of ϵ_* we have $L_y \subset \mathcal{N}(L_*, \epsilon_3)$. Then the holonomy $\mathbf{h}_i^*$ along τ_i^* is represented by a map

$$\mathbf{h}_i^*\colon \mathcal{N}(x_*, \epsilon_*) \to \mathcal{N}(x_*, \epsilon_3) \subset \mathcal{N}(x_1, \epsilon_2)\ . \tag{33}$$

Moreover, the bound $\|\tau_i^*\| \leq D_{L_1}$ implies that the map $\mathbf{h}_i^*$ extends to a map

$$\mathbf{h}_i^* \colon \mathcal{N}(x_*, \epsilon_2) \to \mathcal{N}(x_*, \epsilon_1) \ . \tag{34}$$

As L_y is a compact leaf, its intersection with the transversal $\mathcal{N}(x_*, \epsilon_3)$ is a finite set, denoted by

$$\mathfrak{F}_y = L_y \cap \mathcal{N}(x_*, \epsilon_3) \ . \tag{35}$$

Then for each $1 \leq i \leq k$, by (34) the holonomy map h_i^* satisfies $h_i^*(\mathfrak{F}_y) \subset L_y \cap \mathcal{N}(x_*, \epsilon_3) = \mathfrak{F}_y$. Thus, the finite set of points $\mathfrak{F}_y$ is permuted by the action of a set of generators for $\pi_1(L_*, x_*)$. Thus, compositions of the generators are defined on the set $\mathfrak{F}_y$. That is, for any $w \in \pi_1(L_*, x_*)$ the holonomy h_w^* along w contains the finite set $\mathfrak{F}_y$ in its domain. Let $\mathcal{H}_* \subset \pi_1(L_*, x_*)$ denote the normal subgroup of finite index consisting of all words whose holonomy fixes every point in $\mathfrak{F}_y$.

Let $z \in \mathfrak{F}_y$. For each $w \in \mathcal{H}_*$, the holonomy $\mathbf{h}_w^*$ map is defined at z, and so must be defined on some open neighborhood $z \in V_z^w \subset U$ of z, where the diameter of the set V_z^w depends on z and w. As $y \in U \cap G_e$ the leaf $L_y \subset G_e$ is without holonomy, so the restriction of $\mathbf{h}_w^*$ to the open set V_z^w must fix an open neighborhood in $\mathcal{N}(x_*, \epsilon_1)$ of $z \in U_z^w \subset V_z^w$. Thus, the fix-point set of $\mathbf{h}_w^*$ contains an open neighborhood of $\mathfrak{F}_y$ in $\mathcal{N}(x_*, \epsilon_1)$. Since $y \in L_y \cap \mathcal{N}(x_*, \epsilon_3) = \mathfrak{F}_y$, we have in particular that there is an open neighborhood $y \in U_y^w \subset U \cap B_T(x_*, \epsilon_*)$ contained in the fixed-point set for $\mathbf{h}_w^*$.

We next use these conclusions for the holonomy of the leaf L_* to deduce properties of the holonomy for the leaf L_1. Recall that $\mathcal{N}(L_*, \epsilon_3) \subset \mathcal{N}(L_1, \epsilon_1)$, and each path τ_i^* in L_* is the lift of the path τ_i in L_1 via the covering map $\pi^* \equiv \Pi | L_* \colon L_* \to L_1$. Thus, the holonomy map $\mathbf{h}_i^*$ on $\mathcal{N}(x_*, \epsilon_3)$ is the restriction of the map $\mathbf{h}_i$ to $\mathcal{N}(x_*, \epsilon_3)$. Consequently, the restriction of $\mathbf{h}_i$ to the open set $B_T(x_*, \epsilon_3) \subset \mathcal{N}(x_1, \epsilon_2)$ equals the restriction in (33) of $\mathbf{h}_i^*$ to $\mathcal{N}(x_*, \epsilon_*)$. In particular, $\mathbf{h}_w$ is defined on and fixes the open set $U_y^w \subset B_T(x_*, \epsilon_*)$.

Let $\{w_1, \ldots, w_N\}$ be a set of generators for $\mathcal{H}_*$. Let m_ℓ denote the word length of w_ℓ with respect to the generating set $\{[\tau_1^*], \ldots, [\tau_k^*]\}$, and set $m_* = \max\{m_1, \ldots, m_N\}$.

Fix a choice of $w = w_\ell \in \mathcal{H}_*$. Then the closed path representing w in L_* can be lifted to a path τ_w^y in the leaf L_y, and as $L_y \subset \mathcal{N}(L_1, \epsilon_1)$, its length is bounded above by $\|\tau_w^y\| \leq m_* \cdot 2D_{L_1}$. We show that $\mathbf{h}_w$ is defined on U, and $U \subset \mathrm{Fix}(\mathbf{h}_w)$. This implies that there is a uniform bound on the diameter of the leaves $L_{y'}$ for $y' \in U$, from which it follows that there is an upper bound on the function $\mathrm{vol}\,(L_{y'})$ for $y' \in U$, which contradicts the choice of U.

We first show that $L_{y'}$ is a finite covering of L_1 with the same index as the covering $L_y \to L_1$.

Choose $0 < \delta_* \leq \Delta(2m_* D_{L_1}, \epsilon_*) \leq \epsilon_*$ such that $B_T(y, \delta_*) \subset U_y^w$.

The open set $U \subset \mathcal{N}(x_1, \delta) \setminus Z_1$ is connected, hence is path connected. Thus, given any point $y' \in U$ there is a continuous path $\sigma \colon [0,1] \to U \cap G_e$ such that $\sigma(0) = y$ and $\sigma(1) = y'$. Then choose a sequence of points $0 = t_0 < t_1 < \cdots < t_m = 1$ such that for $y_i = \sigma(t_i)$, we have $\sigma([t_i, t_{i+1}]) \subset B_T(y_i, \delta_*)$. See Figure 2 below.

