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PROBLEM SET

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1 Introduction

This is a collection of problems and related comments concerning the geometry, topology and dynamics of smooth foliations. Most of the problems discussed here were proposed by participants in the conference "Foliations 2005" held in Lodz, Poland during June 14–23, 2005. This problem set also includes a selection of more venerable problems, which were included in the unpublished problem set prepared for the conference "Geometry and Foliations 2003" held at Ryukoku University in Kyoto, Japan from September 10–19, 2003 [72].

"Foliation problem sets" have a long tradition in the study of this subject – they highlight progress in areas of research, and hopes for progress. Smale's celebrated survey in 1967 of dynamical systems [123] might be considered the first foliation problem set, as many of the questions about dynamical systems lead to questions about the properties of foliations associated to dynamical system. Since then, there have been several collections of published problem sets specifically about foliation theory: from Stanford 1976, compiled by Mark Mostow and Paul Schweitzer [106]; from Rio de Janeiro 1976, compiled by Paul Schweitzer [120]; from Rio de Janeiro 1992, compiled by Rémi Langevin [90]; from Santiago do Compostela 1994, compiled by Xosé Masa and Enriqué Macias-Virgós [93]. There was no general problem set published after the meeting Warsaw 2000, although the survey [71] formulated open problems in the area of foliation dynamics and secondary classes, and some of the problems proposed at Warsaw 2000 are included in this current problem set.

These problems were compiled by the author, based on the presenters slides from the problem session and notes taken during the talks. In some cases, tex files were prepared by the presenters. Special thanks go to Ken Richardson for his lengthy tex file produced during the problem session, which helped in the preparation of various problems.

Effort has been made to accurately represent the questions posed; any errors and misrepresentations are the consequence of the author's attempts to reconstruct the precise statements of the questions posed, and are totally the responsibility of the author. References were provided by the author in most cases; inevitable omissions and incorrect citations are again the responsibility of the author.

2 Ergodic theory and dynamics

2.1 Commuting transformations

Suggested by Paweł Walczak

Let X be a compact metric space, $C(X)$ the space of continuous functions in the uniform topology, and $\mathfrak{M}(X)$ the space of Borel probability measures. An application of the Hahn-Banach Theorem yields the following

Lemma: Let $\mathcal{V} \subset \mathfrak{M}(X)$ be a linear subspace. The the following are equivalent:

1. There exists $\mu \in \mathfrak{M}(X)$ such that for all $f \in \mathcal{V}$, we have $\displaystyle\int f \, d\mu = 0$.
2. For all $f \in \mathcal{V}$, there exists $x \in X$ such that $f(x) \geq 0$.

Application 1: Let $T \colon X \to X$ be a continuous transformation. Then there exists $\mu \in \mathfrak{M}(X)$ such that $\mu(f \circ T) = \mu(f)$ for all $f \in C(X)$. This follows by taking $\mathcal{V} = \mathrm{Span}\{f - f \circ T \mid f \in C(X)\}$.

Application 2: Let X be a Riemannian manifold and $\mathcal{F}$ a foliation of X. A measure $\mu \in \mathfrak{M}(X)$ is harmonic if $\displaystyle\int_X (\Delta_F f) \, d\mu = 0$ for all $f \in C^2(X)$. Then there exists a harmonic measure for $\mathcal{F}$..

Application 3: Let $\{T_1, \ldots, T_n\}$ be a finite set of of pairwise commuting continuous transformations of X. Then there exists $\mu \in \mathfrak{M}(X)$ which is T_i-invariant for all $1 \leq i \leq n$, and hence

$$\int_M \sum_{i=1}^{n} (f_i - f_i \circ T_i) d\mu = 0,$$

By the Lemma, for $\mathcal{V} = \mathrm{Span}\Big\{ \sum_{i=1}^{n} f_i - f_i \circ T \mid f_i \in C(X) \Big\}$, this implies

(*) *there exist points x and $y \in X$ for which*

$$\sum_{i=1}^{n} (f_i - f_i \circ T_i)(x) \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} (f_i - f_i \circ T_i)(y) \leq 0.$$

Problem: Find an elementary (that is, constructive) proof of (*).

If $n = 1$, such a proof is obvious: $(f - f \circ T)(x) \geq 0$ and $(f - f \circ T)(y) \leq 0$ at points x and y where f achieves, respectively, its maximum and minimum.

For a reference on this question, see Section 4.1 of [137].

2.2 Dynamics of Hölder homeomorphisms

Suggested by Takashi Tsuboi

The theory of dynamics for actions of subgroups of $\mathrm{Diff}^r(\mathbb{S}^1)$ on the circle $\mathbb{S}^1$, for $r \geq 2$ is considered the "classical" case, as both the Denjoy Theorem and the Koppell Lemma are true for such actions, and they have been extensively studied.

In recent years, there have also been a variety of results concerning group actions on $\mathbb{S}^1$ by $C^{1+\alpha}$ diffeomorphisms, for $0 \leq \alpha < 1$ (so $1 \leq r < 2$) – see [44, 68, 70, 74, 108, 109, 110, 130].

However, for subgroups of $\mathrm{Diff}^r(\mathbb{S}^1)$ in the range between the group $\mathrm{Homeo}(\mathbb{S}^1)$ of homeomorphisms of the circle and the C^1-diffeomorphisms, there are few results.

Introduce two special subgroups of $\mathrm{Homeo}(\mathbb{S}^1)$:

- $\mathrm{Diff}^{1-0}(\mathbb{S}^1) = \{h \in \mathrm{Homeo}(\mathbb{S}^1) \mid h \text{ is } \alpha \text{ Holder for all } \alpha < 1\}$
- $\mathrm{Diff}^{+0}(\mathbb{S}^1) = \{h \in \mathrm{Homeo}(\mathbb{S}^1) \mid h \text{ is } \alpha \text{ Holder for some } \alpha > 0\}$

Problem: What can be proven about the dynamics of subgroups of either $\mathrm{Diff}^{1-0}(\mathbb{S}^1)$ or $\mathrm{Diff}^{+0}(\mathbb{S}^1)$?

2.3 Rotation numbers

Suggested by Shigenori Matsumoto

Theorem: (Katok-Anosov [11]) There exists an area-preserving C^∞-map $f\colon \mathbb{S}^1 \times [0,1] \to \mathbb{S}^1 \times [0,1]$ without a compact f-invariant set in the interior.

Problem: For which $\alpha \in \mathbb{Q}/\mathbb{Z}$ does there exists an f as above with rotation number $\rho(f) = \alpha$?

2.4 Foliation entropy and transverse expansion

Suggested by Steve Hurder

The geometric entropy $h_g(\mathcal{F})$ for a C^1–foliation $\mathcal{F}$ of a closed manifold M remains a mystery some 20 years after its introduction [57] . The geometric entropy $h_g(\mathcal{F})$ measures the exponential rate of growth for (ϵ, n)-separated sets in the analogue of the Bowen metrics for the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{F}$. Thus, $h_g(\mathcal{F})$ is a measure of the complexity of the transverse dynamics of $\mathcal{F}$. The precise value of $h_g(\mathcal{F})$ depends upon a variety of choices, but the property $h_g(\mathcal{F}) = 0$ or $h_g(\mathcal{F}) > 0$ is well-defined. In codimension-one, the relation between $h_g(\mathcal{F}) > 0$ and chaotic leaf dynamics is now well-understood [57, 70, 76]. In particular, $h_g(\mathcal{F}) > 0$ implies $\mathcal{F}$ has a resilient leaf. The goal of obtaining a corresponding theory of foliation measure entropy and a maximum principle are still open.

Ghys, Langevin and Walczak showed in [57] that $h_g(\mathcal{F}) = 0$ implies there exists a transverse invariant measure for $\mathcal{F}$. The absence of a transverse invariant measure implies that $\mathcal{F}$ has no leaves of nonexponential growth, but the converse is false, as there are Riemannian foliations with all leaves of exponential growth.

Problem: Suppose $\mathcal{F}$ is a C^1 (or possibly C^2) foliation of codimension $q > 1$. What does $h_g(\mathcal{F}) > 0$ imply about recurrence properties of the leaves of $\mathcal{F}$. Formulate qualitative dynamical properties of a foliation which are implied by $h_g(\mathcal{F}) > 0$, and are sufficient to imply $h_g(\mathcal{F}) > 0$.

