# Dynamics of expansive group actions on the circle 

Steven Hurder*<br>Department of Mathematics<br>University of Illinois at Chicago

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#### Abstract

In this paper, we study the topological dynamics of $C^{0}$ and $C^{1}$ actions on the circle by a countably generated group $\Gamma$, under the assumption that there is expansive orbit behavior. Our first result is a simple proof that if a $C^{0}$-action $\varphi$ is expansive, then $\Gamma$ must have a free sub-semigroup on two generators, hence has exponential growth. For $C^{1}$-actions, we introduce the set of points $E(\varphi)$ with positive asymptotic exponent. These are the points which are infinitesimally expansive. We prove that the hyperbolic periodic points of elements $\varphi(\gamma)$ are dense in $E(\varphi)$. We then show that if the infinitesimally expansive set $E(\varphi)$ has an accumulation point in itself, then the geometric entropy $h(\varphi)$ must be positive. If $\mathbf{K}$ is a minimal set for a $C^{1}$ action $\varphi$, then either there is an invariant probability measure supported on $\mathbf{K}$, or $K \subset E(\varphi)$. As a corollary of the proof, we give a new proof that a $C^{1}$-action with $h(\varphi)=0$ must have an invariant probability measure. Finally, we use the results of the paper to reformulate a conjecture of Ghys, and give a proof that for a real analytic action $\varphi$, there is either an invariant measure, or there is a nonabelian free subgroup of $\Gamma$.


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## 1 Introduction

Let $\Gamma$ be a finitely-generated group, and $\varphi: \Gamma \times \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ be an effective action of $\Gamma$ on the circle by orientation-preserving homeomorphisms. The topological dynamics of $\varphi$ and the algebraic structure of $\Gamma$ are known to be related in many, often surprising ways (see Ghys [14].) There are also many open questions about such actions. In this paper, we consider some results related to the theme of "expansiveness" and the consequences for the topological dynamics of $\varphi$ and the structure of $\Gamma$.

One motivation for this paper comes from the study of codimension one foliations. The dynamics of codimension one foliations can be modeled by a pseudogroup acting on $\mathbb{S}^{1}[16,33,18,1]$. Conversely, every group action on $\mathbb{S}^{1}$ can be realized as the holonomy pseudogroup of a codimension one foliation. Thus, there is a close relationship between these two areas. However, the study of dynamics of pseudogroup actions requires careful attention to the domains of definition of elements of the pseudogroup, a highly technical issue that often obscures fundamental ideas.

Another advantage of the study of group actions is that the orbit of a point $x$ is given by the quotient space $\{\varphi(\gamma)(x) \mid \gamma \in \Gamma\} \cong \Gamma / \Gamma_{x}$ where $\Gamma_{x}$ is the isotropy subgroup of $x$. Thus, the geometry of the orbit of $x$ is dominated by that of $\Gamma$, and we can formulate properties about the geometry of all orbits using $\Gamma$. For foliations, there is no a priori relationship between the topology of different leaves, even up to coarse geometric equivalence [24].

This paper can be considered as an introduction to the dynamical theory of codimension one foliations, without the complications of pseudgroups. Many results of this paper have corresponding versions for codimension one foliations [23, 25, 27, 29], but the statements are often easier to formulate and more definitive for a group action on $\mathbb{S}^{1}$, and the proofs are more elementary.

In the study of the dynamics of group actions and foliations, "expansive orbit behavior" is often encountered, and it has strong consequences for the topological dynamics of the system. We will use a broad interpretation of expansiveness in this paper, ranging from the classical definition below, to a local form where the action is assumed expansive on some dynamically defined subset of $\mathbb{S}^{1}$. In this generality, expansiveness is one of the key properties in the topological dynamics of a group action. The standard definition of an expansive action is as follows. As we are often considering the restriction of $\varphi$ to invariant subset of $\mathbb{S}^{1}$, we give the definition for a general action $\varphi: \Gamma \times \mathbb{X} \rightarrow \mathbb{X}$ on a metric space ( $\mathbb{X}, d_{X}$ ). For notational convenience, given $\gamma \in \Gamma$ we let $\gamma x=\varphi(\gamma)(x)$.

DEFINITION 1.1 $A$ continuous action $\varphi: \Gamma \times \mathbb{X} \rightarrow \mathbb{X}$ on a metric space $\left(\mathbb{X}, d_{X}\right)$ is expansive if there exists $\epsilon>0$ so that for any pair $x \neq y \in \mathbb{X}$, there exists $\gamma \in \Gamma$ such that $d_{X}(\gamma x, \gamma y)>\epsilon$. If we wish to emphasize the constant $\epsilon$, then we say $\varphi$ is $\epsilon$-expansive.

Expansive actions of $\mathbb{Z}$ on compact Riemannian manifold $M$ are a well-understood area of dynamics [38, 30]. For example, it is simple to show that the topological entropy of an expansive diffeomorphism must be positive. However, an action of $\mathbb{Z}$ on $\mathbb{S}^{1}$ is never expansive, as an expansive map on $\mathbb{S}^{1}$ cannot be invertible. There are also many classes of expansive actions of groups $\Gamma$ acting on shift spaces, which are topologically just Cantor sets [10, 37]. Motivated by these examples, Tom Ward asked might it be possible to find an expansive action of a nilpotent group $\Gamma$ on $\mathbb{S}^{1}$ ? Answering this question provided another motivation for this work.

We now discuss the results of this paper. We will assume $\mathbb{S}^{1}$ has the Riemannian metric induced from the standard embedding in the Euclidean plane as a circle of radius 1 , and let $d: \mathbb{S}^{1} \times \mathbb{S}^{1} \rightarrow[0, \pi]$ denote the path length metric. Recall that a subset $\mathbf{K} \subset \mathbb{S}^{1}$ is minimal if $\mathbf{K}$ is closed and $\Gamma$-invariant, and there is no closed, $\Gamma$-invariant proper subset of $\mathbf{K}$.

Section 2 begins some basic definitions and results about the topological dynamics of group actions on the circle. We include a discussion of the dynamics for actions restricted to minimal sets. For example, we show that if $\varphi$ is expansive when restricted to a minimal set $\mathbf{K}$ then $\mathbf{K}$ has finite type. We also show that an expansive action on $\mathbb{S}^{1}$ must have an open local minimal set. Finally, we define the entropy of an action following [12], define the concept of a "ping-pong game" and "resilient" orbit, and give the relation to entropy.

In section 3, we prove that for an expansive $C^{0}$-action on $\mathbb{S}^{1}, \Gamma$ must have exponential growth. An expansive action $\varphi$ can be suspended to give an expansive topological foliation $\mathcal{F}$ in the sense of Inaba and Tsuchiya, and one of the main results of [29] implies that $\mathcal{F}$ must have a resilient leaf, and hence $\Gamma$ has exponential growth. This provides a negative answer to Ward's question above. Independently, and much later, Connell, Furman, Hurder [7] and Spatzier [39] proved that if $\varphi$ is an expansive action, then $\Gamma$ cannot have a nilpotent subgroup of finite index. Here, we give a simple direct proof of the following optimal result (see Example 8.1 in section 8):

THEOREM 1.2 If $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ is an expansive action, then the entropy of the action $\varphi$ must be positive (or infinite) and $\Gamma$ contains a free sub-semigroup on two generators. In particular, $\Gamma$ must have exponential word growth, and cannot have a nilpotent subgroup of finite index.

The proof uses only elementary topological dynamics, and the techniques are reminiscent of those in [33]. In particular, we show in detail how the expansive hypothesis implies the existence of a "ping-pong game" for the dynamics of $\varphi$, which implies both of the conclusions of Theorem 1.2. The proof for the group action case is analogous to that for foliations given in [29], but is simpler and all details are included, which will hopefully make the ideas from group dynamics more transparent, and convince the reader this is a "basic" result of group dynamics.

Sections 4 through 6 study $C^{1}$-actions. In section 4 , we introduce the $\Gamma$-invariant set $E(\mathcal{F}) \subset \mathbb{S}^{1}$, where a point $x \in \mathbb{S}^{1}$ is in $E(\mathcal{F})$ if the asymptotic exponent of $\Gamma$ at $x$ is positive - the precise definition is given in $\S 4$. Such sets arise in the study of $C^{1}$-group actions on $\mathbb{S}^{1}$ which have positive entropy $[22,23,25]$. One of the main results of this paper is that a $C^{1}$-action satisfies a dichotomy, that there is always either an invariant probability measure, or $E(\mathcal{F})$ contains a minimal set for $\varphi$.

For $a>0$, the subset $E_{a}(\varphi) \subset E(\varphi)$ consists of those points for which the exponent is at least $a$, and $E(\varphi)$ is the union of all such subsets. The set of hyperbolic fixed-points for the diffeomorphisms $\varphi(\gamma)$ form a subset $\mathcal{P}^{h}(\varphi) \subset E(\mathcal{F})$, filtered by the subsets $\mathcal{P}_{b}^{h}(\varphi)$ of points such that $\varphi(\gamma)^{\prime}(x)>b\|\gamma\|$ for some $\gamma \in \Gamma$ with $\gamma x=x$.

THEOREM 1.3 Let $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ be a $C^{1}$-action, and suppose that $E_{a}(\mathcal{F})$ is not empty. Then for all $b<a$ the hyperbolic fixed-points $\mathcal{P}_{b}^{h}(\varphi)$ are dense in $E_{a}(\varphi)$, and $\varphi$ is expansive on $\mathcal{P}_{b}^{h}(\varphi)$.

The proof of Theorem 1.3 follows from Propositions 4.5 and 4.7. The proofs of these two propositions introduce several fundamental techniques for the study of expansive $C^{1}$-actions, which are used repeatedly in later sections.

At every point of $E(\mathcal{F})$ there is an element of holonomy which is locally expanding by an arbitrarily large factor, hence if the set $E(\mathcal{F})$ is "large enough" one expects the action of $\varphi$ to be chaotic, and have positive entropy. We prove this in section 5 :

THEOREM 1.4 Let $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ be a $C^{1}$-action, and suppose that $E_{a}(\mathcal{F})$ has infinite cardinality for some $a>0$. Then there exists a perfect subset $\mathbf{K} \subset E(\mathcal{F})$ on which the restricted action of $\varphi$ is expansive. Moreover, the entropy $h(\varphi)$ of the action must be positive, and $\Gamma$ contains a free sub-semigroup on two generators.

COROLLARY 1.5 If $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ is a $C^{1}$-action with $E(\mathcal{F})$ an uncountable set, then $h(\varphi)>0$.
Section 6 gives criteria for when the set $E(\varphi)$ is non-empty:
THEOREM 1.6 Let $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ be a $C^{1}$-action with minimal set $\mathbf{K}$. Then either every orbit of $\varphi$ on $\mathbf{K}$ has polynomial growth, or $\mathbf{K} \subset E(\varphi)$, and hence there is a hyperbolic ping-pong game for $\varphi$, and the entropy $h(\varphi)>0$.

This result is a generalization of the Sacksteder Theorem [33] applied to $C^{2}$ actions. This implies the following dichotomy for $C^{1}$-actions, which follows from Lemma 6.1 and Theorem 1.6:

COROLLARY 1.7 A $C^{1}$-action $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ either has an invariant probability measure, or there is a hyperbolic ping-pong game for $\varphi$, and the entropy $h(\varphi)>0$.

In section 7, we discuss the application of the ideas of this paper to a conjecture of Ghys that a $C^{0}$-action either has an invariant probability measure, or there exists a nonabelian free subgroup of $\Gamma$. We reformulate the conjecture in terms of the action of $\varphi$ on minimal sets, and show the conjecture is implied by the existence of certain maps with isolated fixed-points. As an application, this proves:

THEOREM 1.8 Let $\varphi: \Gamma \times \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ be a $C^{\omega}$-action. Then either $\varphi$ has an invariant probability measure, or $\Gamma$ has a non-abelian free subgroup on two generators.

Shortly (a few days) after this paper was first circulated, G. Margulis gave a proof of the Ghys Conjecture for $C^{0}$-actions [31].

Finally, in section 8 we discuss three examples which are very helpful in understanding the results of the previous sections, as they illustrate dynamical properties which must be considered (implicitly, at least) in the proofs. The first example is of an expansive, real analytic action of a solvable group on the circle, which shows that the conclusions Theorems 1.2 and 1.4 cannot be strengthened to conclude that $\Gamma$ contains a free non-abelian subgroup on two generators. The second example is an extension of the first, and gives an expansive $C^{1}$-action with an exceptional minimal set $\mathbf{K}$ such that the action is not hyperbolic on $\mathbf{K}$. The third example is simple, but illustrates a group action with a countable family of hyperbolic fixed-points but tame dynamics.

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## 2 Topological dynamics of group actions

We give definitions and basic results concerning the topological dynamics of continuous group actions. Some of these results are just straightforward generalizations of ideas from the dynamics of a single invertible transformation, while other properties are more similar to results from the structure theory of $C^{0}$-foliations [34, 35, 36, 15, 19]. We start with two definitions from topological dynamics of group actions.

DEFINITION 2.1 A continuous action $\varphi: \Gamma \times \mathbb{X} \rightarrow \mathbb{X}$ on a metric space $\left(\mathbb{X}, d_{X}\right)$ is equicontinuous if for all $\epsilon>0$ there exists $\delta(\epsilon)>0$ so that so that for all $x, y \mathbb{X}$, if $d_{X}(x, y) \geq \epsilon$ then $d_{X}(\gamma x, \gamma y) \geq$ $\delta(\epsilon)$ for all $\gamma \in \Gamma$. That is, the family of maps $\{\varphi(\gamma) \mid \gamma \in \Gamma\}$ is equicontinuous.

If $\mathbb{X}=\mathbb{S}^{1}$ and $\varphi$ is an equicontinuous action, then there exists a $\Gamma$-invariant metric $d_{\Gamma}$ on $\mathbb{S}^{1}$. Moreover, $d_{\Gamma}$ defines a $\Gamma$-invariant Borel measure $\mu_{\Gamma}$ on $\mathbb{S}^{1}$ by defining $\mu_{\Gamma}([x, y])=d_{\Gamma}(x, y)$ where $[x, y] \subset \mathbb{S}^{1}$ is a closed interval.

DEFINITION 2.2 A continuous action $\varphi: \Gamma \times \mathbb{X} \rightarrow \mathbb{X}$ on a metric space $\left(\mathbb{X}, d_{X}\right)$ is distal if, for every pair $x \neq y \in \mathbb{X}$, there exists $\delta(x, y)>0$ such that $d_{X}(\gamma x, \gamma y) \geq \delta(x, y)$ for all $\gamma \in \Gamma$.

Clearly, an equicontinuous action is distal, while an expansive action is at the other extreme. There is a dichotomy in the following special case:

LEMMA 2.3 $A$ minimal action $\varphi: \Gamma \times \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ is either equicontinuous or expansive.
Proof: Let $\left\{\sigma_{1}, \ldots, \sigma_{k}\right\}$ be a symmetric generating set for $\Gamma$. Then there exists a modulus of continuity function $\delta(\epsilon)$ so that if $[a, y] \subset \mathbb{S}^{1}$ is a closed interval of length $\delta(\epsilon)<\pi$ then $\sigma_{\ell}([x, y])$ has length at most $\epsilon$. Set $\delta_{0}=\delta(\pi / 2)$.

Suppose that $\varphi$ is not equicontinuous. Then for some $0<\epsilon<\pi$ there exists a sequence of pairs $\left\{\left(x_{n}, y_{n}\right) \mid n=1,2, \ldots\right\}$ with $d\left(x_{n}, y_{n}\right)>\epsilon$ and $\gamma_{n} \in \Gamma$ such that $d\left(\gamma_{n} x_{n}, \gamma_{n} y_{n}\right)<1 / n$. Choose a limit point $z_{*}$ for the set $\left\{\gamma_{n} x_{n} \mid n=1,2, \ldots\right\}$. Passing to a subsequence, we can assume that $d\left(\gamma_{n} x_{n}, z_{*}\right)<1 / n$ and $d_{X}\left(\gamma_{n} x_{n}, \gamma_{n} y_{n}\right)<1 / n$ for all $n$.

Note that $d(x, y)<\epsilon$ implies there is a unique shortest interval $\overline{x, y}$ with endpoints $x$ and $y$. Let $x \neq y \in \mathbb{S}^{1}$ with $d(x, y)<\epsilon$, then there exists $\gamma_{0} \in \Gamma$ such that $\gamma_{0} z_{*}$ lies in the interior of $\overline{x, y}$. Choose $n \gg 0$ such that $\gamma_{0}\left(\overline{\gamma_{n} x_{n}, \gamma_{n} y_{n}}\right)$ is contained in the interior of $\overline{x, y}$.

