# Exceptional minimal sets and the Godbillon-Vey class

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**Abstract**

Let $\mathbf{K}$ be an exceptional minimal set for a $C^1$-action $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ of a finitely generated group $\Gamma$ on the circle. Then $\mathbf{K} \cap E(\varphi)$ has Lebesgue measure zero, where $E(\varphi)$ consists of the points which are infinitesimally expansive. As an application, we obtain that the $GV(\mathcal{F})$ class for a $C^2$-foliation defined by a suspension of a group on a circle vanishes when restricted to an exceptional minimal set of $\mathcal{F}$.

## 1 Introduction

Let $\Gamma$ be a finitely-generated group and $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ an effective action on the circle by orientation-preserving homeomorphisms. Recall that a subset $\mathbf{K} \subset \mathbb{S}^1$ is minimal if it is closed, invariant under the action $\varphi$, and minimal with respect to these two properties. The minimal sets divide into three types: $\mathbf{K}$ is a finite set, or $\mathbf{K} = \mathbb{S}^1$, or $\mathbf{K}$ is a perfect, nowhere dense subset. In this latter case, $\mathbf{K}$ is said to be *exceptional*. If there is an invariant Borel probability measure for the action $\varphi$ supported on the exceptional minimal set $\mathbf{K}$, then $\mathbf{K}$ is said to be of *Denjoy type*. If no such invariant measure exists, then $\mathbf{K}$ is said to be *hyperbolic*. This note concerns the geometry of hyperbolic exceptional minimal sets.

Associated to the $C^1$-action $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is a Borel measurable, $\varphi$-invariant set $E(\varphi) \subset \mathbb{S}^1$ consisting of the points for which the action is infinitesimally expansive. The set $E(\varphi)$ is defined precisely in section 2. The properties of this set are analogous to those of the Pesin set for standard dynamical systems [1, 13, 19, 22]. The set $E(\varphi) \subset \mathbb{S}^1$ arise naturally in the study of the dynamics of foliations [13, 14, 16], especially in relation to the Godbillon-Vey class of $C^2$-foliations and its relation to the geometric entropy of the foliation as defined by Ghys, Langevin and Walczak [8].

The main result of this note is the following theorem:

**THEOREM 1.1** *Assume that $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is a $C^{1+\alpha}$-action by a finitely generated group $\Gamma$ on the circle, for some $\alpha > 0$. Let $\mathbf{K}$ be a minimal set such that $\mathbf{K} \cap E(\varphi)$ has positive Lebesgue measure. Then $\mathbf{K} = \mathbb{S}^1$.*

**COROLLARY 1.2** *Let $\mathbf{K}$ be an exceptional minimal set for a $C^{1+\alpha}$-action $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ of a finitely generated group $\Gamma$ on the circle. Then $\mathbf{K} \cap E(\varphi)$ has Lebesgue measure zero.*

**COROLLARY 1.3** *Let $\mathbf{K}$ be an exceptional minimal set for a $C^{1+\alpha}$-action $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ of a finitely generated group $\Gamma$ on the circle. If $\mathbf{K} \subset E(\varphi)$, then $\mathbf{K}$ has Lebesgue measure zero.*

*Supported in part by NSF grant DMS-0406254. Preprint date: July 24, 2005

If $\mathbf{K}$ is an exceptional minimal set of Denjoy type, then $\mathbf{K} \cap E(\varphi)$ is empty [15]. Moreover, it is standard to construct examples of Denjoy type $C^1$-actions on the circle for which the measure of the unique minimal set is positive. Using a more delicate construction, Bowen gave examples of $C^1$-group actions on the circle which have hyperbolic minimal sets with positive Lebesgue measure in [2], so the Hölder hypothesis in the theorem is necessary.

The Denjoy Theorem [6, 24] implies that for $C^2$-actions, every exceptional minimal set is hyperbolic. Moreover, it is conjectured by G. Hector [9, 10, 20, 25] that every minimal set for a $C^2$-action has Lebesgue measure zero. This has been proven for the special case of Markov minimal sets of $C^2$-actions by S. Matsumoto [21] and J. Cantwell and L. Conlon [5]. It was shown by the author in [15] that if $\mathbf{K}$ is an exceptional minimal set for a $C^1$-action, then the hyperbolic periodic points for the action are dense in $\mathbf{K}$. In particular, $\mathbf{K} \cap E(\varphi)$ is dense in $\mathbf{K}$. This is partial evidence for the following conjecture, which would imply Hector's Conjecture.

**CONJECTURE 1.4** *Let $\mathbf{K}$ be a hyperbolic exceptional minimal set for a $C^{1+\alpha}$-action $\varphi$ of a finitely generated group $\Gamma$ on the circle. Then $\mathbf{K} \cap E(\varphi)$ has full measure in $\mathbf{K}$.*

Theorem 1.1 has a counterpart for minimal sets of codimension one foliations. Associated to a $C^1$-foliation $\mathcal{F}$ of codimension one of a compact manifold $M$ is a saturated, Borel measurable set $E(\mathcal{F}) \subset M$ consisting of the points for which the leafwise infinitesimal holonomy is expansive. Techniques analogous to those in this paper show:

**THEOREM 1.5** *Let $\mathbf{K}$ be an exceptional minimal set of a $C^{1+\alpha}$-foliation $\mathcal{F}$ of codimension one on a compact manifold. Then $\mathbf{K} \cap E(\mathcal{F})$ has Lebesgue measure zero.*

The proof of Theorem 1.5 involves additional technicalities introduced by the need to work with pseudogroups, and is given in a separate monograph [16].