3 Minimal sets

A minimal set for a foliation $\mathcal{F}$ is a closed saturated subset $\mathbf{K} \subset M$, such that there is no proper closed saturated subset $\mathbf{K}' \subset \mathbf{K}$. Equivalently, $\mathbf{K}$ is minimal is every leaf of $\mathcal{F}$ in $\mathbf{K}$ is dense in $\mathbf{K}$.

Minimal sets are fundamental to understanding dynamics. There are many results about the minimal sets for foliations of codimension one, though fundamental questions remain unsolved. For foliations of higher codimension (greater than one) there are very few general results, and many open questions.

3.1 Codimension one

Some venerable problems

Let $\mathbf{K}$ be an exceptional minimal set of a codimension-one, C^2-foliation $\mathcal{F}$ of a closed n-manifold M. The following have proven to be difficult problems to solve, yet their solutions are fundamental for a complete understanding of the dynamics of codimension one minimal sets.

Problem 1: (Dippolito) Let $L \subset \mathbf{K}$ be a semiproper leaf of $\mathcal{F}$, $x \in L$ and let $H_x(L, \mathbf{K})$ be the germinal homology group of L at x relative to $\mathbf{K}$. Prove that $H_x(L, \mathbf{K})$ is infinite cyclic.

Hector proved in his thesis [62] that the infinite jet of holonomy is infinite cyclic. The more precise form of the problem is to show that $H_x(L, \mathbf{K})$ is generated by a contraction.

Problem 2: (Hector) Prove that $M \backslash \mathbf{K}$ has only finitely many components. That is, show that $\mathbf{K}$ has only a finite number of semi-proper leaves.

This is known to be false for C^1 foliations.

Problem 3: Show that the Lebesgue measure of $\mathbf{K}$ is zero.

The measure of $\mathbf{K}$ has is known to be zero for special cases [79, 95, 80].

Problem 4: Show that every leaf of $\mathbf{K}$ has a Cantor set of ends.

Duminy's Theorem [36] shows that the semiproper leaves of $\mathbf{K}$ must have a Cantor set of ends.

Cantwell and Conlon showed that if $\mathbf{K}$ is Markov (i.e, the holonomy pseudogroup $\Gamma \mid K$ is generated by a (1-sided) subshift of finite type), then all four of the above problems are true [34, 35].

3.2 Minimal sets for affine interval exchanges

Suggested by Gilbert Hector

A bijection $f\colon [0,1] \to [0,1]$ is an *Affine Interval Exchange Transformation* (AIET) if there exists a finite sequence

$$0 = a_0 < a_1 < \cdots < a_p = 1$$

such that for each $1 \leq i \leq p$, the restriction $f_i \colon [a_{i-1}, a_i) \to [0,1]$ is affine. If each map f_i is an isometry and the images are disjoint, then f is an *Interval Exchange Transformation* (IET) in the usual sense.

Keane's Theorem [82, 83] characterizes the usual IET's which are minimal.

Carlos Gutierrez constructed in [28, 58] an AIET admitting an exceptional minimal set $\mathbf{K}$, which has Lebesgue measure zero.

Problem: Show that any exceptional minimal set for an AIET must have Lebesgue measure zero.

3.3 Exotic minimal sets

Suggested by Steve Hurder

We say that a minimal set is *exotic* if it is exceptional, and not locally homeomorphic to the product of a manifold and a totally disconnected Cantor set (cf. Kennedy and Yorke [84].) In other words, an exotic minimal set is locally homeomorphic to the product of a manifold with a non-discrete continua.

The closure of every leaf $L \subset \mathbf{K}$ in a minimal set is equal to $\mathbf{K}$ itself, which implies that the orbit of every point is recurrent, and so the dynamics of $\mathbf{K}$ has a type of "non-linear local self-similarity".

Problem 1: Given a closed manifold X and a nowhere dense connected continua $\mathbf{K} \subset M$, is there a foliation $\mathcal{F}$ on a closed manifold M with exceptional minimal set homeomorphic to $\mathbf{K}$?

The only known "obstruction" is that a minimal set must be "locally homogeneous", in that every orbit is dense so any locally-defined property of $\mathbf{K}$ must occur at a dense set of points in $\mathbf{K}$.

In the paper [19], the authors introduced the notion of a pseudogroup generated by a system of étale correspondences $\mathcal{C}$ on X, which generalizes the standard concept of group generated by a system of diffeomorphisms. A variant of Problem 1 is to ask whether a given continua $\mathbf{K}$ is a minimal set for a system of correspondences?

Haefliger has posed the problem of determining which compactly generated pseudogroups can be realized as the pseudogroup of a foliation on a closed manifold [60].

Problem 2: Given a compact manifold M without boundary, is there a general description of the pseudogroups modeled on M which can be realized up to pseudogroup equivalence by a system of étale correspondences?

4 Group actions

4.1 Commutators of diffeomorphisms

Suggested by Hiroki Kodama

Problem 1: Let M be a closed manifold, and $f\colon M \to M$ a diffeomorphism. Does there exists diffeomorphisms $g, h\colon M \to M$ such that $f = [g, h] = g \circ h \circ g^{-1} \circ h^{-1}$?

Problem 2: Suppose that $f_t\colon M \to M$ is a smooth 1-parameter family of diffeomorphisms. Does there exists a smooth family of diffeomorphisms $g_t, h_t\colon M \to M$ such that $f_t = [g_t, h_t]$?

Both problems have solutions if we allow more than one commutator.

4.2 Area-preserving diffeomorphisms

Suggested by Takashi Tsuboi

Let $\mathbb{D}^2 \subset \mathbb{R}^2$ be the closed unit disk with boundary $\partial\mathbb{D}^2$ Let $\Omega = dx \wedge dy$ denote the standard volume form on $\mathbb{R}^2$.

Problem: Study the inclusion $\mathrm{Diff}_\Omega(\mathbb{D}^2, \mathrm{rel}\,\partial\,\mathbb{D}^2) \to \mathrm{Homeo}_\Omega(\mathbb{D}^2, \mathrm{rel}\,\partial\,\mathbb{D}^2)$. In particular, can an area preserving diffeomorphism be written as a product of commutators of measure preserving homeomorphisms?

Note that there exists a Calabi homomorphism [16, 99] $\Phi\colon \mathrm{Diff}_\Omega(\mathbb{D}^2, \mathrm{rel}\,\partial\,\mathbb{D}^2) \to \mathbb{R}$ which is an an obstruction to writing a diffeomorphism h as a product of commutators if $\Phi(h) \neq 0$. It is known that Φ does not extend to the group $\mathrm{Homeo}_\Omega(\mathbb{D}^2, \mathrm{rel}\,\partial\,\mathbb{D}^2)$.

4.3 Cross-sections

Suggested by Sergey Maksymenko

Let G be a Lie group, X a smooth manifold, and $\varphi\colon G \times X \to X$ a smooth action.

For every $x \in X$, let $\mathcal{O}_x$ denote the orbit of x with respect to the action.

Let $h\colon X \to X$ be a smooth mapping such that $h(\mathcal{O}_x) \subset \mathcal{O}_x$ for every $x \in X$. Then, for every $x \in X$ there exists a (not necessarily unique) $g(x) \in G$ such that $h(x) = g(x) \cdot x$. In general, the function $x \mapsto g(x)$ is not even continuous in X. If the action φ is fixed-point free, then $g(x)$ is uniquely determined. The following problem was studied in [91, 92].

Problem: Find conditions on the action φ and the map h such that there exists a *smooth* mapping $g\colon X \to G$ satisfying $h(x) = g(x) \cdot x$.

Example 1: Let $X = G$ and $G \times G \to G$ be the product. Given $h\colon G \to G$, set $g(x) = h(x) \cdot x^{-1}$ then $h(x) = g(x) \cdot x$ trivially.

Example 2: Let $G = \mathbf{SO}(n, \mathbb{R})$ and $X = \mathbb{R}^n$. Let $h\colon \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism such that $\|h(x)\| = \|x\|$ for all $x \in \mathbb{R}^n$. Thus, h preserves the concentric spheres in $\mathbb{R}^n$. In particular, $h(0) = 0$. Then the problem is to find a *smooth* mapping $A\colon \mathbb{R}^n \to \mathbf{SO}(n, \mathbb{R})$ such that $h(x) = A(x) \cdot x$. The difficult point is to choose $A(x)$ so that it is smooth at $x = 0$. This is possible for $n = 2$.