Write $\gamma_{n}=\sigma_{i_{1}} \cdots \sigma_{i_{k}}$ as a product of generators, and define $h_{\ell}=\sigma_{i_{\ell}}^{-1} \cdots \sigma_{i_{1}}^{-1} \cdot \gamma_{0}^{-1}$. Note that $h_{k}(\overline{x, y})$ contains $\overline{\gamma_{n} x_{n}, \gamma_{n} y_{n}}$ in its interior, so is an interval with length at least $\epsilon$. If the length is greater than $\pi$ then there exists $\ell<k$ such that $h_{\ell}(\overline{x, y})$ has length between $\delta_{0}$ and $\pi$. We set $\epsilon_{0}=\min \left\{\epsilon, \delta_{0}\right\}$ and it follows that $\varphi$ is $\epsilon_{0}$-expansive.

### 2.1 Minimal sets

Suppose that $\mathbf{K} \subset \mathbb{S}^{1}$ is a minimal set for $\varphi$. Then either $\mathbf{K}$ is a finite set, or $\mathbf{K}=\mathbb{S}^{1}$, or $\mathbf{K}$ is a perfect, nowhere dense subset.

If $\mathbf{K}$ is finite, the $\Gamma$ acts via permutations of the set of points, so there is a normal subgroup of finite index $\Gamma_{\mathbf{K}} \subset G$ consisting of transformations which fix all the points of $\mathbf{K}$.

If $\mathbf{K}=\mathbb{S}^{1}$ then we saw that $\varphi$ is either equicontinuous or expansive. In the former case, it is well-known that there is a $\Gamma$-invariant probability measure $\mathbf{m}$ on $\mathbb{S}^{1}$ whose support must be $\mathbb{S}^{1}$
(cf. [33]). Renormalize $\mathbf{m}$ to have total mass $2 \pi$, choose a basepoint $x_{0} \in \mathbb{S}^{1}$, then the measure $\mathbf{m}$ defines a homeomorphism $h_{\mathbf{m}}: \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ by setting $h_{\mathbf{m}}(x)=\mathbf{m}\left(\left[x_{0}, x\right]\right)$ modulo $2 \pi$, so that $h_{\mathbf{m}}^{*} d \theta=\mathbf{m}$ where $d \theta$ denotes the standard length measure. Thus, $h_{\mathbf{m}}$ conjugates $\varphi$ to an action $\varphi_{\mathbf{m}}$ on $\mathbb{S}^{1}$ by rotations. As $\Gamma$ acts effectively by orientation-preserving homeomorphisms, this implies $\Gamma$ is free abelian. The case where $\varphi$ is expansive will be discussed in later sections.

A nowhere dense minimal set $\mathbf{K}$ is called exceptional. The dynamical properties of exceptional minimal sets have been studied by many authors $[18,34,19,3,4,5,6,28,29]$ and we recall here just a few properties used in this paper. The complement $\mathbb{S}^{1}-\mathbf{K}=\mathcal{I}$ where $\mathcal{I}=\cup_{n=1}^{\infty} I_{n}$ is a countable union of open connected intervals $I_{n}=\left(a_{n}, b_{n}\right)$, and clearly $\mathcal{I}$ is $\Gamma$-invariant. The intervals $I_{n}$ are called the gaps of $\mathbf{K}$, and the closure $\overline{I_{n}}=\left[a_{n}, b_{n}\right]$ is called a closed gap. An endpoint of $\mathbf{K}$ is a point in the intersection $\mathbf{K} \cap \overline{I_{n}}$ for some $n$.

DEFINITION 2.4 For $x \in \mathbb{S}^{1}$ we say the orbit $\Gamma x$ is semi-proper if there exists $\epsilon>0$ so that for either $\mathcal{I}=(x-\epsilon, x]$ or $\mathcal{I}=[x, x+\epsilon)$, the half-open interval $\mathcal{I}$ has disjoint $\Gamma$-orbit. That is, if $\gamma \neq \tau$ then $\gamma \mathcal{I} \cap \tau \mathcal{I}=\emptyset$.

For example, for an exceptional minimal set $\mathbf{K}$ the orbits $\Gamma a_{n}$ and $\Gamma b_{n}$ are semi-proper. We say that $\mathbf{K}$ has finite type if there are only a finite number of semi-proper orbits determined by the endpoints of $\mathbf{K}$. There are $C^{2}$-actions with an exceptional minimal sets that is not of finite type [3, 4, 28].

LEMMA 2.5 Let $\mathbf{K}$ be an exceptional minimal set of $\varphi$. Then $\varphi$ is expansive on $\mathbf{K}$ if and only if $\mathbf{K}$ has finite type.

Proof: Let $\epsilon>0$ be an expansive constant for the action restricted to $\mathbf{K}$. That is, for every $x \neq y \in \mathbf{K}$ there exists $\gamma$ such that $d(\gamma x, \gamma y)>\epsilon$. Let $I_{i_{1}}, \ldots, I_{i_{n}}$ be the gaps such whose length is greater than $\epsilon$. Then for any gap $I_{\ell}$ there must exist $\gamma$ with $\gamma\left(I_{\ell}\right)$ of length greater than $\epsilon$. As $\gamma\left(I_{\ell}\right)$ is again a gap of $\mathbf{K}, \gamma a_{\ell}$ must be be an endpoint for one of $I_{i_{1}}, \ldots, I_{i_{n}}$. Hence, the orbits of the endpoints $\left\{a_{i_{1}}, \ldots, a_{i_{n}}, b_{i_{1}}, \ldots, b_{i_{n}}\right\}$ contain all endpoints of $\mathbf{K}$.

Conversely, suppose that $\mathbf{K}$ has finite type. Let $\left\{I_{n} \mid n=1,2, \ldots\right\}$ be the gaps of $\mathbf{K}$ and suppose that $I_{1}, \ldots, I_{k}$ are such that the orbits of the endpoints $\left\{a_{1}, b_{1}, \ldots, a_{k}, b_{k}\right\}$ contain all the orbits of the endpoints of $\mathbf{K}$. Choose $0<\epsilon<\max \left\{\left|I_{1}\right|,\left|I_{2}\right|, \ldots\left|I_{k}\right|, \pi / 2\right\}$. Given $x, y \in \mathbf{K}$ with $0<d(x, y)<\epsilon, \mathbf{K}$ is nowhere dense so there exists a gap $I_{n}$ with $I_{n} \subset[x, y]$. Then there exists $\gamma \in \Gamma$ such that $\gamma I_{n}=I_{i}$ for some $1 \leq i \leq k$, and so $d(\gamma x, \gamma y) \geq\left|I_{i}\right|>\epsilon$.

Note that if $\mathbf{K}$ is not of finite type, then the proof shows that action fails to be expansive only for pairs $(x, y)$ which are endpoints of some gap $I_{n}$. Let $\hat{\mathbf{K}}=\mathbf{K}-\cup_{n=1}^{\infty}\left\{a_{n}, b_{n}\right\}$ be $\mathbf{K}$ with all endpoints deleted.

LEMMA 2.6 Let $\mathbf{K}$ be an exceptional minimal set of $\varphi$. Then $\varphi$ is expansive on $\hat{\mathbf{K}}$.
Proof: Let $x \neq y \in \hat{\mathbf{K}}$, then $x$ or $y$ are not endpoints, then for any gap, say $I_{1}$, there is some $\gamma \in \Gamma$ for which $\gamma a_{1} \in(x, y)$. Thus, $\gamma I_{1} \subset(x, y)$ and hence $d\left(\gamma^{-1} x, \gamma^{-1} y\right)>\left|I_{1}\right|$.

Proposition 7.2 below gives another version of this result, proving expansiveness for the action on the "interior" of an exceptional minimal set.

The orbit of every point $x \in \mathbf{K}$ is dense, so for any gap $I_{i}$ there are infinitely many disjoint images $\left\{\gamma I_{i} \mid \gamma \in \Gamma\right\}$ hence there must be a sequence $\gamma_{n} I_{i}$ whose lengths tend to 0 . That is, $\lim _{n \rightarrow \infty} d\left(\gamma_{n} a_{i}, \gamma_{n} b_{i}\right)=0$. Hence, if $\varphi$ has an exceptional minimal set then it cannot be distal.

### 2.2 Local minimal sets and expansiveness

DEFINITION 2.7 $A \Gamma$-invariant set $K$ is a local minimal set if for all $x \in K$ the closure of its orbit, $\overline{\Gamma x}$, equals the closure $\bar{K}$. If $K$ is an open set, then we call $K$ an open local minimal set.

A minimal set $K$ is a local minimal set. An orbit $\Gamma x$ such that each point $\gamma x$ is isolated in $\Gamma x$ is a local minimal set, but is not a minimal set unless the orbit is finite.

LEMMA 2.8 Let $x \in \mathbb{S}^{1}$ and let $K=\operatorname{Int} \overline{\Gamma x}$. Let $V \subset K$ be a non-empty open subset. If for every $y \in V$, the orbit closure $\overline{\Gamma y}$ has non-trivial interior, then the saturation of $V$ is an open local minimal set.

Proof: Suppose that for every $y \in V$, the interior $U_{y}=\operatorname{Int} \overline{\Gamma y}$ is not empty. As $y \in \overline{\Gamma x}$ we have $\overline{\Gamma y} \subset \overline{\Gamma x}$ and hence $U_{y} \subset K=\operatorname{Int} \overline{\Gamma x}$. Thus, there exist $\gamma$ so that $\gamma x \in U_{y}$. The closure of $\Gamma y$ contains $U_{y}$ hence has $\gamma x$ as a limit point. But this implies $\Gamma y=\Gamma \gamma^{-1} y$ has $\gamma^{-1} \gamma x=x$ as a limit point, and thus $\overline{\Gamma y} \supset \overline{\Gamma x}$ so $\overline{\Gamma y}=\overline{\Gamma x}=\bar{K}$. That is, for all $y \in V$ we have $\overline{\Gamma y}=\bar{K}$.

Now consider the saturation $W=\bigcup_{\gamma \in \Gamma} \gamma V$. For each $z \in W$, there is $\gamma \in \Gamma$ with $z=\gamma y$ so the orbit closures satisfy

$$
\overline{\Gamma z}=\overline{\Gamma \gamma y}=\overline{\Gamma y}=\bar{K}=\bar{W}
$$

This elementary lemma has a strong consequence.
PROPOSITION 2.9 An expansive action $\varphi$ has a non-empty open local minimal set.
Proof: Suppose that $\varphi$ has no local open minimal set. Choose $x_{1} \in \mathbb{S}^{1}$. If the orbit closure $\overline{\Gamma x_{1}}$ has no interior, set $X_{1}=\overline{\Gamma x_{1}}$. Otherwise, the interior $U_{1}=\operatorname{Int} \overline{\Gamma x_{1}}$ is non-empty, so that $V_{1}=U_{1} \cap\left\{z \in \mathbb{S}^{1} \mid d\left(z, x_{1}\right)<1 / 100\right.$ is a non-empty open set. By Lemma 2.8 there must exists $y_{1} \in V_{1}$ such that $X_{1}=\overline{\Gamma y_{1}}$ has no interior. Set $K_{1}=Z_{1}$.

The complement $\mathbb{S}^{1}-Z_{1}$ is an open $\Gamma$-invariant set, so consists of a countable union of open intervals. Let $I_{2}$ denote one of the complementary intervals with greatest length. Let $x_{2} \in I_{2}$ be the midpoint, and define $X_{2}=\overline{\Gamma x_{2}}$ if this set has no interior. Otherwise, as before, there exists $y_{2} \in I_{2}$ with $d\left(y_{2}, x_{2}\right)<\left|I_{2}\right| / 100$ such that $X_{2}=\overline{\Gamma y_{2}}$ has no interior. Set $Z_{2}=X_{1} \cup X_{2}$.

Given any $\epsilon>0$, we can repeat the above process a finite number of times to obtain a closed invariant subset $Z_{n}$ with no interior such that the complement $\mathbb{S}^{1}-Z_{n}$ consists of a countable union of intervals each of length less than $\epsilon$.

Assume that $\varphi$ is $\epsilon$-expansive. Choose $Z_{n}$ as above, and let $I \subset \mathbb{S}^{1}-Z_{n}$ be a complementary interval. For all $\gamma \in \Gamma$ the image $\gamma I$ is again a complementary interval, so $|\gamma I|<\epsilon$. But is $x, y \in I$ then this implies $d(\gamma x, \gamma y)<\epsilon$ for all $\gamma$ contradicting the assumption. It follows that is $\varphi$ is $\epsilon$-expansive for some $\epsilon>0$ then $\varphi$ must have a non-empty open local minimal set.

COROLLARY 2.10 Suppose that $\varphi$ has no non-empty open local minimal set. Then there exists a collection of sets $\left\{K_{n} \mid n=1,2, \ldots\right\}$ such that

1. each $K_{n}$ is closed and saturated with no interior
2. the action of $\varphi$ on $K_{n}$ is transitive; that is, there exists $x_{n} \in K_{n}$ whose orbit is dense
3. the union $\bigcup_{n=1}^{\infty} K_{n}$ is dense in $\mathbb{S}^{1}$

The proof of the corollary follows from the same method of proof as above. Note this result is an elementary form of the theory of levels for a topological group action. There is no claim about the dynamics and limiting behavior of the sets $K_{n}$, which a good theory of levels proves $[8,34,35,36,15,19]$. In fact, the sets $K_{n}$ could all be exceptional minimal sets! But the simplicity of the proof above is also notable, and is sufficient for our application to the proof of Theorem 1.2.

### 2.3 Geometric entropy

We recall the definition of the entropy of a group action, following Ghys, Langevin and Walczak [12]. While this definition is usually considered only for $C^{1}$-actions, it still makes sense for topological actions, though the value may be infinite.

Choose a symmetric generating set $\mathcal{S}=\left\{\sigma_{1}, \ldots, \sigma_{k}\right\}$ for $\Gamma$. (We also assume one of the $\sigma_{i}$ is the identity.) An element $\gamma \in \Gamma$ has length $\|\gamma\| \leq N$ if there exists indices $i_{1}, \ldots, i_{N}$ such that $\gamma=\sigma_{i_{1}} \cdots \sigma_{i_{N}}$.

Given $\epsilon>0$ and an integer $N>0$, we say that $x, y \in \mathbb{S}^{1}$ are $(N, \epsilon)$-separated if there exists $\gamma \in \Gamma$ with $\|\gamma\| \leq N$ and $d(\gamma x, \gamma y)>\epsilon$. A finite subset $\left\{x_{1}, \ldots, x_{\nu}\right\} \subset \mathbb{S}^{1}$ is $(N, \epsilon)$-separated if for every $k \neq \ell$ the pair of points $x_{k}, x_{\ell}$ is $(N, \epsilon)$-separated.

Let $\mathcal{S}(\varphi, \epsilon, N)$ denote the maximum cardinality of an $(N, \epsilon)$-separated subset of $\mathbb{S}^{1}$. This is a finite number, as there are at most a finite number of maps in $\{\varphi(\gamma) \mid\|\gamma\| \leq N\}$, hence this is an equicontinuous family. Now define

$$
\begin{equation*}
h(\varphi, \epsilon)=\limsup _{N \rightarrow \infty} \frac{\log \mathcal{S}(\varphi, \epsilon, N)}{N} \tag{1}
\end{equation*}
$$

The geometric entropy of $\varphi$ is the limit

$$
h(\varphi)=\lim _{\epsilon \rightarrow 0} h(\varphi, \epsilon)
$$

This limit is finite if $\varphi$ is a $C^{1}$-action, but may be infinite for topological actions when $\Gamma$ has rank greater than one. In general, $h(\varphi)$ depends upon the choice of the generating set $\mathcal{S}$. The key point is that the dichotomy $h(\varphi) \neq 0$ or $h(\varphi)=0$ is independent of the choices, and depends only on the topological conjugacy class of the action $\varphi$.

The number $h(\varphi)$ is called geometric entropy of $\varphi$ as in the limit (1) the denominator is $N$ which represents the word length. In contrast, to define the "usual" topological entropy from topological dynamics of group actions, the denominator would be the number of elements in the ball of radius $N$ in $\Gamma$ for the word metric on $\Gamma$. The usual topological entropy is always finite, but vanishes if $\varphi$ is a $C^{1}$-action and $\Gamma$ is not rank one abelian.

Given a $\Gamma$-subset $\mathbf{K} \subset \mathbb{S}^{1}$ we can also define the relative geometric entropy $h(\varphi, \mathbf{K})$ where we define $\mathcal{S}(\varphi, \mathbf{K}, \epsilon, N)$ using subsets $\left\{x_{1}, \ldots, x_{\nu}\right\} \subset \mathbf{K}$, and the remainder of the definitions follow the same pattern.

### 2.4 Ping-pong games and resilient orbits

We next recall a dynamical notion which was been colloquially called a "ping-pong game" by de la Harpe [9], though the concept dates from the work of Klein, and it has many uses in the study of dynamical systems.

DEFINITION 2.11 The a pair of maps $\left\{\gamma_{1}: I_{0} \rightarrow I_{1}, \gamma_{2}: I_{0} \rightarrow I_{2}\right\}$ is called a ping-pong game for $\varphi$ if $I_{0} \subset \mathbb{S}^{1}$ is a closed interval, $I_{1}, I_{2} \subset I_{0}$ are closed disjoint subintervals, and $\gamma_{1}, \gamma_{2} \in \Gamma$ satisfy $\gamma_{1} I_{0}=I_{1}$ and $\gamma_{2} I_{0}=I_{2}$. If $\varphi$ is a $C^{1}$-action, and $0<\gamma_{k}^{\prime}(x)<1$ for $k=1,2$ and all $x \in I_{0}$, then we call this a hyperbolic ping-pong game.