Conjecture 1.4 actually has two motivations – one is from dynamics, and the other is based on properties of the Godbillon-Vey invariant $GV(\mathcal{F}) \in H^3(M;\mathbb{R})$ for codimension one $C^2$-foliations. Duminy proved in the unpublished manuscript [3] (see also [4]) that $GV(\mathcal{F})$ non-zero implies that $\mathcal{F}$ must have resilient leaves. The Poincaré-Bendixson theory for codimension one, $C^2$ foliations implies that these resilient leave must be contained in local minimal sets of $\mathcal{F}$. As the Godbillon-Vey class is absolutely continuous with respect to Lebesgue measure, Duminy's Theorem implies that either there exists an exceptional minimal set with positive measure, or there is an open local minimal set for $\mathcal{F}$. The known examples are of the second type, and Matsumoto speculated in [21] that the latter case must always hold.

In the paper [17] by the author with R. Langevin, a new proof of Duminy's Theorem is given, based entirely on methods of ergodic theory, an in particular the proof does not use the Poincaré-Bendixson Theory. One of the key steps is the proof that the Godbillon measure of a $C^1$-foliation is supported on the set $E(\mathcal{F})$. In the particular, for a $C^2$-foliation, the Godbillon-Vey class $GV(\mathcal{F})$ (cf. [3, 11, 12]) localized to a measurable set in the complement of $E(\mathcal{F})$ always vanishes. Combining this result with Theorem 1.5 yields an answer to Matsumoto's question.

**COROLLARY 1.6** *Let $\mathcal{F}$ be a codimension one $C^2$-foliation of a compact manifold $M$. Then the $GV(\mathcal{F})$ class vanishes when restricted to an exceptional minimal set of $\mathcal{F}$. Hence, if $GV(\mathcal{F}) \neq 0$ then $\mathcal{F}$ must have an open local minimal set containing a resilient leaf.*

**Proof:** $\langle GV(\mathcal{F}), \mathbf{K}\rangle = \langle GV(\mathcal{F}), \mathbf{K} \cap E(\mathcal{F})\rangle = 0$. $\square$

Note that if $\mathcal{F}$ is a foliation defined by the suspension of the action of a finitely-presented group acting on the circle, then Corollary 1.6 follows from Corollary 1.2.

The proof of Theorem 1.1 relies on a version of the Denjoy Distortion Lemma for group actions. For each $x \in E(\varphi)$, we show there exists a sequence of uniformly hyperbolic expanding maps with arbitrarily long word length, whose domains contain $x$ in the center and achieve an expansion of the domain to an interval of uniform length $\delta > 0$, where $\delta$ depends upon the group $\Gamma$ and the action $\varphi$ but not the choice of $x$. The construction of these uniform expanders is given in section 4. The $C^{1+\alpha}$ hypothesis is used in the customary way to get strong uniform estimates on the derivatives of these expanding maps. The new idea is that the existence of these uniform expanders implies that every point of $\mathbf{K} \cap E(\varphi)$ has Lebesgue density zero, and hence $\mathbf{K} \cap E(\varphi)$ has measure zero.

The author is indebted to Takashi Inaba, Shigenori Matsumoto and Andrés Navas for very helpful comments, and to Gilbert Hector for discussions about "Hector's Conjecture". The author would also like to thank the faculty and staff of the University of Łodz, Poand for organizing the conference "Foliations 2005" which facilitated these discussions.

## 2 Basic notations

Assume that $\Gamma$ is a finitely generated group. Choose a symmetric generating set $\mathcal{S} = \{\sigma_0, \sigma_1, \ldots, \sigma_k\}$ for $\Gamma$, where $\sigma_0$ is the identity. The symmetric hypothesis means that for each $1 \leq i \leq k$, $\sigma_i^{-1} = \sigma_\ell$ for some $1 \leq \ell \leq k$. An element $\gamma \in \Gamma$ has word length $\|\gamma\| \leq N$ if there exists indices $i_1, \ldots, i_N$ such that $\gamma = \sigma_{i_1} \cdots \sigma_{i_N}$. The word length $\|\gamma\|$ is the least integer $N$ such that $\|\gamma\| \leq N$.

We assume $\mathbb{S}^1$ has the Riemannian metric with total length $2\pi$, and $\theta$ will be used to denote the natural coordinate on $\mathbb{S}^1$. Given a measurable subset $Y \subset \mathbb{S}^1$, let $\mathbf{m}(Y)$ denote Lebesgue measure of $Y$, so that $\mathbf{m}(\mathbb{S}^1) = 2\pi$. If $\chi_Y$ denotes the characteristic function for the set $Y$ then

$$\mathbf{m}(Y) = \int_{\mathbb{S}^1} \chi_Y(\theta)\, d\theta$$

It is convenient to also adopt the notation $[x, y]$ for intervals in $\mathbb{S}^1$, where this means that we consider $x, y \in \mathbb{R}$ and $[x, y] \subset \mathbb{S}^1$ is the image of the usual interval defined by $x, y \in \mathbb{R}$ under the covering mapping $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z} \cong \mathbb{S}^1$.

