Characteristic classes for Riemannian foliations

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The purpose of this paper is to both survey and offer some new results on the non-triviality of the characteristic classes of Riemannian foliations. We give examples where the primary Pontrjagin classes are all linearly independent. The independence of the secondary classes is also discussed, along with their total variation. Finally, we give a negative solution of a conjecture that the map of classifying spaces $FR\Gamma_q \to F\Gamma_q$ is trivial for codimension $q > 1$.

Keywords: Riemannian foliation, characteristic classes, secondary classes, Chern-Simons classes

1. Introduction

The Chern-Simons class[9] of a closed 3-manifold M, considered as foliated by its points, is the most well-known of the secondary classes for Riemannian foliations. Foliations with leaves of positive dimension offer a much richer class to study, and the values of their secondary classes reflect both geometric (metric) and dynamical properties of the foliations. It is known that all of these classes can be realized independently for explicit examples (Theorem 4.3), but there remain a number of open problems to study. In this note, we survey the known results, highlight some of the open problems, and provide a negative answer to an outstanding conjecture.

Let M be a smooth manifold of dimension n, and let $\mathcal{F}$ be a smooth foliation of codimension q. We say that $\mathcal{F}$ is a *Riemannian foliation* if there is a smooth Riemannian metric g on TM which is *projectable* with respect to $\mathcal{F}$. Identify the normal bundle Q with the orthogonal space $T\mathcal{F}^{\perp}$, and let Q have the restricted Riemannian metric $g_Q = g|Q$. For a vector $X \in T_xM$ let $X^{\perp} \in Q_x$ denote its orthogonal projection. Given a leafwise path γ between points x, y on a leaf L, the transverse holonomy h_γ along γ induces a linear transformation $dh_x[\gamma] : Q_x \to Q_y$. The fact that the

Riemannian metric g on TM is projectable is equivalent to the fact that the transverse linear holonomy transformation $dh_x[\gamma]$ is an isometry for all such paths.[16,17,39–41,49]

There are a large variety of examples of Riemannian foliations which arise naturally in geometry. Given a smooth fibration $\pi\colon M \to B$, the connected components of the fibers of π define the leaves of a foliation $\mathcal{F}$ of M. A Riemannian metric g on TM is projectable if there is a Riemannian metric g_B on TB such that the restriction of g to the normal bundle $Q \equiv T\mathcal{F}^{\perp}$ is the lift of the metric g_B. The pair $(\pi\colon M \to B, g)$ is said to be a *Riemannian submersion*. Such foliations provide the most basic examples of Riemannian foliations.

Suspensions of isometric actions of finitely generated groups provide another canonical class of examples of Riemannian foliations. The celebrated Molino Structure Theory for Riemannian foliations of compact manifolds reduces, in a broad sense, the study of the geometry of Riemannian foliations to a mélange of these two types of examples – a combination of fibrations and group actions; see Theorems 6.2 and 6.3 below. When the dimension of M is at most 4, the Molino approach yields a "classification" of all Riemannian foliations. However, in general the structure theory is too rich and subtle to effect a classification for codimension $q \geq 3$ and leaf dimensions $p \geq 2$. The survey by Ghys, Appendix E of[41] gives an overview of the classification problem circa 1988.

The secondary characteristic classes of Riemannian foliations give another approach to a broad classification scheme. Their study focuses attention on various classes of Riemannian foliations, which are investigated in terms of known examples and their Molino Structure Theory, and the values of their characteristic classes, often leading to new insights.

The characteristic classes of a Riemannian foliation are divided into three types: the primary classes, given by the ring generated by the Euler and Pontrjagin classes; the secondary classes; and the blend of these two as defined by the Cheeger-Simons differential characters. Each of these types of invariants have been more or less extensively studied, as discussed below. The paper also includes various new results and unpublished observations, some of which were presented in the author's talk.[22]

The main new result of this paper uses characteristic classes to give a negative answer to Conjecture 3 of the Ghys survey [*op. cit.*]. The proof of the following is given in §3.