Note that the definition of a ping-pong game does not require that either map $\gamma_{i}: I_{0} \rightarrow I_{i}$ have a unique fixed-point. However, for a $C^{1}$ action, if both maps $\varphi\left(\gamma_{1}\right)$ and $\varphi\left(\gamma_{2}\right)$ are hyperbolic contractions on $I_{0}$ then they clearly have unique fixed-points.

Here is one of the standard properties of a ping-pong game.
LEMMA 2.12 If $\left\{\gamma_{1}: I_{0} \rightarrow I_{1}, \gamma_{2}: I_{0} \rightarrow I_{2}\right\}$ is a ping-pong game for $\varphi$, then $\left\{\gamma_{1}, \gamma_{2}\right\}$ generates a free sub-semigroup of $\Gamma$.

Proof: A word $\gamma$ in the free sub-semigroup generated by $\left\{\gamma, \gamma_{2}\right\}$ has the form $\gamma=\gamma_{i_{1}} \cdot \gamma_{i_{2}} \cdots \gamma_{i_{\ell}}$ for indices $i_{j}=1,2$. It suffices to show that $\gamma$ is not the identity, but $\gamma I_{0} \subset \gamma_{i_{1}} I_{0} \subset I_{i_{1}} \neq I_{0}$.

An action $\varphi$ with a ping-pong game has non-zero entropy.
LEMMA 2.13 If $\left\{\gamma_{1}: I_{0} \rightarrow I_{1}, \gamma_{2}: I_{0} \rightarrow I_{2}\right\}$ is a ping-pong game for $\varphi$, then $h(\varphi)>0$.
Proof: Choose a symmetric generating set $\mathcal{S}=\left\{\sigma_{1}, \ldots, \sigma_{k}\right\}$ for $\Gamma$ which contains $\left\{\gamma_{1}, \gamma_{1}^{-1}, \gamma_{2}, \gamma_{2}^{-1}\right\}$. Let $\epsilon>0$ be the distance between $I_{1}$ and $I_{2}$. Choose any $x \in I_{0}$ and set

$$
S_{n}=\left\{\gamma_{i_{1}} \cdot \gamma_{i_{2}} \cdots \gamma_{i_{n}} x \mid i_{j}=1,2\right\}
$$

The set $S_{n}$ has $2^{n}$ elements, and given any pair of distinct points $y=\gamma_{i_{1}} \cdot \gamma_{i_{2}} \cdots \gamma_{i_{n}} x$ and $z=$ $\gamma_{j_{1}} \cdot \gamma_{j_{2}} \cdots \gamma_{j_{n}} x$ there is a least $1 \leq \ell \leq n$ so that $i_{\ell} \neq j_{\ell}$ hence

$$
\gamma_{y, z}=\gamma_{i_{1}} \cdots \gamma_{i_{\ell-1}}=\gamma_{j_{1}} \cdots \gamma_{j_{\ell-1}}
$$

Then $\gamma_{y, z}^{-1} y=\gamma_{i_{\ell}} \cdots \gamma_{i_{n}} x \in I_{i_{\ell}}$ and $\gamma_{y, z}^{-1} z=\gamma_{j_{\ell}} \cdots \gamma_{j_{n}} x \in I_{j_{\ell}}$ hence $d\left(\gamma_{y, z}^{-1} y, \gamma_{y, z}^{-1} z\right)>\epsilon$.
It follows that $\mathcal{S}(\varphi, \epsilon, N) \geq 2^{n} N$ and $h(\varphi) \geq \log (2)$.
We recall a definition from the topological dynamics of foliations:
DEFINITION 2.14 We say $x \in \mathbb{S}^{1}$ is a resilient point for $\varphi$ if there exists $\gamma \in \Gamma$ and an open interval $J=(x-\delta, x+\delta)$ such that $J \cap \Gamma x \neq\{x\}$ and $\gamma: J \rightarrow J$ is a contraction with fixed-point $x$. If $\varphi$ is a $C^{1}$-action and $\varphi(\gamma)^{\prime}(x)<1$, then we say $x$ is a hyperbolic resilient point. An orbit $\Gamma y$ is resilient if there is some $x \in \Gamma y$ which is resilient.

LEMMA 2.15 If there exists a resilient point $x$ for $\varphi$ then there is a ping-pong game for $\varphi$.
Proof: Let $x$ be a resilient point with $\gamma: J \rightarrow J$ a contraction with $x$ as unique fixed-point, and $\tau \in \Gamma$ be such $x \neq \tau x \in J$. Choose $n \gg 0$ such that $\gamma^{n} J \cap \tau \gamma^{n} J=\emptyset$. Then set $\gamma_{1}=\gamma^{n}, \gamma_{2}=\tau \gamma^{n}$, $I_{0}=J, I_{1}=\gamma_{1} J, I_{2}=\gamma_{2} J$.

Conversely, if there is a hyperbolic ping-pong game for a $C^{1}$-action $\varphi$ then it is an exercise to show there is a hyperbolic resilient point also.

## 3 Expansive actions on local minimal sets

PROPOSITION 3.1 Suppose that $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ is an expansive action with an open local minimal set $U$. Then there exists a ping-pong game $\left\{\gamma_{1}: K_{0} \rightarrow K_{1}, \gamma_{2}: K_{0} \rightarrow K_{2}\right\}$ for $\varphi$.

Proof: Let $0<\epsilon$ be the expansive constant for $\varphi$. Set $\delta=\epsilon / 10$.
Given points $x, y \in S^{1}$ with $d(x, y)<\pi$ we let $\overline{x y} \subset S^{1}$ denote the interval (the shortest path in $S^{1}$ ) they determine, and $|\overline{x y}|$ the length of this interval.

The open set $U$ is the disjoint union of open intervals, and since $U$ is invariant and $\varphi$ is expansive, the diameter of at least one interval must be at least $\epsilon$. Choose $x_{1} \in U$ to be the midpoint of a longest connected interval in $M$. (In the case where $U=S^{1}$ select any point.) Let $y_{1}, z_{1} \in U$ be the points with $d\left(x_{1}, y_{1}\right)=d\left(x_{1}, z_{1}\right)=\delta / 4$ and $x_{1} \in \overline{y_{1} z_{1}} \subset U$. Choose $\gamma_{1} \in \Gamma$ with $d\left(\gamma_{1} y_{1}, \gamma_{1} z_{1}\right)>\epsilon$. Let $J_{1}$ denote the interval $\overline{y_{1}, z_{1}}$, and $I_{1}=\gamma_{1} J_{1}$ so $\left|I_{1}\right|>\epsilon$.

Now proceed inductively. Assume 6-tuples $\left\{x_{i}, y_{i}, z_{i}, \gamma_{i}, J_{i}, I_{i}\right\}$ have been chosen for $1 \leq i<n$ and we select a new 6 -tuple $\left\{x_{n}, y_{n}, z_{n}, \gamma_{n}, J_{n}, I_{n}\right\}$. Let $x_{n}$ be the midpoint of $I_{n-1}$ and choose $y_{n}, z_{n} \in I_{n-1}$ be distinct points with $d\left(x_{n}, y_{n}\right)=d\left(x_{n}, z_{n}\right)=\delta / 2^{n+1}$. Choose $\gamma_{n} \in \Gamma$ with $d\left(\gamma_{n} y_{n}, \gamma_{n} z_{n}\right)>\epsilon$. Then set $J_{n}=\overline{y_{n}, z_{n}} \subset I_{n-1}$ and let $I_{n}=\gamma_{n} J_{n}$ which is a subset of $U$. Note that $\left|J_{n}\right|=\delta / 2^{n}$ and $\left|I_{n}\right|>\epsilon$ for all $n \geq 1$.

Let $x_{*}$ be an accumulation point for the set of "midpoints" $\left\{x_{1}, x_{2}, \ldots\right\}$. Note that since all intervals $I_{n}$ have length at least $\epsilon$ the point $x_{*}$ lies in the interior of $U$, and is at least $\epsilon / 2$ distance from the boundary of $U$. By the transitivity of $\varphi$ there exists $\gamma_{*} \in \Gamma$ such that $3 \delta<d\left(x_{*}, \gamma_{*} x_{*}\right)<4 \delta$.

Choose $0<\delta_{1}<\delta / 2$ such that for the closed interval $W=\left\{w \in S^{1} \mid d\left(x_{*}, w\right) \leq \delta_{1}\right\}$, we have $\gamma_{*} W \subset\left\{w \in S^{1} \mid d\left(\gamma_{*} x_{*}, w\right)<\delta_{1}\right\}$. Then both $W$ and its image $\gamma_{*} W$ have diameter less than $\delta$. By the assumption $3 \delta<d\left(x_{*}, \gamma_{*} x_{*}\right)<4 \delta$ it follows that $W \cap \gamma_{*} W=\emptyset$.

Choose $p>0$ so that $\delta / 2^{p}<\delta_{1} / 2$ and $d\left(x_{*}, x_{p}\right)<\delta_{1} / 2$. Then $J_{p} \subset W$.
Choose $q>p$ so that $d\left(x_{*}, x_{q}\right)<\delta_{1} / 2$, so again $J_{q} \subset W$. It follows that $\gamma_{*} J_{p} \cap J_{q}=\emptyset$.
Define

$$
\gamma_{1}=\left(\gamma_{q} \circ \cdots \circ \gamma_{p}\right)^{-1} \quad \& \quad \gamma_{2}=\gamma_{*} \cdot\left(\gamma_{q} \circ \cdots \circ \gamma_{p}\right)^{-1}
$$

Then set $K_{0}=I_{q}$ and $K_{1}=\gamma_{1} I_{q}$. Note that $K_{1} \subset J_{p}$ by our choices. We set $K_{2}=\gamma_{*} K_{1}$ and so $K_{1} \cap K_{2}=\emptyset$. That is, $\gamma_{1}: K_{0} \rightarrow K_{1}$ and $\gamma_{2}: K_{0} \rightarrow K_{2}$ forms a ping-pong table for $\varphi$.

Observe that the proof of Theorem 1.2 now follows from Propositions 2.9 and 3.1, and then applying Lemmas 2.12 and 2.13.

Note also that Example 8.1 gives a real analytic expansive action of a solvable group on the circle. Thus, the conclusion that there is a free sub-semigroup of $\Gamma$ on two generators is best possible. In Example 8.1 there is a unique minimal set which is a fixed-point for the action of $\Gamma$. Thus, the ping-pong game constructed in the proof need not be contained in a minimal set for $\varphi$.

## 4 Infinitesimal expansion and hyperbolic fixed-points

In this section we assume that $\varphi$ is a $C^{1}$-action. We introduce the set of infinitesimally expansive points $E(\varphi)$ and the set of hyperbolic fixed-points $\mathcal{P}^{h}(\varphi)$, then show that each point $x \in E(\varphi)$ is the limit of hyperbolic periodic points.

Choose a symmetric generating set $\mathcal{S}=\left\{\sigma_{1}, \ldots, \sigma_{k}\right\}$ for $\Gamma$, where one of the $\sigma_{i}$ is the identity. An element $\gamma \in \Gamma$ has length $\|\gamma\| \leq N$ if there exists indices $i_{1}, \ldots, i_{N}$ such that $\gamma=\sigma_{i_{N}} \cdots \sigma_{i_{N}}$, and $\|\gamma\|=N$ if this is the least integer such that $\|\gamma\| \leq N$.

DEFINITION 4.1 $A$ point $x \in \mathbb{S}^{1}$ is infinitesimally expansive for $\varphi$ if

$$
\begin{equation*}
\lambda(x)=\limsup _{\|\gamma\| \rightarrow \infty} \frac{\log \left\{\gamma^{\prime}(x)\right\}}{\|\gamma\|}>0 \tag{2}
\end{equation*}
$$

Define $E_{a}(\varphi)=\left\{x \in \mathbb{S}^{1} \mid \lambda(x)>a\right\}$ and let $E(\varphi)=\bigcup_{a>0} E_{a}(\varphi)$.
LEMMA 4.2 $E_{a}(\varphi)$ is $\Gamma$-invariant.
Proof: Let $\lambda(x)>0$ and $\sigma \in \Gamma$. Then

$$
\begin{aligned}
\lambda(\sigma x) & =\limsup _{\|\gamma\| \rightarrow \infty} \frac{\log \left\{\gamma^{\prime}(\sigma x)\right\}}{\|\gamma\|} \\
& =\limsup _{\|\gamma\| \rightarrow \infty} \frac{\log \left\{(\gamma \sigma)^{\prime}(x)\right\}-\log \left\{\left(\sigma^{-1}\right)^{\prime}(\sigma x)\right\}}{\|\gamma \sigma\|} \cdot \frac{\|\gamma \sigma\|}{\|\gamma\|} \\
& =\limsup _{\|\gamma \sigma\| \rightarrow \infty} \frac{\log \left\{(\gamma \sigma)^{\prime}(x)\right\}}{\|\gamma \sigma\|} \\
& =\lambda(x)
\end{aligned}
$$

DEFINITION 4.3 A point $x \in \mathbb{S}^{1}$ is a hyperbolic fixed-point for $\varphi$ if there exists $\gamma \in \Gamma$ with $\gamma x=x$ and $0<\varphi(\gamma)^{\prime}(x)<1$. Let $\mathcal{P}^{h}(\varphi)$ denote the set of all hyperbolic fixed-points for $\varphi$.

For $a \geq 0$, we say a point $x \in \mathcal{P}^{h}(\varphi)$ has exponent greater than $a$ if there exists $\gamma$ with $\gamma x=x$ and $\left.\varphi(\gamma)^{\prime}(x)>\exp \{a \cdot\|\gamma\|)\right\}$. Let $\mathcal{P}_{a}^{h}(\varphi)$ denote the set of all hyperbolic fixed-points with exponent greater than $a$.

Clearly, $\mathcal{P}^{h}(\varphi) \subset E(\mathcal{F})$ and $\mathcal{P}_{a}^{h}(\varphi) \subset E_{a}(\varphi)$ for all $a>0$. We also have:
LEMMA 4.4 $\mathcal{P}_{a}^{h}(\varphi)$ is $\Gamma$-invariant.
Proof: Let $x \in \mathcal{P}_{a}^{h}(\varphi)$ and $\gamma \in \Gamma$ with $\gamma x=x$ and $\log \left\{\varphi(\gamma)^{\prime}(x)\right\} /\|\gamma\|>a$. Given $\sigma \in \Gamma$, choose $n \gg 0$ such that $\log \left\{\varphi(\gamma)^{\prime}(x)\right\} /(\|\gamma\|+2\|\sigma\| / n)>a$. Then note that $\left\|\sigma \cdot \gamma^{n} \cdot \sigma^{-1}\right\| \leq n\|\gamma\|+2\|\sigma\|$, and

$$
\log \left\{\varphi\left(\sigma \cdot \gamma^{n} \cdot \sigma^{-1}\right)^{\prime}(\sigma x)\right\}=n \log \left\{\varphi(\gamma)^{\prime}(x)\right\}
$$

hence $\log \left\{\varphi\left(\sigma \cdot \gamma^{n} \cdot \sigma^{-1}\right)^{\prime}(\sigma x)\right\}>a\left\|\sigma \cdot \gamma^{n} \cdot \sigma^{-1}\right\|$.
The hyperbolic periodic points for each $\varphi(\gamma)$ are isolated, hence at most countable. As $\Gamma$ is also countable, the set $\mathcal{P}^{h}(\varphi)$ is at most countable. Our next result shows that $\mathcal{P}^{h}(\varphi)$ is not empty if $E(\varphi)$ is not empty. The idea of the proof is based on a modification of the method of proof for Proposition 3.1.

PROPOSITION 4.5 For $a>0, \mathcal{P}_{a}^{h}(\varphi)$ is dense in $E_{a}(\varphi)$.
Proof: Let $x \in E_{a}(\varphi)$, set $\lambda=\lambda(x)$, and choose $0<\epsilon<\lambda-a$, and set $\epsilon_{0}=\epsilon / 10$. Define

$$
\|\varphi\|=\max _{1 \leq i \leq h} \max _{x \in \mathbb{S}^{1}}\left|\log \left\{\varphi\left(\sigma_{i}\right)^{\prime}(x)\right\}\right|
$$

Choose $\delta=\delta(\epsilon)>0$ so that if $d(y, z)<\delta$ then

$$
\begin{equation*}
\left|\log \left\{\varphi\left(\sigma_{i}\right)^{\prime}(y)\right\}-\log \left\{\varphi\left(\sigma_{i}\right)^{\prime}(z)\right\}\right|<\epsilon_{0} \text { for all } 1 \leq i \leq k \tag{3}
\end{equation*}
$$

Note that the modulus of continuity $\delta$ strongly depends on the choice of $\epsilon$, and $\delta(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ except in trivial examples.

