Dynamics of expansive group actions on the circle

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Abstract

In this paper, we study the topological dynamics of C^0 and C^1-actions on the circle by a countably generated group Γ, under the assumption that there is expansive orbit behavior. Our first result is a simple proof that if a C^0-action φ is expansive, then Γ must have a free sub-semigroup on two generators, hence has exponential growth. For C^1-actions, we introduce the set of points $E(\varphi)$ which are infinitesimally expansive, and prove that the hyperbolic periodic points are dense in $E(\varphi)$. We show that if the set $E(\varphi)$ has an accumulation point in itself, then the geometric entropy $h(\varphi)$ must be positive. Finally, we prove that if $\mathbf{K}$ is a minimal set for a C^1-action φ, then either there is a Γ-invariant probability measure supported on $\mathbf{K}$, or the hyperbolic periodic points are dense in K.

Contents

*Supported in part by NSF grants DMS-9704768 and DMS-0406254. Preprint date: July 23, 2005

1 Introduction

Let Γ be a finitely-generated group and $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ an effective action on the circle by orientation-preserving homeomorphisms. In this paper, we use "expansiveness" properties of φ to study the topological dynamics of φ and the algebraic structure of Γ.

Let $\varphi\colon \Gamma \times \mathbb{X} \to \mathbb{X}$ denote a continuous action on a metric space $(\mathbb{X}, d_X)$. When the action φ is clear from the context, we use the simplified notation, $\gamma x = \varphi(\gamma)(x)$ for $\gamma \in \Gamma$.

DEFINITION 1.1 *A continuous action $\varphi\colon \Gamma \times \mathbb{X} \to \mathbb{X}$ on a metric space $(\mathbb{X}, d_X)$ is* ϵ–expansive *if, for any pair $x \neq y \in \mathbb{X}$, there exists $\gamma \in \Gamma$ such that $d_X(\gamma x, \gamma y) > \epsilon$. We say that φ is* expansive *if it is ϵ-expansive for some $\epsilon > 0$.*

An expansive action of the group $\mathbb{Z}$ is simply an expansive homeomorphism. The study of expansive diffeomorphisms of compact Riemannian manifolds M is a venerable topic [12, 47]. Many properties of these maps are known; for example, the topological entropy of an expansive diffeomorphism must be positive. (cf. [32, 34, 47, 48]). Actions of groups on shift spaces provide another large class of examples of expansive actions [11, 45]. However, there are no expansive homeomorphisms of the circle, as an expansive map on $\mathbb{S}^1$ cannot be invertible. Thus, for an expansive group action $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$, the group Γ cannot be cyclic.

The study of dynamics of group actions on the circle is a model problem for the study of the dynamical properties of codimension one foliations. In one direction, the transverse pseudogroup of a codimension one foliation $\mathcal{F}$ can be modeled by a pseudogroup acting on $\mathbb{S}^1$ [41, 23, 3], so the dynamics of $\mathcal{F}$ is also modeled by the dynamics of the associated pseudogroup. Conversely, given an action $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ by a finitely-generated group Γ, there is a codimension one foliation $\mathcal{F}_\varphi$ on a compact 3-manifold whose holonomy pseudogroup is equivalent to this action.

Inaba and Tsuchiya studied expansive foliations in [30], where they showed that an expansive foliation of codimension one must have a resilient leaf. Their result applied to the suspension foliation $\mathcal{F}_\varphi$ of an expansive C^0-action $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ implies that Γ has exponential growth. In particular, Γ cannot have a nilpotent subgroup of finite index. This provides a negative answer to a question of Tom Ward [33], which was one of the original motivations for this work: "Can a nilpotent group act expansively on the circle?" Connell, Furman, and the author [8] and Spatzier [49] gave alternate proofs that a nilpotent group cannot act expansively on $\mathbb{S}^1$.

In section 3, we prove the following version of the theorem by Inaba and Tsuchiya [30]:

THEOREM 1.2 *If $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is an expansive C^0-action, then the entropy of the action φ must be positive (or infinite) and Γ contains a free sub-semigroup on two generators. In particular, Γ must have exponential word growth, and cannot have a nilpotent subgroup of finite index.*

The proof of Theorem 1.2 follows from Propositions 2.10 and 3.1, whose proofs use only elementary topological dynamics. These two propositions are roughly equivalent to Lemmas 2.3 and 2.5 of [30]. We show in detail how the expansive hypothesis implies the existence of a "ping-pong game" for the dynamics of φ, which implies both of the conclusions of Theorem 1.2. Example 8.1 exhibits an expansive real analytic action on $\mathbb{S}^1$ by a solvable group, so the conclusion that there is a free sub-semigroup cannot be improved to the existence of a free non-abelian subgroup.

Sections 2 and 3 present some basic topological properties of group actions on the circle. These results are essentially contained in the foliation literature, but we include them here for completeness, as their conclusions and techniques of proof are used in later sections, and also because the statements are easier to formulate and more definitive for a group action on $\mathbb{S}^1$.

Section 2 begins with some basic definitions and results about the topological dynamics of group actions on the circle. We include a discussion of the dynamics for actions restricted to minimal sets. For example, we show that if φ is expansive when restricted to a minimal set $\mathbf{K}$, then $\mathbf{K}$ has finite type. We also show that an expansive action on $\mathbb{S}^1$ must have an open local minimal set. Finally, we define the entropy of a group action following [18], then discuss the concepts of a "ping-pong game" and "resilient orbit" and their relationship with the entropy of an action.

In sections 4 through 7, we assume that $\varphi\colon\Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is a C^1-action and introduce concepts and techniques based on linear approximations. In section 4, we introduce the *infinitesimally expansive* set $E(\varphi) \subset \mathbb{S}^1$, where a point $x \in \mathbb{S}^1$ is in $E(\varphi)$ if the asymptotic infinitesimal exponent of Γ at x is positive. The set $E(\varphi)$ is the union of subsets $E_a(\varphi)$ for $a > 0$, where $E_a(\varphi)$ consists of those points for which the expansion if greater that $\exp(a)$. This is explained precisely in Definition 4.3. The set $E(\varphi)$ is Γ-invariant, and plays a fundamental role in the study of the dynamics of the action. In many ways, $E(\varphi)$ is the analog of the Pesin set for measure-preserving actions, and the heart of this paper is the analysis of how $E(\varphi)$ determines the dynamics of the action.

We say that $x \in \mathbb{S}^1$ is a *hyperbolic fixed-point* for φ if there exists $\gamma \in \Gamma$ such that $\varphi(\gamma)(x) = x$ and $\varphi(\gamma)'(x) > 1$. The set of hyperbolic fixed-points form a subset $\mathcal{P}^h(\varphi) \subset E(\varphi)$, filtered by the subsets $\mathcal{P}^h_b(\varphi)$ of points x such that $\varphi(\gamma)'(x) > \exp(b) \cdot \|\gamma\|$ for some $\gamma \in \Gamma$ with $\gamma x = x$, and $\|\gamma\|$ denotes the word length of γ.

THEOREM 1.3 *Let $\varphi\colon\Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^1-action, $a > 0$, and suppose that $E_a(\varphi)$ is not empty. Then the hyperbolic fixed-points $\mathcal{P}^h_a(\varphi)$ are dense in $E_a(\varphi)$, and φ is expansive on $\mathcal{P}^h_a(\varphi)$.*

The proof of Theorem 1.3 follows from Propositions 4.7 and 4.12. The proofs of these two propositions use a fundamental techniques for the study of expansive C^1-actions which we introduce in this paper, the existence of uniform expanders: Given $x \in E(\varphi)$, there is a sequence of hyperbolic expanding maps with arbitrarily long word length, whose domains contain x in the center and achieve an expansion of the domain to an interval of uniform length $\delta > 0$, where δ depends upon the group Γ and the action φ but not the choice of x. The construction of these uniform expanders is given in section 4, where they are used to construct hyperbolic periodic points. This technique has also been applied in [28] to study the measure of exceptional minimal sets.

In section 5, we show that if $E(\varphi)$ is "large enough" then the action of φ has positive entropy.

THEOREM 1.4 *If $\varphi\colon\Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is a C^1-action with $E(\varphi)$ an uncountable set, then $h(\varphi) > 0$.*

In sections 6 and 7 we study the relationship between the set $E(\varphi)$ and the minimal sets of the action. In particular, we develop criteria for when the set $E(\varphi)$ is non-empty. The following result is a generalization of the Sacksteder Theorem [41] for C^2-actions:

THEOREM 1.5 *Let $\varphi\colon\Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^1-action with minimal set $\mathbf{K}$. Then either there is a Γ-invariant probability measure supported on $\mathbf{K}$, or the hyperbolic periodic points are dense in K. In the latter case, there is a hyperbolic ping-pong game for φ and $h(\varphi) > 0$.*

As a corollary, we obtain that a C^1-action of a group on the circle satisfies the following dichotomy:

COROLLARY 1.6 *Let $\varphi\colon\Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^1-action with minimal set $\mathbf{K}$. Then either every orbit of φ on $\mathbf{K}$ has polynomial growth, or the dynamics of φ on $\mathbf{K}$ contains a hyperbolic ping-pong game and $h(\varphi) > 0$.*

Proof: Apply Theorem 1.5 and Proposition 6.1 to the minimal set $\mathbf{K}$. □

In section 8, we discuss the application of the ideas of this paper to a conjecture of Ghys that a C^0-action on the circle either has a Γ-invariant probability measure, or there exists a non-abelian free subgroup of Γ. This conjecture was solved by G. Margulis in [35], who cites the initial version of this paper in his work. We reformulate Ghys' conjecture in terms of the action of φ on minimal sets, and show the conjecture is implied by the existence of certain maps with isolated fixed-points. As an application, we show how the results of Farb and Shalen in [16] can be used to prove:

THEOREM 1.7 *Let $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^ω-action. Then either φ has an invariant probability measure, or Γ has a non-abelian free subgroup on two generators.*

Finally, in section 9 we discuss three basic examples which are very helpful in understanding the results of the previous sections, as they illustrate dynamical properties which must be considered (implicitly, at least) in the proofs. The first example is of an expansive, real analytic action of a solvable group on the circle with a local minimal set, which shows that the conclusions Theorems 1.2 and 1.4 cannot be strengthened to conclude that Γ contains a free non-abelian subgroup on two generators. The second example is an extension of the first, and gives an expansive C^1-action with an exceptional minimal set $\mathbf{K}$ such that the action is not hyperbolic on $\mathbf{K}$. The third example gives a group action with a countable family of hyperbolic fixed-points but tame dynamics.

The author is especially indebted to C. Connell, A. Furman, E. Ghys, T. Inaba, and G. Margulis for a variety of discussions related to this paper.

2 Topological dynamics of group actions

We assume that Γ is a finitely generated group. Choose a symmetric generating set $\mathcal{S} = \{\sigma_0, \sigma_1, \ldots, \sigma_k\}$ for Γ, where σ_0 is the identity. The symmetric hypothesis means that for each $1 \leq i \leq k$, $\sigma_i^{-1} = \sigma_\ell$ for some $1 \leq \ell \leq k$. An element $\gamma \in \Gamma$ has word length $\|\gamma\| \leq N$ if there exists indices $i_1, \ldots, i_N$ such that $\gamma = \sigma_{i_1} \cdots \sigma_{i_N}$. The word length $\|\gamma\|$ is the least integer N such that $\|\gamma\| \leq N$.

Give $\mathbb{S}^1$ the Riemannian metric with total length 2π, and let $d: \mathbb{S}^1 \times \mathbb{S}^1 \to [0, \pi]$ denote the path length metric.

We give in this section definitions and basic results concerning the topological dynamics of group actions. We start with two definitions from the theory of topological dynamics of group actions.

DEFINITION 2.1 *A continuous action $\varphi: \Gamma \times \mathbb{X} \to \mathbb{X}$ on a metric space $(\mathbb{X}, d_X)$ is* equicontinuous *if for all $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ so that for all $x, y \in \mathbb{X}$, if $d_X(x, y) \geq \epsilon$ then $d_X(\gamma x, \gamma y) \geq \delta(\epsilon)$ for all $\gamma \in \Gamma$. That is, the family of maps $\{\varphi(\gamma) \mid \gamma \in \Gamma\}$ is equicontinuous.*

If $\mathbb{X} = \mathbb{S}^1$ and φ is an equicontinuous action, then there exists a Γ-invariant metric d_Γ on $\mathbb{S}^1$. Moreover, d_Γ defines a Γ-invariant Borel measure μ_Γ on $\mathbb{S}^1$ by defining $\mu_\Gamma([x, y]) = d_\Gamma(x, y)$ where $[x, y] \subset \mathbb{S}^1$ is a closed interval and x, y are sufficiently close.

DEFINITION 2.2 *A continuous action $\varphi: \Gamma \times \mathbb{X} \to \mathbb{X}$ on a metric space $(\mathbb{X}, d_X)$ is* distal *if for every pair $x \neq y \in \mathbb{X}$, there exists $\delta(x, y) > 0$ such that $d_X(\gamma x, \gamma y) \geq \delta(x, y)$ for all $\gamma \in \Gamma$.*

Clearly, an equicontinuous action is distal. There is a dichotomy in the following special case:

LEMMA 2.3 *A minimal action $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is either equicontinuous, or expansive.*

Proof: There exists a modulus of continuity function $\epsilon \mapsto \delta(\epsilon) < \pi$ so that if $I = [x, y] \subset \mathbb{S}^1$ is a closed interval of length less than $\delta(\epsilon)$ then $\sigma_\ell(I)$ has length less than ϵ. Set $\delta_0 = \delta(\pi)$. As $\delta(\epsilon)$ is defined with respect to a symmetric set, it follows that given any generator σ_i and an interval I of length at least π then $\sigma_i(I)$ has length at least δ_0.

Suppose that φ is not equicontinuous. Then for some $0 < \epsilon < \pi$ there exists a sequence of pairs $\{(x_n, y_n) \mid n = 1, 2, \ldots\}$ with $d(x_n, y_n) > \epsilon$ and $\gamma_n \in \Gamma$ such that $d(\gamma_n x_n, \gamma_n y_n) < 1/n$. Choose a limit point z_* for the set $\{\gamma_n x_n \mid n = 1, 2, \ldots\}$. Passing to a subsequence, we can assume that $d(\gamma_n x_n, z_*) < 1/n$ for all n.

Let $x \neq y \in \mathbb{S}^1$ with $d(x, y) < \epsilon$, then there is a unique shortest interval $\overline{x, y}$ with endpoints x, y. By the minimality of the action, there exists $\gamma_0 \in \Gamma$ such that $\gamma_0 z_*$ lies in the interior of $\overline{x, y}$. Thus, for n sufficiently large we can assume that $\gamma_0(\overline{\gamma_n x_n, \gamma_n y_n})$ is contained in the interior of $\overline{x, y}$.

Write $\gamma_n = \sigma_{i_1} \cdots \sigma_{i_k}$ as a product of generators, and define $h_\ell = \sigma_{i_\ell}^{-1} \cdot \cdots \sigma_{i_1}^{-1} \cdot \gamma_0^{-1}$. Note that $h_k(\overline{x, y})$ contains $\overline{x_n, y_n}$ in its interior, so is an interval with length greater than ϵ. If the length of $h_k(\overline{x, y})$ is at most π, then $d(h_k x, h_k y) > \epsilon$.

If the length of $h_k(\overline{x, y})$ is greater than π (so the endpoints are wrapping around on the circle) then we remove some of the generators in the composition h_k so that $h_\ell(\overline{x, y})$ has length less than π. More precisely, by the choice of δ_0, if the length of $h_\ell(\overline{x, y})$ is greater than π, then the length of $h_{\ell-1}(\overline{x, y})$ is greater than δ_0. Thus, if $h_k(\overline{x, y})$ has length greater than π, then starting with $\ell = k$ we can proceed downward inductively until we obtain $\ell > 0$ such that $h_\ell(\overline{x, y})$ has length between δ_0 and π. Then $d(h_\ell x, h_\ell y) > \delta_0$.

Set $\epsilon_0 = \min\{\epsilon, \delta_0\}$, then it follows that φ is ϵ_0-expansive. □

2.1 Minimal sets

Suppose that $\mathbf{K} \subset \mathbb{S}^1$ is a minimal set for φ. Then either $\mathbf{K}$ is a finite set, or $\mathbf{K} = \mathbb{S}^1$, or $\mathbf{K}$ is a perfect, nowhere dense subset. In this latter case, $\mathbf{K}$ is said to be *exceptional.* The case where $\mathbf{K}$ is finite is not discussed in this paper.