Example 3: Let $\varphi(x, t) = \exp(A \cdot t) \cdot x$ be an exponential flow on $\mathbb{R}^n$ for some non-invertible matrix $A \neq 0$. This defines the action $\varphi\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. Then for all $h\colon \mathbb{R}^n \to \mathbb{R}^n$ such that h preserves the trajectories of φ, we have that $h(x) = \exp(A \cdot \alpha(x)) \cdot x$, where $\alpha(x)$ is a smooth function.

5 Holomorphic foliations

5.1 Complex codimension one

Suggested by Taro Asuke

Let M be a closed manifold, and $\mathcal{F}$ a transversally holomorphic foliation of complex codimension one. Ghys, Gómez-Mont, and Saludes [56] introduced a decomposition of M into two disjoint subsets:

• The **Fatou set** *Fatou*$(\mathcal{F})$ is the set of points $x \in M$ where $X(x) \neq 0$ for some $X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$ where this cohomology group is the complex vector space of continuous sections X of the normal bundle $\nu^{1,0}$ which are constant along the leaves, and that have distributional derivatives in $L^2(M)$ with $\overline{\partial} X$ essentially bounded.

• The **Julia set** *Julia*$(\mathcal{F})$ is the closed subset of M defined by $X(x) = 0$ for all $X \in H^0(M, \mathcal{C}_{\mathcal{F}}(\nu^{1,0}))$.

The Fatou set of $\mathcal{F}$ can be thought of as the set of all points where there is some non-trivial, foliation-preserving infinitesimal deformation of $\mathcal{F}$, while the Julia set of $\mathcal{F}$ is all points where $\mathcal{F}$ is essentially rigid under infinitesimal deformations.

It is known [15] that if *Julia*$(\mathcal{F})$ is empty, then the Bott Class $\beta(\mathcal{F}) = 0$ except in some "trivial" cases. This is analogous to Duminy's Theorem [45] that if a C^2-foliation has no exceptional minimal sets, then the Godbillon-Vey class $GV(\mathcal{F}) = 0$.

Problem 1: Are there other dynamical properties which the Julia set

Julia$(\mathcal{F})$ of a complex codimension-one foliation $\mathcal{F}$ possesses, similar to those of exceptional minimal sets for C^2-foliations of real codimension one? For example, must the Lebesgue measure of *Julia*$(\mathcal{F})$ be zero, again except for some trivial examples?

Problem 2: Does there exists a holonomy invariant, non-trivial measurable line field on *Julia*$(\mathcal{F})$?

6 Topology of foliations

6.1 Homology vanishing cycles

Suggested by Paul Schweitzer

Let M^n be a closed m-manifold and $\mathcal{F}$ a codimension-one foliation of M, so that the leaves of $\mathcal{F}$ have dimension $p = m-1$. Recall the following definition from [2]. A *generalized Reeb component* for $\mathcal{F}$ is a compact m-submanifold R with boundary ∂R, such that there is

- R is $\mathcal{F}$-saturated
- ∂R consists of compact leaves of $\mathcal{F}$
- $\mathrm{Int}(R) = R - \partial R$ fibers over $\mathbb{S}^1$ and the fibers are leaves of $\mathcal{F}$
- there exists a transverse vector field $\vec{n}$ to $\mathcal{F}|R$ which is pointing inwards along ∂R.

Miyoshi [102, 103] studies the existence of generalized Reeb components in various foliated manifolds. See also Alcalde Cuesta [42] and Alcalde Cuesta and Hector [1].

The motivation for this definition is the following. Assume there is given an oriented compact connected p-dimensional manifold with non-empty boundary $B = \partial C$, and an embedding $\phi\colon C \hookrightarrow \mathrm{Int}(C)$ of C into its interior. The product manifold $C \times [0,\infty)$ is given the product foliation with leaves $C \times \{t\}$ for $0 \le t < \infty$. Define an equivalence relation $(x,t) \sim (\phi(x), t+1)$ for $x \in C$. Then

$$R_0 = \Big(C \times [0,\infty)\Big)/\sim \tag{1}$$

is an open m-manifold, and one can attach the compact p-manifold

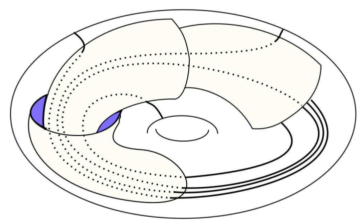
$$L' = \Big((C - \phi(C)) \times [0,\infty)\Big)/x \sim \phi(x) \tag{2}$$

as the boundary of R_0. The union $R(C,\phi) = R = R_0 \cup L'$ has a foliation $\mathcal{F}_R$ whose interior leaves are homeomorphic to the non-compact p-manifold

$L = (C \times \mathbb{N})/(x, n) \sim (\phi(x), n+1)$, and there is one boundary leaf L'. The foliated manifold with boundary $R(C, \phi), \mathcal{F}_R)$ is called a *generalized Reeb component*.

Note that $R(C, \phi)$ fibers over the circle $\mathbb{S}^1 \cong [0, \infty)/t \sim t+1$ and the vector field $\partial/\partial t$ on $C \times [0, \infty)$ is invariant under $\sim$ so descends to a vector field $\vec{n}$ on $R(C, \phi)$ which is inward pointing on the boundary leaf L'.

The standard example of the 3-dimensional Reeb component is obtained by taking $C = \mathbb{D}^2$ the unit disk in the plane, and $\phi \colon \mathbb{D}^2 \to \mathbb{D}^2$ an embedding, such as $\phi(x) = x/2$. Then $R(C, \phi)$ is diffeomorphic to the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$, and L' is diffeomorphic to the boundary torus $\mathbb{S}^1 \times \mathbb{S}^1$.



The point of this construction is that the boundary $B = \partial C$ maps to a $(p-1)$-cycle $Z \subset L'$, and its homology class $[L']$ is non-zero in $H_{p-1}(L')$ as it admits a transverse circle in L' from the construction of L'. However, B is obviously a boundary in C, and thus is also a boundary in each of the leaves in the interior R_0. Thus, $R(C, \phi), \mathcal{F}_R)$ carries a codimension-two vanishing cycle in the more general sense of Sullivan [124].

Recall in dimension 3, Novikov's Theorem [111] states that a 1-dimensional (homotopy) vanishing cycle is contained in a Reeb component. In dimension 4, Alcalde Cuesta, G. Hector, and P. Schweitzer proved:

Theorem: ([2]) Let $\mathcal{F}$ be a codimension-one C^2-foliation of a closed 4-manifold M. If $\mathcal{F}$ has a 2-dimensional homological vanishing cycle, then it belongs to a generalized Reeb component.

Novikov's Theorem actually has two parts: the first asserts the existence of a vanishing cycle, and second part proves that every vanishing cycle lies on the boundary of a Reeb component.

Problem 1: Let M be a closed 4-manifold, and $\mathcal{F}$ a codimension-one foliation. Find geometric or topological conditions on M and the leaves of $\mathcal{F}$ which are sufficient to imply the existence of a 2-dimensional homology vanishing cycle.

Of course, the same problem can be asked for codimension-two homology vanishing cycles for M of dimension n, but this is getting ahead of the problem, as almost nothing is known about the problem in dimension 4.

6.2 Tangential category

Suggested by Elmar Vogt

Let M be a closed manifold, and $\mathcal{F}$ a C^r-foliation of M. A set $U \subset M$ is called *tangentially categorical* if there exists a leafwise homotopy $h_t : U \to M$, $0 \leq t \leq 1$, where $h_0 : U \to M$ is the inclusion, and $h_1 : U \to M$ sends every leaf of $\mathcal{F}|U$ to a point. This notion was introduced by Hellen Colman in her thesis [37, 39].

Problem: Suppose that the leaves of $\mathcal{F}$ are given by the fibers of an $\mathbb{S}^2$-fiber bundle $M \to B$. Assume that the bundle $M \to B$ does not admit a section. Can M be covered by two tangentially categorical open sets?