Theorem 1.1. *For $q \geq 2$, the map $H_{4k-1}(FR\Gamma_q;\mathbb{Z}) \to H_{4k-1}(F\Gamma_q;\mathbb{Z})$ has infinite-dimensional image for all degrees $4k-1 \geq 2q$.*

This paper is an expanded version of a talk given at the joint AMS-RSME Meeting in Seville, Spain in June 2003. The talk was dedicated to the memory of Connor Lazarov, who passed away on February 27, 2003. We dedicate this work to his memory, and especially his fun-loving approach to all things, including his mathematics, which contributed so much to the field of Riemannian foliations.

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2. Classifying spaces

The universal Riemannian groupoid $R\Gamma_q$ is generated by the collection of all local isometries $\gamma\colon (U_\gamma, g'_\gamma) \to (V_\gamma, g''_\gamma)$ where g'_γ and g''_γ are complete Riemannian metrics on $\mathbb{R}^q$, and $U_\gamma, V_\gamma \subset \mathbb{R}^q$ are open subsets. Let $BR\Gamma_q$ denote the classifying space of the groupoid $R\Gamma_q$. The Hausdorff topological space $BR\Gamma_q$ is well-defined up to weak-homotopy equivalence.[14,15] If we restrict to orientation-preserving maps of $\mathbb{R}^q$, then we obtain the groupoid denoted by $R\Gamma_q^+$ with classifying space $BR\Gamma_q^+$.

The universal groupoid Γ_q of $\mathbb{R}^q$ is that generated by the collection of all local diffeomorphisms $\gamma\colon U_\gamma \to V_\gamma$ where $U_\gamma, V_\gamma \subset \mathbb{R}^q$ are open subsets. The realization of the groupoid Γ_q is a non-Hausdorff topological space $B\Gamma_q$, which is well-defined up to weak-homotopy equivalence.

An $R\Gamma_q$–structure on M is an open covering $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ of M and for each $\alpha \in \mathcal{A}$, there is given

- a smooth map $f_\alpha\colon U_\alpha \to V_\alpha \subset \mathbb{R}^q$
- a Riemannian metric g'_α on $\mathbb{R}^q$

such that the pull-backs $f_\alpha^{-1}(T\mathbb{R}^q) \to U_\alpha$ define a smooth vector bundle $Q \to M$ with Riemannian metric $g|Q = g_\alpha = f_\alpha^* g'_\alpha$. An $R\Gamma_q$–structure on M determines a continuous map $M \to |\mathcal{U}| \to BR\Gamma_q$.

Foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ of codimension q of M are *integrably homotopic* if there is a foliation $\mathcal{F}$ of $M \times \mathbb{R}$ of codimension q such that $\mathcal{F}$ is everywhere transverse to the slices $M \times \{t\}$, so defines a foliation $\mathcal{F}_t$ of codimension-q of $M \times \{t\}$, and $\mathcal{F}_t$ of M_t agrees with $\mathcal{F}_t$ of M for $t = 0, 1$. This notion extends to Riemannian foliations, where we require that $\mathcal{F}$ defines a Riemannian foliation of codimension-q of $M \times \mathbb{R}$.

Theorem 2.1 (Haefliger[14,15]). *A Riemannian foliation $(\mathcal{F}, g)$ of M with oriented normal bundle defines an $R\Gamma_q^+$–structure on M. The homotopy class of the composition $h_{\mathcal{F},g}\colon M \to BR\Gamma_q^+$ depends only on the integrable homotopy class of $(\mathcal{F}, g)$.*

The derivative of a local isometry $\gamma\colon (U_\gamma, g'_\gamma) \to (V_\gamma, g''_\gamma)$ takes values in $\mathbf{SO}(q)$, and is functorial, so induces a classifying map $\nu\colon BR\Gamma_q^+ \to B\mathbf{SO}(q)$. The homotopy fiber of ν is denoted by $FR\Gamma_q$. The space $FR\Gamma_q$ classifies $R\Gamma_q^+$–structures with a (homotopy class of) framing for Q. Let $P \to M$ be the bundle of oriented orthonormal frames of $Q \to M$, and $s\colon M \to P$ a choice of framing of Q. Then we have the commutative diagram:

$$
\begin{array}{ccccc}
\mathbf{SO}(q) & = & \mathbf{SO}(q) & = & \mathbf{SO}(q) \\
\downarrow & & \downarrow & & \downarrow \\
P & \xrightarrow{\overline{h^s_{\mathcal{F},g}}} & FR\Gamma_q & \xrightarrow{f} & F\Gamma_q \\
s \uparrow \downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{h_{\mathcal{F},g}} & BR\Gamma_q^+ & \xrightarrow{f} & B\Gamma_q^+ \\
 & & \downarrow \ \nu & & \downarrow \nu \\
 & & B\mathbf{SO}(q) & = & B\mathbf{SO}(q)
\end{array}
$$

where the right-hand column is the sequence of classifying spaces for the groupoid defined by the germs of local diffeomorphisms of $\mathbb{R}^q$. The natural maps $f\colon FR\Gamma_q \to F\Gamma_q$ and $f\colon BR\Gamma_q^+ \to B\Gamma_q^+$ are induced by the natural transformation which "forgets" the normal Riemannian metric data.

The approach to classifying foliations initiated by Haefliger in[14,15] is based on the study of the homotopy classes of maps $[M, BR\Gamma_q^+]$ from a manifold M without boundary to $BR\Gamma_q^+$. Given a homotopy class of an embedding of an oriented subbundle $Q \subset TM$ of dimension q, one studies the homotopy classes of maps $h_{\mathcal{F},g}\colon M \to BR\Gamma_q^+$ such that the composition $\nu \circ h_{\mathcal{F},g}\colon M \to B\mathbf{SO}(q)$ classifies the homotopy type of the subbundle Q. The "Haefliger classification" of Riemannian foliations is thus based on the study of the homotopy types of the spaces $BR\Gamma_q^+$ and $FR\Gamma_q$.

In the case of codimension-one, a Riemannian foliation with oriented normal bundle of M is equivalent to specifying a closed, non-vanishing 1-form ω on M. As $\mathbf{SO}(1)$ is the trivial group, $FR\Gamma_1^+ = BR\Gamma_1^+$, and the classifying map $M \to BR\Gamma_1^+$ is determined by the real cohomology class of ω, which follows from the following result of Joel Pasternack. Let $\mathbb{R}_\delta$ denote the real line, considered as a *discrete* group, and $B\mathbb{R}_\delta$ its classifying space.

Theorem 2.2 (Pasternack[45]). *There is a natural homotopy equivalence* $BR\Gamma_1^+ \simeq FR\Gamma_1 \simeq B\mathbb{R}_\delta$.

For codimension $q \geq 2$, $\mathbf{SO}(q)$ is not contractible, and the homotopy types of $BR\Gamma_q^+$ and $FR\Gamma_q$ are related by the above fibration sequence. For the space $FR\Gamma_q$ there is a partial generalization of Pasternack's Theorem.

Theorem 2.3 (Hurder[18,19]). *The space $FR\Gamma_q$ is $(q-1)$–connected. That is, $\pi_\ell(FR\Gamma_q) = \{0\}$ for $0 \leq \ell < q$. Moreover, the volume form associated to the transverse metric defines a surjection $vol: \pi_q(FR\Gamma_q) \to \mathbb{R}$.*

Proof. We just give a sketch; see[19] for details. Following a remark by Milnor, one observes that by the Phillips Immersion Theorem,[46–48] an $FR\Gamma_q$–structure on $\mathbb{S}^\ell$ for $0 < \ell < q$ corresponds to a Riemannian metric defined on an open neighborhood retract of the ℓ-sphere, $\mathbb{S}^\ell \subset U \subset \mathbb{R}^q$.

Given an $FR\Gamma_q$–structure on the open set $U \subset \mathbb{R}^q$ – which is equivalent to specifying a Riemannian metric on TU – one then constructs an explicit integrable homotopy through framed $R\Gamma_q$-structures on a smaller open neighborhood $\mathbb{S}^\ell \subset V \subset U$. The integrable homotopy starts with the given Riemannian metric on TV, and ends with the standard Euclidean metric on TV, which represents the "trivial" $FR\Gamma_q$–structure on $\mathbb{S}^\ell$. Thus, every $FR\Gamma_q$–structure on $\mathbb{S}^\ell$ is homotopic to the trivial structure.