We say that an exceptional minimal set is *Denjoy type* if there exists a φ-invariant probability measure supported on $\mathbf{K}$. Otherwise, we say that $\mathbf{K}$ is *hyperbolic.*

If $\mathbf{K} = \mathbb{S}^1$, then we saw that φ is either equicontinuous or expansive. In the former case, it is well-known that there is a Γ-invariant probability measure $\mathbf{m}$ on $\mathbb{S}^1$ whose support must be $\mathbb{S}^1$ (cf. [41]). Renormalize $\mathbf{m}$ to have total mass 2π, choose a basepoint $x_0 \in \mathbb{S}^1$, then the measure $\mathbf{m}$ defines a homeomorphism $h_{\mathbf{m}}: \mathbb{S}^1 \to \mathbb{S}^1$ by setting $h_{\mathbf{m}}(x) = \mathbf{m}([x_0, x])$ modulo 2π, so that $h_{\mathbf{m}}^* d\theta = \mathbf{m}$ where $d\theta$ denotes the standard length measure. Thus, $h_{\mathbf{m}}$ conjugates φ to an action $\varphi_{\mathbf{m}}$ on $\mathbb{S}^1$ by rotations. As Γ acts effectively by orientation-preserving homeomorphisms, this implies Γ is free abelian. The case where φ is expansive will be discussed in section 3.

Let $\mathbf{K}$ be an exceptional minimal set. The complement $\mathbb{S}^1 - \mathbf{K} = \mathcal{I}$ is a non-empty Γ-invariant set, and can be written as a disjoint union $\mathcal{I} = \cup_{n=1}^{\infty} I_n$, where each $I_n = (a_n, b_n)$ is an open interval. The intervals I_n are called the *gaps* of $\mathbf{K}$, and the closure $\overline{I_n} = [a_n, b_n]$ is called a *closed gap.* An *endpoint* of $\mathbf{K}$ is a point in the intersection $\mathbf{K} \cap \overline{I_n}$ for some n.

As the set $\mathbf{K}$ is φ-invariant, the action of Γ permutes the gaps of $\mathbf{K}$. This implies one of the basic facts about exceptional minimal sets for circle actions (cf. [17]):

PROPOSITION 2.4 *Let $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^0-action with an exceptional minimal set $\mathbf{K}$. Then $\mathbf{K}$ is unique, and moreover, φ cannot be distal.*

Proof: Suppose $\mathbf{K}_1$ and $\mathbf{K}_2$ are distinct minimal sets for φ. Then there exists a point $x \in I_n \cap \mathbf{K}_2$ where $I_n = (a_n, b_n)$ is a gap for $\mathbf{K}_1$. The orbit of the endpoint a_n is dense in $\mathbf{K}_1$ so the images of the gap I_n under Γ contain an infinite number of disjoint intervals, hence there is some sequence $\{\gamma_i \mid i = 1, 2, \ldots\}$ whose lengths $|\gamma_i I_n| \to 0$. Thus, the sequences $\{\gamma_i x\}$ and $\{\gamma_i a_n\}$ have a common accumulation point x_*, which must be contained in both $\mathbf{K}_1$ and $\mathbf{K}_2$. The closure of the orbit x_* equals both $\mathbf{K}_1$ and $\mathbf{K}_2$ hence they are equal, a contradiction.

Note that for any gap I_n of a minimal set $\mathbf{K}$, there is a sequence $\gamma_i I_n$ whose lengths tend to 0. That is, $\lim_{n\to\infty} d(\gamma_n a_i, \gamma_n b_i) = 0$. Hence, φ is not distal. $\square$

DEFINITION 2.5 *For $x \in \mathbb{S}^1$, we say the orbit Γx is* semi-proper *if there exists $\epsilon > 0$ so that for either $\mathcal{I} = (x - \epsilon, x)$ or $\mathcal{I} = (x, x + \epsilon)$, the orbit of x is disjoint from $\mathcal{I}$. That is, $\Gamma x \cap \mathcal{I} = \emptyset$.*

Given a gap (a_n, b_n) of an exceptional minimal set $\mathbf{K}$, the orbits Γa_n and Γb_n are both semi-proper. We say that an exceptional minimal set $\mathbf{K}$ has *finite type* if there are only a finite number of semi-proper orbits determined by the endpoints of $\mathbf{K}$. Hector asked in his thesis [20, 46] whether every exceptional minimal set in a C^2-foliation has finite type. The question was answered affirmatively for Markov minimal sets [7, 37], and is also known for piecewise projective actions [29]. For the topological case, we have:

LEMMA 2.6 *φ is expansive on the exceptional minimal set $\mathbf{K}$ if and only if $\mathbf{K}$ has finite type.*

Proof: Let $\epsilon > 0$ be an expansive constant for the action restricted to $\mathbf{K}$. That is, for every $x \neq y \in \mathbf{K}$ there exists γ such that $d(\gamma x, \gamma y) > \epsilon$. Let $I_{i_1}, \ldots, I_{i_n}$ be the gaps whose length is greater than ϵ. Then for any gap I_ℓ there must exist γ with $\gamma(I_\ell)$ of length greater than ϵ. As $\gamma(I_\ell)$ is again a gap of $\mathbf{K}$, γa_ℓ must be be an endpoint for one of $I_{i_1}, \ldots, I_{i_n}$, and likewise γb_ℓ must be be an endpoint for one of $I_{i_1}, \ldots, I_{i_n}$. Hence, the orbits of the endpoints $\{a_{i_1}, \ldots, a_{i_n}, b_{i_1}, \ldots, b_{i_n}\}$ contain all endpoints of $\mathbf{K}$.

Conversely, suppose that $\mathbf{K}$ has finite type. Let $\{I_k \mid k = 1, 2, \ldots\}$ be the gaps of $\mathbf{K}$ and suppose that $I_1, \ldots, I_n$ are such that the orbits of the endpoints $\{a_1, b_1, \ldots, a_n, b_n\}$ contain all the orbits of the endpoints of $\mathbf{K}$. Choose $0 < \epsilon < \min\{|I_1|, |I_2|, \ldots |I_n|, \pi/2\}$. Given $x, y \in \mathbf{K}$ with $0 < d(x, y) < \epsilon$, $\mathbf{K}$ is nowhere dense so there exists a gap I_ℓ for $\ell > n$ with $I_\ell \subset [x, y]$. Then there exists $\gamma \in \Gamma$ such that $\gamma I_\ell = I_i$ for some $1 \leq i \leq n$, and so $d(\gamma x, \gamma y) \geq |I_i| > \epsilon$. $\square$

If $\mathbf{K}$ is not of finite type, then the proof above suggests that the action fails to be expansive only for pairs (x, y) which are endpoints of some gap I_n. This is in fact the case. Let $\widehat{\mathbf{K}} = \mathbf{K} - \cup_{n=1}^{\infty}\{a_n, b_n\}$. That is, $\widehat{\mathbf{K}}$ is the set $\mathbf{K}$ with all endpoints deleted.

LEMMA 2.7 *Let $\mathbf{K}$ be an exceptional minimal set of φ. Then φ is expansive on $\widehat{\mathbf{K}}$.*

Proof: Let $x \neq y \in \widehat{\mathbf{K}}$ with $d(x, y) < \pi$. Then (x, y) is an open interval of length $d(x, y)$ whose endpoints lies in $\mathbf{K}$ yet neither x nor y are endpoints of a gap. Thus, $\mathbf{K} \cap (x, y) \neq \emptyset$. Choose any gap of $\mathbf{K}$, say $I_1 = [a_1, b_1]$ and set $\epsilon = |I_1|$. As $\mathbf{K}$ is minimal, there is some $\gamma \in \Gamma$ for which $\gamma a_1 \in (x, y)$. Then $y \in \mathbf{K}$ implies that $\gamma I_1 \subset (x, y)$ and hence $d(\gamma^{-1} x, \gamma^{-1} y) > |I_1| = \epsilon$. $\square$

2.2 Local minimal sets and expansiveness

DEFINITION 2.8 *A Γ-invariant set K is a* local minimal set *if for all $x \in K$ the orbit Γx is dense in K. Equivalently, we require that closure of the orbit, $\overline{\Gamma x}$, equals the closure $\overline{K}$. If K is an open set, then we call K an* open local minimal set*.*

A minimal set K is a local minimal set. An orbit Γx is proper if each point γx is isolated in Γx. A proper orbit Γx is a local minimal set, but it is a minimal set if and only if the orbit is finite.

We prove a technical characterization of open local minimal sets.

LEMMA 2.9 *Let $x \in \mathbb{S}^1$ and let $K = \text{Int}\,\overline{\Gamma x}$. Let $V \subset K$ be a non-empty open subset. If for every $y \in V$, the orbit closure $\overline{\Gamma y}$ has non-trivial interior, then $W = \bigcup_{\gamma \in \Gamma} \gamma V$ is an open local minimal set.*

Proof: Given $y \in V \subset K = \text{Int}\,\overline{\Gamma x}$, the interior $U_y = \text{Int}\,\overline{\Gamma y}$ is not empty.

As $y \in \overline{\Gamma x}$ we have $\overline{\Gamma y} \subset \overline{\Gamma x}$ and hence $U_y \subset \text{Int}\,\overline{\Gamma x}$. Thus, there exist γ so that $\gamma x \in U_y$.

The closure of Γy contains U_y hence has γx as a limit point. But this implies $\Gamma y = \Gamma \gamma^{-1} y$ has $\gamma^{-1}\gamma x = x$ as a limit point, and thus $\overline{\Gamma y} \supset \overline{\Gamma x}$ so $\overline{\Gamma y} = \overline{\Gamma x} = \overline{K}$.

Thus, for all $y \in V$ we have $\overline{\Gamma y} = \overline{K}$.

For each $z \in W$, there is $\gamma \in \Gamma$ with $y = \gamma z \in V$ so the orbit closure satisfies $\overline{\Gamma\, z} = \overline{\Gamma \gamma z} = \overline{\Gamma\, y} = \overline{K}$. As $V \subset K$ and K is Γ-invariant we have that $\overline{W} \subset \overline{K}$, hence for each $z \in W$, $\overline{\Gamma\, z} = \overline{W}$. □

The elementary Lemma 2.9 has a nice application.

PROPOSITION 2.10 *An expansive action φ has a non-empty open local minimal set.*

Proof: Assume that φ is ϵ-expansive, and suppose that φ has no local open minimal set.

Choose $x_1 \in \mathbb{S}^1$. If the orbit closure $\overline{\Gamma x_1}$ has no interior, set $X_1 = \overline{\Gamma x_1}$. Otherwise, the interior $U_1 = \text{Int}\,\overline{\Gamma x_1}$ is non-empty, so that $V_1 = U_1 \cap \{z \in \mathbb{S}^1 \mid d(z, x_1) < 1/100$ is a non-empty open set. By Lemma 2.9 there exists $y_1 \in V_1$ such that $X_1 = \overline{\Gamma y_1}$ has no interior. Set $Z_1 = X_1$.

The complement $\mathbb{S}^1 - Z_1$ is an open Γ-invariant set, so consists of a countable union of open intervals whose lengths tend to zero. Choose an interval I_2 of $\mathbb{S}^1 - Z_1$ with maximum length. Let $x_2 \in I_2$ be the midpoint. If $\overline{\Gamma x_2}$ has no interior, define $X_2 = \overline{\Gamma x_2}$. Otherwise, as before, there exists $y_2 \in I_2$ with $d(y_2, x_2) < |I_2|/100$ such that $X_2 = \overline{\Gamma y_2}$ has no interior. Set $Z_2 = X_1 \cup X_2$.

Repeat the above process a finite number of times, $n > 0$, to obtain a closed Γ-invariant subset Z_n with no interior such that the complement $\mathbb{S}^1 - Z_n$ consists of a countable union of intervals each of length less than ϵ. Let $I \subset \mathbb{S}^1 - Z_n$ be a complementary interval.

For all $\gamma \in \Gamma$ the image γI is again a complementary interval, so $|\gamma I| < \epsilon$. If $x, y \in I$ then this implies $d(\gamma x, \gamma y) < \epsilon$ for all γ contradicting the assumption that φ is ϵ-expansive with no local open minimal set. □

COROLLARY 2.11 *Suppose that φ has no non-empty open local minimal set. Then there exists a collection of sets $\{K_n \mid n = 1, 2, \ldots\}$ such that*

1. *each K_n is closed and saturated with no interior*
2. *the action of φ on K_n is* transitive*; that is, there exists $x_n \in K_n$ whose orbit is dense*

3. the union $\bigcup_{n=1}^{\infty} K_n$ is dense in $\mathbb{S}^1$

The proof of the corollary follows from the same method of proof as above. Note that this result is an elementary form of the theory of levels for a topological group action. There is no claim about the dynamics and limiting behavior of the sets K_n, which a good theory of levels provides [9, 42, 43, 44, 24]. The simplicity of the proof above is notable, and the conclusion is sufficient for our application to the proof of Theorem 1.2.

2.3 Geometric entropy

We recall the definition of the entropy of a group action, following Ghys, Langevin and Walczak [18]. While this definition is usually considered only for C^1-actions, it still makes sense for topological actions, with the understanding that the value of the entropy may be infinite.

Given $\epsilon > 0$ and an integer $N > 0$, we say that $x, y \in \mathbb{S}^1$ are (N, ϵ)–separated if there exists $\gamma \in \Gamma$ with $\|\gamma\| \leq N$ and $d(\gamma x, \gamma y) > \epsilon$. A finite subset $\{x_1, \ldots, x_\nu\} \subset \mathbb{S}^1$ is (N, ϵ)–separated if for every $k \neq \ell$ the pair of points x_k, x_ℓ is (N, ϵ)–separated.

Let $\mathcal{S}(\varphi, \epsilon, N)$ denote the maximum cardinality of an (N, ϵ)-separated subset of $\mathbb{S}^1$. This is a finite number, as the collection of homeomorphisms $\{\varphi(\gamma) \mid \|\gamma\| \leq N\}$ is finite, hence this is an equicontinuous family. Now define

$$h(\varphi, \epsilon) = \limsup_{N \to \infty} \frac{\log \mathcal{S}(\varphi, \epsilon, N)}{N} \geq 0 \tag{1}$$

The *geometric entropy* of φ is the limit

$$h(\varphi) = \lim_{\epsilon \to 0} \; h(\varphi, \epsilon) \geq 0$$

This limit is finite if φ is a C^1-action , but may be infinite for topological actions when Γ has rank greater than one. In general, $h(\varphi)$ depends upon the choice of the generating set $\mathcal{S}$. The key point is that the dichotomy $h(\varphi) \neq 0$ or $h(\varphi) = 0$ is independent of the choices, and depends only on the topological conjugacy class of the action φ.

The *geometric entropy* of φ with respect to the set of generators of Γ is the number $h(\varphi)$. This is not the "usual" topological entropy from topological dynamics for non-cyclic group actions, as in that case the denominator would be the number of elements in the ball of radius N in Γ for the word metric on Γ. The "usual" topological entropy is always finite for C^0-actions, but vanishes if φ is a C^1-action and Γ is not cyclic. In contrast, using the word length N in the denominator of the definition (1) yields a finite limit for a C^1-action of any group Γ.

Given a Γ-subset $\mathbf{K} \subset \mathbb{S}^1$ we can also define the *relative geometric entropy* $h(\varphi, \mathbf{K})$, where now we define $\mathcal{S}(\varphi, \mathbf{K}, \epsilon, N)$ using subsets $\{x_1, \ldots, x_\nu\} \subset \mathbf{K}$, and the remainder of the definitions follow the same pattern.

2.4 Ping-pong games and resilient orbits

We next recall a dynamical concept which is colloquially called a "ping-pong game" [10]. The concept dates from the work of Blaschke, Klein, Schottky and Poincaré, and the result can be viewed as a special case of the Klein-Maskit combination theorem [19, 36]. The idea is ubiquitous in the study of dynamical systems.

DEFINITION 2.12 *A pair of maps* $\{\gamma_1: I_0 \to I_1, \gamma_2: I_0 \to I_2\}$ *is called a* ping-pong game *for* φ *if* $I_0 \subset \mathbb{S}^1$ *is a closed interval,* $I_1, I_2 \subset I_0$ *are closed disjoint subintervals, and* $\gamma_1, \gamma_2 \in \Gamma$ *satisfy* $\gamma_1 I_0 = I_1$ *and* $\gamma_2 I_0 = I_2$*. If* φ *is a* C^1*-action, and* $0 < \gamma_k'(x) < 1$ *for* $k = 1, 2$ *and all* $x \in I_0$*, then we call this a* hyperbolic ping-pong game*.*

Note that the definition of a ping-pong game does not require that either map $\gamma_i: I_0 \to I_i$ be a contraction with a unique fixed-point. We say that $\{\gamma_1: I_0 \to I_1, \gamma_2: I_0 \to I_2\}$ is a *contracting ping-pong game* if the closed intervals I_1 and I_2 are contained in the interior of I_0, and both maps $\gamma_1: I_0 \to I_1$ and $\gamma_2: I_0 \to I_2$ are contractions with unique fixed-points.

Here is one of the standard properties of a ping-pong game.