We assume throughout the paper that $\varphi\colon \Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is a $C^1$-action by orientation-preserving diffeomorphisms. For some results, we require that the action $\varphi$ is $C^{1+\alpha}$, and this will be explicitly stated when needed. Here, $\alpha > 0$ is an exponent of Hölder continuity on the derivatives $\varphi(\sigma_i)'$, $i \leq i \leq k$. That is, there exists $C_\alpha > 0$ so that for all $1 \leq i \leq k$ and $x, y \in \mathbb{R}$ with $\mathbf{m}([x, y]) \leq 1$ then

$$\left|\varphi(\sigma_i)'(y) - \varphi(\sigma_i)'(x)\right| \leq C_\alpha \cdot |y - x|^\alpha \tag{1}$$

It is useful to define a sup norm on the action $\varphi$ by setting

$$\|\varphi\| = \max_{0 \leq i \leq k} \ \max_{x \in \mathbb{S}^1} \left|\log\{\varphi(\sigma_i)'(x)\}\right| \tag{2}$$

## 3 The infinitesimally expansive set

Given $x \in \mathbb{S}^1$ and integer $N \geq 1$, define the function

$$\mu_N(x) = \max\{\varphi'(\gamma)(x) \mid \|\gamma\| \leq N\} \tag{3}$$

The identity transformation has derivative 1, so $\mu_N(x) \geq 1$. The function $\mu_N\colon \mathbb{S}^1 \to \mathbb{R}$ is the maximum of a finite set of continuous functions, so is continuous.

**DEFINITION 3.1** *A point $x \in \mathbb{S}^1$ is* infinitesimally expansive *for $\varphi$ if*

$$\lambda(x) \;=\; \limsup_{N\to\infty} \left\{\frac{\log\{\mu_N(x)\}}{N}\right\} \;=\; \limsup_{||\gamma||\to\infty} \left\{\frac{\log\{\varphi'(\gamma)(x)\}}{||\gamma||}\right\} \;>\; 0 \tag{4}$$

The number $\lambda(x)$ is called the *asymptotic exponent* for $\varphi$ at $x$.

The condition $\lambda(x) > \epsilon$ simply means that there is some sequence of elements $\gamma_n \in \Gamma$ such that $\varphi'(\gamma_n)(x) > \exp(\epsilon \cdot ||\gamma_n||)$ and $||\gamma_n|| \to \infty$. However, we cannot assume that the $\{\gamma_n\}$ are the powers $\gamma_0^n$ of single hyperbolic element $\gamma_0$ as happens in the classical case where $\Gamma$ is cyclic.

**DEFINITION 3.2** *For each $a \geq 0$, $E_a(\varphi) = \{x \in \mathbb{S}^1 \mid \lambda(x) > a\}$ and $E(\varphi) = \bigcup_{a>0} E_a(\varphi)$.*

**LEMMA 3.3** *The function $\lambda\colon \mathbb{S}^1 \to \mathbb{R}$ is Borel measurable, and invariant under the action of $\Gamma$.*

**Proof:** The function $x \mapsto \lambda(x)$ is a limit of continuous functions, hence is Borel.

Let $x \in \mathbb{S}^1$ and $\sigma \in \Gamma$. Then

$$\begin{aligned}
\lambda(\sigma x) &= \limsup_{||\gamma||\to\infty} \frac{\log\{\varphi(\gamma)'(\sigma x)\}}{\|\gamma\|} \\
&= \limsup_{||\gamma||\to\infty} \frac{\log\{\varphi(\gamma\sigma)'(x)\} - \log\{\varphi(\sigma^{-1})'(\sigma x)\}}{\|\gamma\sigma\|} \cdot \frac{\|\gamma\sigma\|}{\|\gamma\|} \\
&= \limsup_{||\gamma\sigma||\to\infty} \frac{\log\{\varphi(\gamma\sigma)'(x)\}}{\|\gamma\sigma\|} = \lambda(x) \qquad \square
\end{aligned}$$

**COROLLARY 3.4** *For each $a \geq 0$, $E_a(\varphi)$ is a Borel measurable, $\Gamma$-invariant subset.* $\square$

The orbits of the points in the set $E(\varphi)$ exhibit chaotic behavior in arbitrarily small neighborhoods of the point [15, 16].

## 4 Uniform expanders

In this section, we introduce the construction of the uniform expanders. There are two forms of the result we need: the first establishes a uniform estimate (6) on the exponential rate of expansion along a well-chosen orbit of a point in $E(\varphi)$ under the assumption that $\varphi$ is a $C^1$-action. The second result shows that if $\varphi$ is $C^{1+\alpha}$, then this additional regularity yields the much more precise estimate (7) of the expansion along an orbit. The first estimate is used in the proof of the more delicate $C^{1+\alpha}$-estimates.

For each generator $\sigma_i \in \Gamma$, the diffeomorphism $\varphi(\sigma_i)$ has continuous derivative on the compact space $\mathbb{S}^1$. Thus, given $\epsilon > 0$ there exists a greatest $\delta = \delta(\epsilon) > 0$ so that if $d(y,z) < \delta$ then

$$|\log\{\varphi(\sigma_i)'(y)\} - \log\{\varphi(\sigma_i)'(z)\}| < \epsilon \quad \text{for all} \quad 0 \leq i \leq k \tag{5}$$

Clearly, $\epsilon' < \epsilon$ implies that $\delta(\epsilon') \leq \delta(\epsilon)$.