Choose a sequence $\gamma_{n} \in \Gamma$ such that $\left\|\gamma_{n}\right\| \rightarrow \infty$ and $\log \left\{\varphi\left(\gamma_{n}\right)^{\prime}(x)\right\} /\left\|\gamma_{n}\right\|>\lambda-\epsilon_{0}$ for $n>0$.
We next introduce a technical condition on each map $\varphi\left(\gamma_{n}\right)$. For each $n>0$, set $\ell_{n}=\left\|\gamma_{n}\right\|$. Then we can choose indices $\left\{i_{1}, \ldots, i_{\ell_{n}}\right\}$ so that $\gamma_{n}^{-1}=\sigma_{i_{\ell_{n}}} \cdots \sigma_{i_{1}}$. For $1 \leq j \leq \ell_{n}$ set $\gamma_{n, j}^{-1}=\sigma_{i_{j}} \cdots \sigma_{i_{1}}$ with $\gamma_{n, 0}^{-1}=I d$.

Set $z_{n}=\gamma_{n} x$ and label the orbit of $z_{n}$ by $z_{n, j}=\varphi\left(\gamma_{n, j}^{-1}\right)\left(z_{n}\right)$. For each $1 \leq j \leq \ell_{n}$ set $\mu_{n, j}=\log \left\{\varphi\left(\sigma_{i_{j}}\right)^{\prime}\left(z_{n, j-1}\right)\right\}$. Then $\log \left\{\varphi\left(\gamma_{n}^{-1}\right)^{\prime}\left(z_{n}\right)\right\}=\mu_{n, 1}+\cdots+\mu_{n, \ell_{n}}$.

An index $1 \leq j \leq \ell_{n}$ is said to be $\epsilon_{0}$-regular if all of the partial sum estimates hold:

$$
\begin{aligned}
\mu_{n, j}+\epsilon_{0} & <0 \\
\mu_{n, j}+\mu_{n, j+1}+2 \epsilon_{0} & <0 \\
& \vdots \\
\mu_{n, j}+\cdots+\mu_{n, \ell_{n}}+\left(\ell_{n}-j+1\right) \epsilon_{0} & <0
\end{aligned}
$$

Regular points always exists, and in fact occur with a density estimated in terms of $\lambda$ and $\|\varphi\|$. Here, we need just the least regular value, which has a simple description. An index $j \leq \ell_{n}$ is $\epsilon_{0}$-irregular if $\mu_{n, 1}+\cdots+\mu_{n, j}+j \epsilon_{0}>0$. There is a greatest irregular value, as

$$
\left(\mu_{n, 1}+\cdots+\mu_{n, \ell_{n}}\right) / \ell_{n} \leq \epsilon_{0}-\lambda \leq-a+-9 \epsilon_{0} / 10<-\epsilon_{0}
$$

If $j_{0}$ is the greatest $\epsilon_{0}$-irregular index, then $j_{0}+1$ is an $\epsilon_{0}$-regular index. If $\mu_{n, j_{0}}>-\epsilon_{0}$ then $j_{1}=j_{0}+1$ is the least $\epsilon_{0}$-regular index. Otherwise, $j_{0}$ is again an $\epsilon_{0}$-regular index, and if $\mu_{n, j_{0}-1}>-\epsilon_{0}$ then $j_{0}$ is the least. Continue in this way until $\mu_{n, i}>-\epsilon_{0}$ then $j_{1}=i+1$ is the least $\epsilon_{0}$-regular index. Note that $i$ is also an $\epsilon_{0}$-irregular index.

For each $n$ let $b_{n}$ be the least $\epsilon_{0}$-regular index for $\gamma_{n}^{-1}$, and define $\tau_{n}=\sigma_{i_{\ell_{n}}} \cdots \sigma_{i_{b_{n}}}$. Set $y_{n}=\tau_{n} x$. The idea of the $\epsilon_{0}$-regular value is that it guarantees that each of the maps $\varphi\left(\sigma_{i_{i}} \cdots \sigma_{i_{b_{n}}}\right)$ for $i \geq i_{b_{n}}$ is a sufficiently strong linear contraction at $y_{n}$ to guarantee that the map is a uniform contraction on the interval $\left(y_{n}-\delta, y_{n}+\delta\right)$ because of the choice of $\delta$. We make this precise in Lemma 4.6 below, but first need some estimates. Note that $\left\|\tau_{n}\right\|=\ell_{n}-b_{n}+1 \leq\left\|\gamma_{n}\right\|$, and

$$
\begin{align*}
\log \left\{\varphi\left(\tau_{n}\right)^{\prime}\left(y_{n}\right)\right\} /\left\|\tau_{n}\right\| & \leq \log \left\{\varphi\left(\tau_{n}\right)^{\prime}\left(y_{n}\right)\right\} /\left\|\gamma_{n}\right\| \\
& \leq \log \left\{\varphi\left(\gamma_{n}\right)^{\prime}\left(z_{n}\right)\right\} /\left\|\gamma_{n}\right\|+\epsilon_{0} \\
& \leq 2 \epsilon_{0}-\lambda \tag{4}
\end{align*}
$$

There is also a uniform lower bound $-\|\varphi\| \leq \log \left\{\varphi\left(\tau_{n}\right)^{\prime}\left(y_{n}\right)\right\} /\left\|\tau_{n}\right\|$.
By passing to a subsequence if necessary, we can assume that $y_{n} \rightarrow y_{*}$ and that $d\left(y_{*}, y_{n}\right)<\delta / 4$ for all $n>0$. Set $I_{n}=\left[y_{n}-\delta / 2, y_{n}+\delta / 2\right]$, and $\mathbf{h}_{n}=\varphi\left(\tau_{n}\right)$. Define $I_{*}=\left[y_{*}-\delta / 4, y_{*}+\delta_{4}\right]$ then for all $n>0$ we have $y_{n} \in I_{*} \subset I_{n}$.

LEMMA 4.6 For each $n>0$,

$$
\begin{equation*}
\delta \exp \left\{-\|\varphi\|\left\|\tau_{n}\right\|\right\} \leq\left|\mathbf{h}_{n}\left(I_{n}\right)\right| \leq \delta \exp \left\{\left(3 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\|\right\} \tag{5}
\end{equation*}
$$

Proof: Fix $n$ and $\tau_{n}=\sigma_{i_{\ell_{n}}} \cdots \sigma_{i_{b_{n}}}$. Then by (3) for $y \in I_{n}$

$$
\left|\log \left\{\varphi\left(\sigma_{i_{b_{n}}}\right)^{\prime}(y)\right\}-\log \left\{\varphi\left(\sigma_{i_{b_{n}}}\right)^{\prime}\left(y_{n}\right)\right\}\right| \leq \epsilon_{0}
$$

Thus, by the definition of $\mu_{n, b_{n}}$ we have for all $y \in I_{n}$

$$
\exp \left\{-\epsilon_{0}+\mu_{n, b_{n}}\right\} \leq \varphi\left(\sigma_{i_{b_{n}}}^{-1}\right)^{\prime}(y) \leq \exp \left\{\epsilon_{0}+\mu_{n, b_{n}}\right\}
$$

hence

$$
\delta \exp \left\{-\epsilon_{0}+\mu_{n, b_{n}}\right\} \leq\left|\varphi\left(\sigma_{i_{b_{n}}}\right)\left(I_{n}\right)\right| \leq \delta \exp \left\{\epsilon_{0}+\mu_{n, b_{n}}\right\}
$$

As $b_{n}$ is $\epsilon_{0}$-regular, $\delta \exp \left\{\epsilon_{0}+\mu_{n, b_{n}}\right\}<\delta$ so we can repeat this argument for $\varphi\left(\sigma_{i_{b_{n}+1}}\right)$ on the interval $\varphi\left(\sigma_{i_{b_{n}}}\right)\left(I_{n}\right)$ to get an estimate

$$
\delta \exp \left\{-2 \epsilon_{0}+\mu_{n, b_{n}}+\mu_{n, b_{n}+1}\right\} \leq\left|\varphi\left(\sigma_{i_{b_{n}+1}} \cdot \sigma_{i_{b_{n}}}\right)\left(I_{n}\right)\right| \leq \delta \exp \left\{2 \epsilon_{0}+\mu_{n, b_{n}}+\mu_{n, b_{n}+1}\right\}
$$

We repeat the above argument $\ell_{n}-b_{n}+1$ times and use (4) to arrive at the upper bound

$$
\begin{aligned}
\left|\varphi\left(\tau_{n}\right)\left(I_{n}\right)\right| & \leq \delta \exp \left\{\left(\ell_{n}-b_{n}+1\right) \epsilon_{0}+\mu_{n, \ell_{n}}+\cdots+\mu_{n, b_{n}}\right\} \\
& \leq \delta \exp \left\{\epsilon_{0}\left\|\tau_{n}\right\|-\left(\lambda-2 \epsilon_{0}\right)\left\|\tau_{n}\right\|\right\} \\
& <\delta \exp \left\{\left(3 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\|\right\}
\end{aligned}
$$

The uniform estimate $\exp \left\{-\|\varphi\|\left\|\tau_{n}\right\|\right\} \leq \varphi\left(\tau_{n}\right)^{\prime}(y)$ for all $y \in \mathbb{S}^{1}$ yields the lower bound $\delta \exp \left\{-\|\varphi\|\left\|\tau_{n}\right\|\right\} \leq\left|\mathbf{h}_{n}\left(I_{n}\right)\right|$.

We now complete the proof of Proposition 4.5. The point $y_{1} \in I_{*}$ is in the interior, and $x=\mathbf{h}_{1}\left(y_{1}\right)$, so there exists $\delta_{1}>0$ such that $\left[x-2 \delta_{1}, x+2 \delta_{1}\right] \subset \mathbf{h}_{1}\left(I_{*}\right)$.

Choose $n>0$ so that $\delta \exp \left\{\left(3 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\|\right\}<\delta_{1}$ and $\left\|\tau_{n}\right\| \geq\|\varphi\|\left\|\tau_{1}\right\| / \epsilon_{0}$ which implies

$$
\begin{equation*}
\left(3 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\|+\|\varphi\|\left\|\tau_{1}\right\| \leq\left(4 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\| \tag{6}
\end{equation*}
$$

Then $y_{1}, y_{n} \in I_{n}$ implies $d\left(\mathbf{h}_{n}\left(y_{1}\right), \mathbf{h}_{n}\left(y_{n}\right)\right)<\delta \exp \left\{\left(3 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\|\right\}<\delta_{1}$ hence $\mathbf{h}_{n}\left(I_{*}\right) \subset \mathbf{h}_{n}\left(I_{n}\right) \subset$ $\mathbf{h}_{1}\left(I_{*}\right)$. Set $\mathbf{g}_{n}=\mathbf{h}_{1}^{-1} \circ \mathbf{h}_{n}, J_{1}=I_{*}$ and $J_{n}=\mathbf{g}_{n}\left(I_{n}\right)$. Thus, $\mathbf{g}_{n}: J_{1} \rightarrow J_{n} \subset J_{1}$ so $\mathbf{g}_{n}$ has a fixed point $x_{n} \in J_{n}$ satisfying $d\left(x, \mathbf{h}_{1}\left(x_{n}\right)\right)<\delta \exp \left\{\left(3 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\|\right\}$ and so

$$
d\left(\tau_{1}^{-1} x, x_{n}\right)<\delta \exp \left\{\left(3 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\|+\|\varphi\|\left\|\tau_{1}\right\|\right\}
$$

which tends to 0 as $n \rightarrow \infty$. Set $\tau_{n}^{*}=\tau_{1}^{-1} \cdot \tau_{n}$, then $\mathbf{g}_{n}=\varphi\left(\tau_{n}^{*}\right)$ and it remains to estimate

$$
\begin{aligned}
\mathbf{g}_{n}^{\prime}\left(x_{n}\right) & =\varphi\left(\tau_{1}^{-1}\right)^{\prime}\left(\tau_{n} x_{n}\right) \cdot \varphi\left(\tau_{n}\right)^{\prime}\left(x_{n}\right) \\
& \leq \exp \left\{\|\varphi\|\left\|\tau_{1}\right\|\right\} \cdot \exp \left\{\left(3 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\|\right\} \\
& \leq \exp \left\{\left(4 \epsilon_{0}-\lambda\right)\left\|\tau_{n}\right\|\right\} \\
& <\exp \left\{-a\left\|\tau_{n}\right\|\right\}
\end{aligned}
$$

Note that while the proof of Proposition 4.5 shows that each point $x \in E(\varphi)$ is the limit of hyperbolic contractions $\mathbf{g}_{n}$ with fixed-points $x_{n} \rightarrow x$, it may happen that for all of these fixedpoints, we have $x_{n}=x$ for all $n$. There is nothing in the proof which implies the fixed-points $x_{n}$ are distinct from the original point $x$. In this case, $\mathcal{P}_{a}^{h}(\varphi)=E_{a}(\varphi)$. Example 8.3 illustrates exactly this case.

Also note that all of the hyperbolic contractions $\mathbf{g}_{n}: J_{1} \rightarrow J_{n} \subset J_{1}$ have domain $J_{1}$ which is an interval of constant width $\delta / 2$. However, the choice of $\delta$ depends on the choice of $x \in E_{a}(\varphi)$, as $\epsilon_{0}=\epsilon / 10$ where $\epsilon$ as chosen to satisfy $0<\epsilon<\lambda-a$ to optimize the estimate on $\mathbf{g}_{n}^{\prime}\left(x_{n}\right)$. The next result uses a simple observation that if $a>0$ then we can bound the choice of $\delta$ for each $x \in \mathcal{P}_{a}^{h}(\varphi)$, and this implies the action is $\delta$-expansive on $\mathcal{P}_{a}^{h}(\varphi)$.

PROPOSITION 4.7 For $a>0, \varphi$ is expansive on $\mathcal{P}_{a}^{h}(\varphi)$.
Proof: Choose $0<\epsilon<a$, and set $\epsilon_{0}=\epsilon / 10$. Then choose $\delta>0$ so that if $d(y, z)<\delta$ then

$$
\begin{equation*}
\left|\log \left\{\varphi\left(\sigma_{i}\right)^{\prime}(y)\right\}-\log \left\{\varphi\left(\sigma_{i}\right)^{\prime}(z)\right\}\right|<\epsilon_{0} \text { for all } 1 \leq i \leq k \tag{7}
\end{equation*}
$$

Given $x \in \mathcal{P}_{a}^{h}(\varphi)$ let $\gamma \in \Gamma$ be such that $\gamma x=x$ and $\varphi(\gamma)^{\prime}(x)<\exp \{-a\|\gamma\|\}$. Set $\ell=\|\gamma\|$ and write $\gamma=\sigma_{i_{\ell}} \cdots \sigma_{i_{1}}$. For $1 \leq j \leq \ell$ set $\gamma_{j}=\sigma_{i_{j}} \cdots \sigma_{i_{1}}$ with $\gamma_{n, 0}=I d$, and $x_{j}=\gamma_{j-1} x$.

For each $1 \leq j \leq \ell$ set $\mu_{j}=\log \left\{\varphi\left(\sigma_{i_{j}}\right)^{\prime}\left(x_{j}\right)\right\}$. Then $\log \left\{\varphi(\gamma)^{\prime}(x)\right\}=\mu_{1}+\cdots+\mu_{\ell}$.
Extend the finite sequence $\left\{\mu_{1}, \ldots, \mu_{\ell}\right\}$ to an infinite periodic sequence

$$
\left\{\mu_{1}, \ldots, \mu_{\ell}, \mu_{1}, \ldots, \mu_{\ell}, \ldots\right\}
$$

and let $S_{m}(n)$ denote the sum of the terms from $m$ to $n$, where $1 \leq m \leq \ell$ and $m \leq n<\infty$. We say $m$ is $\epsilon_{0}$-good if and only if $S_{m}(n)<-(n-m+1) \epsilon_{0}$ for all $n \geq m$.

It is given that $S_{1}(\ell) / \ell \rightarrow-a<-\epsilon_{0}$, so there is a greatest $n_{0}$ with $S_{1}\left(n_{0}\right) \geq-n_{0} \epsilon_{0}$. Set $m_{0}=$ $n_{0}+1$ We claim that $m_{0}$ is $\epsilon_{0}$-good. If not, then there exists $n \geq m_{0}$ with $S_{m_{0}}(n) \geq-\left(n-m_{0}+1\right) \epsilon_{0}$. That is, the sum $\mu_{n_{0}+1}+\cdots+\mu_{n} \geq-\left(n-n_{0}\right) \epsilon_{0}$. By the choice of $n_{0}$, we then have

$$
S_{1}(n)=S_{1}\left(n_{0}\right)+S_{n_{0}}(n) \geq-n_{0} \epsilon_{0}--\left(n-n_{0}\right) \epsilon_{0}=-n \epsilon_{0}
$$

contradicting the choice of $n_{0}$.
Because the sequence of $\mu_{i}$ is periodic with period $\ell$, the sum $S_{m}(n)=S_{m-\ell}(n-\ell)$ hence if $m_{0}>\ell$, then $m_{0}-\ell$ is also an $\epsilon_{0}$-good value. Thus, we can assume $1 \leq m_{0} \leq \ell$.