We show that $\sigma([0,1]) \subset \mathrm{Fix}(\mathbf{h}_w)$ using induction on the index i. For $i = 0$, $y_0 = y$ and by assumption, the disk $B_T(y, \delta_*) \subset U_y^w \subset \mathrm{Fix}(\mathbf{h}_w)$ so $\sigma([0, t_1]) \subset \mathrm{Fix}(\mathbf{h}_w)$.

Now assume $\sigma([0, t_n]) \subset \mathrm{Fix}(\mathbf{h}_w)$, hence $y_n = \sigma(t_n) \in \mathrm{Fix}(\mathbf{h}_w)$. The closed path τ_w^* in L_* representing w is the lift of a closed path τ_w in L_1, which lifts to a closed path $\tau_w^{y_n}$ in L_{y_n}. As $\tau_w^{y_n} \subset \mathcal{N}(L_1, \epsilon_1)$ we have that $\|\tau_w^{y_n}\| \leq 2m_* D_{L_1}$. Then the holonomy map $\mathbf{h}_w$ for w fixes y_n so near y_n it is defined by a map

$$\mathbf{h}_w^{y_n} \colon \mathcal{N}(y_n, \delta_*) \to \mathcal{N}(y_n, \epsilon_*) \ .$$

As the points of $U \cap G_e$ determine leaves without holonomy, the set of fixed-points for $\mathbf{h}_w^{y_n}$ is an open subset of $\mathcal{N}(y_n, \delta_*) \cap U \cap G_e$. The set of fixed-points is also always a (relatively) closed subset, hence $\mathrm{Fix}(\mathbf{h}_w^{y_n})$ contains the connected component of $\mathcal{N}(y_n, \delta_*) \cap U \cap G_e$ which contains the point y_n. By assumption we have that $\sigma([t_n, t_{n+1}]) \subset \mathcal{N}(y_n, \delta_*) \cap U \cap G_e$, hence

$$\sigma([t_n, t_{n+1}]) \subset \mathrm{Fix}(\mathbf{h}_w^{y_n}) \subset \mathrm{Fix}(\mathbf{h}_w) \ . \tag{36}$$

Thus, by induction we conclude that $y' \in \mathrm{Fix}(\mathbf{h}_w)$.

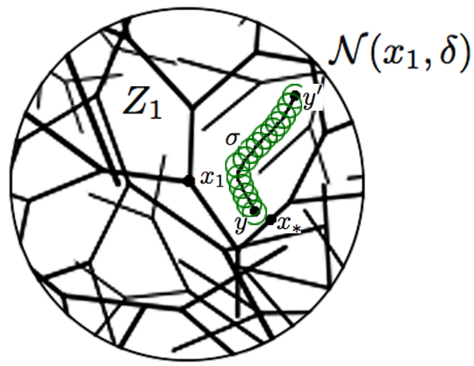


FIGURE 2. A path chain in the good set

The choice of $y' \in U$ was arbitrary, and thus $U \subset \mathrm{Fix}(\mathbf{h}_w)$. We conclude that $L_{y'}$ is a finite covering of L_1 and isotopic to L_y, hence $\mathrm{vol}\,(L_{y'}) \leq 2\,V_{max}$. This completes the proof of Proposition 5.4. □

6. Proof of Main Theorem

In this section, we show that for a compact foliation of a compact manifold, a categorical open set cannot contain a tame point in the bad set. A categorical set must be connected, so we may assume that M is connected. For a connected manifold M, there is a finite covering (of degree $d \leq 4$) $\widetilde{M} \to M$ for which the lifted foliation $\widetilde{\mathcal{F}}$ is again compact, and has oriented tangent and normal bundles. We then apply the following two elementary results to reduce to the oriented case.

LEMMA 6.1. *Let $\widetilde{\pi} \colon \widetilde{M} \to M$ be a finite covering with foliation $\widetilde{\mathcal{F}}$ whose leaves are finite coverings of the leaves of $\mathcal{F}$. Let $U \subset M$ be a transversely categorical saturated open set, and $H \colon U \times [0,1] \to M$ a foliated homotopy to the leaf $L_1 \subset M$ of $\mathcal{F}$. Let $\widetilde{U} \subset \widetilde{M}$ be an open subset such that the restriction $\widetilde{\pi}|\widetilde{U} \to U$ is a homeomorphism. Then there exists a foliated homotopy $\widetilde{H} \colon \widetilde{U} \times [0,1] \to \widetilde{M}$ such that $\widetilde{\pi} \circ \widetilde{H}_t = H_t \circ \widetilde{\pi}$ for all $0 \leq t \leq 1$, where $\widetilde{H}_t(\widetilde{U}) \subset \widetilde{L}_1$ for a finite covering $\widetilde{L}_1$ of L_1.*

Proof. The covering map $\widetilde{\pi}$ has the *unique local lifting of paths* property, so in particular has the *homotopy lifting property*, which yields the existence of the lifted homotopy $\widetilde{H}$. □

LEMMA 6.2. *Let $\widetilde{\pi} \colon \widetilde{M} \to M$ be a finite covering of degree $1 < d < \infty$, with foliation $\widetilde{\mathcal{F}}$ whose leaves are finite coverings of the leaves of $\mathcal{F}$. Let $\widetilde{L} \subset \widetilde{M}$ be a leaf of $\widetilde{\mathcal{F}}$, and let $\widetilde{x} \in \widetilde{L}$ with $x = \widetilde{\pi}(\widetilde{x})$. Then $\widetilde{x}$ is a tame point in the bad set for $\widetilde{\mathcal{F}}$, if and only if x is a tame point in the bad set for $\mathcal{F}$.*