The uniform estimate (5) implies the existence of uniform expanding maps for the points in $E(\varphi)$:

**PROPOSITION 4.1** *Assume that $\varphi$ is a $C^1$-action. Given $a > 0$, there exists $\delta > 0$ so that for all $x \in E_a(\varphi)$ and all $n > 0$, there exists $\omega_n \in \Gamma$ with $\|\omega_n\| \geq n$ such that for*

- $\mathbf{g}_n = \varphi(\omega_n)$, $\mathbf{h}_n = \mathbf{g}_n^{-1}$
- $y_n = \mathbf{g}_n(x)$, $x = \mathbf{h}_n(y_n)$
- $\mathcal{J}_n = [y_n - \delta/2, y_n + \delta/2]$
- $\mathcal{I}_n = \mathbf{h}_n(\mathcal{J}_n)$

*then for all $y \in \mathcal{I}_n$ we have*

$$\mathbf{g}_n'(y) > \exp\{\|\omega_n\| \cdot a/2\} \tag{6}$$

*If the action $\varphi$ is $C^{1+\alpha}$ for some $\alpha > 0$, then a much stronger estimate holds. Let $\epsilon_1 = a/40$ and $\delta_1 = \delta(\epsilon_1)$. Set $\mathcal{L}_n = [y_n - \delta_1/2, y_n + \delta_1/2]$ and $\mathcal{I}_n' = \mathbf{h}_n(\mathcal{L}_n)$. Then there exists a constant $C = C(\alpha, a) > 0$ such that for all $n > 0$ and all $y \in \mathcal{I}_n'$ we have*

$$\mathbf{g}_n'(x)/C \;\leq\; \mathbf{g}_n'(y) \leq C\, \mathbf{g}_n'(x) \tag{7}$$

**Proof:** Set $\epsilon_0 = a/20$ and $\delta = \delta(\epsilon_0)$. Let $x \in E_a(\varphi)$ and set $\lambda = \lambda(x) > a$. Note that

$$a/2 < a/2 + \epsilon_0 < a/2 + 9\epsilon_0 < a < \lambda$$

In particular, note that $\lambda - \epsilon_0 > a/2 + \epsilon_0 > \epsilon_0$ and $\lambda - 3\epsilon_0 > a/2$.

By the definition of $\lambda(x)$, for each $n > 0$ we can choose $\gamma_n \in \Gamma$ such that $\|\gamma_n\| \geq n$ and

$$\log\{\varphi(\gamma_n)'(x)\} > (\lambda - \epsilon_0) \cdot \|\gamma_n\| \tag{8}$$

The sequence of maps $\{\varphi(\gamma_n) \mid n = 1, 2, \ldots\}$ satisfies condition (6) when $y = x$, and we must show the estimate holds for all $y \in \mathcal{I}_n$.

The strategy of the proof is to first consider the inverse maps $\varphi(\gamma_n^{-1})$ which are infinitesimal contractions. We modify these inverse mappings $\varphi(\gamma_n^{-1})$ so that they satisfy uniform estimates for their contraction rates along the orbit of the point $z_n = \varphi(\gamma_n)(x)$. The idea is to truncate the "initial factors" in each map $\varphi(\gamma_n^{-1})$ to remove any initial expansive behavior. This allows us to obtain uniform estimates on the contraction rates of each $\varphi(\gamma_n^{-1})$, starting at some new initial point along the orbit, which when combined with the estimate (5) yields the uniform estimates on the inverse maps. Taking inverses of these truncated maps yields uniformly expanding maps. Though the idea is simple and requires only basic calculus, the process of selecting how to truncate the maps is tedious, as it introduces the technical notion of *$\epsilon_0$-regular indices.*

Fix $n > 0$, let $\ell_n = \|\gamma_n\|$ and choose indices $\{i_1, \ldots, i_{\ell_n}\}$ so that $\gamma_n^{-1} = \sigma_{i_{\ell_n}} \cdots \sigma_{i_1}$.

For $1 \leq j \leq \ell_n$ set $\gamma_{n,j}^{-1} = \sigma_{i_j} \cdots \sigma_{i_1}$ and let $\gamma_{n,0}^{-1}$ denote the identity.

Set $z_n = \varphi(\gamma_n)(x)$ and label the orbit of $z_n$ by $z_{n,j} = \varphi(\gamma_{n,j}^{-1})(z_n)$, so $z_{n,\ell_n} = x$.

For each $1 \leq j \leq \ell_n$ set $\mu_{n,j} = \log\{\varphi(\sigma_{i_j})'(z_{n,j-1})\}$. Then $\lambda - \epsilon_0 > \epsilon_0$ yields

$$\mu_{n,1} + \cdots + \mu_{n,\ell_n} = \log\{\varphi(\gamma_n^{-1})'(z_n)\} < \ell_n \cdot (\epsilon_0 - \lambda) < -\ell_n \cdot \epsilon_0 \tag{9}$$

hence

$$\mu_{n,1} + \cdots + \mu_{n,\ell_n} + \ell_n\,\epsilon_0 < 0 \tag{10}$$

An index $1 \leq j \leq \ell_n$ is said to be *$\epsilon_0$-regular* if all of the partial sum estimates hold:

$$\begin{aligned} \mu_{n,j} + \epsilon_0 &< 0 \\ \mu_{n,j} + \mu_{n,j+1} + 2\epsilon_0 &< 0 \\ &\vdots \\ \mu_{n,j} + \cdots + \mu_{n,\ell_n} + (\ell_n - j + 1)\epsilon_0 &< 0 \end{aligned} \tag{11}$$