Extend the definition of $\gamma_{j}$ for $1 \leq j \leq \ell$ to all $j \geq 1$ by defining $\gamma_{j+\ell}=\gamma \cdot \gamma_{j}$.
Set $x_{0}=\varphi\left(\gamma_{n_{0}}\right)(x)$ and $I_{0}=\left[x_{0}-\delta / 2, x_{0}+\delta / 2\right]$.
Let $\tau=\sigma_{i_{\ell}} \cdots \sigma_{i_{m_{0}}}$ and $\mathbf{g}=\varphi(\tau)$. Then $\mathbf{g}\left(x_{0}\right)=x$ as $\gamma x=x$.
We can now use the same proof as for Lemma 4.6 to obtain
LEMMA 4.8 For each $n>0$,

$$
\begin{equation*}
\delta \exp \left\{-\|\varphi\|\left\|\gamma_{n}\right\|\right\} \leq\left|\varphi\left(\gamma_{n}\right)\left(I_{0}\right)\right| \leq \delta \exp \left\{\left(3 \epsilon_{0}-\lambda\right)\left\|\gamma_{n}\right\|\right\} \tag{8}
\end{equation*}
$$

In particular, note that $\delta \exp \left\{\left(3 \epsilon_{0}-\lambda\right)\left\|\gamma_{n}\right\|\right\} \leq \delta \exp \left\{-a / 2\left\|\gamma_{n}\right\|\right\}$ so the lengths of the interval $I_{n}=\varphi\left(\gamma_{n}\right)\left(I_{0}\right)$ tends to zero as $n \rightarrow \infty$. For each integer $k \geq 0$ the interval $J_{k}=\varphi\left(\gamma^{k}\right) \circ \mathbf{g}\left(I_{0}\right)$ contains $x$ in its interior, and $\left|J_{k}\right| \rightarrow 0$ as $k \rightarrow \infty$.

Given another point $y \in \mathcal{P}_{a}^{h}(\varphi)$ with $y \neq x$ there exists $k \geq 0$ such that $y \notin J_{k}$. Hence,

$$
\varphi\left(\tau^{-1} \gamma^{-k}\right)(y) \notin \varphi\left(\tau^{-1} \gamma^{-k}\right)\left(J_{k}\right)=I_{0}
$$

which implies that $x$ and $y$ can be $\delta / 2$-separated. As the choice of $\delta$ depended only on $a$ and not on the choice of $x \neq y$ this implies $\varphi$ is $\delta / 2$-expansive on $\mathcal{P}_{a}^{h}(\varphi)$.

We point out one corollary of the above proof that will be used in the next section.
COROLLARY 4.9 Let $a>0$ and choose $\delta>0$ as in the proof of Proposition 4.7. Then for each $x \in \mathcal{P}_{a}^{h}(\varphi)$ there exists $\tau_{x}, \gamma_{x} \in \Gamma$ such that for $x_{0}=\tau_{x} x$ and $I_{x}=\left[x_{0}-\delta / 2, x_{0}+\delta / 2\right]$, we have $\gamma_{x} x_{0}=x_{0}$ and $\varphi\left(\gamma_{x}\right): I_{x} \rightarrow I_{x}$ is a hyperbolic contraction.

Proof: With notation as above, set $\tau_{x}=\tau^{-1}=\left(\sigma_{i_{\ell}} \cdots \sigma_{i_{m_{0}}}\right)^{-1}$ and $\gamma_{x}=\tau^{-1} \gamma \tau$.

## 5 Infinitesimal expansion and entropy

We assume that $\varphi$ is a $C^{1}$-action and that $E(\varphi)$ is not empty. Proposition 4.5 then implies that $\mathcal{P}_{a}^{h}(\varphi)$ is not empty for some $a>0$, and by Proposition 4.7 the action of $\varphi$ on $\mathcal{P}_{a}^{h}(\varphi)$ is expansive. By Corollary 4.9 each orbit of $\Gamma$ in $\mathcal{P}_{a}^{h}(\varphi)$ contains a hyperbolic fixed-point point with a uniform length domain, whose length depends only on $a$. In this section, we give conditions on the set $E(\varphi)$ which are sufficient to imply $h(\varphi)>0$ as a consequence of the expansiveness on $\mathcal{P}_{a}^{h}(\varphi)$.

First, we say that $\mathcal{P}_{a}^{h}(\varphi)$ has finite orbit type if there exists $\left\{x_{1}, \ldots, x_{k}\right\} \subset \mathcal{P}_{a}^{h}(\varphi)$ so that for every $x \in \mathcal{P}_{a}^{h}(\varphi)$ there exists $\gamma \in \Gamma$ and $1 \leq i \leq k$ so that $\gamma x_{i}=x$. That is, the quotient $\Gamma \backslash \mathcal{P}_{a}^{h}(\varphi)$ is finite. Otherwise, we say that $\mathcal{P}_{a}^{h}(\varphi)$ has infinite orbit type.

PROPOSITION 5.1 If $\mathcal{P}_{a}^{h}(\varphi)$ has infinite orbit type for some $a>0$, then there exists a ping-pong game $\left\{\tau_{1}: I_{0} \rightarrow I_{1}, \tau_{2}: I_{0} \rightarrow I_{2}\right\}$ for $\varphi$, and hence $h(\varphi)>0$.

Proof: By hypothesis, there exists $a>0$ and an infinite set $\left\{x_{1}, x_{2}, \ldots\right\} \subset \mathcal{P}_{a}^{h}(\varphi)$ such that $\Gamma x_{i} \cap \Gamma x_{k} \neq \emptyset$ implies $i=k$. By Corollary 4.9, there exists $\delta>0$ so that for each orbit $\Gamma x_{k}$ there exists $y_{k} \in \Gamma x_{k}$ and a hyperbolic contraction $\mathbf{h}_{k}=\varphi\left(\gamma_{k}\right): I_{k} \rightarrow I_{k}$ with fixed-point $y_{k}$ where $I_{k}=\left[y_{k}-\delta / 2, y_{k}+\delta / 2\right]$. Moreover, $i \neq k$ implies $y_{i} \neq y_{k}$.

Let $y_{*}$ be a limit point for the set $\left\{y_{1}, y_{2}, \ldots\right\}$ and choose $i, k \gg 0$ so that $d\left(y_{*}, y_{i}\right)<\delta / 10$ and $d\left(y_{*}, y_{k}\right)<\delta / 10$. Then $I_{0}=\left[y_{*}-\delta / 5, y_{*}+\delta / 5\right] \subset I_{i} \cap I_{k}$. Choose $n \gg 0$ so that $\mathbf{h}_{i}^{n}\left(I_{0}\right) \subset I_{*}$, $\mathbf{h}_{k}^{n}\left(I_{0}\right) \subset I_{0}$ and $\mathbf{h}_{i}^{n}\left(I_{0}\right) \subset I_{*} \cap \mathbf{h}_{k}^{n}\left(I_{0}\right) \subset I_{0}=\emptyset$. Set $I_{1}=\mathbf{h}_{i}^{n}\left(I_{0}\right), I_{2}=\mathbf{h}_{k}^{n}\left(I_{0}\right)$ and $\tau_{1}=\gamma_{i}^{k}, \tau_{2}=\gamma_{k}^{n}$, then $\left\{\tau_{1}: I_{0} \rightarrow I_{1}, \tau_{2}: I_{0} \rightarrow I_{2}\right\}$ is a ping-pong game, and by Lemma 2.13, $h(\varphi)>0$.

We next give a condition on $E(\varphi)$ which implies $h(\varphi)>0$. We say $x \in E_{a}(\varphi)$ is an accumulation point for $E_{a}(\varphi)$ if for all $\epsilon>0$ the set $\left\{y \in E_{a}(\varphi) \mid d(x, y)<\epsilon\right\}$ is infinite. Since $E(\varphi)$ is $\Gamma$-invariant, if $x$ is an accumulation point then each point $\gamma x$ for $\gamma \in \Gamma$ is also an accumulation point.

An orbit $\Gamma x$ is proper if each point $\gamma x$ is isolated in the set $\Gamma x$.
LEMMA 5.2 If $E_{a}(\varphi)$ has no accumulation points, then $E_{a}(\varphi)$ is a countable union of proper orbits.

Proof: Suppose that each point $x \in E_{a}(\varphi)$ has an open neighborhood $U_{x}$ such that $U_{x} \cap E_{a}(\varphi)$ is finite. Let $\mathcal{U}$ be the countable collection open intervals $(a-b, a+b)$ where the centers $\left\{a_{n}\right\}$ form a countable dense subset of $\mathbb{S}^{1}$ and $b$ is a rational number. Then for each point $y \in U_{x} \cap E_{a}(\varphi)$ we can find an open set $I_{x, y} \in \mathcal{U}$ with $I_{x, y} \in \mathcal{U}$ and $I_{x, y} \cap E_{a}(\varphi)=\{y\}$. Thus, $E_{a}(\varphi)$ is a countable set.

For $x \in E_{a}(\varphi)$, if $\Gamma x$ is not proper then $x$ is an accumulation point for $E(\varphi)$ as $\Gamma x \subset E_{a}(\varphi)$. Hence, if $E_{a}(\varphi)$ has no accumulation points, it must be countable, and every orbit is proper.

PROPOSITION 5.3 If $E_{a}(\varphi)$ has an accumulation point for some $a>0$, or $E(\varphi)$ is uncountable, then there exists a ping-pong game $\left\{\tau_{1}: I_{0} \rightarrow I_{1}, \tau_{2}: I_{0} \rightarrow I_{2}\right\}$ for $\varphi$, and hence $h(\varphi)>0$.

Proof: Suppose that $E(\varphi)$ is uncountable. Since $E(\varphi)=\bigcup_{n=1}^{\infty} E_{1 / n}(\varphi)$ there must exist $a=1 / n$ for which $E_{a}(\varphi)$ is uncountable. By Lemma 5.2 the set $E_{a}(\varphi)$ has an accumulation point, so it suffices to consider this case.

The idea of the proof is to use the method of proof for Proposition 4.5 to construct an infinite sequence of hyperbolic fixed-points with domains of uniform length $\delta$. These domains have to overlap since $\mathbb{S}^{1}$ is compact, and this will produce the ingredients for the ping-pong game as we have seen before.

The reader is referred to the proof of Proposition 4.5 for details of the steps below which coincide and are omitted here. We start by choosing $0<\epsilon<a$, and set $\epsilon_{0}=\epsilon / 10$. Choose a symmetric generating set $\mathcal{S}=\left\{\sigma_{1}, \ldots, \sigma_{k}\right\}$ for $\Gamma$, where one of the $\sigma_{i}$ is the identity. As before, choose $\delta>0$ so that if $d(y, z)<\delta$ then

$$
\left|\log \left\{\varphi\left(\sigma_{i}\right)^{\prime}(y)\right\}-\log \left\{\varphi\left(\sigma_{i}\right)^{\prime}(z)\right\}\right|<\epsilon_{0} \text { for all } 1 \leq i \leq k
$$

Let $x$ be an accumulation point for $E_{a}(\varphi)$ where $a>0$.
Choose a sequence $\gamma_{n} \in \Gamma$ such that $\left\|\gamma_{n}\right\| \rightarrow \infty$ and $\log \left\{\varphi\left(\gamma_{n}\right)^{\prime}(x)\right\} /\left\|\gamma_{n}\right\|>a-\epsilon_{0}$ for $n>0$. Set $\ell_{n}=\left\|\gamma_{n}\right\|$. Choose indices $\left\{i_{1}, \ldots, i_{\ell_{n}}\right\}$ so that $\gamma_{n}^{-1}=\sigma_{i_{\ell_{n}}} \cdots \sigma_{i_{1}}$. For each $n$ let $b_{n}$ be the least $\epsilon_{0}$-regular index for $\gamma_{n}^{-1}$, and define $\tau_{n}=\sigma_{i_{\ell_{n}}} \cdots \sigma_{i_{b_{n}}}$. Then

$$
\begin{equation*}
-\|\varphi\| \leq \log \left\{\varphi\left(\tau_{n}\right)^{\prime}(x)\right\} /\left\|\tau_{n}\right\| \leq 2 \epsilon_{0}-a \tag{9}
\end{equation*}
$$

For each $n$ set $x_{n}=\tau_{n}^{-1} x$. By passing to a subsequence if necessary, we can assume that $x_{n} \rightarrow x_{*}$ and that $d\left(x_{*}, x_{n}\right)<\delta / 100$ for all $n>0$. Set $I_{n}=\left[x_{n}-\delta / 2, x_{n}+\delta / 2\right]$, and $\mathbf{h}_{n}=\varphi\left(\tau_{n}\right)$. Define $I_{*}=\left[x_{*}-\delta / 4, x_{*}+\delta / 4\right]$ then for all indices $n$ we have $x_{n} \in I_{*} \subset I_{n}$. We can now use the same proof as for Lemma 4.6 to obtain for each $n>0$,

$$
\delta \exp \left\{-\|\varphi\|\left\|\tau_{n}\right\|\right\} \leq\left|\mathbf{h}_{n}\left(I_{n}\right)\right| \leq \delta \exp \left\{\left(3 \epsilon_{0}-a\right)\left\|\tau_{n}\right\|\right\}
$$

Hence, $\left|\mathbf{h}_{n}\left(I_{n}\right)\right| \leq \delta \exp \left\{-a / 2\left\|\gamma_{n}\right\|\right\}$ and so for $n \gg 0$,

$$
\begin{equation*}
\mathbf{h}_{n}\left(I_{*}\right) \subset \mathbf{h}_{n}\left(I_{n}\right) \subset \mathbf{h}_{1}\left(I_{*}\right) \subset \mathbf{h}_{1}\left(I_{1}\right) \tag{10}
\end{equation*}
$$

Thus, (10) implies $\mathbf{h}_{1}^{-1} \circ \mathbf{h}_{n}: I_{1} \rightarrow I_{n} \subset I_{1}$ so $\mathbf{h}_{1}^{-1} \circ \mathbf{h}_{n}$ has a fixed point $y_{n} \in I_{n}$ satisfying $d\left(x_{n}, y_{n}\right)<\delta \exp \left\{(-a / 2)\left\|\tau_{n}\right\|\right\}$. Set $\tau_{n}^{*}=\tau_{1}^{-1} \cdot \tau_{n}$, then $\varphi\left(\tau_{n}^{*}\right)=\mathbf{h}_{1}^{-1} \circ \mathbf{h}_{n}$ has hyperbolic fixedpoint $y_{n}$ which can be chosen arbitrarily close to $x_{n}=\tau_{n}^{-1} x \in \Gamma x$. In particular, we can assume $d\left(y_{n}, x_{n}\right)<\delta / 4$ so the distance from $y_{n}$ to the endpoints of $I_{n}$ is bounded below by $\delta / 4$.

For simplicity of notation, let $\mathbf{g}_{1}=\varphi\left(\tau_{n}^{*}\right), z_{1}=y_{n}, w_{1}=\tau_{n}^{-1} y_{n}, J_{1}=I_{1}$ and $K_{1}=I_{n}$, then $\mathbf{g}_{1}: J_{1} \rightarrow K_{1} \subset J_{1}$ has hyperbolic fixed-point $z_{1}$ which satisfies $d\left(x, w_{1}\right)<\delta \exp \left\{(-a / 2)\left\|\tau_{n}\right\|\right\}$. If $\Gamma z_{1} \cap J_{1}$ contains a point other than $z_{1}$ then $\Gamma z_{1}$ is a resilient orbit, so we are done by Lemma 2.15.

Otherwise, introduce the open set $W_{1}=\operatorname{Int} \mathbf{h}_{1}\left(I_{1}\right)$ with $x \in W_{1}$ and and $W_{1} \cap \Gamma z_{1}=\left\{w_{1}\right\}$. Since $x$ is an accumulation point for $E_{a}(\varphi)$ there exists $x^{2} \in W_{1} \cap E_{a}(\varphi)$ so that $d\left(x, x^{2}\right)<d\left(x, w_{1}\right) / 3$. We then repeat the above construction of a hyperbolic fixed-point using the orbit $\Gamma x^{2}$, to obtain an interval $J_{2}$ of length $\delta$ and map $\mathbf{g}_{2}: J_{2} \rightarrow K_{2} \subset J_{2}$ which has a hyperbolic fixed-point $z_{2}$. As before, introduce the point $w_{2}=\mathbf{h}_{1}\left(z_{1}\right)$ and by choosing $n$ sufficiently large in the construction of $z_{2}$ we can assume $d\left(x^{2}, w_{2}\right)<d\left(x, w_{1}\right) / 3$, hence $d\left(w_{1}, w_{2}\right)>d\left(x, w_{1}\right) / 3$ so $\Gamma w_{2}$ is disjoint from $\Gamma w_{1}$. Again, if $\Gamma z_{2} \cap J_{2}$ contains a point other than $z_{2}$ then $\Gamma z_{2}$ is a resilient orbit, and we are done.