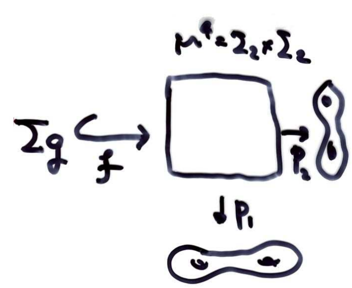
It is known that M can be covered by three, but not by one, tangentially categorical open sets.

The more general problem is to find effective techniques for calculating the tangential category, but this is a very difficult problem.

6.3 Transverse Euler class

Suggested by Yoshihiko Mitsumatsu and Elmar Vogt

Let $M = \Sigma_2 \times \Sigma_2$ be the closed 4-manifold which is the product of two Riemann surfaces, each of genus 2. Let $p_1 : M \to \Sigma_2$ be the projection onto the first factor. This is illustrated below:



Problem 1: Does there exists an embedding $f : \Sigma_g \to M$ of a surface Σ_g of genus g such that $p_1 \circ f : \Sigma_g \to \Sigma_2$ is a covering, and the homological self-intersection $[f(\Sigma_g)] \cap [f(\Sigma_g)] > 4$?

Remark: We know that Σ_3 embeds in such a way that both projections $p_1 \circ f$ and $p_2 \circ f$ are coverings, and $[f(\Sigma_g)] \cap [f(\Sigma_g)] = 4$.

Problem 2: Can we find a foliation $\mathcal{F}$ of M so that $f(\Sigma_g)$ is a leaf of $\mathcal{F}$?

One can also ask for the same, where the base surface has genus $g_1 \geq 2$ and the fiber surface has genus $g_2 \geq 2$. It is not known if using surfaces of higher genus makes the problem any simpler.

6.4 de Rham Theorem for $B\Gamma$

Suggested by Elmar Vogt

Let Γ be an étale groupoid, so that the source and range maps $s, r \colon \Gamma \to M$ are local covering maps. Then Γ determines a semi-simplicial manifold

$$\mathcal{M} = \left\{ M_0 \Longleftarrow M_1 \mathrel{\substack{\longleftarrow\\\longleftarrow\\\longleftarrow}} M_2 \cdots \right\} \tag{3}$$

where all the spaces M_i are (non-Hausdorff) n-manifolds.

Let $A^{p,q} = \Omega^q(M_p)$ denote the q-forms on M_p. The bigraded complex $A^{*,*}$ has a de Rham differential $d \colon A^{*,*} \to A^{*,*+1}$ and a Čech differential $\delta^* \colon A^{*,*} \to A^{*+1,*}$. The sum $d \pm \delta^*$ is the semi-simplicial de Rham differential. The cohomology $H^*_{deR}(\mathcal{M})$ of the total complex $(A^{*,*}, d \pm \delta^*)$ is called the *de Rham cohomology* of $\mathcal{M}$.

Let $S^{p,q}_\infty = S^q_\infty(M_p)$ denote the *smooth* singular q-cochains on M_p. The bigraded complex $S^{*,*}_\infty$ has a cochain differential $\partial^* \colon S^{*,*}_\infty \to S^{*,*+1}_\infty$ and a Čech differential $\delta^* \colon S^{*,*}_\infty \to S^{*+1,*}_\infty$. The sum $\partial^* \pm \delta^*$ is the semi-simplicial cochain differential. The cohomology $H^*_\infty(\mathcal{M})$ of the total complex $(S^{*,*}_\infty, \partial^* \pm \delta^*)$ is called the *smooth singular cohomology* of $\mathcal{M}$.

The natural de Rham map obtained by integration over singular chains, $I^{p,q} \colon A^{p,q} \to S^{p,q}_\infty$, commutes with both differentials, and induces the de Rham homomorphism of the semi-simplical manifold $\mathcal{M}$. For details, see Bott, Shulman and Stasheff [24], Shulman and Stasheff [121], and Dupont [46, 47].

Let $\|\mathcal{M}\|$ denote the thick (or fat) realization of the semi-simplicial manifold $\mathcal{M}$. When Γ is an étale groupoid over a point $\{*\}$, so each $M_i = \{*\}$, then $\|\mathcal{M}\| = B\Gamma$ is the Milnor realization [101] of the topological group Γ. At the other extreme, when $\Gamma = \Gamma_n$ is the groupoid defined by germs of local diffeomorphisms of $\mathbb{R}^n$, then $\|\mathcal{M}\|$ is by definition the classifying space $B\Gamma_n$ for smooth codimension n foliations. Segal gave in [122] several alternate "models" for $B\Gamma_n$ which are all homotopy equivalent to $\|\mathcal{M}\|$. The thick realization has the property that the Γ-structure on $\|\mathcal{M}\|$ is universal in the following sense: for any Γ-structure on a finite simplicial complex K, there exists a map $f\colon K \to \|\mathcal{M}\|$ inducing *exactly* this Γ-structure.

"Smoothing" yields a natural isomorphism $H^*_\infty(\mathcal{M}) \cong H^*(\|\mathcal{M}\|;\mathbb{R})$.

Theorem: (Bott-Shulman-Stasheff [24]) If each manifold M_i is paracompact, then the de Rham map $I^*\colon H^*_{deR}(\mathcal{M}) \to H^*(\|\mathcal{M}\|;\mathbb{R})$ is an isomorphism.

For a topological groupoid Γ the manifolds M_i need not be Hausdorff, and not paracompact.

Problem 1: Prove that $I^*\colon H^*_{deR}(B\Gamma_n) \to H^*(B\Gamma_n;\mathbb{R})$ is an isomorphism.

Here is a more tractable question:

Problem 2: Assume that Γ is an étale groupoid such that $\mathcal{M}$ admits a countable covering by open sets which are Hausdorff. Find conditions on Γ which imply that $I^*\colon H^*_{deR}(\|\mathcal{M}\|) \to H^*(\|\mathcal{M}\|;\mathbb{R})$ is an isomorphism.

For example, suppose that M is a foliated manifold such that all leaves of $\mathcal{F}$ have contractible holonomy covers. Haefliger [59] proved that the natural map $M \to \|\mathcal{M}\|$ is a homotopy equivalence. Let Γ be the étale groupoid defined by the transverse holonomy of $\mathcal{F}$ with respect to a good covering of M by foliation charts.

Problem 3: Prove that $I^*\colon H^*_{deR}(\|\mathcal{M}\|) \to H^*(M;\mathbb{R})$ is an isomorphism, or at least, determine the image of $I^*\colon H^*_{deR}(\|\mathcal{M}\|) \to H^*(M;\mathbb{R})$.

6.5 Exceptional minimal sets and foliations cycles

Suggested by Steve Hurder

Let M be a compact manifold, and $\mathcal{F}$ a C^1-foliation of M with oriented tangent bundle $T\mathcal{F}$.

A minimal set $\mathbf{K} \subset M$ for $\mathcal{F}$ is either all of M, a compact leaf of $\mathcal{F}$, or is nowhere dense, and called an exceptional minimal set.

Let $X \subset M$ be a saturated subset. A homotopy $h_t\colon X \to M$, $0 \le t \le 1$, is foliated if each map $h_t\colon X \to M$ maps leaves of $\mathcal{F}|X$ to leaves of $\mathcal{F}$.

Theorem: [73] Let $\mathbf{K}$ be a minimal set for $\mathcal{F}$, and $h_t\colon \mathbf{K} \to M$, $0 \le t \le 1$, a foliated homotopy with h_0 the inclusion map. Then for each $0 \le t \le 1$, the image $h_t(\mathbf{K})$ is a minimal set for $\mathcal{F}$.

If $\mathbf{K}$ is a compact leaf, then this says that compact leaves are stable under foliated homotopy.

In general, one associates to stability under homotopy some topological property: for example, a compact oriented leaf $\mathbf{K} = L$ defines a non-zero foliation cycle in the sense of Plante [115] and Sullivan [114, 124]. That is, integration over L defines a closed current on the leafwise deRham complex $\Omega^*(\mathcal{F})$. Similarly, a leaf L equipped with a Fölner sequence defines a closed foliation cycle.

Problem: Does an exceptional minimal set $\mathbf{K}$ determine a foliation cycle?