LEMMA 2.13 *If* $\{\gamma_1: I_0 \to I_1, \gamma_2: I_0 \to I_2\}$ *is a* ping-pong game *for* φ*, then* $\{\gamma_1, \gamma_2\}$ *generates a free sub-semigroup of* Γ*.*

Proof: A word γ in the free sub-semigroup generated by $\{\gamma,\gamma_2\}$ has the form $\gamma = \gamma_{i_1} \cdot \gamma_{i_2} \cdots \gamma_{i_\ell}$ for indices $i_j = 1, 2$. It suffices to show that γ is not the identity, and $\gamma I_0 \subset \gamma_{i_1} I_0 \subset I_{i_1} \neq I_0$. □

An action φ with a ping-pong game has non-zero entropy.

LEMMA 2.14 *If* $\{\gamma_1: I_0 \to I_1, \gamma_2: I_0 \to I_2\}$ *is a* ping-pong game *for* φ*, then* $h(\varphi) > 0$*.*

Proof: We can assume without loss of generality that the symmetric generating set $\mathcal{S} = \{\sigma_0, \ldots, \sigma_k\}$ for Γ contains $\{\gamma_1, \gamma_1^{-1}, \gamma_2, \gamma_2^{-1}\}$. Let $\epsilon > 0$ be the distance between I_1 and I_2. Choose any $x \in I_0$ and set

$$S_n = \{\gamma_{i_1}\gamma_{i_2} \cdots \gamma_{i_n} x \mid i_j = 1, 2\}$$

The set S_n has 2^n elements, and given any pair of distinct points $y = \gamma_{i_1}\gamma_{i_2} \cdots \gamma_{i_n} x$ and $z = \gamma_{j_1}\gamma_{j_2} \cdots \gamma_{j_n} x$ there is a least $1 \leq \ell \leq n$ so that $i_\ell \neq j_\ell$ and $\gamma_{i_1} \cdots \gamma_{i_{\ell-1}} = \gamma_{j_1} \cdots \gamma_{j_{\ell-1}}$.
Set $\gamma_{y,z} = \gamma_{i_1} \cdots \gamma_{i_{\ell-1}}$, then

$$\gamma_{y,z}^{-1} y = \gamma_{i_\ell} \cdots \gamma_{i_n} x \in I_{i_\ell} \quad \text{and} \quad \gamma_{y,z}^{-1} z = \gamma_{j_\ell} \cdots \gamma_{j_n} x \in I_{j_\ell}$$

Thus $d(\gamma_{y,z}^{-1} y, \gamma_{y,z}^{-1} z) > \epsilon$. It follows that $\mathcal{S}(\varphi, \epsilon, N) \geq 2^n N$ and $h(\varphi) \geq \log(2)$. □

We recall a definition from the theory of topological dynamics for foliations:

DEFINITION 2.15 *We say that:*

- $x \in \mathbb{S}^1$ *is a* resilient point *for* φ *if there exists* $\gamma \in \Gamma$ *and an open interval* $J = (x - \delta, x + \delta)$ *such that* $J \cap \Gamma x \neq \{x\}$ *and* $\gamma: J \to J$ *is a contraction with fixed-point* x*.*
- $x \in \mathbb{S}^1$ *is a* one-sided resilient point *for* φ *if there exists* $\gamma \in \Gamma$ *and a half-open interval* $J = [x, x + \delta)$ *or* $J = (x - \delta, x]$*, such that* $J \cap \Gamma x \neq \{x\}$ *and* $\gamma: J \to J$ *is a contraction to the fixed-point* x*.*
- *If* φ *is a* C^1*-action and in addition* $\varphi(\gamma)'(x) < 1$*, then* x *is a* hyperbolic *resilient point.*
- *An orbit* Γy *is resilient if there is some* $x \in \Gamma y$ *which is resilient.*

The concepts of "ping-pong game" and "resilient point" turn out to be almost equivalent, though there is a slight asymmetry due to the fact that the maps in the ping-pong game are not assumed to be contractions.

LEMMA 2.16 *There exists a resilient point x for φ if and only if there exists a contracting ping-pong game for φ.*

Proof: Let x be a resilient point. Then there exists an open interval $J = (x - \delta, x + \delta)$, a contraction $\gamma\colon J \to J$ a contraction with x as unique fixed-point, and $J \cap \Gamma x \neq \{x\}$. Choose $\tau \in \Gamma$ such $x \neq \tau x \in J$. Choose $\epsilon < \delta$ such that $\tau x \in (x - \epsilon, x + \epsilon)$, and then set $I_0 = [x - \epsilon, x + \epsilon]$. Note that $I_0 \subset J$ is compact, so the assumption $\gamma\colon J \to J$ is a contraction to x implies there exists $n > 0$ with $\gamma^n J \cap \tau\gamma^n J = \emptyset$. Then set $\gamma_1 = \gamma^n$, $\gamma_2 = \tau\gamma^n$, $I_1 = \gamma_1 I_0$, and $I_2 = \gamma_2 I_0$ and we obtain a contracting ping-pong game for φ.

Conversely, let $\{\gamma_1\colon I_0 \to I_1, \gamma_2\colon I_0 \to I_2\}$ be a contracting ping-pong game for φ. Let $x \in I_1$ be the unique fixed-point for $\gamma_1\colon I_0 \to I_1 \subset I_0$ which by hypothesis is contained in the interior of I_0. Note that $\gamma_2 x \in \gamma_2 I_0 \subset I_2$ and $I_1 \cap I_2 = \emptyset$ implies $\gamma_2 x \neq x$. Let J be the interior of I_0, let γ be the restriction of γ_1 to J, and let τ be the restriction of γ_2 to J. Then these maps satisfy the resilient condition for x. □

COROLLARY 2.17 *Let φ be C^1-action. There exists a hyperbolic resilient point x for φ if and only if there exists a hyperbolic ping-pong game for φ.*

For the general case, there is an implication only in one direction.

LEMMA 2.18 *If φ has a ping-pong game, then it has a one-sided resilient point.*

Proof: Let $\{\gamma_1\colon I_0 \to I_1, \gamma_2\colon I_0 \to I_2\}$ be a ping-pong game for φ. Note that $\gamma_1 I_0$ and $\gamma_2 I_0$ are connected subintervals of I_0. We assume, without loss of generality, that $\gamma_1 I_0$ lies to the left of $\gamma_2 I_0$. Let $K \subset I_0$ be the invariant set for γ_1 defined by

$$K = \bigcap_{n=1}^{\infty} \gamma_1^n I_0$$

Then $K \subset I_1$ is a proper compact subset of $I_0 = [a, b]$ hence $K \subset [a, b)$. We set $x = \sup K$, then x is a fixed-point for the one-sided contraction $\gamma_1\colon [x, b) \to [x, b)$. Set $J = [x, b)$.

Note that $\gamma_2 x \in \gamma_2 I_0 \subset I_2$ and $I_1 \cap I_2 = \emptyset$ implies $\gamma_2 x \subset (x, b)$. Let γ be the restriction of γ_1 to J, and let τ be the restriction of γ_2 to J. □

3 Expansive actions on local minimal sets

PROPOSITION 3.1 *Let $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be an expansive C^0-action with an open local minimal set U. Then there exists a ping-pong game $\{\gamma_1: K_0 \to K_1, \gamma_2: K_0 \to K_2\}$ for φ acting on U.*

Proof: Let $\epsilon > 0$ be an expansive constant for φ. Set $\delta = \epsilon/10$.

Given points $x, y \in \mathbb{S}^1$, we let $[x, y] \subset \mathbb{S}^1$ denote the closed interval defined by the counter-clockwise path in $\mathbb{S}^1$ from x to y. Let (x, y) denote the interior of this interval, and let $|y - x| = |[b, a]|$ denote its length. If $|y - x| \leq \pi$ then $d(x, y) = |y - x|$; otherwise $d(x, y) = 2\pi - |y - x|$, and we always have that $d(x, y) \leq |y - x|$.

If the open local minimal set U is not all of $\mathbb{S}^1$, then it is a countable disjoint union of open intervals,

$$U = \bigcup_{n=1}^{\infty} (a_n, b_n)$$

where each (a_n, b_n) is a gap of $\mathbb{S}^1 - U$. Without loss of generality, we can assume that the intervals are in decreasing order by length, so that $n > m$ implies $|b_n - a_n| \leq |b_m - a_m|$. In particular, there is some $N_\epsilon \geq 0$ so that $n \leq N_\epsilon$ implies $|b_n - a_n| \geq \epsilon$, while $n > N_\epsilon$ implies $|b_n - a_n| < \epsilon$.

For any $\gamma \in \Gamma$ the action of $\varphi(\gamma)$ maps U to itself, so permutes the gaps $\{(a_n, b_n) \mid n = 1, 2, \ldots\}$.

As φ is ϵ-expansive, given any distinct pair of points $x, y \in (a_n, b_n)$ there exists γ with $|\gamma y - \gamma x| > \epsilon$ and hence $|\gamma b_n - \gamma a_n| > \epsilon$. That is, the length of the gap $(\gamma a_n, \gamma b_n)$ must be greater than ϵ. In particular, as there are gaps of length greater than ϵ, we see that $|b_1 - a_1| > \epsilon = 10\,\delta$.

Now, if $U = \mathbb{S}^1$, select any point $x_1 \in \mathbb{S}^1$. Otherwise, with notation as above, let $x_1 \in (a_1, b_1)$ be the midpoint. Let $y_1 \in (a_1, x_1)$ and $z_1 \in (x_1, b_1)$ be the points with $|x_1 - y_1| = |z_1 - x_1| = \delta/4$. Choose $\gamma_1 \in \Gamma$ with $|\gamma_1 z_1 - \gamma_1 y_1| > \epsilon$. Let $J_1 = [y_1, z_1]$, and $I_1 = \gamma_1 J_1$ so that $|I_1| > \epsilon$.

Continue to define a sequence of intervals and maps inductively: Assume that for $1 \leq i < n$, we have chosen 6–tuples $\{x_i, y_i, z_i, \gamma_i, J_i, I_i\}$ so that

- $J_i = [y_i, z_i] \subset U$, with x_i the midpoint of J_i and $|J_i| = \delta/2^i$
- $I_i = \gamma_i(J_i) = [\gamma_i y_i, \gamma_i z_i]$ with $|I_i| > \epsilon$

We then select a new 6–tuple $\{x_n, y_n, z_n, \gamma_n, J_n, I_n\}$ satisfying these conditions for $i = n$.

Let x_n be the midpoint of I_{n-1} and let

$$y_n \in (\gamma_{n-1} y_{n-1}, x_n) \quad , \quad z_n \in (x_n, \gamma_{n-1} z_{n-1})$$

be the points with $|x_n - y_n| = |z_n - x_n| = \delta/2^{n+1}$. Choose $\gamma_n \in \Gamma$ with $|\gamma_n z_n - \gamma_n y_n| > \epsilon$. Then set $J_n = [y_n, z_n] \subset I_{n-1} \subset U$ and let $I_n = \gamma_n J_n$.

For each $n > 0$, there is an index i_n so that the interval $I_n \subset (a_{i_n}, b_{i_n})$. As I_n has length at least ϵ, the length $|b_{i_n} - a_{i_n}| > \epsilon$, hence $i_n \leq N_\epsilon$. By choice, $J_1 \subset (a_1, b_1)$, and $J_\ell \subset I_{\ell-1}$ for all $\ell > 1$, hence the domains and ranges of all the elements γ_n chosen above are contained in a finite union of open intervals. After suitable preliminaries, we will construct two new maps whose domains and ranges are contained in the same open interval, and moreover define a ping-pong game there.

Let x_* be an accumulation point for the set of midpoints $\{x_1, x_2, \ldots\}$. Note that since $x_n \in I_{n-1}$ is the midpoint, and $|I_{n-1}| > \epsilon$, the distance from each x_n to the complement $\mathbb{S}^1 - U$ is bounded below by $\epsilon/2$. Thus, the distance from x_* to $\mathbb{S}^1 - U$ is also bounded below by $\epsilon/2$, so $x_* \in U$. Also, as $\epsilon/2 = 5\delta$ we have that $(x_* - 5\delta, x_* + 5\delta) \subset U$.

We now recall that U is a local minimal set, so that U is contained in the closure of the orbit of any point in U. In particular, the orbit of x_* is dense in U, so there exists $\gamma_* \in \Gamma$ such that $\gamma_* x_* \in (x_* + 3\delta, x_* + 4\delta)$ and hence $3\delta < d(x_*, \gamma_* x_*) < 4\delta$.

Choose $0 < \delta_1 < \delta/2$ such that for the closed interval $W = \{w \in \mathbb{S}^1 \mid d(x_*, w) \leq \delta_1\}$, we have $\gamma_* W \subset \{w \in \mathbb{S}^1 \mid d(\gamma_* x_*, w) < \delta/2\}$. Then both W and its image $\gamma_* W$ have diameter less than δ, and the assumption $3\delta < d(x_*, \gamma_* x_*) < 4\delta$ implies that $W \cap \gamma_* W = \emptyset$.

Choose $p > 0$ so that $\delta/2^p < \delta_1/2$ and $d(x_*, x_p) < \delta_1/2$. Then

$$J_p = [x_p - \delta/2^p, x_p + \delta/2^p] \subset [x_p - \delta_1/2, x_p + \delta_1/2] \subset [x_* - \delta_1, x_* + \delta_1] = W$$

Next choose $q > p$ so that $d(x_*, x_q) < \delta_1/2$. Then $\delta/2^q < \delta/2^p < \delta_1/2$, so we also have that $J_q \subset W$. Since $W \cap \gamma_* W = \emptyset$, it follows that $J_q \cap \gamma_* J_p = \emptyset$.

There is one more step in the preparation for playing ping-pong. For each $p \leq n \leq q$ we have maps $\gamma_n \colon J_n \to I_n$ where the domain J_n of γ_n is contained in the range I_{n-1} of the previous map. Thus, the composition of their inverses is well-defined,

$$(\gamma_q \circ \cdots \circ \gamma_p)^{-1} = \gamma_p^{-1} \circ \cdots \circ \gamma_q^{-1} \colon I_q \to J_p$$

Moreover, $J_p \subset W$ so the range of this map is contained in W. Now set

$$\gamma_1 = (\gamma_q \circ \cdots \circ \gamma_p)^{-1} \quad \text{and} \quad \gamma_2 = \gamma_* \cdot (\gamma_q \circ \cdots \circ \gamma_p)^{-1}$$

Now set $K_0 = I_q$ and $K_1 = \gamma_1(I_q) \subset J_p \subset W$. We set $K_2 = \gamma_* K_1$ and so $K_1 \cap K_2 = \emptyset$.

Then $(\gamma_1 \colon K_0 \to K_1, \gamma_2 \colon K_0 \to K_2)$ is a ping-pong game for φ. □

The proof of Theorem 1.2 follows immediately from this result. It is given that $\varphi \colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is an expansive C^0–action. By Proposition 2.10, the action φ has a non-empty open local minimal set. Then by Proposition 3.1, there exists a ping-pong game $\{\gamma_1 \colon K_0 \to K_1, \gamma_2 \colon K_0 \to K_2\}$ for φ. Lemma 2.13 implies that then $\{\gamma_1, \gamma_2\}$ generates a free semigroup in Γ, and Lemma 2.14 implies that $h(\varphi) > 0$.

Note that Example 8.1 describes a real analytic, expansive action of a solvable group on the circle. Thus, the conclusion above that there is a free sub-semigroup of Γ on two generators is best possible. This example also has a unique minimal set which is a fixed-point for the action of Γ, so the ping-pong game constructed in the above proof need not be contained in a minimal set for φ. In section 8, we consider the case where the action has a ping-pong game that is contained in a minimal set.

4 Infinitesimal expansion and hyperbolic fixed-points

In the next three sections, we consider a C^1-action $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$, and examine the relationships between expansiveness, "Lyapunov exponents", hyperbolic periodic points and positive entropy for the action φ. First, we introduce the set of infinitesimally expansive points $E(\varphi)$, which plays the role of the Pesin set for partially hyperbolic diffeomorphisms (cf. [1, 31, 38, 40]). We also introduce the set of hyperbolic fixed-points $\mathcal{P}^h(\varphi)$, and show that each point $x \in E(\varphi)$ is the limit of hyperbolic periodic points. We use these results to relate properties of the set $E(\varphi)$ with expansiveness of φ and the geometric entropy of φ.

Recall that $\{\sigma_0, \ldots, \sigma_k\}$ is a symmetric generating set for Γ. Define

$$\|\varphi\| = \max_{0 \leq i \leq k} \max_{x \in \mathbb{S}^1} \left|\log\{\varphi(\sigma_i)'(x)\}\right|$$

Choose $\delta = \delta(\epsilon) > 0$ so that if $d(y, z) < \delta$ then

$$|\log\{\varphi(\sigma_i)'(y)\} - \log\{\varphi(\sigma_i)'(z)\}| < \epsilon \quad \text{for all} \quad 0 \leq i \leq k \tag{2}$$

The "modulus of continuity" function $\delta(\epsilon)$ depends on the choice of generating set for Γ and on the action φ. This function will be assumed given throughout this section.