Iterating this procedure, we either arrive at a resilient leaf, or we obtain a series of hyperbolic contractions $\mathbf{g}_{i}: J_{i} \rightarrow K_{i} \subset J_{i}$ where $\left|J_{i}\right|=\delta$ and $\mathbf{g}_{i}$ has a fixed-point $z_{i} \in J_{i}$ which is bounded away from the endpoints of $J_{i}$ by at least $\delta / 4$. Then there exists $i<k$ so that $\left\{z_{i}, z_{k}\right\} \subset I_{0}=J_{i} \cap J_{k}$ and $n \gg 0$ so that for $I_{1}=\mathbf{g}_{i}^{n}\left(J_{i}\right), I_{2}=\mathbf{g}_{k}^{n}\left(J_{k}\right), \mathbf{h}_{1}=\mathbf{g}_{i}^{n}$ and $\mathbf{h}_{2}=\mathbf{g}_{k}^{n},\left\{\mathbf{h}_{1}: I_{0} \rightarrow I_{1}, \mathbf{h}_{2}: I_{0} \rightarrow I_{2}\right\}$ is a ping-pong game.

## 6 Infinitesimal expansion and minimal sets

It is generally difficult to decide when the set $E(\varphi)$ is non-empty. The two known cases are when $\mathcal{P}^{h}(\varphi)$ is non-empty, and when the Godbillon-Vey class of $C^{2}$-action is non-zero [27]. In this section, we establish a dichotomy for a minimal set $\mathbf{K}$, that either there is a $\varphi$-invariant probability measure supported on $\mathbf{K}$, or $\mathbf{K} \subset E(\varphi)$. If $\mathbf{K}$ is finite, this is trivial. If $\mathbf{K}$ is exceptional and some $x \in \mathbf{K}$ has exponential orbit growth, then $\mathbf{K} \subset E_{a}(\varphi)$ for $a>0$ the growth rate of an endpoint. If $\mathbf{K}=\mathbb{S}^{1}$ and $\mathbf{K} \not \subset E(\varphi)$, then we use a new $\epsilon$-cocycle tempering procedure (cf. [21, 26]) to show there is an invariant probability measure on $\mathbb{S}^{1}$.

Choose a symmetric generating set $\mathcal{S}=\left\{\sigma_{1}, \ldots, \sigma_{k}\right\}$ for $\Gamma$, where one of the $\sigma_{i}$ is the identity, and define the corresponding word metric on $\Gamma$. Define a set $\Gamma_{N}=\{\gamma \mid\|\gamma\| \leq N\}$, which is the ball of radius $N$ about the identity for the word metric on $\Gamma$. Let $\# \Gamma_{N}$ denote the cardinality of the set $\Gamma_{N}$. The growth rate of $\Gamma$ is the number

$$
g r(\Gamma, \mathcal{S})=\limsup _{N \rightarrow \infty} \frac{\log \# \Gamma_{N}}{N}
$$

Given $x \in \mathbb{S}^{1}$ we define the growth rate of the action $\varphi$ at $x$ to be

$$
g r(\Gamma, \mathcal{S}, \varphi, x)=\limsup _{N \rightarrow \infty} \frac{\log \#\left\{\Gamma_{N} x\right\}}{N}
$$

Clearly, $\operatorname{gr}(\Gamma, \mathcal{S}, \varphi, x) \leq \operatorname{gr}(\Gamma, \mathcal{S})$. If $\operatorname{gr}(\Gamma, \mathcal{S}, \varphi, x)>0$ we sat that the orbit $\Gamma x$ has exponential growth, and subexponential growth otherwise. We recall a well-known fact.

LEMMA 6.1 Let $\varphi$ be a $C^{0}$-action with a minimal set $\mathbf{K}$. Suppose that some $x \in \mathbf{K}$ has $\operatorname{gr}(\Gamma, \mathcal{S}, \varphi, x)=0$. Then there is a $\varphi$-invariant probability measure $\mathbf{m}$ on $\mathbb{S}^{1}$ with support on $\mathbf{K}$. Hence, there exists a constant $C>0$ and integer $r \geq 1$ so that for every $y \in \mathbf{K}, \#\left\{\Gamma_{N} \cdot y\right\} \leq C \cdot N^{r}$. In other words, either every orbit of $\Gamma$ on $\mathbf{K}$ has exponential growth, or all orbits have polynomial growth.

Proof: The measure $\mathbf{m}$ is a weak-* limit of the probability measures $\left\{\mathbf{m}_{N} \mid N=1,2, \ldots\right\}$ defined by, for a continuous function $g: \mathbb{S}^{1} \rightarrow \mathbb{R}$,

$$
\mathbf{m}_{N}(g)=\frac{1}{\#\left\{\Gamma_{N} \cdot x\right\}} \sum_{y \in \Gamma_{N} \cdot x} g(y)
$$

The measure $\mathbf{m}$ defines a continuous map $\pi_{\mathbf{m}}: \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ by $\pi_{\mathbf{m}}(y)=2 \pi \mathbf{m}([y, 0]) \bmod 2 \pi \mathbb{Z}$, where [ $y, 0]$ is the counter-clockwise interval from 0 to $y$. The map $\pi_{\mathrm{m}}$ is monotone increasing on $\mathbf{K}$, as $\mathbf{K}$ minimal implies that every relatively open set in $\mathbf{K}$ has positive $\mathbf{m}$-measure. If $\mathbf{K}$ is exceptional, $\pi_{\mathrm{m}}$ is constant on the gaps of $\mathbf{K}$, thus, is at most 2 -to- 1 on $\mathbf{K}$ as it identifies the endpoints of a common gap. Thus, $\pi_{\mathrm{m}}$ is injective on the orbits $\Gamma_{y}$ for any $y \in \mathbf{K}$.

Since $\mathbf{K}$ is invariant, $\pi_{\mathbf{m}}$ defines a semi-conjugacy of $\varphi$ to an action of $\rho_{\mathbf{m}}: \Gamma \times \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ which is a group of rotations of the circle. If $r$ is the rank of this abelian group of rotations, then there exists a constant $C>0$ so that for every $\theta \in \mathbb{S}^{1}$ the orbit growth $\#\left\{\rho_{\mathbf{m}}\left(\Gamma_{N}\right)(\theta)\right\} \leq C \cdot N^{r}$. Then for any $y \in \mathbf{K}$ we have

$$
\#\left\{\varphi\left(\Gamma_{N}\right)(y)\right\}=\#\left\{\rho_{\mathbf{m}}\left(\Gamma_{N}\right)\left(\pi_{\mathbf{m}}(y)\right)\right\} \leq C \cdot N^{r}
$$

There is also a dichotomy for $E(\varphi)$ and minimal sets:

LEMMA 6.2 Let $\mathbf{K}$ be a minimal set for $\varphi$, and suppose that $x \in \mathbf{K}$ satisifies $\lambda(x)=a>0$. Then for all $0<b<a, \mathbf{K} \subset E_{b}(\varphi)$ and $\mathcal{P}_{b}^{h}(\varphi) \cap \mathbf{K}$ is dense in $\mathbf{K}$. Hence, either $\mathbf{K}$ and $E(\varphi)$ are disjoint, or $\mathbf{K} \subset E(\varphi)$.

Proof: Apply Proposition 4.5 to the point $x \in E_{a}(\varphi) \cap \mathbf{K}$, then following the notation of the proof, there exists a hyperbolic contraction $\mathbf{g}_{n}=\varphi\left(\tau_{n}^{*}\right): J_{1} \rightarrow J_{n} \subset J_{1}$ with fixed point $x_{n} \in J_{1}$ so that $\tau_{1}^{-1} x \in J_{1}$, and $x_{n} \in \mathcal{P}_{b}^{h}(\varphi)$. This implies that

$$
d\left(x_{n}, \mathbf{g}_{n}^{i}\left(\tau_{1}^{-1} x\right)\right)=d\left(\mathbf{g}_{n}^{i}\left(x_{n}\right), \mathbf{g}_{n}^{i}\left(\tau_{1}^{-1} x\right)\right) \rightarrow 0 \quad \text { as } \quad i \rightarrow \infty
$$

As $x \in \mathbf{K}$ and $\mathbf{K}$ is $\Gamma$-invariant, $\mathbf{g}_{n}^{i}\left(\tau_{1}^{-1} x\right) \in \mathbf{K}$ for all $i$, so the fixed-point $x_{n}$ is in the closure of $\mathbf{K}$. But $\mathbf{K}$ is closed, so $x_{n} \in \mathbf{K}$. We have constructed one hyperbolic fixed-point in $\mathbf{K}$, but the orbit of $x_{n}$ under $\Gamma$ is dense in $\mathbf{K}$, so $\mathcal{P}_{b}^{h}(\varphi) \cap \mathbf{K}$ is dense in $\mathbf{K}$.

Let $y \in \mathbf{K}$. Then as $\mathbf{K}$ is minimal, there exists $\sigma \in \Gamma$ such that $\sigma y \in J_{1}$. The sequence of diffeomorphisms $\varphi\left(\tau_{n}^{*}\right)^{-i} \varphi(\sigma)$ has asymptotic exponent at $y$ the same as for $\mathbf{g}_{n}^{-i}$ at $x_{n}$, so $\lambda(y)>b$. This shows $\mathbf{K} \subset E_{b}(\varphi)$.

Now assume that $\mathbf{K}$ is an exceptional minimal set with gaps $\left\{I_{i}=\left(a_{i}, b_{i}\right) \mid i=1,2, \ldots\right\}$. The orbit of every point $x \in \mathbf{K}$ is dense, so for any gap $I_{i}$ there are infinitely many disjoint images $\left\{\gamma I_{i} \mid \gamma \in \Gamma\right\}$ for which the sum of the lengths of is at most $2 \pi$. If $\Gamma$ has exponential growth, this observation suffices to show $\mathbf{K} \subset E(\varphi)$.

PROPOSITION 6.3 Let $\varphi$ be a $C^{1}$-action with exceptional minimal set $\mathbf{K}$. If $\operatorname{gr}(\Gamma, \mathcal{S}, \varphi, x)=$ $a>0$ for some endpoint $x \in \mathbf{K}$, then for all $b<a, \mathbf{K} \subset E_{b}(\varphi)$. Conversely, if $\lambda(x)>0$ for some point $x \in \mathbf{K}$, then every orbit of $\mathbf{K}$ has exponential growth.

Proof: Suppose $\mathbf{K}$ is an exceptional minimal set, and $x \in \mathbf{K}$ is an endpoint with $\operatorname{gr}(\Gamma, \mathcal{S}, \varphi, x)=$ $a>0$. We will exhibit a point $a_{*} \in \mathbf{K} \cap E_{b}\left(\varphi_{b}\right)$ for all $b<a$. Then by Lemma 6.2 we have $\mathbf{K} \subset E_{b}(\varphi)$ for all $b<a$.

Let $I_{x}=\left(a_{x}, b_{x}\right)$ be the gap for $\mathbf{K}$ with endpoint $x$, and assume $x=a_{x}$. (The case where $x=b_{x}$ proceeds identically.) Set $\mathcal{O}(N, x)=\#\left\{\Gamma_{N} \cdot x\right\}$. Choose $0<\epsilon<a$ and set $\epsilon_{0}=\epsilon / 10$. Then there exists $C>0$ and a sequence $N_{k} \rightarrow \infty$ such that $\mathcal{O}\left(N_{k}, x\right) \geq C \exp \left\{N_{k}\left(a-\epsilon_{0}\right)\right\}$ for all $k \geq 1$.

Choose $\delta=\delta(\epsilon)>0$ so that if $d(y, z)<\delta$ then

$$
\begin{equation*}
\left|\log \left\{\varphi\left(\sigma_{i}\right)^{\prime}(y)\right\}-\log \left\{\varphi\left(\sigma_{i}\right)^{\prime}(z)\right\}\right|<\epsilon_{0} \quad \text { for all } 1 \leq i \leq k \tag{11}
\end{equation*}
$$

For $\gamma_{1}, \gamma_{2} \in \Gamma$ either $\gamma_{1} I_{x} \cap \gamma_{2} I_{x}=\emptyset$ or $\gamma_{1} I_{x}=\gamma_{2} I_{x}$. Thus, there are $\mathcal{O}\left(N_{k}, x\right)$ disjoint intervals in the collection $\left\{\gamma I_{x} \mid\|\gamma\| \leq N_{k}\right\}$. As the sum of the lengths of these intervals is less than $2 \pi$, for each $k$ there exists $\gamma_{k} \in \Gamma_{N}$ with $\left|\gamma_{k} I_{x}\right| \leq 2 \pi / \mathcal{O}\left(N_{k}, x\right) \leq C_{2} \exp \left\{\left(\epsilon_{0}-a\right) N_{k}\right\}$. Denote $\gamma_{k} I_{x}=\left(a_{k}, b_{k}\right)$. Let $a_{*} \in \mathbf{K}$ be an accumulation point of the set of endpoints $\left\{a_{k}\right\}$. Passing to a subsequence, we can assume that $d\left(a_{*}, a_{k}\right)<\delta / 4 k$.

Let $N_{0}$ be such that $\|\gamma\| \geq N_{0}$ implies $\left|\gamma I_{x}\right|<\delta / 4$. For each $k$ set $\ell_{k}=\left\|\gamma_{k}\right\|$ and write $\gamma_{k}=\sigma_{i_{\ell_{n}}} \cdots \sigma_{i_{1}}$. Let $j_{0}<N_{0}$ be the least index so that $j>j_{0}$ implies $\left|\sigma_{i_{j}} \cdots \sigma_{i_{1}} I_{x}\right|<\delta / 4$. Set $\sigma_{k}=\sigma_{i_{j_{0}}} \cdots \sigma_{i_{1}}$ and

$$
\tau_{k}=\gamma_{k} \cdot \sigma_{k}^{-1}=\sigma_{i_{\ell_{k}}} \cdots \sigma_{i_{j_{0}+1}}
$$

so that $\left\|\gamma_{k}\right\|-N_{0}<\left\|\tau_{k}\right\| \leq\left\|\gamma_{k}\right\|$. Set $J_{k}=\sigma_{k} I_{x}$. Then $\left|I_{x}\right| \exp \left\{-\|\varphi\| N_{0}\right\} \leq\left|J_{k}\right| \leq\left|I_{x}\right| \exp \left\{\|\varphi\| N_{0}\right\}$.

We can now apply the mean value theorem to the map $\varphi\left(\tau_{k}^{-1}\right): I_{k} \rightarrow J_{k}$ to obtain a point $y_{k} \in I_{k}$ such that

$$
\varphi\left(\tau_{k}^{-1}\right)^{\prime}\left(y_{k}\right) \geq \frac{\left|J_{k}\right|}{\left|I_{k}\right|} \geq \frac{\left|I_{x}\right| \exp \left\{-\|\varphi\| N_{0}\right\}}{C_{2} \exp \left\{\left(\epsilon_{0}-a\right) N_{k}\right\}} \geq C_{3} \exp \left\{\left(a-\epsilon_{0}\right) N_{k}-\|\varphi\| N_{0}\right\}
$$

For all $k \geq 1$ we have

$$
d\left(a_{*}, y_{k}\right) \leq d\left(a_{*}, a_{k}\right)+d\left(a_{k}, y_{k}\right) \leq \delta / 4 k+\delta / 4<\delta / 2
$$

so that $y_{k} \in\left(a_{*}-\delta / 2, a_{*}+\delta / 2\right)$. By the choice of $j_{0}$, the same argument implies that for $j \geq j_{0}$

$$
d\left(\sigma_{i_{j}} \cdots \sigma_{i_{j_{0}+1}} a_{*}, \sigma_{i_{j}} \cdots \sigma_{i_{j_{0}+1}} y_{k}\right) \leq \delta / 2
$$

so that we can apply the estimate (11) along the orbit of $a_{*}$ and $y_{k}$ determined by the generators of $\tau_{k}$ to conclude

$$
\varphi\left(\tau_{k}^{-1}\right)^{\prime}\left(a_{*}\right) \geq C_{3} \exp \left\{\left(a-2 \epsilon_{0}\right) N_{k}-\|\varphi\| N_{0}\right\}
$$

As $N_{k} \rightarrow \infty$ this implies $a_{*} \in E_{b}(\varphi)$ for $b<a-2 \epsilon_{0}$. The choice of the limit point $a_{*}$ was independent of the choice of $\epsilon$ and $\delta$, and as $\epsilon>0$ can be made as small as desired, this implies $a_{*} \in E_{b}(\varphi)$ for all $b<a$.

To prove the converse, we note that by Lemma 6.1 it suffices to show that there exists one orbit in $\mathbf{K}$ with exponential growth. Suppose $x \in \mathbf{K}$ with $\lambda(x)>0$, then by Lemma 6.2 there exists a hyperbolic contraction $\mathbf{g}_{n}=\varphi\left(\tau_{n}^{*}\right): J_{1} \rightarrow J_{n} \subset J_{1}$ with fixed point $x_{n} \in J_{1} \cap \mathbf{K}$. As $\mathbf{K}$ is minimal, there exists $\sigma \in \Gamma$ such that $\sigma x \in J_{1}$ but $x \neq \sigma x$. Thus, $x$ is a resilient point, so by Lemma 2.15 there is a ping-pong game for $\varphi$ with $x$ as one of the fixed-points, hence the orbit of $x$ has exponential growth.