Proof. Suppose that x is a tame point for $\mathcal{F}$, then for $\epsilon > 0$ there exists a continuous path $\gamma \colon [0,1] \to M$ with $\gamma(1) = x$, as in Definition 5.1. The map $\widetilde{\pi}$ has the unique local lifting of paths property, so there exists a unique path $\widetilde{\gamma} \colon [0,1] \to \widetilde{M}$ with $\widetilde{\gamma}(1) = \widetilde{x}$. Moreover, for each $0 \leq t < 1$ the leaf $\widetilde{L}_t$ containing $\widetilde{\gamma}(t)$ is a finite covering of the leaf $L_t \subset M$ containing $\gamma(t)$ where the degree $\widetilde{\pi} \colon \widetilde{L}_t \to L_t$ has degree at most d. Thus, the volume $vol(L_t)$ tends to infinity as $t \to 1$, and thus the same holds for the volume function $\widetilde{vol}(\widetilde{L}_t)$ in $\widetilde{M}$. Thus, $\widetilde{x}$ is a tame point in the bad set for $\widetilde{\mathcal{F}}$. Conversely, if $\widetilde{x}$ is tame point for $\widetilde{\mathcal{F}}$ then the proof that x is a tame point for $\mathcal{F}$ follows similarly. □

Here is the main result of this section.

PROPOSITION 6.3. *Let $\mathcal{F}$ be a compact C^1-foliation of a compact manifold M. If $V \subset M$ is a saturated open set which contains a tame point, then V is not transversely categorical.*

Proof. As a consequence of Lemma 6.2, we can assume in the following that both the tangent bundle $T\mathcal{F}$ and the normal bundle Q to $\mathcal{F}$ are oriented.

Let $x_1 \in X_1$ be a tame point, $V \subset M$ an open set with $x_1 \in V$, and suppose there exists a leafwise homotopy $H: V \times [0,1] \to M$ with H_0 the inclusion map, and $H_1(V) \subset L_*$ for some leaf L_*. We show that this yields a contradiction.

Recall that for $x \in M$, we let $v(x)$ denote the volume of the leaf L_x containing x.

As x_1 is a tame point, there is a smooth path $\gamma: [0,1] \to V$ such that $\gamma(1) = x_1$, $\gamma(t) \in G_e$ for $0 \leq t < 1$, and the volume $v(\gamma(t))$ of the leaf L_t containing the point $\gamma(t)$ satisfies $\lim_{t\to 1} v(\gamma(t)) = \infty$.

Define a map $\phi: [0,1] \times [0,1] \to M$ by setting $\phi_s(t) = \phi(s,t) = H_s(\gamma(t))$. Denote by $L_{s,t}$ the leaf containing $\phi(s,t)$. The key to obtaining a contradiction is to analyze the behavior of the leaf volume function $v(\phi(s,t)) = \mathrm{vol}\,(L_{s,t})$.

Set $x_t = \gamma(t)$. Then $x_t \in G_e$ for $0 < t \leq 1$, while $x_1 \in X_1$ is the given tame point.

As remarked after Definition 5.1, the restricted path $\gamma: [0,1) \to G_e$ lies in the set of leaves without holonomy, hence for the leaf L_0 containing $x_0 = \gamma(0)$, there is a foliated isotopy $\Gamma: L_0 \times [0,1) \to G_e$ such that $\Gamma_t(x_0) = x_t$. In particular, each map $\Gamma_t: L_0 \to L_t$ has homological degree 1.

Also note that for $t = 0$, and each $0 \leq s \leq 1$, the map $H_s: L_0 = L_{0,0} \to L_{s,0}$ is surjective by Theorem 4.1. Let $d_{s,0}$ denote its homological degree. The path of leaves $s \mapsto L_{s,0}$ starting at L_0 has an upper bound D_{L_0} on their volumes by Proposition 4.2, and moreover, there is an upper bound

$$d_0 = \sup\{d_{s,0} \mid 0 \leq s \leq 1\} \ . \tag{37}$$

For L_1 the leaf containing the tame point $x_1 = \gamma(1) \in X_1$, and each $0 \leq s \leq 1$, the map $H_s: L_1 = L_{0,1} \to L_{s,1}$ is also surjective by Theorem 4.1. Let $d_{s,1}$ denote its homological degree. The path of leaves $s \mapsto L_{s,1}$ starting at L_1 has an upper bound D_{L_1} on their volumes by Proposition 4.2, and moreover, there is an upper bound

$$d_1 = \sup\{d_{s,1} \mid 0 \leq s \leq 1\} \ . \tag{38}$$

Set

$$D_L = \max\{D_{L_0}, D_{L_1}\} \ . \tag{39}$$

The set V is saturated, so for each $0 \leq t < 1$, the leaf $L_t \subset V$ as $\gamma(t) \in V$. Thus, we can define a continuous 2–parameter family of maps $\Phi: [0,1] \times [0,1) \times L_0 \to M$ by setting $\Phi_{s,t}(y) = H_s(\Gamma_t(y))$ for $y \in L_0$. It is important to recall the usual caution with the study of compact foliations: the path of leaves $t \mapsto L_t$ with unbounded volumes cannot limit on a compact leaf in the bad set. Thus, the paths $s \mapsto L_{s,t}$ must become more chaotic as $t \to 1$, and correspondingly, the family of maps $\Phi_{s,t}$ is not defined for $t = 1$. On the other hand, we are given that the path $\gamma(t)$ limits on x_1 and so the trace $\Phi_{s,t}(x_0)$ extends to the continuous map $\phi(s,t) = H_s(\gamma(t))$ for $t = 1$. We use this extension of $\Phi_{s,t}(y)$ for $y = x_0$ to show that the map extends for all $y \in L_0$ which gives a contradiction.