**LEMMA 4.2** *There exists an $\epsilon_0$-regular index $0 < b_n < \ell_n$ for $\gamma_n^{-1}$ such that*

$$\mu_{n,b_n} + \cdots + \mu_{n,\ell_n} < \ell_n(2\epsilon_0 - \lambda) \tag{12}$$

**Proof:** We say that an index $k \leq \ell_n$ is *$\epsilon_0$-irregular* if $\mu_{n,1} + \cdots + \mu_{n,k} + k\,\epsilon_0 \geq 0$. If there does not exist an $\epsilon_0$-irregular value, then this means that all of the conditions (11) hold starting at $j = 1$, hence $k = 1$ is $\epsilon_0$-regular, and we set $b_n = 1$.

On the other hand, if there exists an $\epsilon$-irregular value $k$, then (10) implies $k < \ell_n$.

Let $k_0$ be the greatest $\epsilon_0$-irregular index. Then $k_0 + 1$ is an $\epsilon_0$-regular index. To see this, let $\ell > k_0$, then we have $\mu_{n,1} + \cdots + \mu_{n,\ell} + \ell\,\epsilon_0 < 0$ and so by (10) and the definition of $\epsilon_0$-irregular we have

$$\begin{aligned} \mu_{n,k_0+1} + \cdots + \mu_{n,\ell} + (\ell - k_0)\,\epsilon_0 &= (\mu_{n,1} + \cdots + \mu_{n,\ell} + \ell\,\epsilon_0) - (\mu_{n,1} + \cdots + \mu_{n,k_0} + k_0\,\epsilon_0) \\ &\leq (\mu_{n,1} + \cdots + \mu_{n,\ell} + \ell\,\epsilon_0) < 0 \end{aligned}$$

It remains to establish the estimate (12). If $b_n = 1$ then by (9)

$$\mu_{n,1} + \cdots + \mu_{n,\ell_n} < \ell_n(\epsilon_0 - \lambda) < \ell_n(2\epsilon_0 - \lambda) \tag{13}$$

If $b_n > 1$ where $k = b_n - 1$ is $\epsilon_0$-irregular, then by (9) and the definition of $\epsilon_0$-irregular we have

$$\begin{aligned}
\mu_{n,k_0+1} + \cdots + \mu_{n,\ell_n} + (\ell - k_0)\,\epsilon_0 &= (\mu_{n,1} + \cdots + \mu_{n,\ell_n} + \ell_n\,\epsilon_0) - (\mu_{n,1} + \cdots + \mu_{n,k_0} + k_0\,\epsilon_0) \\
&\leq (\mu_{n,1} + \cdots + \mu_{n,\ell_n} + \ell_n\,\epsilon_0) \\
&\leq \ell_n(\epsilon_0 - \lambda) + \ell_n\,\epsilon_0 \\
&= \ell_n(2\epsilon_0 - \lambda)
\end{aligned}$$

as was to be shown. □

For each $n \geq 1$ let $b_n = k_n + 1$ be an $\epsilon_0$-regular index for $\gamma_n^{-1}$ satisfying (12). Define

$$\tau_n = \sigma_{i_{\ell_n}} \cdots \sigma_{i_{b_n}}\ ,\ \omega_n = \tau_n^{-1}\ ,\ \mathbf{h}_n = \varphi(\tau_n)\ ,\ \mathbf{g}_n = \varphi(\omega_n) = \mathbf{h}_n^{-1} \tag{14}$$

Note that $\|\omega_n\| = \|\tau_n\| = \ell_n - k_n \leq \|\gamma_n\|$. Set $y_n = \mathbf{g}_n(x)$, so $\mathbf{h}_n(y_n) = x$.

**LEMMA 4.3** *The word length $\|\tau_n\| = \ell_n - k_n$ tends to infinity as $n \to \infty$. More precisely, there is an estimate*

$$\ell_n - k_n \geq \ell_n(\lambda - 2\epsilon_0)/\|\varphi\| \tag{15}$$

**Proof:** By the definition of $\|\varphi\|$ we have $|\log\{\varphi(\sigma_i)'\}| \leq \|\varphi\|$ hence

$$|\log\{\varphi(\tau_n)'| \leq (\ell_n - k_n) \cdot \|\varphi\|$$

Note that $\lambda > 0$ implies $\|\varphi\| > 0$. Then evaluate $\varphi(\tau_n)'$ at $y_n$ and apply (12) to obtain (15). □

The purpose of introducing the concept of the $\epsilon_0$-regular value is that it implies each of the maps $\varphi(\sigma_{i_i} \cdots \sigma_{i_{b_n}})$ for $i \geq i_{b_n}$ is a sufficiently strong linear contraction at $y_n$ so that the map $\mathbf{h}_n$ is a uniform contraction on the interval $[y_n - \delta/2, y_n + \delta/2]$. We make this precise in Lemma 4.4 below.

Set $\mathcal{J}_n = [y_n - \delta/2, y_n + \delta/2]$ and $\mathcal{I}_n = \mathbf{h}_n(\mathcal{J}_n)$.