The surjection $\mathrm{vol}: \pi_q(FR\Gamma_q) \to \mathbb{R}$ is well-known, and is realized by varying the total volume of a Riemannian metric on $\mathbb{S}^q$, considered as foliated by points. □

Associated to the classifying map $\nu\colon BR\Gamma_q^+ \to B\mathbf{SO}(q)$ is the Puppe sequence

$$\cdots \longrightarrow \Omega FR\Gamma_q \xrightarrow{\Omega\nu} \Omega BR\Gamma_q^+ \longrightarrow \mathbf{SO}(q) \xrightarrow{\delta} FR\Gamma_q \longrightarrow BR\Gamma_q^+ \xrightarrow{\nu} B\mathbf{SO}(q) \qquad (1)$$

In the case of codimension $q = 2$, $\mathbf{SO}(2) = \mathbb{S}^1$ and $FR\Gamma_2$ is 1-connected, so the map $\delta\colon \mathbf{SO}(2) \to FR\Gamma_2$ is contractible. This yields as an immediate consequence:

Theorem 2.4 (Hurder[21]). *$\Omega BR\Gamma_2^+ \cong \mathbf{SO}(2) \times \Omega FR\Gamma_2$.*

It is noted in[21] that the homotopy equivalence in Theorem 2.4 is not an H-space equivalence, as this would imply that map $\nu^*\colon H^*(B\mathbf{SO}(2);\mathbb{R}) \to H^*(BR\Gamma_2;\mathbb{R})$ is an injection, which is false. In contrast, we have the following result:

Theorem 2.5. *The connecting map $\delta\colon \mathbf{SO}(q) \to FR\Gamma_q$ in (1) is not homotopic to a constant for $q \geq 3$.*

Note that the map $\delta\colon \mathbf{SO}(q) \to FR\Gamma_q$ classifies the Riemannian foliation with standard framed normal bundle on $\mathbf{SO}(q) \times \mathbb{R}^q$, obtained via the pull-back of the standard product foliation of $\mathbf{SO}(q) \times \mathbb{R}^q$ via the action of $\mathbf{SO}(q)$ on $\mathbb{R}^q$. Theorem 2.5 asserts that the canonical twisted foliation of $\mathbf{SO}(q) \times \mathbb{R}^q$ is not integrably homotopic through framed Riemannian foliations to the standard product foliation. This will be proven in section 4, using basic properties of the secondary classes for Riemannian foliations. For the non-Riemannian case, it is conjectured that the connecting map $\delta\colon \mathbf{SO}(q) \to F\Gamma_q$ is homotopic to the constant map.[23]

To close this discussion of general properties of the classifying spaces of Riemannian foliations, we pose a problem particular to codimension two:

Problem 2.1. *Prove that the map induced by the volume form* $vol\colon \pi_2(FR\Gamma_2) \to \mathbb{R}$ *is an isomorphism. That is, given two* $R\Gamma_2$*-structures* $\mathcal{F}_0$ *and* $\mathcal{F}_1$ *on* $M = \mathbb{R}^3 - \{0\}$*, with homotopic normal bundles, prove that* $\mathcal{F}_0$ *and* $\mathcal{F}_1$ *are homotopic as* $R\Gamma_2$*-structures if and only if they have cohomologous transverse volume forms.*

One can view this as asking for a "transverse uniformization theorem" for Riemannian foliations of codimension two. Note that Example 5.2 below shows the conclusion of Problem 2.1 is false for $q = 3$.

3. Primary classes

The primary classes of a Riemannian foliation are those obtained from the cohomology of the classifying space of the normal bundle $Q \to M$, pulled-back via the classifying map $\nu\colon M \to B\mathbf{SO}(q)$. Recall[38] that the cohomology groups of $\mathbf{SO}(q)$ are isomorphic to free polynomial ring:

$$
\begin{aligned}
H^*(B\mathbf{SO}(2); \mathbb{Z}) &\cong \mathbb{Z}[E_1] \\
H^*(B\mathbf{SO}(q); \mathbb{Z}) &\cong \mathbb{Z}[E_m, P_1, \ldots, P_{m-1}]\ ,\ q = 2m \geq 4 \\
H^*(B\mathbf{SO}(q); \mathbb{Z}) &\cong \mathbb{Z}[P_1, \ldots, P_m]\ ,\ q = 2m + 1 \geq 3
\end{aligned}
$$

As usual, P_j denotes the Pontrjagin cohomology class of degree $4j$, E_m denotes the Euler class of degree $2m$, and the square $E_m^2 = P_m$ is the top degree generator of the Pontrjagin ring.