The idea of the proof of the existence of this extension is to use the techniques for studying a homotopy of compact leaves introduced in Section 4, to control the degrees of the maps on the fundamental classes of the leaves, induced by the maps $\Phi_{s,t}$. This will in turn yield bounds on the volumes of these leaves, which yields bounds on their diameters. We can thus use Proposition 2.6 to conclude that for $t_* < 1$ sufficiently close to $t = 1$, for each $0 \leq s \leq 1$, the image $\Phi_{s,t_*}(L_0)$ is contained in a uniform normal neighborhood of $H_s(L_1)$. Then the conclusion (43) of Lemma 6.4 for $s = 0$ contradicts the assumption that $\lim_{t\to\infty} \mathrm{vol}\,(L_{0,t}) = \infty$. The proof of these assertions in the next subsection completes the proof of Proposition 6.3.

6.1. **Details of the proof.** We now give the details required to fill out the above sketch of the proof of Proposition 6.3. First, observe that $\Phi_{1,t}: L_0 \to L_*$, for $0 \leq t < 1$, is a family of homotopic maps, hence its homological degree is constant. Thus, for all $0 \leq t < 1$,

$$
\begin{aligned}
\deg(H_1\colon L_0 \to L_*) &= \deg(\Phi_{1,0}\colon L_0 \to L_{1,0}) \\
&= \deg(\Phi_{1,t}\colon L_0 \to L_{1,t}) \\
&= \deg(\Gamma_t\colon L_0 \to L_t) \cdot \deg(H_1\colon L_t \to L_{1,t} = L_*) \\
&= \deg(H_1\colon L_t \to L_{1,t})
\end{aligned}
$$

It follows that

$$\deg(H_1\colon L_t \to L_{1,t}) \leq d_0 \ , \ \forall\ 0 \leq t < 1 \ . \tag{40}$$

Let $\mathfrak{D} = R(2d_0 d_1 D_L)$ be the maximum diameter of a leaf with volume at most $2\, d_0\, d_1\, D_L$, where we recall that d_0 is defined in (37), d_1 is defined in (38), and D_L is defined in (39).

For each $0 \leq s \leq 1$, recall that $L_{s,1}$ is the leaf containing $H_s(x_1)$, and let $0 < \epsilon'_s = \epsilon_{L_{s,1}} \leq \epsilon_0$ be such that the normal projection $\Pi_{L_{s,1}}\colon \mathcal{N}(L_{s,1}, \epsilon'_s) \to L_{s,1}$ is well-defined. Set $\delta'_s = \Delta(\mathfrak{D}, \epsilon'_s)$.

Let L be a compact leaf such that $\mathrm{vol}\,(L) \leq 2\, d_0\, d_1\, D_L$ and $L \cap \mathcal{N}(L_{s,1}, \delta'_s) \neq \emptyset$, then by the choice of $\mathfrak{D}$ and δ'_s, Proposition 2.6 implies that $L \subset \mathcal{N}(L_{s,1}, \epsilon'_s)$. Thus, the restriction $\Pi_{L_{s,1}}\colon L \to L_{s,1}$ is well-defined and a covering map, and moreover by Corollary 2.7 we have the estimate

$$\mathrm{vol}\,(L) \leq 2 \deg(\Pi_{L_{s,1}}\colon L \to L_{s,1}) \cdot \mathrm{vol}\,(L_{s,1}) \leq 2 \deg(\Pi_{L_{s,1}}\colon L \to L_{s,1}) \cdot D_L \ . \tag{41}$$

The next step is to choose a finite covering of the trace of the path $x_{s,1} = H_s(x_1)$ with respect to the constants δ'_s. For each s, $\mathcal{N}(L_{s,1}, \delta'_s)$ is an open neighborhood of $L_{s,1}$, so for $\phi(s,t) = H_s(\gamma(t))$ there exists $\lambda_s > 0$ such that

$$\phi([s - \lambda_s, s + \lambda_s] \times [1 - \lambda_s, 1]) \subset \mathcal{N}(L_{s,1}, \delta'_s) \ . \tag{42}$$

Choose a sequence $0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1$ of points such that for $\lambda_n = \lambda_{s_n}$ the collection of open intervals $\{\mathcal{I}_n = (s_n - \lambda_n, s_n + \lambda_n) \mid n = 0, 1, \ldots, N\}$ is an open covering of $[0,1]$.

Set $\delta''_n = \delta'_{s_n}$ and $\epsilon''_n = \epsilon'_{s_n}$ for $0 \leq n \leq N$, and $\lambda_* = \min\{\lambda_n \mid n = 0, 1, \ldots, N\} > 0$.

Here is the key result:

LEMMA 6.4. *For $0 \leq s \leq 1$ and $1 - \lambda_* \leq t < 1$ we have that*

$$\mathrm{vol}\,(L_{s,t}) \leq 2\, d_0\, d_1\, D_L \ . \tag{43}$$

Proof. For each $1 \leq n \leq N$, set $\xi_0 = 0$ and $\xi_{N+1} = 1$, and for $1 \leq n \leq N$ choose points

$$\xi_n \in (s_{n-1}, s_{n-1} + \lambda_{n-1}) \cap (s_n - \lambda_n, s_n) \ .$$

Then the closed intervals $\{[\xi_0, \xi_1], [\xi_1, \xi_2], \ldots, [\xi_{N-1}, \xi_N], [\xi_N, \xi_{N+1}]\}$ form a closed cover $[0,1]$.

Let μ satisfy $1 - \lambda_* \leq \mu < 1$, and let $L_\mu = \Gamma_\mu(L_0)$ be the leaf through $\gamma(\mu)$. The technical idea of the proof of (43) is to compare the homological degrees of the maps

$$H_{\xi_i}|L_\mu \quad : \quad L_\mu = L_{0,\mu} \to L_{\xi_i,\mu} \tag{44}$$

$$H_{\xi_i}|L_1 \quad : \quad L_1 = L_{0,1} \to L_{\xi_i,1} \tag{45}$$

using a downward induction argument on n, starting with $n = N$, and showing there is a uniform bound on the ratios of their degrees for all $1 - \lambda_* \leq \mu < 1$.