For $x \in \mathbb{S}^1$ and each integer $N \geq 1$, define the function

$$\mu_N(x) = \max\{\varphi'(\gamma)(x) \mid \|\gamma\| \leq N\} \tag{3}$$

The identity transformation has derivative 1, so $\mu_N(x) \geq 1$. The function $\mu_N\colon \mathbb{S}^1 \to \mathbb{R}$ is the maximum of a finite set of continuous functions, so is continuous.

DEFINITION 4.1 *Let φ be a C^1-action. A point $x \in \mathbb{S}^1$ is* infinitesimally expansive *for φ if*

$$\lambda(x) \;=\; \limsup_{N\to\infty} \left\{\frac{\log\{\mu_N(x)\}}{N}\right\} \;=\; \limsup_{||\gamma||\to\infty} \left\{\frac{\log\{\varphi'(\gamma)(x)\}}{||\gamma||}\right\} \;>\; 0 \tag{4}$$

The number $\lambda(x)$ is called the *asymptotic exponent* for φ at x. Note that the condition $\lambda(x) > \epsilon$ simply means that there is some sequence of elements $\gamma_n \in \Gamma$ such that $\varphi'(\gamma_n)(x) > \exp(\epsilon \cdot ||\gamma_n||)$ and $||\gamma_n|| \to \infty$. However, we cannot assume that the $\{\gamma_n\}$ are the powers γ_0^n of single hyperbolic element γ_0 as happens in the classical case where Γ is cyclic.

LEMMA 4.2 *The function $\lambda\colon \mathbb{S}^1 \to \mathbb{R}$ is Borel measurable, and invariant under the action of Γ.*

Proof: The function $x \mapsto \lambda(x)$ is a limit of continuous functions, hence is Borel.

Let $x \in \mathbb{S}^1$ and $\sigma \in \Gamma$. Then

$$\begin{aligned}
\lambda(\sigma x) &= \limsup_{\|\gamma\|\to\infty} \frac{\log\{\varphi(\gamma)'(\sigma x)\}}{\|\gamma\|} \\
&= \limsup_{\|\gamma\|\to\infty} \frac{\log\{\varphi(\gamma\sigma)'(x)\} - \log\{\varphi(\sigma^{-1})'(\sigma x)\}}{\|\gamma\sigma\|} \cdot \frac{\|\gamma\sigma\|}{\|\gamma\|} \\
&= \limsup_{\|\gamma\sigma\|\to\infty} \frac{\log\{\varphi(\gamma\sigma)'(x)\}}{\|\gamma\sigma\|} \\
&= \lambda(x) \qquad \square
\end{aligned}$$

DEFINITION 4.3 *For each $a \geq 0$, $E_a(\varphi) = \{x \in \mathbb{S}^1 \mid \lambda(x) > a\}$ and $E(\varphi) = \bigcup_{a>0} E_a(\varphi)$.*

COROLLARY 4.4 *For each $a \geq 0$, $E_a(\varphi)$ is a Borel measurable, Γ-invariant subset.*

Next consider the fixed-points for individual elements $\gamma \in \Gamma$ and their relationship to $E(\varphi)$.

DEFINITION 4.5 *A point $x \in \mathbb{S}^1$ is a* hyperbolic fixed-point *for φ if there exists $\gamma \in \Gamma$ with $\varphi(\gamma)(x) = x$ and $0 < \varphi(\gamma)'(x) < 1$. Let $\mathcal{P}^h(\varphi)$ denote the set of all hyperbolic fixed-points for φ.*

For $a \geq 0$, we say a point $x \in \mathcal{P}^h(\varphi)$ has *exponent greater than* a if there exists γ with $\gamma x = x$ and $\varphi(\gamma)'(x) > \exp\{a \cdot \|\gamma\|\}$. Let $\mathcal{P}^h_a(\varphi)$ denote the set of all hyperbolic fixed-points with exponent greater than a. Clearly, $\mathcal{P}^h(\varphi) \subset E(\varphi)$ and $\mathcal{P}^h_a(\varphi) \subset E_a(\varphi)$ for all $a > 0$. We also have:

LEMMA 4.6 *$\mathcal{P}^h_a(\varphi)$ is Γ-invariant.*

Proof: Let $x \in \mathcal{P}^h_a(\varphi)$, $\gamma \in \Gamma$ with $\gamma x = x$ and $\varphi(\gamma)'(x) > \exp\{a \cdot \|\gamma\|\}$, so $\log\{\varphi(\gamma)'(x)\} > a \cdot \|\gamma\|$. Given $\sigma \in \Gamma$, choose $n \gg 0$ such that $\log\{\varphi(\gamma)'(x)\} > a \cdot (\|\gamma\| + 2\|\sigma\|/n)$. Then note that $\|\sigma \cdot \gamma^n \cdot \sigma^{-1}\| \leq n\|\gamma\| + 2\|\sigma\|$, and

$$\log\{\varphi(\sigma \cdot \gamma^n \cdot \sigma^{-1})'(\sigma x)\} = n \log\{\varphi(\gamma)'(x)\}$$

hence $\log\{\varphi(\sigma \cdot \gamma^n \cdot \sigma^{-1})'(\sigma x)\} > a \cdot \|\sigma \cdot \gamma^n \cdot \sigma^{-1}\|$. □

The hyperbolic fixed points for each power $\varphi(\gamma^n)$ are isolated, hence the hyperbolic periodic points for $\varphi(\gamma^n)$ are at most countably infinite. As Γ is countable, the set $\mathcal{P}^h(\varphi)$ is countable.

PROPOSITION 4.7 *For $a > 0$, $\mathcal{P}^h_a(\varphi)$ is dense in $E_a(\varphi)$.*

Proof: Let $x \in E_a(\varphi)$, set $\lambda = \lambda(x)$, and choose $0 < \epsilon < \lambda - a$, and set $\epsilon_0 = \epsilon/10$. Then

$$a < a + \epsilon_0 < a + 9\epsilon_0 < \lambda - \epsilon_0 < \lambda$$

In particular, $\lambda - \epsilon_0 > a + \epsilon_0 > \epsilon_0$. Let $\delta = \delta(\epsilon_0)$ so that (2) holds for $\epsilon/10$.

Choose a sequence $\gamma_n \in \Gamma$ such that $\|\gamma_n\| \to \infty$ and $\log\{\varphi(\gamma_n)'(x)\} > (\lambda - \epsilon_0) \cdot \|\gamma_n\|$ for $n > 0$.

Our goal is to construct a sequence of hyperbolic contraction mappings with ranges contained in their domains, which will give rise to a sequence of hyperbolic fixed-points $\{x_n\}$, that we then show limit to x. This is accomplished is several steps. We first consider the inverse maps $\varphi(\gamma_n^{-1})$ which are infinitesimal contractions, and modify the mappings $\varphi(\gamma_n^{-1})$ so that they satisfy uniform estimates for their contraction rates along the orbit of a point. The idea is to truncate the "initial factors" in each map $\varphi(\gamma_n^{-1})$ to remove any initial expansive behavior. This allows us to obtain uniform estimates on the contraction rates of each $\varphi(\gamma_n^{-1})$ which is used to obtain the sequence of hyperbolic fixed-points $\{x_n\}$. Though the idea is simple, the process is very technical.

Fix $n > 0$, let $\ell_n = \|\gamma_n\|$ and choose indices $\{i_1, \ldots, i_{\ell_n}\}$ so that $\gamma_n^{-1} = \sigma_{i_{\ell_n}} \cdots \sigma_{i_1}$.

For $1 \leq j \leq \ell_n$ set $\gamma_{n,j}^{-1} = \sigma_{i_j} \cdots \sigma_{i_1}$ and let $\gamma_{n,0}^{-1}$ denote the identity.

Set $z_n = \varphi(\gamma_n)(x)$ and label the orbit of z_n by $z_{n,j} = \varphi(\gamma_{n,j}^{-1})(z_n)$, so $z_{n,\ell_n} = x$.

For each $1 \leq j \leq \ell_n$ set $\mu_{n,j} = \log\{\varphi(\sigma_{i_j})'(z_{n,j-1})\}$. Then $\lambda - \epsilon_0 > \epsilon_0$ yields

$$\mu_{n,1} + \cdots + \mu_{n,\ell_n} = \log\{\varphi(\gamma_n^{-1})'(z_n)\} < \ell_n \cdot (\epsilon_0 - \lambda) < -\ell_n \cdot \epsilon_0 \tag{5}$$

An index $1 \leq j \leq \ell_n$ is said to be ϵ_0-*regular* if all of the partial sum estimates hold:

$$\begin{aligned} \mu_{n,j} + \epsilon_0 &< 0 \\ \mu_{n,j} + \mu_{n,j+1} + 2\epsilon_0 &< 0 \\ &\vdots \\ \mu_{n,j} + \cdots + \mu_{n,\ell_n} + (\ell_n - j + 1)\epsilon_0 &< 0 \end{aligned} \tag{6}$$

LEMMA 4.8 *There exists an ϵ_0-regular index $0 < b_n < \ell_n$ for γ_n^{-1} such that*

$$\mu_{n,b_n} + \cdots + \mu_{n,\ell_n} < \ell_n(2\epsilon_0 - \lambda) \tag{7}$$

Proof: We say that an index $k \leq \ell_n$ is ϵ_0-*irregular* if $\mu_{n,1} + \cdots + \mu_{n,k} + k\epsilon_0 \geq 0$. If there does not exist an ϵ_0-irregular value, then this means that all of the conditions (6) hold starting at $j = 1$, hence $j = 1$ is ϵ_0-regular, and we set $b_n = 1$.

On the other hand, if there exists an ϵ-irregular value k, then (5) implies $k < \ell_n$.

Let k_0 be the greatest ϵ_0-irregular index. Then $k_0 + 1$ is an ϵ_0-regular index. To see this, let $\ell > k_0$, then we have $\mu_{n,1} + \cdots + \mu_{n,\ell} + \ell\epsilon_0 < 0$ and so

$$\begin{aligned} \mu_{n,j_0+1} + \cdots + \mu_{n,\ell} + (\ell - j_0)\epsilon_0 &= (\mu_{n,1} + \cdots + \mu_{n,\ell} + \ell\epsilon_0) - (\mu_{n,1} + \cdots + \mu_{n,k_0} + k_0\epsilon_0) \\ &\leq (\mu_{n,1} + \cdots + \mu_{n,\ell} + \ell\epsilon_0) < 0 \end{aligned}$$

It remains to establish the estimate (7). If $b_n = 1$ then by (5)

$$\mu_{n,1} + \cdots + \mu_{n,\ell_n} < \ell_n(\epsilon_0 - \lambda) < \ell_n(2\epsilon_0 - \lambda) \tag{8}$$

If $b_n > 1$ where $k = b_n - 1$ is ϵ_0-irregular, then by (5) and the definition of ϵ_0-irregular we have

$$\begin{aligned} \mu_{n,k_0+1} + \cdots + \mu_{n,\ell_n} + (\ell - k_0)\,\epsilon_0 &= (\mu_{n,1} + \cdots + \mu_{n,\ell_n} + \ell_n\,\epsilon_0) - (\mu_{n,1} + \cdots + \mu_{n,k_0} + k_0\,\epsilon_0) \\ &\leq (\mu_{n,1} + \cdots + \mu_{n,\ell_n} + \ell_n\,\epsilon_0) \\ &\leq \ell_n(\epsilon_0 - \lambda) + \ell_n\,\epsilon_0 \\ &= \ell_n(2\epsilon_0 - \lambda) \end{aligned}$$

as was to be shown. $\square$

For each $n \geq 1$ let $b_n = k_n + 1$ be an ϵ_0-regular index for γ_n^{-1} satisfying (7). Define

$$\tau_n = \sigma_{i_{\ell_n}} \cdots \sigma_{i_{b_n}}\ ,\ \omega_n = \tau_n^{-1}\ ,\ \mathbf{h}_n = \varphi(\tau_n)\ ,\ \mathbf{g}_n = \varphi(\omega_n) = \mathbf{h}_n^{-1} \tag{9}$$

Note that $\|\omega_n\| = \|\tau_n\| = \ell_n - k_n \leq \|\gamma_n\|$. Set $y_n = \mathbf{g}_n(x)$, so $\mathbf{h}_n(y_n) = x$. By passing to a subsequence if necessary, we can assume that there is a limit point $y_* \in \mathbb{S}^1$ such that $y_n \to y_*$ and $d(y_*, y_n) < \delta/4$ for all $n > 0$.

LEMMA 4.9 *The word length* $\|\tau_n\| = \ell_n - k_n$ *tends to infinity as* $n \to \infty$. *More precisely, there is an estimate*

$$\ell_n - k_n \geq \ell_n(\lambda - 2\epsilon_0)/\|\varphi\| \tag{10}$$

Proof: By the definition of $\|\varphi\|$ we have $|\log\{\varphi(\sigma_i)'\}| \leq \|\varphi\|$ hence

$$|\log\{\mathbf{h}'_n\}| \leq (\ell_n - k_n) \cdot \|\varphi\|$$

Note that $\lambda > 0$ implies $\|\varphi\| > 0$. Then evaluate $\mathbf{h}'_n$ at y_n and apply (7) to obtain (10). $\square$

The purpose of introducing the concept of the ϵ_0-regular value is that it implies each of the maps $\varphi(\sigma_{i_i} \cdots \sigma_{i_{b_n}})$ for $i \geq i_{b_n}$ is a sufficiently strong linear contraction at y_n so that the map $\mathbf{h}_n$ is a uniform contraction on the interval $[y_n - \delta/2, y_n + \delta/2]$. We make this precise in Lemma 4.10 below.

Set $\mathcal{J}_n = [y_n - \delta/2, y_n + \delta/2]$ and $\mathcal{I}_n = \mathbf{h}_n(\mathcal{J}_n)$.

LEMMA 4.10 *For each* $n > 0$ *and* $z \in \mathcal{J}_n$

$$\exp\{-\|\varphi\|\|\tau_n\|\} \leq \mathbf{h}'_n(z) \leq \exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\} \tag{11}$$

$$\delta \exp\{-\|\varphi\|\|\tau_n\|\} \leq |\mathbf{h}_n(\mathcal{J}_n)| \leq \delta \exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\} \tag{12}$$

Proof: Fix n and let $\tau_n = \sigma_{i_{\ell_n}} \cdots \sigma_{i_{b_n}}$ be as above.

The chain rule yields the lower bound estimate $-\|\varphi\|\|\tau_n\| \leq \log\{\mathbf{h}'_n(z)\}$ for all $z \in \mathbb{S}^1$, so in particular for $z \in \mathcal{J}_n$. This implies the lower bound estimate in (11).

For $z \in \mathcal{J}_n$, the uniform continuity hypothesis (2) yields

$$\left|\log\{\varphi(\sigma_{i_{b_n}})'(z)\} - \log\{\varphi(\sigma_{i_{b_n}})'(y_n)\}\right| \leq \epsilon_0$$

Thus, by the definition of μ_{n,b_n} we have for all $z \in \mathcal{J}_n$

$$\exp\{-\epsilon_0 + \mu_{n,b_n}\} \leq \varphi(\sigma_{i_{b_n}})'(z) \leq \exp\{\epsilon_0 + \mu_{n,b_n}\} \tag{13}$$

As $|\mathcal{J}_n| = \delta$ this implies that

$$\delta \exp\{-\epsilon_0 + \mu_{n,b_n}\} \leq |\varphi(\sigma_{i_{b_n}})(\mathcal{J}_n)| \leq \delta \exp\{\epsilon_0 + \mu_{n,b_n}\}$$

By the assumption that b_n is ϵ_0-regular, we have that $\epsilon_0 + \mu_{n,b_n} < 0$ hence $\delta \exp\{\epsilon_0 + \mu_{n,b_n}\} < \delta$.

For $z_1 = \varphi(\sigma_{i_{b_n}})(z) \in \varphi(\sigma_{i_{b_n}})(\mathcal{J}_n)$ we estimate $\varphi(\sigma_{i_{b_n+1}})'(z_1)$ as above to obtain the estimates

$$\exp\{-2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\} \leq \varphi(\sigma_{i_{b_n+1}} \cdot \sigma_{i_{b_n}})'(z) \leq \exp\{2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\}$$

$$\delta \exp\{-2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\} \leq |\varphi(\sigma_{i_{b_n+1}} \cdot \sigma_{i_{b_n}})(\mathcal{J}_n)| \leq \delta \exp\{2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\}$$

Then b_n is ϵ_0-regular implies that $2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1} < 0$, hence $\delta \exp\{2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\} < \delta$.