If $\mathbf{K}$ contains a leaf of subexponential growth, then the answer is yes, and the question is then about the "placement problem" [128], which is how the cycle it determines is "positioned" in the topology of M. In general, it is not clear why there should be a foliation cycledefined by $\mathcal{F}$, but there should be some homotopy invariant associated to the minimal set $\mathbf{K}$. What is this invariant?

6.6 The space of foliations

Suggested by Paul Schweitzer

Let Σ_g be the closed oriented surface of genus g. Consider the space $\mathrm{Fol}_c^r(\Sigma_g \times [0,1])$ of compact, codimension-one, C^∞-foliations tangent to the boundary, with the C^r-topology for $1 \leq r \leq \infty$. Here, compact means that each leaf of $\mathcal{F}$ is a compact submanifold.

Problem: Show that $\mathrm{Fol}_c^r(\Sigma_g \times [0,1])$ is contractible.

Remark 1: For genus $g = 0$, $\Sigma_0 = \mathbb{S}^2$, and this is equivalent to the Smale Conjecture that the inclusion $\mathbf{O}(4) \hookrightarrow \mathrm{Diff}^\infty(\mathbb{S}^3)$ is a homotopy equivalence. This proof of this by Allen Hatcher in [61] was remarkably difficult.

Remark 2: The Reeb Stability Theorem implies that each foliation $\mathcal{F} \in \mathrm{Fol}_c^r(\Sigma_g \times [0,1])$ is diffeomorphic (and smoothly isotopic) to the product foliation.

Remark 3: Might it be possible to find an analytic proof? For example, using the evolution methods of Hamilton and Perelman [105].

7 Harmonic measures

7.1 Harmonic measures in codimension one

Suggested by Andrés Navas

Let $\Gamma \subset \mathrm{Diff}^1(\mathbb{S}^1)$ be a finitely-generated subgroup, with generators $\{h_1, \ldots, h_n\}$. For each $1 \leq i \leq n$ choose a weight $0 < p_i < 1$ with $p_1 + \cdots p_n = 1$, and let $\vec{p} = (p_1, \ldots, p_n)$. The $\vec{p}$-Lyapunov exponent of the action is the sum

$$\lambda(\vec{p}) = \sum_{i=1}^{n} p_i \cdot \int_{\mathbb{S}^1} \ln(h_i'(x))\, dx \tag{4}$$

Let $\mathfrak{M}$ denote the space of Borel probability measures on $\mathbb{S}^1$. A measure $\mu \in \mathfrak{M}$ is Γ-invariant if $h_i^*(\mu) = \mu$ for all $1 \leq i \leq n$. Peter Baxendale proved the following in [17]

Theorem: Suppose there is no $\mu \in \mathfrak{M}$ which is Γ-invariant. Then $\lambda(\vec{p}) < 0$.

For $\mu \in \mathfrak{M}$ define the weighted Laplacian $\Delta_{\vec{p}}(\mu) = \sum_{i=1}^{n} p_i\, h_i^*(\mu)$. Then there exists a unique $\mu(\vec{p}) \in \mathfrak{M}$ such that $\Delta_{\vec{p}}(\mu(\vec{p})) = \mu(\vec{p})$. The measure $\mu(\vec{p})$ is called the harmonic measure for the Laplacian $\Delta_{\vec{p}}$. The Kakutani Ergodic Theorem implies that the action of Γ is ergodic with respect to $\mu(\vec{p})$. (See [44].)

Problem 1: When is $\mu(\vec{p})$ equivalent to Lebesgue measure?

Problem 2: If the action $\Gamma \times \mathbb{S}^1 \to \mathbb{S}^1$ is minimal, can we choose $\vec{p}$ so that $\mu(\vec{p})$ is absolutely continuous?

Note this should be compared to the recent results of Chris Connell and Roman Muchnik [41] which solves Probem 2 for a word-hyperbolic group acting on its boundary at infinity.

7.2 Pointwise convergence

Suggested by Vadim Kaimanovich

Let M be a compact Riemannian manifold, and $\mathcal{F}$ a C^∞-foliation of M. Let $\mathfrak{m}$ be a harmonic measure on M; that is, if $\mathcal{D}^t \colon L^1(M, \mathfrak{m}) \to L^1(M, \mathfrak{m})$ is the leafwise diffusion operator, then for any $f \in L^1(M, \mathfrak{m})$ and all $t \geq 0$,

$$\langle \mathcal{D}^t f, \mathfrak{m} \rangle = \int_M \mathcal{D}^t\, f\, d\mathfrak{m} = \int_M f\, d\mathfrak{m} = \langle f, \mathfrak{m} \rangle$$

Lucy Garnett proved in [52, 53] that $\mathcal{D}^t\, f \to f^*$ in $L^1(M, \mathfrak{m})$, where f^* is constant on the ergodic components of $\mathfrak{m}$, and moreover there is an ergodic theorem

$$\frac{1}{T} \int_0^T \mathcal{D}^t\, f\, dt \longrightarrow \langle f, \mathfrak{m} \rangle$$

The following problem was raised by the work [81].

Problem: Find conditions which guarantee that $\mathcal{D}^t f \longrightarrow f^*$ a.e. $\mathfrak{m}$.
See Section 7 of [29].

8 Geometry of foliations

8.1 Conformal geometry of surfaces

Suggested by Paweł Walczak

Let Σ be a surface in $\mathbb{R}^3$ (or possibly the Euclidean 3-sphere $\mathbb{S}^3$ or hyperbolic space $\mathbb{H}^3$.)

Let $\kappa_1, \kappa_2 \colon \Sigma \to \mathbb{R}$ denote the principal curvature functions. Assume that $\kappa_1(x) \neq \kappa_2$ for all $x \in \Sigma$. Let X_1, X_2 be corresponding unit vector fields tangent to their corresponding lines of curvatures.

The functions $\Theta_i = \dfrac{\kappa_i}{(\kappa_1 - \kappa_2)^2} \cdot X_i$ are called the *conformal principal curvatures*. It is known that if $\Theta_1 \cdot \Theta_2$ is constant, then $\Theta_1 \cdot \Theta_2$ is identically zero.

Problem 1: Classify the surfaces Σ with Θ_1 and Θ_2 constant.

It is known that if $\Theta_1 = 0 = \Theta_2$ then Σ is a Dupin cyclid, which is a conformal image of a torus of revolution, or of a cylinder, or a cone over $\mathbb{S}^1$.

Problem 2: Let M be a compact 3-manifold of constant sectional curvature $\kappa = 1$ (or $\kappa = 0$, $\kappa = -1$.) Establish the existence (or non-existence) of 2-dimensional foliations by leaves with constant conformal principal curvatures on M.

8.2 Ridge curves

Suggested by Rémi Langevin

Let $\Sigma \subset \mathbb{R}^3$ (or possibly $\Sigma \subset \mathbb{S}^3$) be an embedded surface with no umbilical points.

Let $\kappa_1, \kappa_2 \colon \Sigma \to \mathbb{R}$ denote the principal curvature functions, then $\kappa_1(x) \neq \kappa_2$ for all $x \in \Sigma$. Let X_1, X_2 be corresponding unit vector fields tangent to their corresponding lines of curvatures, and $\mathcal{F}_1$ and $\mathcal{F}_2$ the corresponding 1-dimensional foliations of Σ whose leaves are the lines of curvature.

The points of a leaf of $\mathcal{F}_1$ where κ_1 is maximal (or minimal) are conformally invariant, up to the action of the Möbius group on $\mathbb{R}^3$ or $\mathbb{S}^3$. The set of these points are form closed curves in Σ, called the ridge curves; see the illustration Figure 1.

Problem 1: Must some ridge curve be non-homologous to zero in $H_1(\Sigma)$?

Let $\mathbb{T}^2$ be the flat torus, and $\mathcal{F}$ a 1-dimensional smooth foliation. Define the ridge curves of $\mathcal{F}$ to be the points where the curvature of the leaves is maximal (or minimal) on the leaf.

Problem 2: For which foliations of the flat torus $\mathbb{T}^2$ is there necessarily a ridge curve which is non-homologous to zero in $H_1(\mathbb{T}^2)$?