For $n = N$, by (42) we have that

$$\phi([1 - \xi_N, 1] \times [\mu, 1]) \subset \phi([1 - \lambda_N, 1] \times [1 - \lambda_N, 1]) \subset \mathcal{N}(L_{1,1}, \delta''_N)$$

and thus for each $1 - \xi_N \leq s \leq 1$ the point $\phi(s, \mu) \in \mathcal{N}(L_{1,1}, \delta''_N)$.

Note that $L_{1,\mu} = L_{1,1} = L_*$, thus for $s < 1$ sufficiently close to 1 we have $H_s(L_\mu) \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$ as the homotopy H_s is uniformly continuous when restricted to the compact leaf L_μ.

Let r_N be the infimum of r such that $r \leq s \leq 1$ implies $L_{s,\mu} \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$. The above remark implies $r_N < 1$. We claim that $r_N < \xi_N$.

Assume, to the contrary, that $r_N \geq \xi_N$. Let $r_N < r < 1$. Then for $r \leq s \leq 1$, $L_{s,\mu} \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$ and so the normal projection $\Pi_{L_{1,1}} \colon L_{s,\mu} \to L_{1,1}$ is well-defined and a covering map. The restriction

$$H \colon L_\mu \times [r,1] \to \mathcal{N}(L_{1,1}, \epsilon''_N)$$

yields a homotopy between $H_r \colon L_\mu \to L_{r,\mu}$ and $H_1 \colon L_\mu \to L_{1,\mu} = L_{1,1}$. Thus,

$$\deg(\Pi_{L_{1,1}} \circ H_r \colon L_\mu \to L_{r,\mu} \to L_{1,1}) = \deg(\Pi_{L_{1,1}} \circ H_1 \colon L_\mu \to L_{1,\mu} \to L_{1,1}) = \deg(H_1 \colon L_\mu \to L_{1,1})$$

as $\Pi_{L_{1,1}} \colon L_{1,\mu} \to L_{1,1}$ is the identity. The upper bound (40) implies $\deg(H_1 \colon L_\mu \to L_{1,1}) \leq d_0$, hence the covering degree of $\Pi_{L_{1,1}} \colon L_{r,\mu} \to L_{1,1}$ is bounded above by d_0, as it is an integer which divides $\deg(H_1 \colon L_\mu \to L_{1,1})$. By Corollary 2.7 it follows that

$$\text{vol}\,(L_{r,\mu}) \leq 2\, d_0 \cdot \text{vol}\,(L_{1,1}) \leq 2\, d_0 \cdot D_L \ . \tag{46}$$

The leaf volume function is lower semi-continuous, hence we also have that

$$\text{vol}\,(L_{r_N,\mu}) \ \leq \ \lim_{r \to r_N+} \ \text{vol}\,(L_{r,\mu}) \ \leq \ 2\, d_0 \cdot D_L \ .$$

Thus, the estimate (46) holds for all $r_N \leq r \leq 1$ and $1 - \lambda_* \leq \mu < 1$.

As we assumed that $r_N \geq \xi_N \geq \lambda_N$ we have that $\phi(r_N, \mu) \in \mathcal{N}(L_{1,1}, \delta''_N)$, hence Proposition 2.6 implies that $L_{r_N,\mu} \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$. By the uniform continuity of H_s restricted to L_μ at $s = r_N$, there is $r < r_N$ such that $r < s \leq r_N$ implies $L_{s,\mu} \subset \mathcal{N}(L_{1,1}, \epsilon''_N)$. This contradicts the choice of r_N as the infimum of such r, hence we must have that $r_N < \xi_N$.

This proves the first statement of the inductive hypothesis for $n = N$, which is that the estimate (46) holds for all $\xi_N \leq r \leq 1$ and $1 - \lambda_* \leq \mu < 1$.

We next consider the ratios of covering degrees for a pair of leaves in adjacent normal neighborhoods. For $\xi_N \leq s \leq 1$, we have $\phi(s,1) \in \mathcal{N}(L_{1,1}, \delta''_N)$ and $\text{vol}\,(L_{s,1}) \leq D_L$ hence $L_{s,1} \subset \mathcal{N}(L_{1,1}, \epsilon''_{N-1})$, and so the normal projection restricts to a covering map $\Pi_{L_{1,1}} \colon L_{s,1} \to L_{1,1}$. Moreover, this implies that both $L_{\xi_N,\mu}$ and $L_{\xi_N,1}$ are coverings of $L_{1,1}$, and their homological degrees are denoted by

$$\alpha^\mu_N \quad = \quad \deg(\Pi_{L_{1,1}} \colon L_{\xi_N,\mu} \to L_{1,1}) \tag{47}$$

$$a_N \quad = \quad \deg(\Pi_{L_{1,1}} \colon L_{\xi_N,1} \to L_{1,1}) \tag{48}$$

Note that as $s_{N-1} < \xi_N$, the leaves $L_{\xi_N,\mu}$ and $L_{\xi_N,1}$ are also coverings of $L_{s_{N-1},1}$. We compare their homological degrees. By the uniform continuity of H_s restricted to the curve $\gamma(t)$, for $0 \leq s \leq 1$, the path $t \mapsto \phi(s,t)$ has limit $x_{s,1} = H_s(x_1)$. By Proposition 2.6, the volume bound (46) for $1 - \xi_N \leq s \leq 1$ and $1 - \lambda_* \leq t < 1$ implies that

$$H_s(L_t) = L_{s,t} \subset \mathcal{N}(L_{s,1}, \epsilon'_s) \ . \tag{49}$$

Thus, there is a well-defined limit

$$\deg\,(\Phi_{s,1} \colon L_0 \to L_{s,1}) \equiv \lim_{t \to 1} \left\{ \deg\left(\Pi_{L_{s,1}} \circ \Phi_{s,t} \colon L_0 \to \mathcal{N}(L_{s,1}, \epsilon'_s) \to L_{s,1}\right) \right\} \ .$$

The terminology $\deg\,(\Phi_{s,1} \colon L_0 \to L_{s,1})$ is a small abuse of notation, as given $y \in L_0$ there is no assurance that $t \mapsto \Phi_{s,t}(y)$ has a limit at $t = 1$; it is only given that the image is trapped in the open neighborhood $\mathcal{N}(L_{s,1}, \epsilon'_s)$, and the images are homotopic for t sufficiently close to 1.