There are three main results concerning the universal map $\nu^*\colon H^\ell(B\mathbf{SO}(q); \mathcal{R}) \to H^\ell(BR\Gamma_q^+; \mathcal{R})$, where $\mathcal{R}$ is a coefficient ring, which we discuss in detail below.

Theorem 3.1 (Pasternack[45]). $\nu^*\colon H^\ell(B\mathbf{SO}(q); \mathbb{R}) \to H^\ell(BR\Gamma_q^+; \mathbb{R})$ *is trivial for* $\ell > q$.

Theorem 3.2 (Bott, Heitsch[5]). $\nu^*\colon H^*(B\mathbf{SO}(q);\mathbb{Z}) \to H^*(BR\Gamma_q^+;\mathbb{Z})$ *is injective.*

Theorem 3.3 (Hurder[19,22]). $\nu^*\colon H^\ell(B\mathbf{SO}(q);\mathbb{R}) \to H^\ell(BR\Gamma_q^+;\mathbb{R})$ *is injective for* $\ell \leq q$.

The contrast between Theorems 3.1 and 3.2 is one of the themes of this section, while the proof of Theorem 3.3 is based on an observation.

Let ∇_g denote the Levi-Civita connection on $Q \to M$ associated to the projectable metric g for $\mathcal{F}$. The Chern-Weil construction associates to each universal class P_j the closed Pontrjagin form $p_j(\nabla_g) \in \Omega^{4j}(M;\mathbb{R})$. For $q = 2m$, as Q is assumed to be oriented, there is also the Euler form $e_m(\nabla_g) \in \Omega^{2m}(M;\mathbb{R})$ whose square $e_m(\nabla_g)^2 = p_m(\nabla_g)$. The universal map $\nu^*\colon H^*(B\mathbf{SO}(q);\mathbb{R}) \to H^*(BR\Gamma_q^+;\mathbb{R})$ is defined by its values on foliated manifolds, where $\nu^*(P_j) = [p_j(\nabla_g)] \in H^{4j}(M;\mathbb{R})$, and $[\beta]$ represents the de Rham cohomology class of a closed form β.

Let m be the least integer such that $q \leq 2m+2$. Given $J = (j_1, j_2, \ldots, j_m)$ with each $j_\ell \geq 0$, set $p_J = p_1^{j_1} \cdot p_2^{j_2} \cdots p_m^{j_m}$, which has degree $4|J| = 4(j_1 + \cdots + j_m)$. Let $\mathcal{P}$ denote a basis monomial: for $q = 2m+1$, it has the form $\mathcal{P} = p_J$ with $\deg(\mathcal{P}) = 4|J|$. For $q = 2m$, either $\mathcal{P} = p_J$ with $\deg(\mathcal{P}) = 4|J|$, or $\mathcal{P} = e_m \cdot p_J$ with $\deg(\mathcal{P}) = 4|J| + 2m$.

Pasternack[44] first observed in his thesis that the proof of the Bott Vanishing Theorem[4] can be strengthened in the case of Riemannian foliations, as the adapted metric ∇_g is projectable. He showed that on the level of differential forms, an analogue of the Bott Vanishing Theorem holds.

Theorem 3.4 (Pasternack[44,45]). *If* $\deg(\mathcal{P}) > q$ *then* $\mathcal{P}(\nabla_g) = 0$.

Theorem 3.1 follows immediately. Today, this result is considered "obvious", but that is due to the later extensive development of this field in the 1970's.