Problem 3: Let $\mathbb{T}^2 \subset \mathbb{R}^3$ be an embedded torus with no umbilical points. Can the foliations of $\mathbb{T}^2$ by the lines of curvature admit a Reeb annulus?

Figure 1. Ridge curves

8.3 The Anosov-Weil problem

Suggested by Viacheslav Grines

Let $\Sigma = \mathbb{H}^2/\Gamma$ be a closed surface of genus $g \geq 2$. The hyperbolic plane $\mathbb{H}^2$ is identified with the open unit disk $U \subset \mathbb{R}^2$ with the Poincaré metric, so that the space S_∞ of asymptotic directions at infinity in $\mathbb{H}^2$ is identified with the boundary of the open disk, $S_\infty \cong \partial U \cong \mathbb{S}^1$.

Let $\mathcal{F}$ be a foliation with singularities on Σ, with singularity set $\Lambda \subset \Sigma$.

Let L be a leaf of $\mathcal{F}$, and $\widetilde{L} \subset U$ the lift of L. Pupko [116] showed that if $\widetilde{L}$ is not bounded, then it has an asymptotic direction at infinity, and hence a limit point on the boundary $\sigma \sim z_L \in \mathbb{S}^1$. If we parametrize the lift as a unit speed curve $\ell(t)$, then $\lim_{t\to\infty} \ell(t) = z_L \in \mathbb{S}^1$. If $g\colon \mathbb{R} \to U$ is a geodesic for the hyperbolic metric such that we also have $\lim_{t\to\infty} g(t) = z_L$ then one can ask how the hyperbolic distance $d(t) = d(\ell(t), g(t))$ behaves as $t \to \infty$. (See Figure 2 below.) This is called the Anosov-Weil problem, and the history of it was discussed by Anosov in [9, 10]. See also [12, 13, 14].

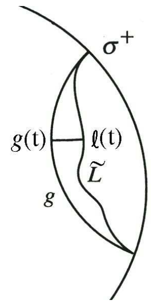


Figure 2.

The leaf $\widetilde{L}$ is said to have the *restricted deviation property* if there exists $k > 0$ such that $d(t) \leq k$ for all $k \in [0, \infty)$. It is known that if $\widetilde{L}$ has an asymptotic direction and the set of singularities Λ is finite, then $\widetilde{L}$ has the restricted deviation property [13]. Moreover, the authors Aranson, Grines and Zhuzhoma give a general construction of counter-examples which shows that this is false if the singular set Λ has the power of the continuum.

Problem: Do there exists counter-examples where Λ is countably infinite? That is, find a foliation $\mathcal{F}$ on Σ with countable singular set Λ, such that the restricted deviation property fails for some unbounded leaf $\widetilde{L}$.

8.4 Extending analytic foliations

Suggested by Rémi Langevin

Let $M = \mathbb{H}^3/\Gamma$ be a hyperbolic 3-manifold without boundary. Walczak [135] proved that there are no totally geodesic codimension-one foliations on compact M, and Ghys [55] extended this result to the case where M has finite volume.

A basic technique for the study of a codimension-one foliation $\mathcal{F}$ on a hyperbolic 3-manifold M is to lift the given foliation to a foliation $\widetilde{\mathcal{F}}$ on the universal cover of M which is identified with the hyperbolic plane $\mathbb{H}^3$. If $\mathcal{F}$ has no Reeb components, then each leaf of $\mathcal{F}$ lifts to a planar leaf of $\widetilde{\mathcal{F}}$. If we identify $\mathbb{H}^3$ with the unit disk in $\mathbb{R}^3$ with the Poincaré metric, then we can consider the asymptotic behavior of the leaves $\widetilde{L}$ of $\widetilde{\mathcal{F}}$, and ask whether the leaves extend continuously to the boundary $\partial\mathbb{H}^3 \cong \mathbb{S}^2$. This is the approach used by Fenley [48, 49, 50, 51] in his studies of taut foliations of hyperbolic 3-manifolds. Czarnecki [40] has studied the properties of extensions in the case of variable negative curvature.

A foliation $\mathcal{F}$ on $M = \mathbb{H}^3/\Gamma$ is *strongly analytic* if the lift $\widetilde{\mathcal{F}}$ on $\mathbb{H}^3$ admits an analytic extension across the boundary $\partial\mathbb{H}^3$ of the hyperbolic ball.

Problem: Classify the strongly analytic codimension-one foliations of $M = \mathbb{H}^3/\Gamma$.

The assumption that all leaves of $\widetilde{\mathcal{F}}$ extend analytically across the boundary $\partial\mathbb{H}^3$ is a very strong hypothesis, so one expects the answer to be very restrictive.

8.5 Constructing Foliations

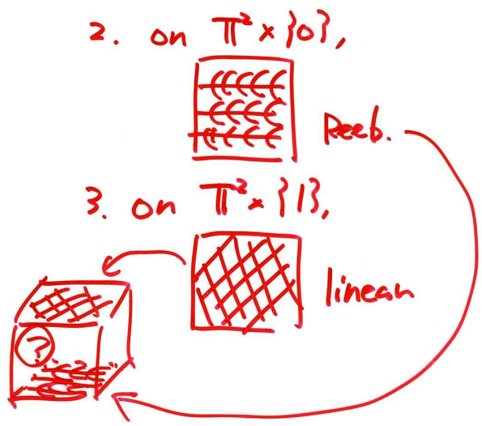
Suggested by Masayuki Asaoka

Let $M = \mathbb{T}^2 \times [0,1]$ be the 3-manifold with two boundary components, $\mathbb{T}^2 \times \{0\}$ and $\mathbb{T}^2 \times \{1\}$.

Problem: Does there exists a pair $\mathcal{F}$,$\mathcal{G}$ of mutually transverse codimension-one foliations on M such that both $\mathcal{F}$ and $\mathcal{G}$

- are transverse to both boundary components of M,
- intersect $\mathbb{T}^2 \times \{0\}$ in a pair of Reeb foliations,
- intersect $\mathbb{T}^2 \times \{1\}$ in a pair of linear foliations?

This is illustrated below:



8.6 Transverse flows and length spectra

Suggested by Fabian Kopei

Let $f\colon \mathbb{N} \to \mathbb{R}^+ = (0, \infty)$ a given positive function on the natural numbers.

Problem: Does there exists a compact 3-manifold M with a codimension-one foliation $\mathcal{F}$ and a transverse flow $\varphi_t\colon M \to M$ such that there is a bijection ϕ between the closed orbits γ of φ and $\mathbb{N}$, and $f(\phi(\gamma))$ is the length of the closed orbit γ?

In particular, for $f(n) = n$, this asks whether there is a flow such that the closed orbits of φ are countable, and their lengths assume every positive integer with multiplicity exactly one.

Another example to consider, is to let p_n denote the n^{th} prime in succession, then set $f(n) = \ln(p_n)$.

8.7 Almost compact foliations

Suggested by Steve Hurder

Let M be a closed manifold, and $\mathcal{F}$ is a C^1 foliation with leaf dimension p and codimension q. The following was problem A.3.1 in the Rio 1992 problem set [90]:

Problem: Suppose that $\mathcal{F}$ is a topological (or C^1, C^2, etc.) foliation of a compact manifold M. Is it possible that $\mathcal{F}$ has exactly one non-compact leaf, with all of the remaining leaves compact?

The answer is no in codimension one, as the set of the compact leaves is a compact set. Elmar Vogt proved that for a topological foliation in codimension two, $\mathcal{F}$ must be either a Seifert fibration, or has uncountably many non-compact leaves [134].

A foliation with at most countable number of non-compact leaves is called *almost compact* in [69]. It is known that every leaf of an almost compact foliation must be proper. If an almost compact foliation admits a cross-section (a closed transverse submanifold which intersects every leaf of the foliation) then every leaf must be compact and the foliation is a generalized Seifert fibration [69]. The current formulation of the problem is thus, does there exists a foliation of codimension greater than two on a compact manifold M with at most countable number of non-compact proper leaves, and all of the remaining leaves compact?

9 (Singular) Riemannian foliations

9.1 Riemannian foliations

Suggested by Ken Richardson

Problem 1: Find geometric or topological obstructions to the existence of Riemannian foliations on a given closed Riemannian manifold.