Then for $1 - \lambda_s \leq t < 1$, define

$$\Xi(s,t) = \frac{\deg\,(\Phi_{s,1} \colon L_0 \to L_{s,1})}{\deg\,(\Phi_{s,t} \colon L_0 \to L_{s,t})} \ . \tag{50}$$

We now apply this discussion in the case $s = \xi_N$ where we have the volume bound (46). It again follows from Proposition 2.6 that for $1 - \lambda_* \leq t < 1$, and noting that $s_N = 1$,

$$H_{\xi_N}(L_t) = L_{\xi_N,t} \subset \mathcal{N}(L_{s_N,1}, \epsilon''_N) \cap \mathcal{N}(L_{s_{N-1},1}, \epsilon''_{N-1}) \ . \tag{51}$$

Thus, for $1 - \lambda_* \leq \mu \leq t < 1$ the maps

$$\Pi_{L_{1,1}} \circ \Phi_{\xi_N,\mu} \sim \Pi_{L_{1,1}} \circ \Phi_{\xi_N,t} \colon L_0 \to \mathcal{N}(L_{1,1}, \epsilon''_N)$$

are homotopic, hence

$$\deg(\Pi_{L_{1,1}} \circ \Phi_{\xi_N,\mu}) = \deg(\Pi_{L_{1,1}} \circ \Phi_{\xi_N,t}) \ . \tag{52}$$

For t sufficiently close to 1 the map $\Pi_{L_{1,1}} \circ \Phi_{\xi_N,t}$ on the left-hand-side of (52) factors

$$\Pi_{L_{1,1}} \circ \iota \circ \Pi_{L_{\xi_N,1}} \circ \Phi_{\xi_N,t} \colon L_0 \to \mathcal{N}(L_{\xi_N,1}, \epsilon'_{\xi_{N-1}}) \to L_{\xi_N,1} \subset \mathcal{N}(L_{1,1}, \epsilon''_N) \to L_{1,1}$$

while the map $\Pi_{L_{1,1}} \circ \Phi_{\xi_N,\mu}$ on right-hand-side of (52) factors

$$\Pi_{L_{1,1}} \circ \Phi_{\xi_N,\mu} \colon L_0 \to L_{\xi_N,\mu} \to L_{1,1} \ .$$

Identifying the degrees of these maps in our terminology, we obtain from (52) that

$$\deg(\Phi_{\xi_N,\mu} \colon L_0 \to L_{\xi_N,\mu}) \cdot \alpha_N^\mu = \deg(\Pi_{L_{1,1}} \circ \Phi_{\xi_N,\mu}) = \deg(\Pi_{L_{1,1}} \circ \Phi_{\xi_N,t}) = \deg(\Phi_{\xi_N,1} \colon L_0 \to L_{\xi_N,1}) \cdot a_N$$

and so

$$\alpha_N^\mu = \Xi(\xi_N, \mu) \cdot a_N \ . \tag{53}$$

Thus, the ratio (50) gives the relation between the homological degrees of the maps in (47) and (48). This completes the proof of the first stage of the induction for the proof of Lemma 6.4.

The general inductive hypotheses involves two statements: Given $0 \leq n \leq N$, we first assume that:

$$\text{for all } 0 \leq n \leq N, \text{ for all } \xi_n \leq s \leq 1 \text{ and } 1 - \lambda_* \leq t \leq 1, \text{ then } \mathrm{vol}\,(L_{s,t}) \leq 2\, d_0\, d_1 \cdot D_L \ . \tag{54}$$

Given (54), then for $n \leq \ell \leq N$ and $1 - \lambda_* \leq \mu \leq 1$ define the integers $a_\ell, b_\ell, \alpha_\ell^\mu, \beta_\ell^\mu$.

$$\begin{array}{lclcrcl}
L_{\xi_\ell,1} & \subset & \mathcal{N}(L_{s_\ell,1}, \epsilon''_\ell) & , & a_\ell & = & \deg\left(\Pi_{L_{s_\ell,1}} \colon L_{\xi_\ell,1} \to L_{s_\ell,1}\right) \\
L_{\xi_\ell,1} & \subset & \mathcal{N}(L_{s_{\ell-1},1}, \epsilon''_{\ell-1}) & , & b_\ell & = & \deg\left(\Pi_{L_{s_{\ell-1},1}} \colon L_{\xi_\ell,1} \to L_{s_{\ell-1},1}\right) \\
L_{\xi_\ell,\mu} & \subset & \mathcal{N}(L_{s_\ell,1}, \epsilon''_\ell) & , & \alpha_\ell^\mu & = & \deg\left(\Pi_{L_{s_\ell,1}} \colon L_{\xi_\ell,\mu} \to L_{s_\ell,1}\right) \\
L_{\xi_\ell,\mu} & \subset & \mathcal{N}(L_{s_{\ell-1},1}, \epsilon''_{\ell-1}) & , & \beta_\ell^\mu & = & \deg\left(\Pi_{L_{s_{\ell-1},1}} \colon L_{\xi_\ell,\mu} \to L_{s_{\ell-1},1}\right)
\end{array}$$

For notational convenience, set $b_{N+1} = \beta_{N+1}^\mu = 1$ and $a_0 = \alpha_0^\mu = 1$. Second, we assume that:

$$\text{for all } n \leq \ell \leq N, \text{ and } 1 - \lambda_* \leq \mu \leq 1, \text{ then } \frac{\alpha_\ell^\mu}{a_\ell} = \Xi(\xi_\ell, \mu) = \frac{\beta_\ell^\mu}{b_\ell} \ . \tag{55}$$

We show that if (54) and (55) are true for n, then the corresponding statements are true for $n-1$.