When $\mathbf{K}=\mathbb{S}^{1}$, we require a separate method of proof to show if $\mathbf{K} \cap E(\varphi)=\emptyset$ then there exists an invariant measure on $\mathbf{K}$. This sort of problem requires a "tempering" procedure, as the hypothesis is that the Lebesgue volume growth along orbits is subexponential, and we need to show there is an equivalent measure which has uniformly slow growth. There are two types of tempering procedures in the literature, one which works when $\Gamma$ has subexponential growth [21, 26], and the other applies when the volume growth along orbits is bounded [41]. The proof of the following result introduces a new tempering procedure, which combines the techniques of both methods.

PROPOSITION 6.4 Let $\varphi$ be a minimal $C^{1}$-action. Then either $E(\varphi)=\mathbb{S}^{1}$, or $\varphi$ has an invariant probability measure supported on $\mathbb{S}^{1}$ and $E(\varphi)=\emptyset$.

Proof: Suppose that $\mathbb{S}^{1} \not \subset E(\varphi)$, then by Lemma 6.2 we can assume $\lambda(x)=0$ for all $x \in \mathbb{S}^{1}$. We will construct a sequence of probability measures $\left\{\mathbf{m}_{n} \mid n=1,2, \ldots\right\}$ so that for any continuous $\phi: \mathbb{S}^{1} \rightarrow \mathbb{R}$ we have $\left|\mathbf{m}_{n}(\phi)-\mathbf{m}_{n}\left(\phi \circ \sigma_{i}\right)\right| \leq\|\phi\| / n$ for all $1 \leq i \leq k$. The weak-* limit $\mathbf{m}_{*}$ of the set of measures $\left\{\mathbf{m}_{n}\right\}$ is invariant under $\Gamma$, and as $\varphi$ is minimal, the support $\left|\mathbf{m}_{*}\right|=\mathbb{S}^{1}$.

For each $N>0$ define

$$
\begin{equation*}
\mu(\varphi, N, x)=\max \left\{\varphi(\gamma)^{\prime}(x) \mid\|\gamma\| \leq N\right\} \tag{12}
\end{equation*}
$$

Note that $\mu(\varphi, 0, x)=1$, and $\mu(\varphi, N, x) \geq 1$ for all $N$. As $\lambda(x)=0$, for all $\epsilon>0$,

$$
\lim _{N \rightarrow \infty} \exp \{-N \epsilon\} \mu(\varphi, N, x)=0
$$

For each $\epsilon>0$, define a Borel function $f_{\epsilon}$ on $\mathbf{K}$ by $f_{\epsilon}(x)=\sum_{N=0}^{\infty} \exp \{-N \epsilon\} \cdot \mu(\varphi, N, x)$ and introduce the measure $\mathbf{m}_{\epsilon}=f_{\epsilon} d \theta$ where $d \theta$ denote the standard Lebesgue measure on $\mathbb{S}^{1}$.

LEMMA 6.5 Let $\phi: \mathbb{S}^{1} \rightarrow \mathbb{R}$ be continuous. Then $\left|\varphi\left(\sigma_{i}\right)^{*} \mathbf{m}_{\epsilon}(\phi)-\mathbf{m}_{\epsilon}(\phi)\right| \leq \exp (-\epsilon)\|\phi\|$.
Proof: We calculate

$$
\begin{aligned}
\left.\varphi\left(\sigma_{i}\right)^{*} \mathbf{m}_{\epsilon}\right|_{x} & =\left.f_{\epsilon}\left(\sigma_{i} x\right)\left(\varphi\left(\sigma_{i}\right)^{*} d \theta\right)\right|_{x} \\
& =\left\{\sum_{N=0}^{\infty} \exp \{-N \epsilon\} \cdot \mu\left(\varphi, N, \sigma_{i} x\right)\right\} \varphi\left(\sigma_{i}\right)^{\prime}(x) d \theta \\
& =\left\{\sum_{N=0}^{\infty} \exp \{-N \epsilon\} \cdot \mu\left(\varphi, N, \sigma_{i} x\right) \varphi\left(\sigma_{i}\right)^{\prime}(x)\right\} d \theta
\end{aligned}
$$

Observe that $\mu\left(\varphi, N, \sigma_{i} x\right) \varphi\left(\sigma_{i}\right)^{\prime}(x) \leq \mu(\varphi, N+1, x)$ as $\left\|\gamma \sigma_{i}\right\| \leq N+1$, hence

$$
\begin{align*}
\left.\varphi\left(\sigma_{i}\right)^{*} \mathbf{m}_{\epsilon}\right|_{x} & \leq \sum_{N=0}^{\infty} \exp \{-N \epsilon\} \cdot \mu(\varphi, N+1, x) \\
& \leq \exp \{\epsilon\} \cdot \sum_{N=1}^{\infty} \exp \{-N \epsilon\} \cdot \mu(\varphi, N, x) \\
& <\left.\exp \{\epsilon\} \cdot \mathbf{m}_{\epsilon}\right|_{x} \tag{13}
\end{align*}
$$

Note that if $\|\gamma\| \leq N-1$ then $\left\|\gamma \sigma_{i}^{-1}\right\| \leq N$, so $\mu\left(\varphi, N, \sigma_{i} x\right) \varphi\left(\sigma_{i}\right)^{\prime}(x) \geq \mu(\varphi, N-1, x)$, hence

$$
\begin{align*}
\left.\varphi\left(\sigma_{i}\right)^{*} \mathbf{m}_{\epsilon}\right|_{x} & \geq \sum_{N=1}^{\infty} \exp \{-N \epsilon\} \cdot \mu(\varphi, N-1, x) \\
& \geq \exp \{-\epsilon\} \cdot \sum_{N=0}^{\infty} \exp \{-N \epsilon\} \cdot \mu(\varphi, N, x) \\
& >\left.\exp \{-\epsilon\} \cdot \mathbf{m}_{\epsilon}\right|_{x} \tag{14}
\end{align*}
$$

The estimates (13) and (14) imply $\left|\varphi\left(\sigma_{i}\right)^{*} \mathbf{m}_{\epsilon}(\phi)-\mathbf{m}_{\epsilon}(\phi)\right| \leq \exp (\epsilon)\|\phi\|$.

For each integer $n>0$, choose $\epsilon$ with $\exp (\epsilon) \leq 1+1 / n$ and set $\mathbf{m}_{n}=\mathbf{m}_{\epsilon} / \mathbf{m}_{\epsilon}(1)$. The sequence of probability measures $\left\{\mathbf{m}_{n} \mid n=1,2, \ldots\right\}$ have uniformly slow growth, so this complete the proof of Proposition 6.4.

Theorem 1.6 and Corollary 1.7 of the introduction now follow from Lemma 6.1, Propositions 6.3 and 6.4, and Proposition 5.3.

Note that in the proof of Proposition 6.4, we used the hypothesis that the action is minimal only to conclude that the invariant measure $\mathbf{m}_{*}$ had support on all of $\mathbb{S}^{1}$. Moreover, $\lambda(x)=0$ was required only for a set of full measure. Thus, if we combine the above proof with Proposition 5.3, we obtain a new proof of Theorem 5.1, [12] in the case of groups acting on the circle:

COROLLARY 6.6 A $C^{1}$-action $\varphi$ with $h(\varphi)=0$ must have an invariant probability measure.

## 7 The free subgroup conjecture

In a lecture at the Dynamical Systems Symposium at the University of Paris, VI in June 1998, Etienne Ghys made the following conjecture:

CONJECTURE 7.1 (Ghys) Let $\varphi: \Gamma \times \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ be a $C^{0}$-action. Then either $\varphi$ has an invariant probability measure, or $\Gamma$ has a non-abelian free subgroup on two generators.

In this section, we make some observations about this conjecture and the results of this paper. As an application, we give a proof of the conjecture for real analytic actions.

Recall that a topological action $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ admits a minimal set, which is either a periodic orbit, the entire circle $\mathbb{S}^{1}$, in which case the action is minimal, or an exceptional minimal set. A periodic orbit supports an invariant probability measure, so it suffices to consider the other two cases.

If the action is minimal, then by Lemma 6.1 either there exists an invariant measure with full support, or every orbit of $\varphi$ has exponential growth. In the latter case, by Lemma 2.3 the action on $\mathbb{S}^{1}$ must be expansive. Proposition 3.1 implies that the action admits a ping-pong game.

If the action preserves an exceptional minimal set $\mathbf{K}$, then again there is the dichotomy that either there is an an invariant measure with full support on $\mathbf{K}$, or the orbit of every point in $\mathbf{K}$ has exponential growth. The latter case, where the orbits have exponential growth, can be reduced to the minimal action case:

PROPOSITION 7.2 Suppose that $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ has an exceptional minimal set $\mathbf{K}$, and there is no invariant probability measure with support $\mathbf{K}$. Then there exists a ping-pong game $\left\{\gamma_{1}: I_{0} \rightarrow\right.$ $\left.I_{1}, \gamma_{2}: I_{0} \rightarrow I_{2}\right\}$ for $\varphi$ restricted to $\mathbf{K}$.

Proof: Define a continuous map $\rho: \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ which is constant on the gaps of $\mathbf{K}$, and monotone on $\mathbf{K}$. The action $\varphi$ descends to a quotient action $\bar{\varphi}: \Gamma \times \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ which is semi-conjugate to $\varphi$ via $\rho$. The action $\bar{\varphi}$ is then minimal, with no invariant measure. By Lemma 2.3 the action $\bar{\varphi}$ on $\mathbb{S}^{1}$ must be expansive. Proposition 3.1 implies that $\bar{\varphi}$ admits a ping-pong game $\left\{\bar{\varphi}\left(\gamma_{1}\right): \overline{I_{0}} \rightarrow \overline{I_{1}}, \bar{\varphi}\left(\gamma_{2}\right): \overline{I_{0}} \rightarrow \overline{I_{2}}\right\}$. We set $I_{0}=\rho^{-1}\left(\overline{I_{0}}\right), I_{1}=\rho^{-1}\left(\overline{I_{1}}\right), I_{2}=\rho^{-1}\left(\overline{I_{2}}\right)$, and obtain a ping-pong game for $\varphi$.

There is no statement that the maps $\varphi\left(\gamma_{1}\right)$ and $\varphi\left(\gamma_{2}\right)$ have unique fixed-points. Even if one of the maps $\bar{\varphi}\left(\gamma_{i}\right)$ has a unique fixed point, it could be the image of a gap under $\rho$, hence its preimage consist of the endpoints of a gap of $\mathbf{K}$.

These remarks show that Conjecture 7.1 is implied by the more concrete problem:
CONJECTURE 7.3 Let $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ be a $C^{0}$-action with a minimal set $K$, and suppose that there is a ping-pong game for $\varphi$ on $\mathbf{K}$. Then $\Gamma$ has a non-abelian free subgroup on two generators.

For $C^{1}$-actions, the existence of hyperbolic fixed-points for the action of $\varphi$ on minimal sets lets us strengthen the above remarks. Assume that $\varphi$ is a $C^{1}$-action with no invariant probability measure. Then there exists a minimal set $\mathbf{K}$ which is either $\mathbb{S}^{1}$ or exceptional. By Corollary 1.7 there is a hyperbolic ping-pong game for $\varphi$ on $\mathbf{K}$. That is, there exists elements $\gamma_{1}, \gamma_{2} \in \Gamma$ and a domain $I_{0}$ with $I_{0} \cap \mathbf{K} \neq \emptyset$ so that the pair of restricted maps $\left\{f: I_{0} \rightarrow I_{1}, g: I_{0} \rightarrow I_{2}\right\}$ is a ping-pong game. Moreover, $f$ and $g$ are hyperbolic contractions with fixed points $f(x)=x \in \mathbf{K}$ and $g(y)=y \in \mathbf{K}$ for $x \neq y$.

Thus, for $C^{1}$-actions, we can add to the hypotheses of Conjecture 7.3 the statement that the maps defining the ping-pong game are hyperbolic contractions on the domain $I_{0}$ with unique fixedpoints there. This is the setting for applying some version of the "Tits alternative" [40, 9]. One possible approach is the formulation of the Tits alternative in section 3 of Farb-Shalen [11]. We give the statement in the present context:

THEOREM 7.4 Let $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ be a $C^{0}$-action with a minimal set $\mathbf{K}$ and no invariant probability measure on $\mathbf{K}$. Suppose there exists $\gamma_{1}, \gamma_{2} \in \Gamma$ so that, for $f=\varphi\left(\gamma_{1}\right), g=\varphi\left(\gamma_{2}\right)$, $\operatorname{Fix}(f)=\left\{x \in \mathbb{S}^{1} \mid f(x)=x\right\}$, and $\operatorname{Fix}(g)=\left\{x \in \mathbb{S}^{1} \mid g(x)=x\right\}$, we have

1. $f \mid \mathbf{K}$ and $g \mid \mathbf{K}$ have infinite order
2. $\operatorname{Fix}(f) \cap \mathbf{K} \neq \emptyset$ and $\operatorname{Fix}(g) \cap \mathbf{K} \neq \emptyset$
3. $\operatorname{Fix}(f) \cap \operatorname{Fix}(g) \cap \mathbf{K}=\emptyset$

Then for some $n>0$, the group generated by $f^{n}$ and $g^{n}$ is a nonabelian free group.
Proof: If $\mathbf{K}=\mathbb{S}^{1}$ this is exactly what is proved in [11]. If $\mathbf{K}$ is exceptional, define a minimal action $\bar{\varphi}$ on $\mathbb{S}^{1}$ via the semi-conjugacy of the proof of Proposition 7.2 , then proceed as before.

We use this result to prove the following:
THEOREM 7.5 Let $\varphi: \Gamma \times S^{1} \rightarrow S^{1}$ be a $C^{0}$-action with a minimal set $K$, and suppose that there is a ping-pong game $\left\{\gamma_{1}: I_{0} \rightarrow I_{1}, \gamma_{2}: I_{0} \rightarrow I_{2}\right\}$ for $\varphi$ on $\mathbf{K}$ such that for $f=\varphi\left(\gamma_{1}\right), g=\varphi\left(\gamma_{2}\right)$, the sets $\operatorname{Fix}(f) \cap \mathbf{K}$ and $\operatorname{Fix}(g) \cap \mathbf{K}$ are discrete. Then $\Gamma$ has a non-abelian free subgroup.

Proof: Let $\operatorname{Fix}(f) \cap \mathbf{K}=\left\{x_{1}, \ldots, x_{n}\right\}$ and $\operatorname{Fix}(g) \cap K=\left\{y_{1}, \ldots, y_{m}\right\}$. If $\operatorname{Fix}(f) \cap \operatorname{Fix}(g)=\emptyset$ then we are done by Theorem 7.4. Otherwise, it suffices to choose an element $\tau \in \Gamma$ such that

$$
\begin{equation*}
\left\{x_{1}, \ldots, x_{n}\right\} \cap\left\{\tau y_{1}, \ldots, \tau y_{n}\right\} \cap \mathbf{K}=\emptyset \tag{15}
\end{equation*}
$$

for we then replace $g$ with $h \circ g \circ h^{-1}$ where $h=\varphi(\tau)$, and condition (7.4.3) is satisfied since $\mathbf{K}$ is invariant. Following an observation of Farb and Shalen in section 3 of [11], we note that if no such $\tau \in \Gamma$ exists, then for every $\tau$ there exists $x_{i}$ and $y_{k}$ with $\tau y_{k}=x_{i}$, or $\tau \in T i, k$ where $T_{i, k}=\left\{\sigma \in \Gamma \mid \sigma y_{k}=x_{i}\right\}$. Let $\Gamma_{y_{k}} \subset \Gamma$ denote the stabilizer of $y_{k}$, then $T_{i, k}=\tau \Gamma_{y_{k}}$ for any element $\tau \in T_{i, k}$. Thus, $\Gamma$ is a finite union of cosets of stabilizers of points in $\operatorname{Fix}(g)$. By the "Coset Lemma" 3.1 of [11], one of the stabilizers $\Gamma_{y_{k}}$ must have finite index in $\Gamma$. But this implies that the orbit $\Gamma y_{k}$ is finite, which contradicts the fact that the orbit of every point in $\mathbf{K}$ is dense. Thus, there must exist some $\tau$ satisfying (15).

For a $C^{1}$-action $\varphi$ with minimal set $\mathbf{K}$ and no invariant probability measure supported on $\mathbf{K}$, there is always a ping-pong game on $\mathbf{K}$ as remarked previously. Thus, Conjecture 7.1 would follow if the maps $f$ and $g$ can be chosen with isolated fixed-points in $\mathbf{K}$. The fixed-points of a real analytic action are always isolated, so Theorem 1.8 follows from the remarks of this section and Theorem 7.5.
$N B$. In the manuscript [31], G. Margulis has given a proof of Conjecture 7.1 for $C^{0}$-actions. The methods are similar to those discussed in this paper, but Margulis considers the action of $\Gamma$ on $\epsilon$-nets, not just on individual points, and so avoids the need for assuming isolated fixed-points.