The above problem is difficult and must involve global instead of local obstructions. The reason for this is that locally, Riemannian foliations always exist. Given any point of a Riemannian manifold, consider a local hypersurface through that point. Then using the normal exponential map, one can construct a family of hypersurfaces that are equidistant.

Note that some obstructions have already been found, although most of these involve determining whether or not an existing foliation can be given a holonomy invariant transverse metric.

For example, the Bott vanishing theorem states that given a distribution of codimension q, the Pontryagin classes of degree $> 2q$ all vanish if the distribution is involutive, and the Pontryagin classes of degree $> q$ all vanish if the distribution is the tangent bundle of a Riemannian foliation. So, in some sense we do have a global obstruction, because one could theoretically compute the Pontryagin classes of all possible distributions, and if it were impossible for this vanishing to occur, then the manifold does not admit a Riemannian foliation in that dimension.

It is also known that closed hyperbolic 3-manifolds have no Riemannian foliations [135].

Problem 2: Determine if the small time asymptotic expansion for the trace of the basic heat kernel of a Riemannian foliation can ever have logarithmic terms.

It is known that the expansion has no logarithmic terms in many cases (examples: codimension 3 and below, when all the leaf closures have the same dimension, when the dimension of the leaf closures differ by no more than 2). This problem is closely related to the problem of determining if the trace of the equivariant heat kernel (of a compact Lie group action) has logarithmic terms. Again, in this case, nothing has been proven, and there is no example whose trace has been calculated that has logarithmic terms.

An example of an integral that comes up is:

$$\begin{aligned}&\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{B}^n}\int_{\theta\in[0,2\pi]^k}\exp\left(-\frac{1}{t}\sum_{j=1}^{n}\sin^2\left(\mathbf{a}_j\cdot\theta\right)x_j^2\right)d\theta\ dx\\ \sim\ &\frac{1}{(4\pi t)^{q/2}}\sum_{a\geq 0,b\geq 0}c_{ab}t^{a/2}\left(\log t\right)^b,\end{aligned}$$

where $\mathbf{a}_1,\mathbf{a}_2,...,\mathbf{a}_n$ are in $\mathbb{Z}^k$. Here q is the dimension of the quotient of $\mathbb{B}^n$ by the action generated by $x_j\to e^{i\mathbf{a}_j\cdot\theta}x_j$.

Problem 3: Find the c_{ab}, and determine whether c_{ab} must be 0 if $b>0$.

Problem 4: Determine relationships between the spectrum of the basic Laplacian of a Riemannian foliation and the global geometry of the manifold. Some information is known; for example, it is known that the spectrum of the basic Laplacian on functions determines the transverse volume V_{tr} and the codimension of the leaf closures in the principal stratum. Note that

$$V_{\mathrm{tr}}=\int_M\frac{1}{\mathrm{vol}\left(\overline{L_x}\right)}d\mathrm{vol}\left(x\right),$$

where $\overline{L_x}$ is the leaf closure containing x. Also, it is known that under restrictions on the first eigenvalue of the basic Laplacian and transverse Ricci curvature, one can prove that the foliation is the suspension of a sphere. But there is much more work to be done here.

Problem 5: What conditions imply a zero basic index for basic Dirac operators?

Known examples: The existence of a basic vector field that is never tangent to the foliation implies that the basic Euler characteristic vanishes (by the Belfi-Park-Richardson Hopf index theorem [18]). Also, if the foliation is transversally spin, then if the transverse Scalar curvature (found by Seoung Dal Jung) is positive, then the basic index of the basic spin Dirac operator vanishes.

Problem 6: Apply the basic index theorem to obtain corresponding theorems about basic signature and basic Dirac operators.

Problem 7: Use the basic index theorem to find obstructions to the splitting of bundles over manifolds equipped with Riemannian foliations.

For example, one may determine an obstruction for the the foliation to be transversally spin.

Problem 8: Develop scattering theory for Riemannian foliations.

Problem 9: Can metrics on Riemannian foliations be uniformized?

For example, if the foliation is dimension two, is it always possible to choose a metric such that each leaf is constant curvature? What about transverse metrics on codimension 2 Riemannian foliations?

9.2 Singular Riemannian foliations

Suggested by Dirk Töben and Ken Richardson

The standard definition of a singular Riemannian foliation is that in Chapter 6 of Molino [104]. The two canonical examples are the orbits of a connected Lie group acting by isometries on a compact manifold, and the partition $\overline{\mathcal{F}}$ of a compact manifold obtained by taking the closures of the leaves of a standard Riemannian foliation $\mathcal{F}$. It also includes the useful (non-singular) Riemannian foliations, and there are also more exotic constructions, such as the singular foliation of $\mathbb{R}^n$ induced by an isoparametric submanifold [6, 8, 25, 127].

Problem 1: Find geometric or topological obstructions to the existence of singular Riemannian foliations on a closed Riemannian manifold.

This problem is either easier or more difficult than the corresponding problem for Riemannian foliations, as singular foliations include the usual class of Riemannian foliations, but much less is known about their structure. For example, the following Molino Conjecture remains open, although Marcos Alexandrino reports recent progress on this in [7]

Problem 2: Show that the closures of the leaves of a singular Riemannian foliation form a singular Riemannian foliation.

Problem 3: Find geometric and topological differences between singular, orbit-like Riemannian foliations and those that are not orbit-like, that are not simple restatements of definitions.

Problem 4: Develop basic index theory for singular Riemannian foliations, as a generalization of equivariant index theory.

9.3 Hodge decomposition

Suggested by Jesús Álvarez López

Let M be a closed manifold, and $\mathcal{F}$ a Riemannian C^∞-foliation of M. The leafwise Laplacian acting on the leafwise deRham complex is denoted by $\Delta_{\mathcal{F}}\colon \Omega^*(\mathcal{F}) \to \Omega^*(\mathcal{F})$.

The leafwise deRham complex admits a Hodge Decomposition [4, 5]

$$\Omega^*(\mathcal{F}) = \ker \Delta_{\mathcal{F}} \oplus \overline{\mathrm{im}\ d_{\mathcal{F}}} \oplus \overline{\mathrm{im}\ \delta_{\mathcal{F}}} \tag{5}$$

Álvarez López and Candel study equicontinuous foliations in [3] and show they are the topological version of Riemannian foliations. The leaves of an equicontinuous are smoothly immersed $C^{0,\infty}$-submanifolds, so one can define the complex of leafwise forms $\Omega^*_{0,\infty}(\mathcal{F})$ and a leafwise Laplacian $\Delta_{\mathcal{F}}$ as for Riemannian foliations.

Problem: Show that the Hodge decomposition (5) is a topological result. That is, show that for an equicontinuous foliation there is a Hodge decomposition

$$\Omega^*_{0,\infty}(\mathcal{F}) = \ker \Delta_{\mathcal{F}} \oplus \overline{\mathrm{im}\ d_{\mathcal{F}}} \oplus \overline{\mathrm{im}\ \delta_{\mathcal{F}}} \tag{6}$$

The difficulty in proving (6) is that the techniques for showing (5) require the use of the transverse Laplacian for a Riemannian foliation, which does not exists in the topological category, so a new approach is needed. The problem can be considered as asking for an alternate proof of (5) which does not use the transverse structure of $\mathcal{F}$, but only requires the dynamical properties of an equicontinuous or Riemannian foliation.

9.4 Wave equation and quantum entropy

Suggested by Yuri Kordyukov

The following problems based on the papers [85, 86, 87, 88, 89] by Kordyukov, which should be consulted for more details.

Let (M, g) be a Riemannian manifold. Let $\varphi_t \colon T^*M \to T^*M$ be the geodesic flow of g, and $\varphi_t \colon S^*M \to S^*M$ the restriction of φ_t to the unit cotangent sphere bundle. The flow φ_t induces a flow on function spaces $f_t = \varphi_t^* \colon C^\infty(S^*(M)) \to C^\infty(S^*(M))$.