Next consider the injectivity of $\nu^*\colon H^k(B\mathbf{SO}(q);\mathbb{R}) \to H^k(BR\Gamma_q^+;\mathbb{R})$. We recall a basic observation of Thom.[38]

Theorem 3.5. *There is a compact, orientable Riemannian manifold B of dimension q such that all of the Pontrjagin and Euler classes up to degree q are independent in $H^*(B;\mathbb{R})$. If q is odd, then B can be chosen to be a connected manifold.*

Proof. For q even, let B equal the disjoint union of all products of the form $\mathbf{CP}^{i_1} \times \cdots \times \mathbf{CP}^{i_k} \times S^1 \times \cdots \times S^1$ with dimension q. For q odd, B is the connected sum of all products of the form $\mathbf{CP}^{i_1} \times \cdots \times \mathbf{CP}^{i_k} \times S^1 \times \cdots \times S^1$

with dimension q. The claim then follows by the Splitting Principle[38] for the Pontrjagin classes. □

Proof of Theorem 3.3. This now follows from the universal properties of $BR\Gamma_q^+$, as we endow the manifold B with the foliation $\mathcal{F}$ by points, with the standard Riemannian metric on B. □

The proof of Theorem 3.3 in[19] used the fact that $\nu\colon BR\Gamma_q^+ \to B\mathbf{SO}(q)$ is q-connected.

Next, we discuss the results of Bott and Heitsch.[5] Let $K \subset \mathbf{SO}(q)$ be a closed Lie subgroup, and let $\Gamma \subset K$ be a finitely-generated subgroup. Suppose that B is a closed connected manifold, with basepoint $b_0 \in B$. Assume there is a surjection $\rho\colon \Lambda = \pi_1(B, b_0) \to \Gamma \subset K \subset \mathbf{SO}(q)$. Then via the natural action of $\mathbf{SO}(q)$ on $\mathbb{R}^q$ we obtain an action of Λ on $\mathbb{R}^q$. Let $\widetilde{B} \to B$ denote the universal covering of B, equipped with the right action of Λ by deck transformations. Then form the flat bundle

$$\mathbb{E}_\rho = \widetilde{B} \times \mathbb{R}^q/(b\cdot\gamma, \vec{v}) \sim (b, \rho(\gamma)\cdot\vec{v}) \xrightarrow{\pi} \widetilde{B}/\Lambda = B \qquad (2)$$

As the action of Λ on $\mathbb{R}^q$ preserves the standard Riemannian metric, we obtain a Riemannian foliation $\mathcal{F}_\rho$ on $\mathbb{E}_\rho$ whose leaves are the integral manifolds of the flat structure. The classifying map of the foliation $\mathcal{F}_\rho$ is given by the composition of maps

$$\mathbb{E}_\rho \to B\Lambda \to B(K_\delta) \to B(\mathbf{SO}(q)_\delta) \to BR\Gamma_q^+ \qquad (3)$$

where K_δ and $\mathbf{SO}(q)_\delta$ denotes the corresponding Lie groups considered with the discrete topology, and $B(K_\delta)$ and $B(\mathbf{SO}(q)_\delta)$ are the corresponding classifying spaces.

The Bott-Heitsch examples take K to be a maximal torus, so that for $q = 2m$ or $q = 2m+1$, we have $K = \mathbb{T}^m = \mathbf{SO}(2) \times \cdots \times \mathbf{SO}(2)$ with m factors. Consider first the case $q = 2$. For an odd prime p, let $\Gamma = \mathbb{Z}/p\mathbb{Z}$, embedded as the p-th roots of unity in $K = \mathbf{SO}(2)$. Let $B = \mathbb{S}^{2\ell+1}/\Gamma$ be the quotient of the standard odd-dimensional sphere, and consider the composition

$$\nu\circ\rho\colon B \to \mathbb{E}_\rho \to B(\mathbb{Z}/p\mathbb{Z}) \to B(\mathbf{SO}(2)_\delta) \to BR\Gamma_2^+ \xrightarrow{\nu} B\mathbf{SO}(2) \qquad (4)$$

The composition $\nu\circ\rho$ classifies the Euler class of the flat bundle $\mathbb{E}_\rho \to B$, which is torsion. The map in cohomology with $\mathbb{Z}/p\mathbb{Z}$-coefficients,

$$(\nu\circ\rho)^*\colon H^*(B\mathbf{SO}(2); \mathbb{Z}/p\mathbb{Z}) \to H^*(B; \mathbb{Z}/p\mathbb{Z}) \qquad (5)$$

is injective for $* \leq 2\ell$. It follows that the map

$$\nu^* \colon H^*(B\mathbf{SO}(2); \mathbb{Z}/p\mathbb{Z}) \to H^*(BR\Gamma_2^+; \mathbb{Z}/p\mathbb{Z}) \tag{6}$$

is injective in all degrees. As this holds true for all odd primes, it is also injective for integral cohomology. □

Theorem 3.2 is a striking result, as Theorem 3.1 implies that $\nu^* \colon H^*(B\mathbf{SO}(2); \mathbb{Q}) \to H^*(BR\Gamma_2^+; \mathbb{Q})$ is the trivial map for $* > 2$. One thus concludes from the Universal Coefficient Theorem for cohomology[5] that the homology groups $H_*(BR\Gamma_2^+; \mathbb{Z})$ cannot be finitely generated in all odd degrees $* \geq 3$.