## 8 Examples

We give three examples which illustrate the ideas developed in this paper. The first example is a real analytic expansive action of a solvable group. The second second example embeds the first example into the gaps of a Denjoy example to produce an expansive $C^{1}$-action of a solvable group with a Cantor type minimal set. The third example is a $C^{\infty}$-action with a countable collection of hyperbolic fixed-points, but is not expansive. None of these examples is new, but understanding each of them provides a foundation for understanding the ideas of this paper. They also provide "counter-examples" to extending the conclusions of the theorems of the paper, as discussed in comments after each example.

EXAMPLE 8.1 An expansive $C^{\omega}$-action of a solvable group on $S^{1}$ with minimal set a point.

Define maps of the real line by $f(u)=2 u$ and $g(u)=u+1$. Embed the real line into the circle $S^{1}=\left\{z=x+i y \mid x^{2}+y^{2}=1\right\} \subset \mathbb{C}$ by the linear fractional map $h(u)=(1+i u) /(1-i u)$. This map conjugates $f$ to $\alpha=h \circ f \circ h^{-1}$ and $\beta=h \circ g \circ h^{-1}$, which are both real analytic linear fractional transformations of $\mathbb{S}^{1}$. Let $\Gamma$ be the solvable subgroup of Diff ${ }^{\omega}\left(\mathbf{S}^{1}\right)$ they generate, and $\varphi: \Gamma: \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ the action they determine.

The action of $f, g$ on $\mathbb{R}$ has every orbit dense, hence the action of $\Gamma$ on $S^{1}$ admits a unique fixed-point $(-1,0)$, and every other orbit is dense. In particular, the fixed-point $(-1,0)$ for the full action is the unique minimal set.

Given $x \neq y \in S^{1}$, at least one of these cannot be the fixed-point $(-1,0)$. Hence, there exists $\ell$ such that $g^{\ell}\left(h^{-1}(x)\right)$ and $g^{\ell}\left(h^{-1}(y)\right)$ lie on opposite sides of the origin in $\mathbb{R}$. Applying a suitable power $k>0$ of $f$ we can ensure that their images $f^{k} \circ g^{\ell}\left(h^{-1}(x)\right)$ and $f^{k} \circ g^{\ell}\left(h^{-1}(y)\right)$ span an interval containing either $[0,1]$ or $[-1,0]$ in $\mathbb{R}$, hence $h \circ f^{k} \circ g^{\ell}\left(h^{-1}(x)\right)$ and $h \circ f^{k} \circ g^{\ell}\left(h^{-1}(y)\right)$ are $\epsilon$ separated in $S^{1}$ for $\epsilon<1 / 2$.

This example is interesting for two reasons. First, it shows that expansiveness is not sufficient to imply there is a free subgroup on two generators in $\Gamma$, even for analytic actions, as the group $\Gamma$ is solvable. Secondly, the action of $\Gamma$ does have a ping-pong table, which is given in terms of $f$ and $g$ by the two maps $h^{-1} \circ \mathbf{h}_{1} \circ h(u)=g^{-2} \circ f^{-2}(u)=-2+u / 4$ and $h^{-1} \circ \mathbf{h}_{2} \circ h(u)=g^{2} \circ f^{-2}(u)=2+u / 4$. We can take $h-1\left(I_{0}\right)=[-4,4], h-1\left(I_{1}\right)=[-3,-1]$ and $h-1\left(I_{2}\right)=[1,3]$. The forward iterates of $I_{0}$ define a Cantor set $\mathbf{K}$ which is invariant under the semigroup generated by $\left\{\mathbf{h}_{1}, \mathbf{h}_{2}\right\}$. But the set $\mathbf{K}$ is not invariant for the full group $\Gamma$ as the only minimal set for $\Gamma$ is $(-1,0)$. Thus, properties of the subdynamics generated by a ping-pong game need not carry over to the full group action.

EXAMPLE 8.2 An expansive action of a solvable group on $S^{1}$ with exceptional minimal set.
Let $\gamma: \mathbb{S}^{1} \rightarrow \mathbb{S}^{1}$ be a (Denjoy) $C^{1}$-diffeomorphism with an invariant exceptional minimal set $K$. The complement of $K$ consists of a disjoint union of open intervals $\left\{U_{1}, U_{2}, \ldots\right\}$ and $\gamma$ acts transitively on the set of intervals. Thus, we can index the open sets by $\mathbb{Z}$ where $U_{\ell}=\gamma^{\ell} U_{0}$.

Let $\mathbf{h}_{1}$ and $\mathbf{h}_{2}$ be the linear fractional maps of $\mathbb{S}^{1}$ constructed in Example 8.1. Let $\mathbf{h}_{3}$ be a modification of the map $\mathbf{h}_{1}$ so that $\mathbf{h}_{3}$ agrees with $\mathbf{h}_{1}$ away from a small neighborhood of the fixed-point $(-1,0)$. Near $(-1,0)$, we require that $\mathbf{h}_{3}$ also have $(-1,0)$ as a unique sink, but such that $\mathbf{h}_{3}^{\prime}(-1,0)=1$, and all higher derivatives vanish at $(-1,0)$. That is, we make $\mathbf{h}_{1}$ flat at $(-1,0)$.

Let $\phi: \mathbb{S}^{1}-(-1,0) \rightarrow U_{0}$ be an affine diffeomorphism (which is unique).

Define $\alpha \in \operatorname{Diff}^{1}\left(\mathbb{S}^{1}\right)$ with fixed-point set $\mathbf{K}$, and on $U_{\ell}$ we define

$$
\alpha \mid U_{\ell}=\gamma^{\ell} \circ \phi \circ \mathbf{h}_{3} \circ \phi^{-1} \circ \gamma^{-\ell}
$$

Define $\beta \in \operatorname{Diff}^{1}\left(\mathbb{S}^{1}\right)$ with fixed-point set $\mathbf{K}$, and on $U_{\ell}$ we define

$$
\beta \mid U_{\ell}=\gamma^{\ell} \circ \phi \circ \mathbf{h}_{2} \circ \phi^{-1} \circ \gamma^{-\ell}
$$

Let $\Gamma \subset \operatorname{Diff}^{1}\left(\mathbb{S}^{1}\right)$ be the subgroup generated by $\{\alpha, \beta, \gamma\}$. Clearly, $\mathbf{K}$ is the unique minimal set for the action of $\Gamma$, and every point in the complement of $\mathbf{K}$ has dense orbit in $S^{1}$.

The action of $\Gamma$ is expansive. There are two cases to consider. If $x \neq y \in S^{1}$, and both points lie in the same connected component of the complement of $\mathbf{K}$, then a suitable power $\gamma^{\ell}(x) \in U_{0}$ so we can proceed as in Example 8.1. If $x \neq y$ do not lie in the closure of the same connected component of the complement of $\mathbf{K}$, then there exists some $U_{\ell}$ contained in the interval $\overline{x y}$, thus $\gamma^{-\ell}(x)$ and $\gamma^{-\ell}(y)$ contain $U_{0}$ in the interval they determine, so the points are again $\epsilon$-separated.

This example is a little more subtle than the first, as it has a unique exceptional minimal set, which is obviously of finite type, and the action is expansive on $\mathbf{K}$. The set of hyperbolic periodic points $\mathcal{P}_{\log 2}^{h}(\varphi)$ is dense in the complement of $\mathbf{K}$. (The point $(1,0)$ is fixed by $\alpha$, where $\alpha^{\prime}(1,0)=2$, and the conjugates of $\alpha$ provide a dense set of fixed-points.) However, $\mathbf{K} \cap E(\varphi)=\emptyset$ as there is an invariant measure for the action on $\mathbf{K}$. This shows that in general, $E(\varphi)$ is not a closed set. Also, the orbits in the complement of $\mathbf{K}$ have exponential growth, so that in Theorem 6.3, it is not sufficient that the exponential growth orbit limit to $K$ - it must be an orbit of a point in $\mathbf{K}$.

EXAMPLE 8.3 A proper $C^{\infty}$-action of a solvable group on $S^{1}$ with countably many hyperbolic fixed-points and no ping-pong games.

Let $\mathbf{h}_{1}$ be the hyperbolic linear fractional map as in Example 8.1. Choose a fundamental domain $I=[a, b]$ for $\mathbf{h}_{1}$ in the invariant open set $\mathbb{S}_{+}^{1}=\left\{z=x+i y \mid x^{2}+y^{2}=1 \& y>0\right\}$.

Let $\phi: \mathbb{S}^{1}-(-1,0) \rightarrow I$ be an affine diffeomorphism (which is unique) and us it to define $\mathbf{h}_{4}$ on $I$ as $\mathbf{h}_{4}=\phi \circ \mathbf{h}_{1} \circ \phi^{-1}$. Extend $\mathbf{h}_{4}$ to all of $\mathbb{S}^{1}$ as the identity outside of $I$. On the interior of the interval $I, \mathbf{h}_{4}$ has a unique hyperbolic fixed-point $y_{0} \in I$ with derivative 2 , and all other points are asymptotic to the endpoints of $I$.

Let $\Gamma$ be the subgroup of $\operatorname{Diff}^{\infty}\left(\mathbb{S}^{1}\right)$ generated by $\left\{\mathbf{h}_{1}, \mathbf{h}_{4}\right\}$, and $\varphi$ the action it generates. The group $\Gamma$ is solvable, but its commutator subgroup is infinitely generated, so $\Gamma$ must have exponential growth. Note that $\varphi$ has countably many hyperbolic fixed-points $\left\{y_{k}=\mathbf{h}_{1}^{k}\left(y_{0}\right) \mid k \in \mathbb{Z}\right\}$, where $y_{k}$ is fixed by all of the maps $\mathbf{g}_{k}^{\ell}=\mathbf{h}_{1}^{k} \circ \mathbf{h}_{4}^{\ell} \circ \mathbf{h}_{1}^{-k}$. Thus, $\lambda\left(y_{k}\right)>\ell \cdot \log (2) /(2 k+\ell)$ for all $\ell$.

On the other hand, the orbits of $\varphi$ consist of either a singleton (for $\mathrm{x}=(-1,0)$ ); or copy of $\mathbb{Z}$ (for the translates $\mathbf{h}_{1}^{\ell}\left(y_{0}\right)$, for the endpoints of $I$, and for all points in $\mathbb{S}_{-}^{1}$ ); or they are isomorphic to $\mathbb{Z}^{2}$. That is, all orbits have quadratic growth.

Obviously, $\varphi$ has no ping-pong table, in spite of the existence of many hyperbolic fixed-points. The issue is that the domains of contractions of these fixed-points do not overlap, so do not generate the complicated dynamics of a ping-pong table. It is cautionary, for many of the constructions made in this paper require producing contractions with overlapping domains, and it is not sufficient to just exhibit the contractions, as the domains must be controlled as well.

The suspension of this example is one of the most basic in the study of foliations, as it is "depth two" and the leaves have quadratic growth, even though the group $\Gamma$ has exponential growth. More examples and constructions can be found in Hector [17], and also in the book [1].

## References

[1] A. Candell and L. Conlon. Foliations 1. Amer. Math. Soc., Providence, RI, 2000.
[2] J. Cantwell and L. Conlon. Poincaré-Bendixson theory for leaves of codimension one. Transactions Amer. Math. Soc., 265:181-209, 1981.
[3] J. Cantwell and L. Conlon. Foliations and subshifts. Tohoku Math. J., 40:165-187, 1988.
[4] J. Cantwell and L. Conlon. Leaves of Markov local minimal sets in foliations of codimension one. Publications Matematiques, 33:461-484, 1989. Published by the Universitat Autòmata de Barcelona.
[5] J. Cantwell and L. Conlon. Endsets of exceptional leaves; a theorem of G. Duminy. Notes for informal circulation, 1989.
[6] J. Cantwell and L. Conlon. Markov minimal sets have hyperbolic leaves. Ann. Global Anal. Geom., 9:13-25, 1991.
[7] C. Connell, A. Furman, and S. Hurder. Expansive Maps of the Circle. UIC preprint, March 2000.
[8] P. Dippolito. Codimension one foliations of closed manifolds. Annals of Math.,107:403-457, 1978.
[9] P. de la Harpe. Free groups in linear groups. Enseign. Math (2), 29:129-144, 1983.
[10] M. Einsiedler, D. Lind, R. Miles and T. Ward. Expansive subdynamics for algebraic $\mathbb{Z}^{d}$-actions. preprint, April 2000.
[11] B. Farb and P. Shalen. Groups of real-analytic diffeomorphisms of the circle. preprint, April 1998.
[12] É. Ghys, R. Langevin, and P. Walczak. Entropie geometrique des feuilletages. Acta Mathematica, 168:105-142, 1988.
[13] É. Ghys. Actions de réseaux sue le cercle. Invent. Math, 137:199-231, 1999.
[14] É. Ghys. Groups acting on the circle. preprint, IMCA, Lima, June 1999.
[15] C. Godbillon. Feuilletages, Études géométriques II. Publication de C.N.R.S., Université de L. Pasteur, Strasbourg 1986.
[16] A. Haefliger. Structures feulletées et cohomologie à valeur dans un faisceau de groupoïdes. Comment. Math. Helv., 32:248-329, 1958.
[17] G. Hector. Quleques exemples de feuilletages espèces rares. Ann. Inst. Fourier, Grenoble, 26:239-264, 1976.
[18] G. Hector. Architecture of $C^{2}$-foliations. Astérisque, 107-108:243-258, Société Mathématique de France 1983.
[19] G. Hector and U. Hirsch. Introduction to the Theory of Foliations. Part A \& B. Vieweg and Sohn, Braunschweig/Wiesbaden 1986.
[20] M. Hirsch. A stable analytic foliation with only exceptional minimal sets. in Dynamical Systems, Warwick, 1974, Lect. Notes in Math., 468:9-10, Springer-Verlag 1975.
[21] S. Hurder. The Godbillon measure of amenable foliations. J. Differential Geom., 23:347-365, 1986.
[22] S. Hurder. Ergodic theory of foliations and a theorem of Sacksteder. In Dynamical Systems: Proceedings, University of Maryland 1986-87. Lect. Notes in Math. volume 1342, pages 291-328, New York and Berlin, 1988. Springer-Verlag.
[23] S. Hurder. Exceptional minimal sets for $C^{1+\alpha}$-group actions on the circle. Ergodic Theory Dynamical Systems, 11:455-467, 1991.
[24] S. Hurder. Coarse geometry of foliations. Geometric Study of Foliations, Tokyo 1993 (eds. Mizutani et al). World Scientific, 1995, 35-96.
[25] S. Hurder. Entropy and Dynamics of $C^{1}$ Foliations. preprint, August 2000. Available at http://www.math.uic.edu/~hurder/publications.
[26] S. Hurder and A. Katok. Ergodic Theory and Weil measures for foliations. Annals of Math., 126:221275, 1987.
[27] S. Hurder and R. Langevin. Dynamics and the Godbillon-Vey Class of $C^{1}$ Foliations. preprint, September 2000. Available at http://www.math.uic.edu/~hurder/publications.
[28] T. Inaba and S. Matsumoto. Resilient leaves in transversely projective foliations. Journal of Faculty of Science, University of Tokyo, 37:89-101, 1990.
[29] T. Inaba and N. Tsuchiya. Expansive Foliations. Hokkaido Math. Journal, 21:39-49, 1992.
[30] A. Katok and B. Hasselblatt. Introduction to the Modern Theory of Dynamical Systems Cambridge Univeristy Press, 1995.
[31] G. Margulis. Free subgroups of the homeomorphism group of the circle. preprint, August 30, 2000.
[32] Y. B. Pesin. Dimension Theory in Dynamical Systems. Contemporary Views and Applications Chicago Lectures in Mathematics, University of Chicago Press, 1997.
[33] R. Sacksteder. Foliations and pseudogroups. American Jour. Math., 87:79-102, 1965.
[34] E. Salhi. On local minimal sets. C. R. Acad. Sci., Paris., 295:691-694, 1982.
[35] E. Salhi. Sur un théorme de structure des feuilletages de codimension 1. C. R. Acad. Sci., Paris., 300:635-638, 1985.
[36] E. Salhi. Niveau des feuilles. C. R. Acad. Sci., Paris., 301:219-222, 1985.
[37] K. Schmidt. Dynamical Systems of Algebraic Origin. Progress in Mathematics vol. 128, Birkháuser Verlag, Basel-Boston-Berlin (1995).
[38] M. Shub. Global Stability of Dynamical Systems. Springer-Verlag, New York and Berlin (1987)
[39] R. Spatzier. preprint, May 2000. (1 page in TeX)
[40] J. Tits. Free subgroups in linear groups. J. of Algebra, 20:250-270, 1972.
[41] R. Zimmer. Ergodic Theory and Semisimple Groups. Birkhäuser, Boston, Basel, Stuttgart, 1984.

## Steven Hurder

Department of Mathematics (m/c 249)
University of Illinois at Chicago
851 S. Morgan St.
CHICAGO, IL 60607-7045 USA
Email: hurder@uic.edu
Web: http://www.math.uic.edu/~hurder/


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