The choice of $\lambda_s > 0$ so that (42) holds implies that

$$\phi([s_{n-1} - \lambda_{n-1}, s_{n-1} + \lambda_{n-1}] \times [1 - \lambda_*, 1]) \subset \mathcal{N}(L_{s_{n-1},1}, \delta''_{n-1})$$

and hence $\phi(s,t) \in \mathcal{N}(L_{s_{n-1},1}, \delta''_{n-1})$ for all $\xi_{n-1} \leq s \leq \xi_n$ and $1 - \lambda_* \leq t < 1$.

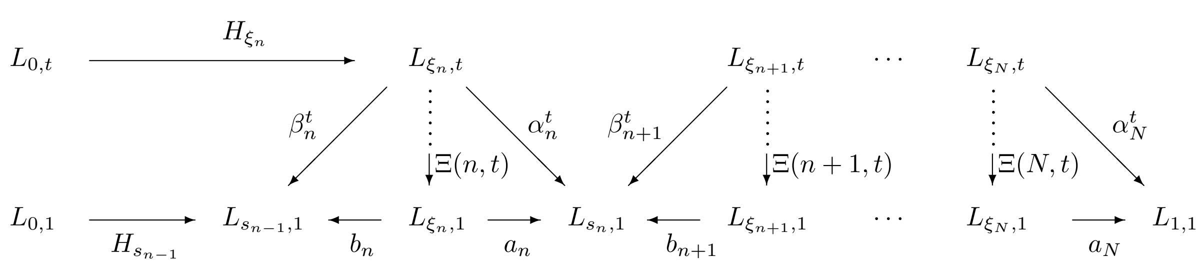
For $s = \xi_n$ the hypothesis (54) implies that for all $1 - \lambda_* \leq t < 1$,

$$\mathrm{vol}\,(L_{\xi_n,t}) \leq 2\, d_0\, d_1 \cdot D_L \ \text{ and hence } L_{\xi_n,t} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1}) \ . \tag{56}$$

Thus, the restriction $\Pi_{L_{s_{n-1},1}} \colon L_{\xi_n,t} \to L_{s_{n-1},1}$ is a covering map. The key to the proof of the inductive step is to obtain a uniform estimate for the homological degree of this covering map.

LEMMA 6.5. *For all* $1 - \lambda_* \leq t < 1$, $\beta_n^t \cdot \deg\left(H_{\xi_n} \colon L_{0,t} \to L_{\xi_n,t}\right) \ \leq \ d_0\, d_1$.

Proof. Consider the diagram



where the integer next to a covering map indicates its homological degree.

The maps $H_{\xi_n}: L_{0,1} \to L_{\xi_n,1}$ and $H_{s_{n-1}}: L_{0,1} \to L_{s_{n-1},1}$ are homotopic through maps into $\mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$, hence

$$\deg\left(H_{s_{n-1}}: L_{0,1} \to L_{s_{n-1},1}\right) = b_n \cdot \deg\left(H_{\xi_n}: L_{0,1} \to L_{\xi_n,1}\right) \ . \tag{57}$$

As $\deg\left(H_{s_{n-1}}: L_{0,1} \to L_{s_{n-1},1}\right) = d_{s,1} \leq d_1$ and the degrees of the maps are positive integers, it follows that $1 \leq b_n \leq d_1$.

For $n \leq \ell < N$ and $1 - \lambda_* \leq t < 1$, the maps $H_{\xi_\ell}: L_{0,t} \to L_{\xi_\ell,t}$ and $H_{\xi_{\ell+1}}: L_{0,t} \to L_{\xi_{\ell+1},t}$ are homotopic through maps into $\mathcal{N}(L_{s_\ell,1}, \epsilon''_\ell)$, hence

$$\alpha^t_\ell \cdot \deg\left(H_{\xi_\ell}: L_{0,t} \to L_{\xi_\ell,t}\right) = \beta^t_{\ell+1} \cdot \deg\left(H_{\xi_{\ell+1}}: L_{0,t} \to L_{\xi_{\ell+1},t}\right) \ . \tag{58}$$

Likewise, for $n \leq \ell < N$, the maps $H_{\xi_\ell}: L_{0,1} \to L_{\xi_\ell,1}$ and $H_{\xi_{\ell+1}}: L_{0,1} \to L_{\xi_{\ell+1},1}$ are homotopic through maps into $\mathcal{N}(L_{s_\ell,1}, \epsilon''_\ell)$, hence

$$a_\ell \cdot \deg\left(H_{\xi_\ell}: L_{0,1} \to L_{\xi_\ell,1}\right) = b_{\ell+1} \cdot \deg\left(H_{\xi_{\ell+1}}: L_{0,1} \to L_{\xi_{\ell+1},1}\right) \ . \tag{59}$$