Repeat this argument $\ell_n - b_n + 1$ times and use (25) to arrive at the upper bound

$$\begin{aligned}
\mathbf{h}'_n(z) &\leq \exp\{(\ell_n - b_n + 1)\epsilon_0 + \mu_{n,\ell_n} + \cdots + \mu_{n,b_n}\} \\
&\leq \exp\{\epsilon_0\|\tau_n\| - (\lambda - 2\epsilon_0)\|\tau_n\|\} \\
&< \exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\}
\end{aligned}$$

The estimate (12) follows immediately from (11). □

We now complete the proof of Proposition 4.7. We have defined intervals $\mathcal{J}_n = [y_n - \delta/2, y_n + \delta/2]$ of uniform length δ on which there are hyperbolic contractions $\mathbf{h}_n\colon \mathcal{J}_n \to \mathcal{I}_n$. While all of the intervals $\mathcal{I}_n$ contain the common point x, the domains are not *a priori* controlled. However, compactness was used to select a subsequence such that $y_n \to y_*$ with $d(y_*, y_n) < \delta/4$ for all $n > 0$, hence the domains $\mathcal{J}_n$ are forced to overlap. This forces the existence of fixed points.

Introduce $\mathcal{J}_* = [y_* - \delta/4, y_* + \delta/4]$, then for all $n > 0$, we have that $y_n \in \mathcal{J}_* \subset \mathcal{J}_n$.

The point $y_1 \in \mathcal{J}_*$ is in the interior, and $x = \mathbf{h}_1(y_1)$, so we can choose $\delta_1 > 0$ such that

$$[x - 2\delta_1, x + 2\delta_1] \subset \mathbf{h}_1(\mathcal{J}_*) \tag{14}$$

Choose $n > 0$ so that $\delta \exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\} < \delta_1$ and $\|\tau_n\| \geq \|\varphi\|\|\tau_1\|/\epsilon_0$ which implies

$$(3\epsilon_0 - \lambda)\|\tau_n\| + \|\varphi\|\|\tau_1\| \leq (4\epsilon_0 - \lambda)\|\tau_n\| \tag{15}$$

Then as $x = \mathbf{h}_n(y_n)$ for all $y \in \mathcal{J}_n$ we have

$$d(\mathbf{h}_n(y), x) = d(\mathbf{h}_n(y), \mathbf{h}_n(y_n)) \leq |\mathbf{h}_n(\mathcal{J}_n)| < \delta \exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\} < \delta_1 \tag{16}$$

Since $\mathcal{J}_* \subset \mathcal{J}_n$ by (14) we have

$$\mathbf{h}_n(\mathcal{J}_*) \subset \mathbf{h}_n(\mathcal{J}_n) \subset \mathbf{h}_1(\mathcal{J}_*)$$

Set $\mathbf{g}_n = \mathbf{h}_1^{-1} \circ \mathbf{h}_n$, so that $\mathbf{g}_n(\mathcal{J}_*) \subset \mathcal{J}_*$. Define $\mathcal{I}_1 = \mathcal{J}_*$ and $\mathcal{I}_n = \mathbf{g}_n(\mathcal{J}_n)$, then $\mathbf{g}_n\colon \mathcal{I}_1 \to \mathcal{I}_n \subset \mathcal{I}_1$. Thus, $\mathbf{g}_n$ has a fixed point $x_n \in \mathcal{I}_n$ which by (16) satisfies $d(\mathbf{h}_1(x_n), x) < \delta \exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\}$.

Then calculate

$$d(\tau_1^{-1}x, x_n) < \delta \exp\{(3\epsilon_0 - \lambda)\|\tau_n\| \ + \|\varphi\|\|\tau_1\|\}$$

which tends to 0 as $n \to \infty$. Set $\tau_n^* = \tau_1^{-1} \cdot \tau_n$, then $\mathbf{g}_n = \varphi(\tau_n^*)$ and it remains to estimate

$$\begin{aligned} \mathbf{g}_n'(x_n) &= \varphi(\tau_1^{-1})'(\tau_n x_n) \cdot \varphi(\tau_n)'(x_n) \\ &\leq \exp\{\|\varphi\|\|\tau_1\|\} \cdot \exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\} \\ &\leq \exp\{(4\epsilon_0 - \lambda)\|\tau_n\|\} \\ &< \exp\{-a\|\tau_n\|\} \qquad \square \end{aligned}$$

Note that while the proof of Proposition 4.7 shows that each point $x \in E(\varphi)$ is the limit of hyperbolic contractions $\mathbf{g}_n$ with fixed-points $x_n \to x$, it may happen that $x_n = x$ for all n. There is nothing in the proof which implies the fixed-points x_n are distinct from the original point x. For example, this is the case when $\mathcal{P}_a^h(\varphi) = E_a(\varphi)$ is discrete, as illustrated by Example 8.3.

The hyperbolic contractions $\mathbf{g}_n\colon \mathcal{I}_1 \to \mathcal{I}_1$ constructed in the above proof all have domain $\mathcal{I}_1$ which is an interval of constant width $\delta/2$. The choice of δ depends on the choice of $x \in E_a(\varphi)$, to the extent that $x \in E_a(\varphi)$ determines the exponent $\lambda > a$ of the action at x, and we then choose $\epsilon > 0$ to satisfy $0 < \epsilon < \lambda - a$. Given $\epsilon_0 = \epsilon/10$ then δ is chosen based on the modulus of continuity for φ needed to obtain ϵ_0–control. This produces a contraction with exponent greater than a. We will show later that if we relax this requirement on the strength of the contractions, so that the exponent in only at least $a/2$, then the choice of δ can be made uniform in x.

Next, we consider the intersection of a minimal set with $E(\varphi)$:

PROPOSITION 4.11 *Let* $\mathbf{K}$ *be a minimal set for* φ*, and suppose that* $x \in \mathbf{K}$ *satisifies* $\lambda(x) > 0$*. Then for all* $0 < a < \lambda(x)$*, the intersection* $\mathcal{P}^h_a(\varphi) \cap \mathbf{K}$ *is dense in* $\mathbf{K}$*. Hence, either* $\mathbf{K}$ *and* $E(\varphi)$ *are disjoint, or the hyperbolic periodic points are dense in* $\mathbf{K}$*.*

Proof: Let $x \in \mathbf{K}$ satisfy $\lambda(x) > 0$. Set $\lambda = \lambda(x)$ and choose $0 < a < \lambda$. As in the proof of Proposition 4.7, there exists

- $y_* \in \mathbb{S}^1$, $\delta > 0$ and interval $\mathcal{I}_1 = \mathcal{J}_* = [y_* - \delta/4, y_* + \delta/4]$
- $\tau_1 \in \Gamma$ such that $\tau_1^{-1}x \in \mathcal{I}_1$
- for each $n > 0$, $\tau_n \in \Gamma$ with $\|\tau_n\| \to \infty$ such that
- for $\tau_n^* = \tau_1^{-1} \cdot \tau_n$, $\mathbf{g}_n = \varphi(\tau_n^*) : \mathcal{I}_1 \to \mathcal{I}_1$ is a hyperbolic contraction with
- fixed point $x_n \in (y_* - \delta/4, y_* + \delta/4) \subset \mathcal{I}_1$ such that $\mathbf{g}_n'(x_n) < \exp\{(4\epsilon_0 - \lambda)\|\tau_n\|\}$

As $4\epsilon_0 - \lambda < -a$ and $\|\tau_n^*\| \leq \|\tau_1\| + \|\tau_n\|$, for $n \gg 0$ we have that $(4\epsilon_0 - \lambda)\|\tau_n\| \leq -a\|\tau_n^*\|$, and hence $x_n \in \mathcal{P}^h_a(\varphi)$. Moreover, as $\mathbf{g}_n$ is a contraction on $\mathcal{I}_1$ with fixed-point x_n, we have

$$d(x_n, \mathbf{g}_n^i(\tau_1^{-1}x)) = d(\mathbf{g}_n^i(x_n), \mathbf{g}_n^i(\tau_1^{-1}x)) \to 0 \quad \text{as} \quad i \to \infty \tag{17}$$

As $\mathbf{K}$ is Γ-invariant and $x \in \mathbf{K}$, we see that $\mathbf{g}_n^i(\tau_1^{-1}x) \in \mathbf{K}$ for all i. Thus, (17) implies x_n is in the closure of $\mathbf{K}$, hence $x_n \in \mathbf{K}$. This shows that $x_n \in \mathcal{P}^h_a(\varphi) \cap \mathbf{K}$. The orbit of x_n under Γ is dense in $\mathbf{K}$, so $\mathcal{P}^h_a(\varphi) \cap \mathbf{K}$ is dense in $\mathbf{K}$. □

The next result uses a simple observation that if we fix $a > 0$ and require only that the map we construct has expansion rate $a/2 > 0$, then we can choose the width δ of the domain of the contractions uniformly for all $x \in \mathcal{P}^h_a(\varphi)$. Given a uniform choice of δ one can then show that the action of φ is expansive on $\mathcal{P}^h_a(\varphi)$. This is the technical idea underlying the proof of the next result.

PROPOSITION 4.12 *For* $a > 0$*,* φ *is expansive on* $\mathcal{P}^h_a(\varphi)$*.*

Proof: Set $\epsilon_0 = a/10$ and $\delta = \delta(\epsilon_0)$.
Given $x \in \mathcal{P}^h_a(\varphi)$, let $\gamma \in \Gamma$ be such that $\gamma x = x$ and $\varphi(\gamma)'(x) < \exp\{-a\|\gamma\|\}$.
Set $\ell = \|\gamma\|$ and $\lambda = -\log\{\varphi(\gamma)'(x)\}/\ell > a$.
Choose a sequence of generators $\{\sigma_{i_j}\}$ so that $\gamma = \sigma_{i_\ell} \cdots \sigma_{i_1}$.
For $1 \leq j \leq \ell$, set $z_j = \varphi(\sigma_{i_j} \cdots \sigma_{i_1})(x)$ and $z_0 = x$. Set $\mu_j = \log\{\varphi(\sigma_{i_j})'(z_{j-1})\}$.
Then

$$\log\{\varphi(\gamma)'(x)\} = \mu_1 + \cdots + \mu_\ell = -\ell\lambda < -\ell a \tag{18}$$

As in the proof of Proposition 4.7, we need to modify the map $\varphi(\gamma)$ so that it is a contraction with domain of uniform length δ. We first show the existence of what is essentially an ϵ-regular index, as in the proof of Lemma 4.8, but modified to account for the fact that $\varphi(\gamma)$ is periodic at x.

Extend the finite sequence $\{\mu_1, \ldots, \mu_\ell\}$ to an infinite periodic sequence $\{\mu_1, \ldots, \mu_\ell, \mu_1, \ldots, \mu_\ell, \ldots\}$ and similarly extend the indexing of the generators $\{\sigma_{i_1}, \ldots, \sigma_{i_\ell}, \sigma_{i_1}, \ldots, \sigma_{i_\ell}, \ldots\}$. For $j \geq 1$ set $\gamma_j = \sigma_{i_j} \cdots \sigma_{i_1}$, with $\gamma_0 = Id$, and so $\gamma_\ell = \gamma$. Let

$$S_m(n) = \sum_{j=m}^{n} \mu_{i_j} = \mu_{i_m} + \cdots + \mu_{i_n} \tag{19}$$

denote the sum of the terms from m to n, where $1 \le m \le \ell$ and $m \le n < \infty$.

We say m is ϵ_0-good if $S_m(n) < -(n-m+1)\epsilon_0$ for all $n \ge m$. That is, for all $n > m$

$$\varphi(\sigma_{i_n} \cdots \sigma_{i_m})'(z_{m-1}) < \exp\{-(n-m+1) \cdot \epsilon_0\}$$

By (18), we have that $S_1(\ell)/\ell < -a$, hence

$$\lim_{n \to \infty} S_1(n)/n = S_1(\ell)/\ell = -\lambda < -a < -\epsilon_0$$

Thus, there is a greatest n_0 with $S_1(n_0) \ge -n_0\epsilon_0$. Set $m_0 = n_0 + 1$.

We claim that m_0 is ϵ_0-good. If not, then by definition there exists $n \ge m_0$ with

$$S_{m_0}(n) \ge -(n - m_0 + 1)\epsilon_0$$

and given that $m_0 = n_0 + 1$ we have

$$\mu_{n_0+1} + \cdots + \mu_n \ge -(n - n_0)\epsilon_0$$

By the choice of n_0,

$$S_1(n) = S_1(n_0) + S_{m_0}(n) \ge -n_0\epsilon_0 - (n - n_0)\epsilon_0 = -n\epsilon_0$$

contradicting the maximality in the choice of n_0.

Because the extended sequence of μ_i is periodic with period ℓ, $S_m(n) = S_{m-\ell}(n-\ell)$. Hence, if $m_0 > \ell$, then $m_0 - \ell$ is also an ϵ_0-good value. Thus, we can assume $1 \le m_0 \le \ell$.

For each $n > n_0$ set $\tau_n = \gamma_n \cdot \gamma_{n_0}^{-1} = \sigma_{i_n} \cdots \sigma_{i_{m_0}}$. Then $\tau_n \gamma_{n_0} = \gamma_n$ and $\tau_\ell \gamma_{n_0} = \gamma$.

Set $x_0 = \gamma_{n_0} x$, $\mathcal{J}_0 = [x_0 - \delta/2, x_0 + \delta/2]$, and $\mathcal{I} = \varphi(\tau_\ell)(\mathcal{J}_0)$.

Then $\tau_\ell x_0 = \tau_\ell \gamma_{n_0} x = \gamma x = x$ hence $x \in \mathcal{I}$.

LEMMA 4.13 *For all n sufficiently large,*

$$\delta \exp\{-n\ell \cdot \|\varphi\|\} \le |\varphi(\gamma^n)(\mathcal{I})| \le \delta \exp\{n\ell \cdot (3\epsilon_0 - \lambda)\} \tag{20}$$

Proof: The proof is a technical modification of that of Lemma 4.10, so is omitted. □

We now conclude the proof of Proposition 4.12. Note that $\lambda - 3\epsilon_0 > a/2$ as $\lambda > a$ and $\epsilon_0 < a/10$. Thus $\delta \exp\{n\ell \cdot (3\epsilon_0 - \lambda)\} \le \delta \exp\{-n\ell \cdot a/2\}$ so the lengths of the intervals $\mathcal{I}_n = \varphi(\gamma^n)(\mathcal{I})$ tend to zero as $n \to \infty$.

Given a point $y \in \mathcal{P}_a^h(\varphi)$ with $y \ne x$ there exists $n \ge 0$ such that $y \notin \mathcal{I}_n$. Hence,

$$\varphi(\tau_\ell^{-1}\gamma^{-n})(y) \notin \varphi(\tau_\ell^{-1}\gamma^{-n})(\mathcal{I}) = \mathcal{J}_0 \ , \ \varphi(\tau_\ell^{-1}\gamma^{-n})(x) = x_0$$

which implies that x and y can be $\delta/2$-separated. As the choice of δ depended only on a and not on the choice of $x \ne y$ this implies φ is $\delta/2$-expansive on $\mathcal{P}_a^h(\varphi)$. □

We point out one corollary of the above proof that will be used in the next section.

COROLLARY 4.14 *Let $a > 0$ and choose $\delta > 0$ as in the proof of Proposition 4.12. Then for each $x \in \mathcal{P}^h_a(\varphi)$ there exists $\tau_x, \gamma_x \in \Gamma$ such that for $x_0 = \tau_x x$ and $\mathcal{J}_x = [x_0 - \delta/2, x_0 + \delta/2]$, we have $\gamma_x x_0 = x_0$ and $\varphi(\gamma_x): \mathcal{J}_x \to \mathcal{J}_x$ is a hyperbolic contraction.*

Proof: Set $\tau_x = \tau_\ell^{-1} = (\sigma_{i_\ell} \cdots \sigma_{i_{m_0}})^{-1}$ and $\gamma_x = \tau_\ell^{-1} \gamma \tau_\ell = \sigma_{i_{n_0}} \cdots \sigma_{i_1} \sigma_{i_\ell} \cdots \sigma_{i_{m_0}}$. □.

Note that the map $\varphi(\gamma)$ is also a contraction on some interval about x, but the width of the domain of the contraction does not have a uniform estimate, while the contraction $\varphi(\gamma_x)$ has domain $\mathcal{J}_x$ whose length has length δ, where δ depends upon $a > 0$ but not on x or γ.

5 Infinitesimal expansion and entropy

In this section, we give conditions on the set $E(\varphi)$ which are sufficient to imply $h(\varphi) > 0$.

If $E(\varphi)$ is not empty, then Proposition 4.7 implies that $\mathcal{P}^h_a(\varphi)$ is not empty for some $a > 0$. By Proposition 4.12, the action of φ on $\mathcal{P}^h_a(\varphi)$ is expansive. Moreover, by Corollary 4.14 each orbit of Γ in $\mathcal{P}^h_a(\varphi)$ contains a hyperbolic fixed-point point with domain of uniform length δ, where δ depends only on φ and a. These are strong conclusions, but are not sufficient to imply $h(\varphi) > 0$, as it is possible that $E(\varphi)$ consists of a finite number of isolated hyperbolic fixed–points. We formulate two hypothesis which avoid this possibility, and yield the conclusion that the entropy is positive.