**LEMMA 4.4** *For each $n > 0$ and $z \in \mathcal{J}_n$*

$$\exp\{-\|\varphi\|\|\tau_n\|\} \leq \mathbf{h}'_n(z) \leq \exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\} \tag{16}$$

$$\delta\exp\{-\|\varphi\|\|\tau_n\|\} \leq |\mathbf{h}_n(\mathcal{J}_n)| \leq \delta\exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\} \tag{17}$$

**Proof:** Fix $n$ and let $\tau_n = \sigma_{i_{\ell_n}} \cdots \sigma_{i_{b_n}}$ be as above.

The chain rule yields the lower bound estimate $-\|\varphi\|\|\tau_n\| \leq \log\{\mathbf{h}'_n(z)\}$ for all $z \in \mathbb{S}^1$, so in particular for $z \in \mathcal{J}_n$. This implies the lower bound estimate in (16).

For $z \in \mathcal{J}_n$, the uniform continuity hypothesis (5) yields

$$\left|\log\{\varphi(\sigma_{i_{b_n}})'(z)\} - \log\{\varphi(\sigma_{i_{b_n}})'(y_n)\}\right| \leq \epsilon_0$$

Thus, by the definition of $\mu_{n,b_n}$ we have for all $z \in \mathcal{J}_n$

$$\exp\{-\epsilon_0 + \mu_{n,b_n}\} \leq \varphi(\sigma_{i_{b_n}})'(z) \leq \exp\{\epsilon_0 + \mu_{n,b_n}\} \tag{18}$$

As $|\mathcal{J}_n| = \delta$ this implies that

$$\delta\exp\{-\epsilon_0 + \mu_{n,b_n}\} \leq |\varphi(\sigma_{i_{b_n}})(\mathcal{J}_n)| \leq \delta\exp\{\epsilon_0 + \mu_{n,b_n}\}$$

By the assumption that $b_n$ is $\epsilon_0$-regular, we have that $\epsilon_0 + \mu_{n,b_n} < 0$ hence $\delta \exp\{\epsilon_0 + \mu_{n,b_n}\} < \delta$.

For $z_1 = \varphi(\sigma_{i_{b_n}})(z) \in \varphi(\sigma_{i_{b_n}})(\mathcal{J}_n)$ we estimate $\varphi(\sigma_{i_{b_n+1}})'(z_1)$ as above to obtain the estimates

$$\exp\{-2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\} \leq \varphi(\sigma_{i_{b_n+1}} \cdot \sigma_{i_{b_n}})'(z) \leq \exp\{2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\}$$

$$\delta \exp\{-2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\} \leq |\varphi(\sigma_{i_{b_n+1}} \cdot \sigma_{i_{b_n}})(\mathcal{J}_n)| \leq \delta \exp\{2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\}$$

Then $b_n$ is $\epsilon_0$-regular implies that $2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1} < 0$, hence $\delta \exp\{2\epsilon_0 + \mu_{n,b_n} + \mu_{n,b_n+1}\} < \delta$.

Repeat this argument $\ell_n - b_n + 1$ times and use (12) to arrive at the upper bound

$$\begin{aligned} \varphi(\tau_n)'(z) &\leq \exp\{(\ell_n - b_n + 1)\epsilon_0 + \mu_{n,\ell_n} + \cdots + \mu_{n,b_n}\} \\ &\leq \exp\{\epsilon_0\|\tau_n\| - (\lambda - 2\epsilon_0)\|\tau_n\|\} \\ &< \exp\{(3\epsilon_0 - \lambda)\|\tau_n\|\} \end{aligned}$$

The estimate (17) follows immediately from (16). □

Now observe that as $\lambda - 3\epsilon_0 > a/2$ the estimate (16) yields the uniform estimate (6) for the inverse map $\mathbf{g}_n$. This completes the proof of the first part of Proposition 4.1.

Note that for all $n > 0$, $x \in \mathcal{I}_n$ is in the interor, and (6) implies the estimate

$$|\mathcal{I}_n| < \delta \cdot \exp\{-\|\omega_n\| \cdot a/2\} \tag{19}$$

The $C^1$-hypothesis is sufficient to obtain the hyperbolic expansion estimate (6), but it should be noted this is not very precise control over the growth rate along the orbit. For example, if $a = 4$ then it estimates that some points in $\mathcal{I}_n$ expand at the rate $4^n$ while others may expand at the rate $2^n$, hence the ratio of the variation over the interval $\mathcal{I}_n$ is itself exponential.

The assumption that the diffeomorphisms generating the action $\varphi$ are $C^{1+\alpha}$ for some $\alpha > 0$ allows for a much more precise estimate on the variation of the derivative along hyperbolically contracting orbits, which will yield the estimate (7)

$$\mathbf{g}_n'(x)/C \;\leq\; \mathbf{g}_n'(y) \leq C\, \mathbf{g}_n'(x)$$

which we must show. First note that the composition of a $C^\infty$-function with a $C^\alpha$-function is $C^\alpha$, so that the functions $\log(\varphi(\sigma_i)')$ also satisfy an $\alpha$-Hölder estimate. That is, there exists $C_\alpha > 0$ so that for all $1 \leq i \leq k$ and $y, z \in \mathbb{R}$ with $|z - y| \leq 1$ then

$$|\log\{\varphi(\sigma_i)'(z)\} - \log\{\varphi(\sigma_i)'(y)\}| < C_\alpha \cdot |z - y|^\alpha \tag{20}$$

Also, in order to get a uniform estimate of rates of contraction along the points $z_{n,j}$ we need to restrict the domain of $\mathbf{h}_n$ further. Set $\epsilon_1 = \epsilon_0/2 = a/40$ and $\delta_1 = \delta(\epsilon_1)$.