It follows from equation (58) that

$$\begin{aligned}
\deg\left(H_1: L_{0,t} \to L_{1,t}\right) &= \frac{\alpha^t_N}{\beta^t_{N+1}} \cdot \deg\left(H_{\xi_N}: L_{0,t} \to L_{\xi_N,t}\right) \\
&= \frac{\alpha^t_{N-1}\alpha^t_N}{\beta^t_N\beta^t_{N+1}} \cdot \deg\left(H_{\xi_{N-1}}: L_{0,t} \to L_{\xi_{N-1},t}\right) \\
&\vdots \\
&= \frac{\alpha^t_n \cdots \alpha^t_{N-1}\alpha^t_N}{\beta^t_{n+1} \cdots \beta^t_N\beta^t_{N+1}} \cdot \deg\left(H_{\xi_n}: L_{0,t} \to L_{\xi_n,t}\right) \\
&= \frac{\alpha^t_n \cdots \alpha^t_N}{\beta^t_n \cdots \beta^t_N} \cdot \beta^t_n \cdot \deg\left(H_{\xi_n}: L_{0,t} \to L_{\xi_n,t}\right)
\end{aligned}$$

so that by the inductive hypothesis (55) we have

$$\begin{aligned}
\beta^t_n \cdot \deg\left(H_{\xi_n}: L_{0,t} \to L_{\xi_n,t}\right) &= \frac{\beta^t_n \cdots \beta^t_N}{\alpha^t_n \cdots \alpha^t_N} \cdot \deg\left(H_1: L_{0,t} \to L_{1,t}\right) \\
&= \frac{b_n \cdots b_N}{a_n \cdots a_N} \cdot \deg\left(H_1: L_{0,t} \to L_{1,t}\right) \qquad (60)
\end{aligned}$$

Using (57) we obtain

$$\deg\left(H_1: L_{0,1} \to L_{1,1}\right) = \frac{a_n \cdots a_N}{b_n \cdots b_N} \cdot \deg\left(H_{s_{n-1}}: L_{0,1} \to L_{s_{n-1},1}\right) \ . \tag{61}$$

so that

$$\frac{b_n \cdots b_N}{a_n \cdots a_N} = \frac{\deg\left(H_{s_{n-1}}: L_{0,1} \to L_{s_{n-1},1}\right)}{\deg\left(H_1: L_{0,1} \to L_{1,1}\right)} \leq d_1 \ . \tag{62}$$

and hence combining (40) , (60) and (62) we obtain

$$\beta^t_n \cdot \deg\left(H_{\xi_n}: L_{0,t} \to L_{\xi_n,t}\right) \leq d_1 \cdot \deg\left(H_1: L_{0,t} \to L_{1,t}\right) \leq d_0\, d_1 \ . \tag{63}$$

This completes the proof of Lemma 6.5. □

Fix $1 - \lambda_* \leq \mu < 1$. Let $r_{n-1} \leq \xi_n$ be the infimum of r satisfying $r \leq \xi_n$ such that $r \leq s \leq \xi_n$ implies that $L_{s,\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$. As $L_{\xi_n,\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$, the continuity of H_s at $s = \xi_n$ implies that $r_{n-1} < \xi_n$. We claim that $r_{n-1} < \xi_{n-1}$.

Assume, to the contrary, that $r_{n-1} \geq \xi_{n-1}$. Let $r_{n-1} < r < \xi_n$, then for $r \leq s \leq \xi_n$, $L_{s,\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$ and so the normal projection $\Pi_{L_{s_{n-1},1}} \colon L_{s,\mu} \to L_{s_{n-1},1}$ is well-defined and a covering map. The restriction

$$H \colon L_\mu \times [r, \xi_n] \to \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$$

yields a homotopy between $H_r \colon L_\mu \to L_{r,\mu}$ and $H_{\xi_n} \colon L_\mu \to L_{\xi_n,\mu}$. Thus,

$$\deg(\Pi_{L_{s_{n-1},1}} \circ H_r \colon L_\mu \to L_{r,\mu} \to L_{s_{n-1},1}) = \deg(\Pi_{L_{\xi_{n-1},1}} \circ H_{\xi_n} \colon L_\mu \to L_{\xi_n,\mu} \to L_{\xi_{n-1},1}) \ .$$

It follows from the estimate (63) that

$$\deg(\Pi_{L_{s_{n-1},1}} \colon L_{r,\mu} \to L_{s_{n-1},1}) \leq \deg(\Pi_{L_{s_{n-1},1}} \circ H_r \colon L_\mu \to L_{r,\mu} \to L_{s_{n-1},1}) \leq d_0\, d_1 \tag{64}$$

hence

$$\mathrm{vol}\,(L_{r,\mu}) \leq 2\, d_0\, d_1 \cdot \mathrm{vol}\,(L_{s_{n-1},1}) \leq 2\, d_0\, d_1 \cdot D_L \ . \tag{65}$$

The leaf volume function is lower semi-continuous, hence we also have that

$$\mathrm{vol}\,(L_{r_{n-1},\mu}) \ \leq \ \lim_{r \to r_{n-1}+} \mathrm{vol}\,(L_{r,\mu}) \ \leq \ 2\, d_0\, d_1 \cdot D_L \ . \tag{66}$$

Thus, the estimate (65) holds for all $r_{n-1} \leq r \leq 1$ and $1 - \lambda_* \leq \mu < 1$.

As we assumed that $r_{n-1} \geq \xi_{n-1} \geq s_{n-1} - \lambda_{n-1}$ we have that $\phi(r_{n-1}, \mu) \in \mathcal{N}(L_{s_{n-1},1}, \delta''_{n-1})$ hence $L_{r_{n-1},\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$. By the continuity of H_s at $s = r_{n-1}$ there is $r < r_{n-1}$ such that $r < s \leq r_{n-1}$ implies $L_{s,\mu} \subset \mathcal{N}(L_{s_{n-1},1}, \epsilon''_{n-1})$. This contradicts the choice of r_{n-1} as the infimum of such r, hence we must have that $r_{n-1} < \xi_{n-1}$. This proves the first statement of the inductive hypothesis for $n - 1$.

The second inductive statement (55) follows exactly as before.

Thus, we conclude by downward induction that (43) holds for all $1 - \lambda_* \leq t < 1$ and all $0 \leq s \leq 1$.

This completes the proof of Lemma 6.4, and so completes the proof of Proposition 6.3. □

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