We say that $\mathcal{P}^h_a(\varphi)$ has *finite orbit type* if there exists a finite set $\{x_1, \ldots, x_k\} \subset \mathcal{P}^h_a(\varphi)$ whose orbit is all of $\mathcal{P}^h_a(\varphi)$. That is, for all $x \in \mathcal{P}^h_a(\varphi)$, there exists $\gamma \in \Gamma$ and $1 \leq i \leq k$ so that $\gamma x_i = x$. Otherwise, we say that $\mathcal{P}^h_a(\varphi)$ has *infinite orbit type*.

PROPOSITION 5.1 *Suppose that there exists $a > 0$ such that $\mathcal{P}^h_a(\varphi)$ has infinite orbit type. Then there exists a ping-pong game $\{\tau_1: I_0 \to I_1, \tau_2: I_0 \to I_2\}$ for φ, and hence $h(\varphi) > 0$.*

Proof: By hypothesis, there exists $a > 0$ and an infinite set $\{x_1, x_2, \ldots\} \subset \mathcal{P}^h_a(\varphi)$ such that $i \neq k$ implies the orbits satisfy $\Gamma x_i \cap \Gamma x_k = \emptyset$. By Corollary 4.14, there exists $\delta > 0$ so that for each $k > 0$, there exists a hyperbolic contraction $\mathbf{h}_k = \varphi(\gamma_k): \mathcal{J}_k \to \mathcal{J}_k$ with fixed-point $y_k \in \Gamma \cdot x_k$ where $\mathcal{J}_k = [y_k - \delta/2, y_k + \delta/2]$. Moreover, we can assume that the orbits of y_i and y_k are disjoint for $i \neq k$, and in particular $y_i \neq y_k$.

Let y_* be a limit point for the set $\{y_1, y_2, \ldots\}$. Choose $i, k > 0$ sufficiently large so that

$$d(y_*, y_i) < \delta/10 \ , \quad d(y_*, y_k) < \delta/10$$

Then $I_0 = [y_* - \delta/5, y_* + \delta/5] \subset \mathcal{J}_i \cap \mathcal{J}_k$. Choose n sufficiently large so that

$$\mathbf{h}_i^n(I_0) \subset I_0 \ , \ \mathbf{h}_k^n(I_0) \subset I_0 \ , \ \mathbf{h}_i^n(I_0) \cap \mathbf{h}_k^n(I_0) = \emptyset$$

Set $I_1 = \mathbf{h}_i^n(I_0)$, $I_2 = \mathbf{h}_k^n(I_0)$ and $\tau_1 = \gamma_i^k$, $\tau_2 = \gamma_k^n$, then $\{\tau_1: I_0 \to I_1, \tau_2: I_0 \to I_2\}$ is a ping-pong game. By Lemma 2.14, $h(\varphi) > 0$. □

We next give a topological condition on $E(\varphi)$ sufficient to imply that $h(\varphi) > 0$. We say $x \in E_a(\varphi)$ is an *accumulation point* for $E_a(\varphi)$ if for all $\epsilon > 0$ the set $\{y \in E_a(\varphi) \mid d(x, y) < \epsilon\}$ is infinite. Since $E(\varphi)$ is Γ-invariant, if x is an accumulation point then each point γx for $\gamma \in \Gamma$ is also an accumulation point.

A point $x \in X$ is *isolated in X* if there exists $\epsilon > 0$ such that $\{y \in X \mid d(x, y) < \epsilon\} = \{x\}$.

An orbit $\Gamma \cdot x$ is *proper* if each point γx is isolated in the set $\Gamma \cdot x$.

LEMMA 5.2 *If $E_a(\varphi)$ has no accumulation points, then $E_a(\varphi)$ is a countable union of proper orbits.*

Proof: Suppose that each point $x \in E_a(\varphi)$ has an open neighborhood U_x such that $U_x \cap E_a(\varphi)$ is finite. Let $\mathcal{U}$ be the countable collection of open intervals $(a_n - r_n, a_n + r_n)$ where the centers $\{a_n\}$ form a countable dense subset of $\mathbb{S}^1$ and r_n is a rational number. Then for each $y \in U_x \cap E_a(\varphi)$ there is an open set $I_{x,y} \in \mathcal{U}$ with $I_{x,y} \cap E_a(\varphi) = \{y\}$. Thus, $E_a(\varphi)$ is a countable set.

For $x \in E_a(\varphi)$, if $\Gamma \cdot x$ is not proper then x is an accumulation point for $E(\varphi)$ as $\Gamma \cdot x \subset E_a(\varphi)$. Hence, if $E_a(\varphi)$ has no accumulation points, it must be countable, and every orbit is proper. □

In the analogous case of codimension one foliations, a proper orbit corresponds to a proper leaf. The proper leaves can be ordered by their "depth", where compact leaves have depth 0. A leaf L at "infinite level" has proper leaves at all levels in its closure $\overline{L}$. Hector gave a remarkable construction of C^2-foliations with leaves at infinite level in [22]; see also the works of Cantwell and Conlon [4, 5, 6]. While such a leaf exhibits complicated dynamics, it still does not contain a resilient leaf in its closure, and so does not force the foliation to have positive entropy [18].

PROPOSITION 5.3 *Let $a > 0$ and suppose that $E_a(\varphi)$ has an accumulation point. Then there exists a ping-pong game $\{\tau_1 : I_0 \to I_1, \tau_2 : I_0 \to I_2\}$ for φ, and hence $h(\varphi) > 0$.*

Proof: The idea of the proof is to construct an infinite sequence of hyperbolic contractions with domains of uniform length δ, as in the proof of Proposition 4.7. Then either the orbit of the fixed-point for one of the hyperbolic contractions is not isolated, so that we have a resilient orbit, or the domains of the collection of contractions must overlap since $\mathbb{S}^1$ is compact. In either case, we show this yields a ping-pong game. As many of the details below are the same as in the proof of Proposition 4.7, we refer back to that when appropriate.

Assume there is given $a > 0$, x an accumulation point for $E_a(\varphi)$, $0 < \epsilon < a$, $\epsilon_0 = \epsilon/10$ and $\delta = \delta(\epsilon_0)$.

Choose a sequence $\gamma_n \in \Gamma$ such that $\ell_n = \|\gamma_n\| \to \infty$ and $\log\{\varphi(\gamma_n)'(x)\}/\|\gamma_n\| > a - \epsilon_0$ for $n > 0$.

For a each n, choose indices $\{i_1, \ldots, i_{\ell_n}\}$ so that $\gamma_n^{-1} = \sigma_{i_{\ell_n}} \cdots \sigma_{i_1}$. Let b_n be the least ϵ_0-regular index for γ_n^{-1}, and define $\tau_n = \sigma_{i_{\ell_n}} \cdots \sigma_{i_{b_n}}$. Then for $x_n = \varphi(\tau_n)^{-1}(x)$ we have

$$-\|\varphi\| \leq \log\{\varphi(\tau_n)'(x_n)\}/\|\tau_n\| \leq -(a - 2\epsilon_0) \tag{21}$$

By passing to a subsequence if necessary, we can assume that $x_n \to x_*$ and that $d(x_*, x_n) < \delta/100$ for all $n > 0$. Set $\mathcal{J}_n = [x_n - \delta/2, x_n + \delta/2]$. Define $\mathcal{J}_* = [x_* - \delta/4, x_* + \delta/4]$ so that $x_n \in \mathcal{J}_* \subset \mathcal{J}_n$.

We can now use the same proof as for Lemma 4.10 to obtain for each $n > 0$,

$$\delta \exp\{-\|\varphi\|\|\tau_n\|\} \leq |\varphi(\tau_n)(\mathcal{J}_n)| \leq \delta \exp\{-(a - 3\epsilon_0)\|\tau_n\|\}$$

Hence, $|\varphi(\tau_n)(\mathcal{J}_n)| \leq \delta \exp\{-(a/2)\|\tau_n\|\}$ and so for n sufficiently large,

$$\varphi(\tau_n)(\mathcal{J}_1) \subset \varphi(\tau_n)(\mathcal{J}_n) \subset \varphi(\tau_1)(cJ_*) \subset \varphi(\tau_1)(\mathcal{J}_1) \tag{22}$$

Thus, for $\tau_n^* = \tau_1^{-1} \cdot \tau_n$, the inclusions (22) imply that

$$\varphi(\tau_n^*) = \varphi(\tau_1)^{-1} \circ \varphi(\tau_n) : \mathcal{J}_1 \to \mathcal{J}_n \subset \mathcal{J}_1$$

and hence $\varphi(\tau_n^*)$ has a hyperbolic fixed point $y_n \in \mathcal{J}_n$ satisfying $d(x_n, y_n) < \delta \exp\{(-a/2)\|\tau_n\|\}$. In particular, by passing to a subsequence if necessary, we can assume $d(y_n, x_n) < \delta/4$ for all n. Thus, the distance from y_n to the endpoints of $\mathcal{J}_n$ is bounded below by $\delta/4$.

As this is the first stage of an iterative construction, we set $\mathbf{g}_1 = \varphi(\tau_n^*)$, $z_1 = y_n$, $w_1 = \tau_n^{-1} y_n$, $J_1 = \mathcal{J}_1$ and $K_1 = \mathcal{J}_n$, then $\mathbf{g}_1 \colon J_1 \to K_1 \subset J_1$ has hyperbolic fixed-point z_1 so that its image w_1 satisfies $d(x, w_1) < \delta \exp\{(-a/2)\|\tau_n\|\}$. If $\Gamma \cdot z_1 \cap J_1$ contains a point other than z_1 then $\Gamma \cdot z_1$ is a resilient orbit, so we are done by Lemma 2.16.

Otherwise, let W_1 denote the interior of $\varphi(\tau_1)(\mathcal{J}_1)$. Then $\xi_1 = \tau_1 x_1 = x \in W_1$ and $W_1 \cap \Gamma \cdot z_1 = \{w_1\}$.

As x is an accumulation point for $E_a(\varphi)$, there exists $\xi_2 \in W_1 \cap E_a(\varphi)$ so that

$$0 < d(\xi_1, \xi_2) < d(x, w_1)/3$$

We then repeat the above construction of a hyperbolic fixed-point using the orbit $\Gamma \cdot \xi_2$, to obtain an interval J_2 of length δ and map $\mathbf{g}_2 \colon J_2 \to K_2 \subset J_2$ which has a hyperbolic fixed-point z_2. As before, introduce the point $w_2 = \mathbf{h}_1(z_1)$ and by choosing n sufficiently large in the construction of z_2 we can assume $d(\xi_2, w_2) < d(x, w_1)/3$, hence $d(w_1, w_2) > d(x, w_1)/3$ so $\Gamma \cdot w_2$ is disjoint from $\Gamma \cdot w_1$. If $\Gamma \cdot z_2 \cap J_2$ contains a point other than z_2 then $\Gamma \cdot z_2$ is a resilient orbit, and we are done.

Iterating this procedure, we either arrive at a resilient leaf, or we obtain a series of hyperbolic contractions $\mathbf{g}_i \colon J_i \to K_i \subset J_i$ where $|J_i| = \delta$ and $\mathbf{g}_i$ has a fixed-point $z_i \in J_i$ which is bounded away from the endpoints of J_i by at least $\delta/4$. Then there exists $i < k$ so that $\{z_i, z_k\} \subset I_0 = J_i \cap J_k$ and $n \gg 0$ so that for $I_1 = \mathbf{g}_i^n(J_i)$, $I_2 = \mathbf{g}_k^n(J_k)$, $\mathbf{h}_1 = \mathbf{g}_i^n$ and $\mathbf{h}_2 = \mathbf{g}_k^n$, $\{\mathbf{h}_1 \colon I_0 \to I_1, \mathbf{h}_2 \colon I_0 \to I_2\}$ is a ping-pong game. □

Proof of Theorem 1.4: Suppose that $E(\varphi)$ is uncountable. Since $E(\varphi) = \bigcup_{n=1}^{\infty} E_{1/n}(\varphi)$ there must exist $a = 1/n$ for which $E_a(\varphi)$ is uncountable. By Lemma 5.2 the set $E_a(\varphi)$ has an accumulation point. Then by Proposition 5.3 we have $h(\varphi) > 0$. □

6 Infinitesimal expansion and minimal sets

In this section, we establish a criterium for when the set $E(\varphi)$ is non-empty, based on the topological dynamics of the action and the metric geometry of the group Γ.

The first results relate the growth rates of orbits of points in $\mathbf{K}$ and properties of the set $E(\varphi)$. We briefly recall the definition of orbit growth rates, and some standard consequences of the definitions.

Let $\Gamma_N = \{\gamma \mid \|\gamma\| \le N\}$ denote the ball of radius N about the identity for the word metric on Γ. Let $\#\Gamma_N$ denote the cardinality of the set Γ_N. The *growth rate* of Γ is the number

$$gr(\Gamma, \mathcal{S}) = \limsup_{N \to \infty} \frac{\log\{\#\Gamma_N\}}{N}$$

Given $x \in \mathbb{S}^1$ we define the upper and lower growth rates of the action φ at x to be

$$gr^+(\Gamma, \mathcal{S}, \varphi, x) = \limsup_{N \to \infty} \frac{\log \#\{\Gamma_N \cdot x\}}{N} \ , \qquad gr^-(\Gamma, \mathcal{S}, \varphi, x) = \liminf_{N \to \infty} \frac{\log \#\{\Gamma_N \cdot x\}}{N}$$

Clearly, $gr^-(\Gamma, \mathcal{S}, \varphi, x) \le gr^+(\Gamma, \mathcal{S}, \varphi, x) \le gr(\Gamma, \mathcal{S})$. We recall some basic facts about actions on $\mathbb{S}^1$ with an invariant measure (cf. [25, 39, 41]) which are summarized in the following statement:

PROPOSITION 6.1 *Let φ be a C^0-action with a minimal set $\mathbf{K}$. Suppose that some $x \in \mathbf{K}$ has $gr^-(\Gamma, \mathcal{S}, \varphi, x) = 0$. Then there is a Γ-invariant probability measure $\mathbf{m}$ on $\mathbb{S}^1$ with support on $\mathbf{K}$. Hence, there exists a constant $C > 0$ and an integer $r \geq 1$ so that for every $y \in \mathbf{K}$, $\#\{\Gamma_N \cdot y\} \leq C \cdot N^r$. Thus, either every orbit of Γ on $\mathbf{K}$ satisfies $gr^-(\Gamma, \mathcal{S}, \varphi, x) > 0$, or all orbits have polynomial growth.*

Proof: By hypothesis, there exists a sequence $0 < N_1 < N_2 < \cdots$ of integers for which

$$\lim_{i \to \infty} \frac{\log \#\{\Gamma_{N_i} \cdot x\}}{N_i} = 0$$

For each $i > 0$, define a Borel probability measure, where for a continuous function $g \colon \mathbb{S}^1 \to \mathbb{R}$,

$$\mathbf{m}_i(g) = \frac{1}{\#\{\Gamma_{N_i} \cdot x\}} \sum_{\gamma \in \Gamma_{N_i}} g(\gamma x)$$

Since $x \in \mathbf{K}$, each measure $\mathbf{m}_i$ has support on $\mathbf{K}$. Let $\mathbf{m}$ be a weak-* limit of the probability measures $\{\mathbf{m}_i \mid i = 1, 2, \ldots\}$, then $\mathbf{m}$ is a Γ–invariant Borel probability measure supported on the minimal set $\mathbf{K}$.

The measure $\mathbf{m}$ defines a continuous map $\pi_{\mathbf{m}} \colon \mathbb{S}^1 \to \mathbb{S}^1$ by $\pi_{\mathbf{m}}(y) = 2\pi\mathbf{m}([y, 0]) \bmod 2\pi\mathbb{Z}$, where $[y, 0]$ is the counter-clockwise interval from 0 to y. The map $\pi_{\mathbf{m}}$ is monotone increasing on $\mathbf{K}$, as every relatively open set in $\mathbf{K}$ has positive $\mathbf{m}$-measure. If $\mathbf{K}$ is exceptional, $\pi_{\mathbf{m}}$ is constant on the gaps of $\mathbf{K}$. In this case, $\pi_{\mathbf{m}}$ identifies the endpoints of a common gap, hence the map is at most 2-to-1 on $\mathbf{K}$. The action of Γ is orientation preserving, so the endpoints of a gap cannot be mapped to one another by the action. Thus, $\pi_{\mathbf{m}}$ is injective on the orbits Γ_y for any $y \in \mathbf{K}$.