Set $\mathcal{L}_n = [y_n - \delta_1/2, y_n + \delta_1/2] \subset \mathcal{J}_n$. Then for $z \in \mathcal{L}_n$ we have $|z - y_n| \leq \delta_1$ hence (20) yields

$$\mu_{n,b_n} - C_\alpha \cdot \delta_1^\alpha \;\leq\; \log\{\varphi(\sigma_{i_{b_n}})'(z)\} \;\leq\; \mu_{n,b_n} + C_\alpha \cdot \delta_1^\alpha \tag{21}$$

We can also use the restricted domain $z \in \mathcal{L}_n$ to improve the estimate (18) by replacing $\epsilon_0$ with $\epsilon_1$

$$\exp\{-\epsilon_1 + \mu_{n,b_n}\} \leq \varphi(\sigma_{i_{b_n}})'(z) \leq \exp\{\epsilon_1 + \mu_{n,b_n}\} \tag{22}$$

The assumption that $b_n$ is $\epsilon_0$-regular implies

$$\epsilon_1 + \epsilon_1 + \mu_{n,b_n} = \epsilon_0 + \mu_{n,b_n} < 0$$

hence $\epsilon_1 + \mu_{n,b_n} < -\epsilon_1$. That is, (22) implies $\varphi(\sigma_{i_{b_n}})'(z) \leq \exp\{-\epsilon_1\}$ for all $z \in \mathcal{L}_n$. From this we conclude that $\varphi(\sigma_{i_{b_n}})(\mathcal{L}_n)$ is an interval of length at most $\delta_1 \cdot \exp\{-\epsilon_1\}$.

For $z \in \mathcal{L}_n$ set $z_1 = \varphi(\sigma_{i_{b_n}})(z)$ so that for $y_{n-1} = \varphi(\sigma_{i_{b_n}})(y_n)$ we have $|z_1 - y_{n-1}| \leq \delta_1 \cdot \exp\{-\epsilon_1\}$. Thus,

$$\mu_{n,b_{n-1}} - C_\alpha\,(\delta_1 \,\exp\{-\epsilon_1\})^\alpha \;\leq\; \log\{\varphi(\sigma_{i_{b_{n-1}}})'(z_1)\} \;\leq\; \mu_{n,b_{n-1}} + C_\alpha\,(\delta_1 \,\exp\{-\epsilon_1\})^\alpha \tag{23}$$

As in the proof of Lemma 4.4, after iterating the above two-step argument $\ell_n - b_n + 1$ times we obtain

$$(\mu_{n,\ell_n} + \cdots + \mu_{n,b_n}) - \log\{C\} \;\leq\; \log\{\mathbf{h}_n'(z)\} \;\leq\; (\mu_{n,\ell_n} + \cdots + \mu_{n,b_n}) + \log\{C\} \tag{24}$$

where $C = C(\alpha, a) > 0$ is the sum of the estimates on the variations of the derivatives for each map in the product of gerators defining $\mathbf{h}_n$ so

$$\begin{aligned}\log\{C\} &= C_\alpha\,\{(\delta_1)^\alpha + (\delta_1 \exp\{-\epsilon_1\})^\alpha + (\delta_1 \exp\{-2\epsilon_1\})^\alpha + \cdots\} \\ &= C_\alpha \delta_1^\alpha\,\{1 + \exp\{-\alpha\epsilon_1\}) + \exp\{-2\alpha\epsilon_1\} + \cdots\}\end{aligned}$$

Since $\log\{\mathbf{h}_n'(y_n)\} = (\mu_{n,\ell_n} + \cdots + \mu_{n,b_n})$ this yields the estimate

$$\mathbf{h}_n'(y_n)/C \leq \mathbf{h}_n'(z) \leq C\mathbf{h}_n'(y_n)$$

Noting that $\mathbf{h}_n(y_n) = x$ and $\mathbf{h}_n^{-1} = \mathbf{g}_n$ we thus obtain (7), completing the proof of Proposition 4.1.

## 5 Proof of Theorem 1.1

Assume that $\varphi$ is a $C^{1+\alpha}$-action. We show that if $\mathbf{K}$ is a minimal set such that $\mathbf{K} \,\cap\, E(\varphi)$ has positive Lebesgue measure, then $\mathbf{K} = \mathbb{S}^1$.

The minimal set $\mathbf{K}$ cannot be a finite set of points as it has positive measure, nor can it be of Denjoy type as $\mathbf{K} \cap E(\varphi) \neq \emptyset$ implies that all orbits of $\varphi$ in $\mathbf{K}$ have exponential growth [15].

Suppose that $\mathbf{K}$ is a hyperbolic exceptional minimal set. We show that in this case, the assumption $\mathbf{K} \cap E(\varphi)$ has positive Lebesgue measure leads to a contradiction. Recall that $\mathbf{m}$ denotes the standard Lebesgue measure on $\mathbb{S}^1$. We need an elementary fact.