Let Ψ_k denote the algebra of k^{th}-order pseudodifferential operators on $C^\infty M$. Associated to each $P \in \Psi^k(M)$ is its principal symbol $\sigma(P)\colon T^*M \to \mathbb{R}$. The restriction of the principal symbol to the unit cotangent bundle defines a mapping $\sigma \colon \Psi^k(M) \to C^\infty(S^*M)$. For example, the Laplacian of g is a second order differential operator, $\Delta_g \colon C^\infty(M) \to C^\infty(M)$, so $\Delta \in \Psi^2(M)$, and $\sigma(\Delta_g) = 1$, the constant function. (Actually, it is the restriction of the norm squared, $\|\cdot\| \colon T^*(M) \to \mathbb{R}$, but this is identically 1 on $S^*(M)$.)

The "square root" of the Laplacian $P = \sqrt{\Delta_g} \in \Psi^1(M)$ also has constant symbol $\sigma(\sqrt{\Delta_g}) = 1$.

Let $e^{itP}\colon L^2(M) \to L^2(M)$ be the wave operator associated to $\sqrt{\Delta_g}$, which is a unitary operator defined by the spectral theorem. For $A \in \Psi^0(M)$, set $F_t(A) = e^{itP} A e^{-itP}$. Then $F_t\colon \Psi^0(M) \to \Psi^0(M)$ is a smooth 1-parameter flow on $\Psi^0(M)$, which is the wave propagation of A. The waves of the Laplacian "propagate along geodesics", which is the import of the commutative diagram:

$$\begin{array}{ccc} \Psi^0(M) & \xrightarrow{F_t} & \Psi^0(M) \\ \sigma\downarrow & & \downarrow\sigma \\ C^\infty(S^*M) & \xrightarrow{f_t} & C^\infty(S^*M) \end{array}$$

Problem 1: Can one define a type of "quantum entropy of the non-commutative geodesic flow F_t? If so, is there a relationship with topological entropy of the geodesic flow f_t?

This is the test case for Problem 2. Let (M, g) be a closed Riemannian manifold foliation with a smooth foliation $\mathcal{F}$. The tangent bundle splits $TN = T\mathcal{F} \oplus H$ where $H = TF^{\perp}$, and the metric decomposes $g = g_{\mathcal{F}} + g_H$, where $g_H = g|H$ is a metric on the normal bundle H. We assume that g_H is bundle-like, so that $\mathcal{F}$ is a Riemannian foliation. It is well-known that the bundle H is then totally geodesic [104].

The splitting $TM = T\mathcal{F} \oplus H$ induces a bigrading on the algebra of smooth forms, $\Omega^{p,q}(M) \cong \Omega^p(\mathcal{F}) \wedge C^\infty(\wedge^q H^*)$, and a decomposition of the exterior differential on smooth forms into components $d = d_{\mathcal{F}} + d_H + Q$, where

$$\begin{aligned} d_{\mathcal{F}}\colon \Omega^{p,q}(M) &\to \Omega^{p+1,q}(M)\ , \\ d_H\colon \Omega^{p,q}(M) &\to \Omega^{p,q+1}(M)\ , \\ Q\colon \Omega^{p,q}(M) &\to \Omega^{p-1,q+2}(M) \end{aligned}$$

and $d_{\mathcal{F}}$ and d_H are first-order linear operators, and Q has order 0. Define the transverse Laplacian:

$$\Delta_H = d_H d_H^* + d_H^* d_H\colon C^\infty(\wedge^* H^*) \to C^\infty(\wedge^* H^*)$$

Then Δ_H is essentially self-adjoint as an unbounded operator on $L^2(\wedge^* H^*)$, so as above, we can define the unitary operator $e^{it\Delta_H}\colon L^2(\wedge^* H^*) \to L^2(\wedge^* H^*)$.

For $A \in \Psi^0(H)$, set $F_t(A) = e^{it\Delta_H} A e^{-it\Delta_H}$. Then $F_t\colon \Psi^0(H) \to \Psi^0(H)$ is a smooth 1-parameter flow on $\Psi^0(H)$, which is the "transverse wave propagation" of Δ_H. The waves of the transverse Laplacian "propagate along transverse geodesics", which are well-defined since H is totally geodesic.

Problem 2: Can one define the "quantum entropy of the non-commutative tranverse geodesic flow F_t? If so, what is the relationship with the transverse geometry of $\mathcal{F}$?

9.5 Transverse zeta functions

Suggested by Yuri Kordyukov

Consider the holonomy groupoid $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{F}$, which is a Hausdorff manifold as $\mathcal{F}$ is Riemannian [138]. Let $C_c^\infty(\mathcal{G}_{\mathcal{F}})$ be the space of smooth functions on $\mathcal{G}_{\mathcal{F}}$ with compact support. This is an algebra under the convolution product. Each $\phi \in C_c^\infty(\mathcal{G}_{\mathcal{F}})$ defines a linear operator $k_\phi \colon C^\infty(\wedge^* H^*) \to C^\infty(\wedge^* H^*)$ which extends to a compact operator on $L^2(\wedge^* H^*)$. This yields a representation $\rho \colon C_c^\infty(\mathcal{G}_{\mathcal{F}}) \to \mathcal{B}(L^2(\wedge^* H^*))$. Note that the representation ρ does not necessarily extend to the usual (reduced) C^*-algebra $C^($\mathcal{F}$)$ of $\mathcal{F}$, which is the closure of $C_c^\infty(\mathcal{G}_{\mathcal{F}})$ in $C^($\mathcal{F}$)$. The one exception is when $\mathcal{F}$ is an amenable foliation.

Assume that $\mathcal{F}$ has codimension n, then for complex $z \in \mathbb{C}$ with $\mathrm{Re}(z) > n/2$ and $k \in C_c^\infty(\mathcal{G}_{\mathcal{F}})$ there is a well-defined trace

$$\zeta_k(z) = \mathrm{Tr}\Big\{\rho(k) \circ \Big(\Delta_h + \mathrm{Id}\Big)^{-z}\Big\} \tag{7}$$

and the function $z \mapsto \zeta_k(z)$ extends to meromorphic function on $\mathbb{C}$ with at most simple poles.

Problem 3: What is the largest class of kernels k for which the zeta-function ζ_k can be defined?

Clearly, if $k \colon \mathcal{G}_{\mathcal{F}} \to \mathbb{R}$ has very rapid decay, then the ζ_k is well-defined. The question is to find a natural class of kernels for which ζ_k is defined, say those with exponential decay at a rate depending upon the geometry of $\mathcal{F}$.

10 Godbillon-Vey classes

Let $\mathcal{F}$ be a C^2 foliation on a manifold M without boundary, and let $GV(\mathcal{F}) \in H^{2q+1}(M)$ denotes its Godbillon-Vey class, where q is the codimension of $\mathcal{F}$.

10.1 Geometry of Godbillon-Vey class

A classic problem

For foliations of codimension-one, Moussu and Pelletier [107] and Sullivan [124] conjectured $GV(\mathcal{F}) \neq 0$ implies $\mathcal{F}$ must have a leaf of exponential growth. Duminy [45] proved that $GV(\mathcal{F}) \neq 0$ implies $\mathcal{F}$ has a resilient leaf, and hence there are uncountably many leaves with uniformly exponential growth. Hurder and Langevin [78] gave a new proof of this result using techniques of measurable ergodic theory. Recently, Hurder [75] showed that $GV(\mathcal{F}) \neq 0$ implies that $\mathcal{F}$ has a resilient leaf which is not contained in an exceptional minimal set, hence there must be a resilient leaf contained in an open local minimal set of $\mathcal{F}$.

The Reinhart-Wood formula [117] gave a *pointwise* geometric interpretation of $GV(\mathcal{F})$ for 3-manifolds. What is needed is a more global geometric property of $\mathcal{F}$ which is measured by $GV(\mathcal{F})$.

Problem: Give a *geometric* interpretation of the Godbillon-Vey invariant.

The helical wobble description by Thurston [125] is a first attempt at such a result, and the Reinhart-Wood formula suitably interprets this idea locally. Langevin has suggested that possibly the Godbillon-Vey invariant can be interpreted in the context of integral geometry and conformal invariants [27] as a measure in some suitable sense. The goal for any such an interpretation, is that it should provide *sufficient* conditions for $GV(\mathcal{F}) \neq 0$.

10.2 Godbillon-Vey class in higher codimension

A classic problem

Problem: Let $\mathcal{F}$ be a codimension $q > 1$ foliation with $GV(\mathcal{F}) \neq 0$. What can be said about the geometry and dynamics of $\mathcal{F}$?

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