Since $\mathbf{K}$ is Γ-invariant, $\pi_{\mathbf{m}}$ defines a semi-conjugacy of φ to an action $\rho_{\mathbf{m}} \colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ which is a subgroup of the abelian group of orientation preserving rotations of the circle. Let r be the rank of the image $\rho_{\mathbf{m}}(\Gamma)$, then there exists a constant $C > 0$ so that for every $\theta \in \mathbb{S}^1$ the orbit growth $\#\{\rho_{\mathbf{m}}(\Gamma_N)(\theta)\} \leq C \cdot N^r$. Then for any $y \in \mathbf{K}$ we have

$$\#\{\varphi(\Gamma_N)(y)\} = \#\{\rho_{\mathbf{m}}(\Gamma_N)(\pi_{\mathbf{m}}(y))\} \leq C \cdot N^r \qquad \square$$

COROLLARY 6.2 *Let $\varphi \colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^0-action with a minimal set $\mathbf{K}$. Then for each $x \in \mathbf{K}$, if the lower growth rate $gr^-(\Gamma, \mathcal{S}, \varphi, x) = 0$, then also the upper growth rate $gr^+(\Gamma, \mathcal{S}, \varphi, x) = 0$. Thus both the upper and lower growth rates are either zero, or positive.* $\square$

If $gr^+(\Gamma, \mathcal{S}, \varphi, x) > 0$, we say that the orbit $\Gamma \cdot x$ has exponential growth.

Assume that $\mathbf{K}$ is an exceptional minimal set in $\mathbb{S}^1$ with gaps labeled by $\{I_i = (a_i, b_i) \mid i = 1, 2, \ldots\}$. If a gap I_i has an endpoint that has exponential growth, then the collection of images of the gap $\{\gamma I_i \mid \gamma \in \Gamma\}$ also grows in number exponentially. We will show this implies that $\mathbf{K} \cap E(\varphi) \neq \emptyset$. The idea is that an application of the Mean Value Theorem yields orbits with exponential expansion, and then, as in the previous sections, we use the collection of these orbits to construct hyperbolic periodic orbits in K. The proof below also gives a new proof of the main theorem of [27].

PROPOSITION 6.3 *Let φ be a C^1-action with exceptional minimal set $\mathbf{K}$. If there exists an endpoint $x \in \mathbf{K}$ for which $\lambda = gr^+(\Gamma, \mathcal{S}, \varphi, x) > 0$, then for all $0 < a < \lambda$ there exists $y_* \in \mathbf{K} \cap \mathcal{P}_a(\varphi)$. Conversely, if $\lambda(x) > 0$ for some point $x \in \mathbf{K}$, then every orbit of $\mathbf{K}$ has exponential growth.*

Proof: Let $x \in \mathbf{K}$ be an endpoint with $\lambda = gr^+(\Gamma, \mathcal{S}, \varphi, x) > 0$. Given $0 < a < b < \lambda$, we will construct $y_* \in \mathbf{K} \cap E_b(\varphi)$.

Let $I_x = (a_x, b_x)$ be the gap of $\mathbf{K}$ containing x. Assume that x is the left endpoint a_x for the gap; the case where $x = b_x$ proceeds identically.

Choose $0 < \epsilon < \lambda - b$, and set $\epsilon_0 = \epsilon/10$. Note that $\lambda - 3\epsilon_0 > b$.

Choose $\delta = \delta(\epsilon_0) > 0$ a modulus of continuity for the derivatives $\varphi(\sigma_j)'$, so that if $d(y,z) < \delta$ then

$$|\log\{\varphi(\sigma_j)'(y)\} - \log\{\varphi(\sigma_j)'(z)\}| < \epsilon_0 \quad \text{for all} \quad 1 \leq j \leq k \tag{23}$$

Set $\mathcal{O}(N, x) = \#\{\Gamma_N \cdot x\}$. Then by the definition of $\lambda = gr^+(\Gamma, \mathcal{S}, \varphi, x)$, there exists $C_1 > 0$ and an increasing sequence of integers $N_n \to \infty$ such that $\mathcal{O}(N_n, x) \geq C_1 \exp\{N_n(\lambda - \epsilon_0)\}$ for all $n \geq 1$.

For $\gamma_1, \gamma_2 \in \Gamma$, either $\varphi(\gamma_1)(I_x) \cap \varphi(\gamma_2)(I_x) = \emptyset$ or $\varphi(\gamma_1)(I_x) = \varphi(\gamma_2)(I_x)$. The collection of gaps $\mathcal{I}_n = \{\varphi(\gamma)(I_x) \mid \|\gamma\| \leq N_n\}$ contains exactly $\mathcal{O}(N_n, x)$ disjoint intervals, as the action φ is orientation preserving.

The sum of the lengths of the intervals in $\mathcal{I}_n$ is less than 2π, so for each n there exists $\gamma_n \in \Gamma_{N_n}$ such that

$$|\varphi(\gamma_n)(I_x)| \leq 2\pi/\mathcal{O}(N_n, x) \leq C_2 \exp\{-N_n(\lambda - \epsilon_0)\} \tag{24}$$

where $C_2 = 2\pi/C_1$ is independent of n. Moreover, we can assume that γ_n has the least length $\ell_n = \|\gamma_n\| \leq N_n$ for which (24) holds. Set $\gamma_n = \sigma_{i_{\ell_n}} \cdots \sigma_{i_1}$, then for each $1 \leq j < k \leq \ell_n$ we have

$$\varphi(\sigma_{i_j} \cdots \sigma_{i_1})(I_x) \cap \varphi(\sigma_{i_k} \cdots \sigma_{i_1})(I_x) = \emptyset$$

Otherwise,

$$\varphi(\gamma_n)(I_x) = \varphi(\sigma_{\ell_n} \cdots \sigma_{i_{k+1}} \cdot \sigma_{i_j} \cdots \sigma_{i_1})(I_x)$$

contradicting the choice of γ_n as the least length element satisfying (24). Set $I_n = \varphi(\gamma_n)(I_x)$.

Fix an integer $N_0 > 8\pi/\delta$. As N_n is an increasing sequence, by passing to a subsequence, we can assume that the lengths ℓ_n are also an increasing sequence, and $\ell_n > N_0$ for all $n \geq 1$.

LEMMA 6.4 *Given integers $1 \leq a < b \leq \ell_n$ with $b - a \geq N_0$, there exist index c with $a \leq c \leq b$ such that $|\varphi(\sigma_{i_c} \cdots \sigma_{i_1})(I_x)| < \delta/4$.*

Proof: The intervals $\varphi(\sigma_{i_k} \cdots \sigma_{i_1})(I_x)$ for $a \leq k \leq b$ are all pairwise disjoint, so have total length at most 2π. As there are at least N_0 such intervals, there must be at least one value $k = c$ with $|\varphi(\sigma_{i_c} \cdots \sigma_{i_1})(I_x)| < 2\pi/N_0 = \delta/4$ □

The estimate (24) and the Mean Value Theorem applied to the map $\varphi(\gamma_n)\colon I_x \to I_n$ implies there exists a point $w_n \in I_n$ such that

$$\varphi(\gamma_n)'(w_n) \;\leq\; \frac{|I_n|}{|I_x|} \;\leq\; \frac{C_2 \exp\{-N_n(\lambda - \epsilon_0)\}}{|I_x|} \;\leq\; C_3 \exp\{\ell_n(\epsilon_0 - \lambda)\} \tag{25}$$

where $C_3 = C_2/|I_x| > 0$ is independent of n. Set $y_n = \varphi(\gamma_n)(w_n) \in I_n$.

As $\ell_n \to \infty$, by passing to a subsequence if necessary, we can assume that for all $n > 0$

$$\log\{\varphi(\gamma_n)'(w_n)\}/\ell_n < 2\epsilon_0 - \lambda < -b - \epsilon_0 \tag{26}$$

Thus, the points y_n, which lie in the gaps of $\mathbf{K}$, have expansion rate at least $\lambda - 2\epsilon_0 > b + \epsilon$. We now proceed as in the proof of Proposition 4.11, to construct hyperbolic a fixed point $y_* \in \mathbf{K} \cap \mathcal{P}_a(\varphi)$.

By the same methods as in the proofs of Proposition 4.7 and Lemmas 4.8 and 4.10, for each $n \geq 1$ there exists an index $0 < b_n < \ell_n$ so that for $\tau_n = \sigma_{i_{\ell_n}} \cdots \sigma_{b_n+1}$ and $\kappa_n = \sigma_{i_{b_n}} \cdots \sigma_1$

1. $x_n = \varphi(\kappa_n)(w_n)$, $J_n = \varphi(\kappa_n)(I_x)$, hence for $\mathbf{h}_n = \varphi(\tau_n)$, $\mathbf{h}_n \colon J_n \to I_n$ and $\mathbf{h}_n(x_n) = y_n$
2. $|\varphi(\sigma_{i_j} \cdots \sigma_{i_{b_n+1}})([x_n - \delta/2, x_n + \delta/2])| < \delta$ for each $b_n < j \leq \ell_n$
3. $\log\{\mathbf{h}_n'(w)\}/\|\tau_n\| < 3\epsilon_0 - \lambda$ for all $w \in (x_n - \delta/2, x_n + \delta/2)$

The gap J_n is the image of the gap I_x but we do not have an *a priori* estimate on its length. However, Lemma 6.4 implies that at least one of the intervals $\varphi(\sigma_{i_k} \cdots \sigma_1)(I_x)$ for $b_n \leq k < b_n + N_0$ has length less than $\delta/4$. Thus, if we modify the proof of Lemma 4.10 so that after we find the least ϵ_0-regular index b_n we then select the index $b_n \leq k < b_n + N_0$ for which the estimate holds on the gap length, then the rest of the estimates go through as before, with the possible need to pass to a subsequence to ensure that the degree of contraction is at least $3\epsilon_0 - \lambda$. We continue to label the new index as b_n and can thus assume

4. $|J_n| = |\varphi(\kappa_n)(I_x)| < \delta/4$

Let $a_n, b_n \in \mathbf{K}$ denote the endpoints of $J_n = (a_n, b_n)$. Let $x_* \in \mathbf{K}$ be an accumulation point of the set of endpoints $\{a_n\}$. Passing to a subsequence if necessary, we can assume that $d(x_*, a_n) < \delta/4n$ for all $n \geq 1$. Then for all $n \geq 2$ we have

$$d(x_*, x_n) \leq d(x_*, a_n) + d(a_n, x_n) \leq \delta/4n + \delta/4 < 3\delta/8$$

so that $(x_* - \delta/8, x_* + \delta/8) \subset (x_n - \delta/2, x_n + \delta/2)$. Thus, for all $n \geq 2$ we have the estimate

$$\mathbf{h}_n'(w) < \exp\{-b\|\tau_n\|\} \ \text{ for all } \ w \in (x_* - \delta/8, x_* + \delta/8) \tag{27}$$

Recall that $x_* \in \mathbf{K}$ so that for each $n \geq 2$, $z_n = \mathbf{h}_n(x_*) \in \mathbf{K}$. Let $z_* \in \mathbf{K}$ be an accumulation point for the set $\{z_n \mid n \geq 2\}$. The Γ-orbit of every point of $\mathbf{K}$ is dense, so there exists $\gamma^* \in \Gamma$ for which $\varphi(\gamma^*)(z_*) \in (x_* - \delta/8, x_* + \delta/8)$. Let $\epsilon_* > 0$ be such that $\varphi(\gamma^*)([z_* - \epsilon_*, z_* + \epsilon_*]) \subset (x_* - \delta/8, x_* + \delta/8)$. Now chose n sufficiently large so that

1. $\varphi(\tau_n)(x_n - \delta/2, x_n + \delta/2) \subset (z_* - \epsilon_*, z_* + \epsilon_*)$
2. $\log\{\varphi(\gamma^*\tau_n)'(y)\} < (4\epsilon_0 - \lambda)\|\gamma^*\tau_n\|$

As $(x_* - \delta/8, x_* + \delta/8) \subset (x_n - \delta/2, x_n + \delta/2)$ we obtain a uniform hyperbolic contraction

$$\varphi(\gamma^*\tau_n) \colon (x_* - \delta/8, x_* + \delta/8) \to (x_* - \delta/8, x_* + \delta/8)$$

hence $\varphi(\gamma^*\tau_n)$ has a hyperbolic fixed-point y_*. Clearly, $x_* \in \mathbf{K}$ is in the domain of the contraction $\varphi(\gamma^*\tau_n)$, so y_* lies in the closure of $\mathbf{K}$. As $\mathbf{K}$ is closed, we must have that $y_* \in \mathbf{K}$. Then the inverse map $\varphi(\gamma^*\tau_n)^{-1}$ also has fixed-point y_* with exponent of expansion at least $\lambda - 4\epsilon_0 > b$ and hence $y_* \in \mathbf{K} \cap E_b(\varphi) \subset \mathbf{K} \cap E_a(\varphi)$ as was to be shown.

To prove the converse, it suffices to show that there exists one orbit in $\mathbf{K}$ with exponential growth. Suppose $x \in \mathbf{K}$ with $\lambda(x) > 0$, then by Proposition 4.11 there exists a hyperbolic contraction

$\mathbf{g}_n = \varphi(\tau_n^*): J_1 \to J_n \subset J_1$ with fixed point $x_n \in J_1 \cap \mathbf{K}$. As $\mathbf{K}$ is minimal, there exists $\sigma \in \Gamma$ such that $\varphi(\sigma)(x) \in J_1$ but $x \neq \varphi(\sigma)(x)$. Thus, x is a resilient point, so by Lemma 2.16 there is a ping-pong game for φ with x as one of the fixed-points, hence the orbit of x has exponential growth. Thus, each orbit of φ in $\mathbf{K}$ has exponential growth by Proposition 6.1.

This completes the proof of Proposition 6.3. □

COROLLARY 6.5 *Let $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^1-action with exceptional minimal set $\mathbf{K}$. Then either there is a Γ-invariant probability measure supported on $\mathbf{K}$, or the hyperbolic periodic points are dense in K.*

Proof: Combine Proposition 6.3 with Proposition 6.1 and Lemma 4.6. □

Note that this proves Theorem 1.5 in the case of an exceptional minimal set.

It remains to analyze the case where φ is a minimal action on $\mathbb{S}^1$, so there is a unique minimal set $\mathbf{K} = \mathbb{S}^1$. If $E(\varphi)$ is not empty, then by Proposition 4.11 and Lemma 4.6 the hyperbolic periodic-points are dense in $\mathbb{S}^1$. To complete the proof of Theorem 1.5, we need to show that if the action has no φ-invariant Borel probability measure, then $E(\varphi)$ is non-empty. By Lemma 2.3, the action of φ must be expansive. By Proposition 3.1, the action must contain a ping-pong game $\{\gamma_1: K_0 \to K_1, \gamma_2: K_0 \to K_2\}$.

Consider the dynamical system on $\mathbb{S}^1$ defined by the free group $F_2 = \mathbb{Z} * \mathbb{Z}$ acting via the elements $\{\gamma_1, \gamma_2\}$. By choice of the generators, this has an Cantor set which is invariant under the free semi-group generated by $\{\gamma_1, \gamma_2\}$, for which the endpoints have exponential orbit growth. Thus, we can repeat the argument above to conclude that there exists a hyperbolic periodic orbit for this subaction. Hence, $E(\varphi)$ is not empty.

7 Expansive actions on minimal sets

In a lecture at the Dynamical Systems Symposium at the University of Paris, VI in June 1998, Étienne Ghys made the following conjecture:

CONJECTURE 7.1 (Ghys) *Let $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^0-action. Then either φ has a Γ-invariant probability measure, or Γ has a non-abelian free subgroup on two generators.*

The point of the conjecture is the assertion that there exists a non-abelian free subgroup of the action, a much stronger conclusion than the existence of a non-abelian free semi-group. In this section, we examine the relation between this conjecture and expansiveness for the action, and derive a proof of Conjecture 7.1 for real analytic actions using results of Farb and Shalen [16].

Conjecture 7.1 is trivial for actions with periodic orbits, so we may assume the C^0-action φ has no periodic orbit. Hence, there exists an exceptional minimal set $\mathbf{K}$, or the action is minimal.

If the action is minimal, then by Proposition 6.1 either there exists a Γ-invariant measure with full support, or every orbit of φ has exponential growth. In the latter case, by Lemma 2.3 the action on $\mathbb{S}^1$ must be expansive. Proposition 3.1 implies that the action admits a ping-pong game.

If the action preserves an exceptional minimal set $\mathbf{K}$, then again there is the dichotomy that either there is a Γ-invariant measure with full support on $\mathbf{K}$, or the orbit of every point in $\mathbf{K}$ has exponential growth.