**LEMMA 5.1** *Let $\mathbf{K}$ be a hyperbolic exceptional minimal set. Then for any $\delta > 0$, there exists $\mu(\delta) < \delta$ such that for all $x \in \mathbb{S}^1$,*

$$\mathbf{m}([x - \delta/2, x + \delta/2] \cap \mathbf{K}) \leq \mu(\delta)$$

**Proof:** Let $\chi_{\mathbf{K}}$ denote the characteristic function of the set $\mathbf{K}$. Then the function

$$\mu(\delta, x) = \int_{x-\delta/2}^{x+\delta/2} \chi_{\mathbf{K}}(y)\, dy$$

is continuous and has range in the interval $[0, \delta]$. Let $\mu(\delta)$ denote the maximum value of $\mu(\delta, x)$ for $x \in \mathbb{S}^1$. Then $\mu(\delta) \leq \delta$, and if equality holds then there is $x$ such that $[x - \delta/2, x + \delta/2] \cap \mathbf{K}$ has measure $\delta$. This is impossible as the open interval $(x - \delta/2, x + \delta/2)$ must intersect an open set in the complement of $\mathbf{K}$, so $(x - \delta/2, x + \delta/2) \cap (\mathbb{S}^1 - \mathbf{K})$ has positive measure. Thus, $\mu(\delta) < \delta$. □

Suppose $\mathbf{m}(\mathbf{K} \cap E(\varphi)) > 0$, then there exists $a > 0$ such that $\mathbf{m}(\mathbf{K} \cap E_a(\varphi)) > 0$. Let $x \in \mathbf{K} \cap E_a(\varphi)$ be a point of Lebesgue density 1. That is, for all $\epsilon > 0$ there exists a $\nu(\epsilon) > 0$ so that if $[a, b] \subset \mathbb{S}^1$ is an interval with $x \in [a, b]$ and $\mathbf{m}([a, b]) \leq \nu(\epsilon)$, then

$$\mathbf{m}([a, b] \cap \mathbf{K} \cap E_a(\varphi)) \geq (1 - \epsilon) \cdot \mathbf{m}([a, b]) \tag{25}$$

In particular, $\mathbf{m}([a, b] \cap \mathbf{K}) \geq (1 - \epsilon) \cdot \mathbf{m}([a, b])$ and hence

$$\mathbf{m}([a, b] \cap (\mathbb{S}^1 - \mathbf{K})) \leq \epsilon \cdot \mathbf{m}([a, b]) \tag{26}$$

By the $C^{1+\alpha}$-case of Proposition 4.1, there exists $\delta_1 > 0$, $\alpha > 0$ and $C = C(\alpha, a) > 0$ so that we can choose for each $n > 0$ an element $\omega_n \in \Gamma$ with $\|\omega_n\| \geq n$ such that for

- $\mathbf{g}_n = \varphi(\omega_n)$, $\mathbf{h}_n = \mathbf{g}_n^{-1}$
- $y_n = \mathbf{g}_n(x)$,
- $\mathcal{J}_n = [y_n - \delta_1/2, y_n + \delta_1/2]$
- $\mathcal{I}_n = \mathbf{h}_n(\mathcal{J}_n)$

so $x \in \mathcal{I}_n$, $\mathbf{g}_n : \mathcal{I}_n \to \mathcal{J}_n$, and for all $y \in \mathcal{I}_n$ we have

$$\mathbf{g}_n'(y) > \exp\{\|\omega_n\| \cdot a/2\} \tag{27}$$

and

$$\mathbf{g}_n'(x)/C \ \leq \ \mathbf{g}_n'(y) \leq C \, \mathbf{g}_n'(x) \tag{28}$$

Then by (27) we have

$$\mathbf{m}(\mathcal{I}_n) = \mathbf{m}(\mathbf{h}_n(\mathcal{J}_n)) \leq \delta_1 \cdot \exp\{-\|\omega_n\| \cdot a/2\} \leq \delta_1 \cdot \exp\{-na/2\}$$

Hence, for all $\epsilon > 0$ there exists $n$ such that $\mathbf{m}(\mathcal{I}_n) \leq \nu(\epsilon)$ and therefore by (26)

$$\mathbf{m}(\mathcal{I}_n \cap (\mathbb{S}^1 - \mathbf{K})) \leq \epsilon \cdot \mathbf{m}(\mathcal{I}_n)$$

By the estimate (28) we have that

$$\mathbf{m}(\mathbf{g}_n(\mathcal{I}_n \cap (\mathbb{S}^1 - \mathbf{K}))) \leq \epsilon \, C \, \mathbf{g}_n'(x) \cdot \mathbf{m}(\mathcal{I}_n) \tag{29}$$

The estimate (28) also implies that

$$\delta_1 = \mathbf{m}(\mathcal{J}_n) \geq \mathbf{g}_n'(x) \cdot \mathbf{m}(\mathcal{I}_n)/C \tag{30}$$

As the set $\mathbf{K}$ is invariant, we combine the estimates (29) and (30) to obtain

$$\mathbf{m}(\mathcal{J}_n \cap (\mathbb{S}^1 - \mathbf{K}))) = \mathbf{m}(\mathbf{g}_n(\mathcal{I}_n \cap (\mathbb{S}^1 - \mathbf{K}))) \leq \epsilon \, \delta_1 \, C^2$$

This implies

$$\mu(\delta_1) \geq \mathbf{m}(\mathcal{J}_n \cap \mathbf{K}) \geq (1 - \epsilon \, C^2)\delta_1$$

As $\epsilon > 0$ was arbitrary and $C$ is independent of $\epsilon$, this implies $\mu(\delta_1) = \delta_1$, a contradiction.

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