PROPOSITION 7.2 *Suppose that $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is a C^0-action with an exceptional minimal set* $\mathbf{K}$*, and no invariant probability measure with support* $\mathbf{K}$*. Then there exists a ping-pong game $\{\gamma_1: I_0 \to I_1, \gamma_2: I_0 \to I_2\}$ for φ restricted to* $\mathbf{K}$*.*

Proof: Define a continuous map $\rho: \mathbb{S}^1 \to \mathbb{S}^1$ which is constant on the gaps of $\mathbf{K}$, and monotone on $\mathbf{K}$. The action φ descends to a quotient action $\overline{\varphi}: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ which is semi-conjugate to φ via ρ. The action $\overline{\varphi}$ is then minimal, with no invariant measure. By Lemma 2.3 the action $\overline{\varphi}$ on $\mathbb{S}^1$ must be expansive. Proposition 3.1 implies that $\overline{\varphi}$ admits a ping-pong game $\{\overline{\varphi}(\gamma_1): \overline{I_0} \to \overline{I_1}, \overline{\varphi}(\gamma_2): \overline{I_0} \to \overline{I_2}\}$. We set $I_0 = \rho^{-1}(\overline{I_0})$, $I_1 = \rho^{-1}(\overline{I_1})$, $I_2 = \rho^{-1}(\overline{I_2})$, and obtain a ping-pong game for φ. □

These remarks show that Conjecture 7.1 is equivalent to the alternate problem:

CONJECTURE 7.3 *Let $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^0-action with a minimal set K, and suppose that there is a ping-pong game for φ on* $\mathbf{K}$*. Then Γ contains a non-abelian free subgroup on two generators.*

Thus, a direct approach to the conjecture is consider a free semigroup defined by the elements $\{\gamma_1, \gamma_2\} \subset \Gamma$ defining the maps in a ping-pong game, and find conditions which imply that $\{\gamma_1, \gamma_2\}$ generates a free subgroup. That is, if there is some non-trivial word w_0 in the generators such that w_0 acts trivially on $\mathbb{S}^1$, then this should imply the action admits a Γ-invariant measure.

For C^1-actions, the results of the second part of this paper allow us to sharpen the above remarks. If φ is a C^1-action with no invariant probability measure, then there exists a minimal set $\mathbf{K}$ which is either exceptional or all of $\mathbb{S}^1$, and Corollary 1.6 implies there is a hyperbolic ping-pong game for φ on $\mathbf{K}$. Thus, for C^1-actions, we can add to the hypotheses of Conjecture 7.3 the statement that the maps defining the ping-pong game are hyperbolic contractions on the domain I_0 with unique fixed-points there. This is the setting for applying some version of the "Tits alternative" [50, 10].

One possible approach to a proof of Conjecture 7.3 is to use the formulation of the Tits alternative in section 3 of Farb-Shalen [16]. We give a slight generalization of their statement adapted to the present context:

THEOREM 7.4 *Let $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^0-action with a minimal set* $\mathbf{K}$ *and no invariant probability measure on* $\mathbf{K}$*. Suppose there exists $\gamma_1, \gamma_2 \in \Gamma$ so that, for $f = \varphi(\gamma_1)$, $g = \varphi(\gamma_2)$, $\mathrm{Fix}(f) = \{x \in \mathbb{S}^1 \mid f(x) = x\}$, and $\mathrm{Fix}(g) = \{x \in \mathbb{S}^1 \mid g(x) = x\}$, we have*

1. *$f|\mathbf{K}$ and $g|\mathbf{K}$ have infinite order*
2. *$\mathrm{Fix}(f) \cap \mathbf{K} \neq \emptyset$ and $\mathrm{Fix}(g) \cap \mathbf{K} \neq \emptyset$*
3. *$\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \cap \mathbf{K} = \emptyset$*

Then for some $n > 0$, the group generated by f^n and g^n is a nonabelian free group.

Proof: If $\mathbf{K} = \mathbb{S}^1$ this is exactly what is proved in [16]. If $\mathbf{K}$ is exceptional, define a minimal action $\overline{\varphi}$ on $\mathbb{S}^1$ via the semi-conjugacy of the proof of Proposition 7.2, then proceed as before. □

We can apply the Farb-Shalen version of the Tits alternative to obtain a partial proof Conjecture 7.3:

THEOREM 7.5 *Let $\varphi: \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ be a C^0-action with a minimal set K, and suppose that there is a ping-pong game $\{\gamma_1: I_0 \to I_1, \gamma_2: I_0 \to I_2\}$ for φ on* $\mathbf{K}$ *such that for $f = \varphi(\gamma_1)$, $g = \varphi(\gamma_2)$, the sets $\mathrm{Fix}(f) \cap \mathbf{K}$ and $\mathrm{Fix}(g) \cap \mathbf{K}$ are discrete. Then Γ has a non-abelian free subgroup.*

Proof: Let $\text{Fix}(f) \cap \mathbf{K} = \{x_1, \ldots, x_n\}$ and $\text{Fix}(g) \cap K = \{y_1, \ldots, y_m\}$. If $\text{Fix}(f) \cap \text{Fix}(g) = \emptyset$ then we are done by Theorem 7.4. If not, suppose there exists an element $\tau \in \Gamma$ such that

$$\{x_1, \ldots, x_n\} \cap \{\tau y_1, \ldots, \tau y_n\} \cap \mathbf{K} = \emptyset \tag{28}$$

Replace g with $h \circ g \circ h^{-1}$ where $h = \varphi(\tau)$, and condition (7.4.3) is satisfied since $\mathbf{K}$ is Γ-invariant.

Suppose that no such $\tau \in \Gamma$ exists, then following an observation of Farb and Shalen in section 3 of [16], note that for every $\tau \in \Gamma$ there exists x_i and y_k with $\tau y_k = x_i$. That is, τ belongs to the set $T_{i,k} = \{\sigma \in \Gamma \mid \sigma y_k = x_i\}$. Let $\Gamma_{y_k} \subset \Gamma$ denote the stabilizer of y_k, then $T_{i,k} = \tau \Gamma_{y_k}$ for any element $\tau \in T_{i,k}$. Thus, Γ is a finite union of cosets of stabilizers of points in $\text{Fix}(g)$. By the "Coset Lemma" 3.1 of [16], one of the stabilizers Γ_{y_k} must have finite index in Γ. But this implies that the orbit Γy_k is finite, which contradicts the fact that the orbit of every point in $\mathbf{K}$ is dense. Thus, there must exist some τ satisfying (28). □

Proof of Theorem 1.7: For a C^1-action φ with minimal set $\mathbf{K}$ and no Γ-invariant probability measure supported on $\mathbf{K}$, there is always a ping-pong game on $\mathbf{K}$ as remarked previously. Thus, Conjecture 7.3 is true if the maps f and g can be chosen with isolated fixed-points in $\mathbf{K}$. The fixed-points of a real analytic action are always isolated, so Theorem 1.7 follows from the remarks of this section and Theorem 7.5. □

Margulis' proof of Conjecture 7.1 in [35] used methods similar to those discussed above, but Margulis considered the action of Γ on ϵ-nets, not just on individual points, and which avoids the need for assuming there are isolated fixed-points.

8 Examples

We give three examples which illustrate the ideas developed in this paper. The first example is a real analytic expansive action of a solvable group. The second second example embeds the first example into the gaps of a Denjoy example to produce an expansive C^1-action of a solvable group with a Cantor type minimal set. The third example is a C^∞-action with a countable collection of hyperbolic fixed-points, but is not expansive. None of these examples is new, but each provides illustrations of the ideas of this paper, and also provide "counter-examples" to extending some of the results of the paper.

EXAMPLE 8.1 *An expansive C^ω-action of a solvable group on $\mathbb{S}^1$ with minimal set a point.*

Define maps of the real line by $f(u) = 2u$ and $g(u) = u + 1$. Embed the real line into the circle $\mathbb{S}^1 = \{z = x + iy \mid x^2 + y^2 = 1\} \subset \mathbb{C}$ by the linear fractional map $h(u) = (1 + iu)/(1 - iu)$. This map conjugates f to $\alpha = h \circ f \circ h^{-1}$ and $\beta = h \circ g \circ h^{-1}$, which are both real analytic linear fractional transformations of $\mathbb{S}^1$. Let Γ be the solvable subgroup of $\text{Diff}^\omega(\mathbb{S}^1)$ they generate, and $\varphi\colon \Gamma \colon \mathbb{S}^1 \to \mathbb{S}^1$ the action they determine.

The action of f, g on $\mathbb{R}$ has every orbit dense, hence the action of Γ on $\mathbb{S}^1$ admits a unique fixed–point $(-1, 0)$, and every other orbit is dense. In particular, the fixed-point $(-1, 0)$ for the full action is the unique minimal set.

Given $x \neq y \in \mathbb{S}^1$, at least one of these cannot be the fixed–point $(-1, 0)$. Hence, there exists ℓ such that $g^\ell(h^{-1}(x))$ and $g^\ell(h^{-1}(y))$ lie on opposite sides of the origin in $\mathbb{R}$. Applying a suitable power $k > 0$ of f we can ensure that their images $f^k \circ g^\ell(h^{-1}(x))$ and $f^k \circ g^\ell(h^{-1}(y))$ span an interval containing either [0,1] or [-1,0] in $\mathbb{R}$, hence $h \circ f^k \circ g^\ell(h^{-1}(x))$ and $h \circ f^k \circ g^\ell(h^{-1}(y))$ are ϵ separated in $\mathbb{S}^1$ for $\epsilon < 1/2$.

This example is interesting for two reasons. First, it shows that expansiveness is not sufficient to imply there is a free subgroup on two generators in Γ, even for analytic actions, as the group Γ is solvable. Secondly, the action of Γ does have a ping-pong table, which is given in terms of f and g by the two maps $h^{-1} \circ \mathbf{h}_1 \circ h(u) = g^{-2} \circ f^{-2}(u) = -2 + u/4$ and $h^{-1} \circ \mathbf{h}_2 \circ h(u) = g^2 \circ f^{-2}(u) = 2 + u/4$. We can take $h-1(I_0) = [-4, 4]$, $h-1(I_1) = [-3, -1]$ and $h-1(I_2) = [1, 3]$. The forward iterates of I_0 define a Cantor set $\mathbf{K}$ which is invariant under the semigroup generated by $\{\mathbf{h}_1, \mathbf{h}_2\}$. But the set $\mathbf{K}$ is not invariant for the full group Γ as the only minimal set for Γ is $(-1, 0)$. Thus, properties of the subdynamics generated by a ping-pong game need not carry over to the full group action.

EXAMPLE 8.2 *An expansive action of a solvable group on $\mathbb{S}^1$ with exceptional minimal set.*

Let $\gamma\colon \mathbb{S}^1 \to \mathbb{S}^1$ be a (Denjoy) C^1-diffeomorphism with a Γ-invariant exceptional minimal set K. The complement of K consists of a disjoint union of open intervals $\{U_1, U_2, \ldots\}$ and γ acts transitively on the set of intervals. Thus, we can index the open sets by $\mathbb{Z}$ where $U_\ell = \gamma^\ell U_0$.

Let $\mathbf{h}_1$ and $\mathbf{h}_2$ be the linear fractional maps of $\mathbb{S}^1$ constructed in Example 8.1. Let $\mathbf{h}_3$ be a modification of the map $\mathbf{h}_1$ so that $\mathbf{h}_3$ agrees with $\mathbf{h}_1$ away from a small neighborhood of the fixed-point $(-1, 0)$. Near $(-1, 0)$, we require that $\mathbf{h}_3$ also have $(-1, 0)$ as a unique sink, but such that $\mathbf{h}_3'(-1, 0) = 1$, and all higher derivatives vanish at $(-1, 0)$. That is, we make $\mathbf{h}_1$ flat at $(-1, 0)$.

Let $\phi\colon \mathbb{S}^1 - (-1, 0) \to U_0$ be an affine diffeomorphism (which is unique).

Define $\alpha \in \mathrm{Diff}^1(\mathbb{S}^1)$ with fixed–point set $\mathbf{K}$, and on U_ℓ we define $\alpha | U_\ell = \gamma^\ell \circ \phi \circ \mathbf{h}_3 \circ \phi^{-1} \circ \gamma^{-\ell}$.

Define $\beta \in \mathrm{Diff}^1(\mathbb{S}^1)$ with fixed–point set $\mathbf{K}$, and on U_ℓ we define $\beta | U_\ell = \gamma^\ell \circ \phi \circ \mathbf{h}_2 \circ \phi^{-1} \circ \gamma^{-\ell}$.

Let $\Gamma \subset \mathrm{Diff}^1(\mathbb{S}^1)$ be the subgroup generated by $\{\alpha, \beta, \gamma\}$. Clearly, $\mathbf{K}$ is the unique minimal set for the action of Γ, and every point in the complement of $\mathbf{K}$ has dense orbit in $\mathbb{S}^1$.

The action of Γ is expansive. There are two cases to consider. If $x \neq y \in \mathbb{S}^1$, and both points lie in the same connected component of the complement of $\mathbf{K}$, then there exists ℓ so that $\gamma^\ell(x) \in U_0$ and we can then proceed as in Example 8.1. If $x \neq y$ do not lie in the closure of the same connected component of the complement of $\mathbf{K}$, then there exists some U_ℓ contained in the interval $\overline{xy}$, thus $\gamma^{-\ell}(x)$ and $\gamma^{-\ell}(y)$ contain U_0 in the interval they determine, so the points are again ϵ-separated.

This example is a little more subtle than the first, as it has a unique exceptional minimal set, which is obviously of finite type, and the action is expansive on $\mathbf{K}$. The set of hyperbolic periodic points $\mathcal{P}^h_{\log 2}(\varphi)$ is dense in the complement of $\mathbf{K}$. (The point $(1, 0)$ is fixed by α, where $\alpha'(1, 0) = 2$, and the conjugates of α provide a dense set of fixed-points.) However, $\mathbf{K} \cap E(\varphi) = \emptyset$ as there is a Γ-invariant measure for the action on $\mathbf{K}$. This shows that in general, $E(\varphi)$ is not a closed set. Also, the orbits in the complement of $\mathbf{K}$ have exponential growth, so that in Theorem 6.3, it is not sufficient that the exponential growth orbit limit to K - it must be an orbit of a point in $\mathbf{K}$.

EXAMPLE 8.3 *A proper C^∞-action of a solvable group on $\mathbb{S}^1$ with countably many hyperbolic fixed-points and no ping-pong games.*

Let $\mathbf{h}_1$ be the hyperbolic linear fractional map as in Example 8.1. Choose a fundamental domain $I = [a, b]$ for $\mathbf{h}_1$ in the invariant open set $\mathbb{S}^1_+ = \{z = x + iy \mid x^2 + y^2 = 1 \text{ and } y > 0\}$.

Let $\phi\colon \mathbb{S}^1 - (-1, 0) \to I$ be an affine diffeomorphism (which is unique) and use it to define $\mathbf{h}_4$ on I as $\mathbf{h}_4 = \phi \circ \mathbf{h}_1 \circ \phi^{-1}$. Extend $\mathbf{h}_4$ to all of $\mathbb{S}^1$ as the identity outside of I. On the interior of the interval I, $\mathbf{h}_4$ has a unique hyperbolic fixed-point $y_0 \in I$ with derivative 2, and all other points are asymptotic to the endpoints of I.

Let Γ be the subgroup of $\mathrm{Diff}^\infty(\mathbb{S}^1)$ generated by $\{\mathbf{h}_1, \mathbf{h}_4\}$, and φ the action it generates. The group Γ is solvable, but its commutator subgroup is infinitely generated, so Γ must have exponential growth. Note that φ has countably many hyperbolic fixed-points $\{y_k = \mathbf{h}_1^k(y_0) \mid k \in \mathbb{Z}\}$, where y_k is fixed by all of the maps $\mathbf{g}_k^\ell = \mathbf{h}_1^k \circ \mathbf{h}_4^\ell \circ \mathbf{h}_1^{-k}$. Thus, $\lambda(y_k) > \ell \cdot \log(2)/(2k+\ell)$ for all ℓ.

On the other hand, the orbits of φ consist of either a singleton (for x = $(-1,0)$); or copy of $\mathbb{Z}$ (for the translates $\mathbf{h}_1^\ell(y_0)$, for the endpoints of I, and for all points in $\mathbb{S}^1_-$); or they are isomorphic to $\mathbb{Z}^2$. That is, all orbits have quadratic growth.

Obviously, φ has no ping-pong table, in spite of the existence of many hyperbolic fixed-points. The issue is that the domains of contractions of these fixed-points do not overlap, so do not generate the complicated dynamics of a ping-pong table. It is cautionary, for many of the constructions made in this paper require producing contractions with overlapping domains, and it is not sufficient to just exhibit the contractions, as the domains must be controlled as well.

The suspension of this example is one of the most basic in the study of foliations, as it is "depth two" and the leaves have quadratic growth, even though the group Γ has exponential growth.

More examples and constructions of group actions and codimension-one foliations with special dynamics can be found in Hector's seminal paper [21], and in the books [3]. The series of papers by Farb and Franks [13, 14, 15] show that every finitely generated nilpotent group acts effectively on the circle, via a C^1-action but not necessarily via a C^2-action. The work of Burslem and Wilkenson [2] give an elegant classification of analytic actions of solvable groups on the circle.

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