The treatment of the cases where $q = 2m > 2$ and $q = 2m + 1 > 2$ follows similarly, where one takes $\Gamma = (\mathbb{Z}/p\mathbb{Z})^m \subset \mathbb{T}^m \subset \mathbf{SO}(q)$, and let $p \to \infty$. An application of the splitting theorem for the Pontrjagin classes of vector bundles then yields Theorem 3.2.

The fibration sequence $FR\Gamma_q \to BR\Gamma_q^+ \to B\mathbf{SO}(q)$ yields a spectral sequence converging to the homology groups $H^*(BR\Gamma_q^+; \mathbb{Z})$ with E^2-term

$$E^2_{r,s} \cong H_r(B\mathbf{SO}(q); H_s(FR\Gamma_q; \mathbb{Z})) \tag{7}$$

It follows that the groups $H_s(FR\Gamma_q; \mathbb{Z})$ cannot all be finitely generated for odd degrees $* \geq q$. In fact, we will see that this follows from the results of Pasternack and Lazarov discussed in the next section on secondary classes, but the homology classes being detected via the torsion classes above seem to be of a different "sort" than those detected via the secondary classes.

Recall that the universal classifying map $f \colon FR\Gamma_q \to F\Gamma_q$ "forgets" the added structure of a holonomy-invariant transverse Riemannian metric for the foliation. It has been conjectured (see page 308,[41]) that this map induces the trivial map in homotopy.

Conjecture 3.1. $f_\# \colon \pi_k(FR\Gamma_q) \to \pi_k(F\Gamma_q)$ *is trivial for all* $k > 0$.

The ideas of the proof of Theorem 3.2 imply that Conjecture 3.1 is false.

Theorem 3.6. *The image of* $f_* \colon H_{4k-1}(BR\Gamma_q; \mathbb{Z}) \to H_{4k-1}(B\Gamma_q; \mathbb{Z})$ *is infinite-dimensional for* $4k > 2q \geq 4$.

Proof. Our approach uses the homological methods of the proof of Theorem 3.2[5] and especially the commutative diagram from page 144.

Let $\mathcal{P} \in H^{4k}(B\mathbf{SO}(q); \mathbb{Z})$ for $4k > q$ be a generating monomial. The Bott-Heitsch Theorem 3.2 implies that the image $f^* \circ \nu^*(\mathcal{P}) \in H^{4k}(BR\Gamma_q^+; \mathbb{Z})$ is not a torsion class under the composition

$$H^{4k}(B\mathbf{SO}(q);\mathbb{Z}) \xrightarrow{\nu^*} H^{4k}(B\Gamma_q^+;\mathbb{Z}) \xrightarrow{f^*} H^{4k}(BR\Gamma_q^+;\mathbb{Z}) \qquad (8)$$

Let $\mathcal{A}_{4k-1} = \text{image}\{H_{4k-1}(BR\Gamma_q^+;\mathbb{Z}) \to H_{4k-1}(B\Gamma_q^+;\mathbb{Z})\}$. Suppose that $\mathcal{A}_{4k-1}$ is finite-dimensional, then $\text{Ext}(\mathcal{A}_{4k-1},\mathbb{Z})$ is a torsion group. Consider the commutative diagram:

$$\begin{array}{ccccc}
 & & H^{4k}(B\mathbf{SO}(q);\mathbb{Z}) & & \\
 & & \big\downarrow \nu^* & & \\
\text{Ext}(H_{4k-1}(B\Gamma_q^+;\mathbb{Z}),\ \mathbb{Z}) & \xrightarrow{\tau} & H^{4k}(B\Gamma_q^+;\mathbb{Z}) & \xrightarrow{e} & \text{Hom}(H_{4k}(B\Gamma_q^+;\mathbb{Z}),\ \mathbb{Z}) \\
\big\downarrow \iota^* & & & & \\
\text{Ext}(\mathcal{A}_{4k-1},\ \mathbb{Z}) & & \big\downarrow f^* & & \big\downarrow f^* \\
\big\downarrow \sigma^* & & & & \\
\text{Ext}(H_{4k-1}(BR\Gamma_q^+;\mathbb{Z}),\ \mathbb{Z}) & \xrightarrow{\tau} & H^{4k}(BR\Gamma_q^+;\mathbb{Z}) & \xrightarrow{e} & \text{Hom}(H_{4k}(BR\Gamma_q^+;\mathbb{Z}),\ \mathbb{Z})
\end{array}$$

In the diagram, e is the evaluation map of cohomology on homology, and τ maps onto its kernel. The inclusion $\iota\colon \mathcal{A}_{4k-1} \subset H_{4k-1}(B\Gamma_q^+;\mathbb{Z})$ induces the map ι^*, and the surjection $\sigma\colon H_{4k-1}(BR\Gamma_q^+;\mathbb{Z}) \to \mathcal{A}_{4k-1}$ induces σ^*.

The Bott Vanishing Theorem implies that the class

$$e \circ \nu^*(\mathcal{P}) \in \text{Hom}(H_{4k}(B\Gamma_q^+;\mathbb{Z}),\ \mathbb{Z}) \subset \text{Hom}(H_{4k}(B\Gamma_q^+;\mathbb{Z}),\ \mathbb{Q})$$

is trivial for $\deg(\mathcal{P}) > 2q$. Thus, there exists $\mathcal{P}_\tau \in \text{Ext}(H_{4k-1}(B\Gamma_q^+;\mathbb{Z}),\ \mathbb{Z})$ such that $\tau(\mathcal{P}_\tau) = \nu^*(\mathcal{P})$. The class $\iota^*(\mathcal{P}_\tau) \in \text{Ext}(\mathcal{A}_{4k-1},\ \mathbb{Z})$ is torsion, by the assumption on $\mathcal{A}_{4k-1}$. Thus, $f^* \circ \nu^*(\mathcal{P}) = \tau \circ \sigma^* \circ \iota^*(\mathcal{P}_\tau)$ is a torsion class, which contradicts the Bott-Heitsch results. Thus, $\mathcal{A}_{4k-1}$ cannot be finite-dimensional for $4k > 2q$. □

Corollary 3.1. *The image of* $f_*\colon H_{4k-1}(FR\Gamma_q;\mathbb{Z}) \to H_{4k-1}(F\Gamma_q;\mathbb{Z})$ *is infinite-dimensional for* $4k > 2q \geq 4$.

Proof. This follows from the commutative diagram following Theorem 2.1, the functorial properties of the spectral sequence (7), the fact that $H_*(B\mathbf{SO}(q);\mathbb{Z})$ is finitely generated in all degrees, and Theorem 3.6. □

Problem 3.1. *Find geometric interpretations of the cycles in the image of the map* $f_*\colon H_{4k-1}(FR\Gamma_q;\mathbb{Z}) \to H_{4k-1}(F\Gamma_q;\mathbb{Z})$.

The construction of foliations with solenoidal minimal sets in[10,26] give one realization of some of the classes in the image of this map, as discussed in the talk by the author[25] at the conference of these Proceedings. Neither these examples,[26] nor the situation overall, is understood in sufficient depth.

4. Secondary classes

Assume that $(\mathcal{F}, g)$ is a Riemannian foliation of codimension q. We also assume that there exists a framing $s\colon M \to P$ of the normal bundle. Then the data $(\mathcal{F}, g, s)$ yields a classifying map $h^s_{\mathcal{F},g}\colon M \to FR\Gamma_q$. In this section, we discuss the construction of the secondary characteristic classes of such foliations, constructed using the Chern-Weil method,[8] and some of the results about these classes.

Recall that ∇_g denotes the Levi-Civita connection of the projectable metric g on Q.

Let $\mathcal{I}(\mathbf{SO}(q))$ denote the ring of Ad-invariant polynomials on the Lie algebra $\mathfrak{so}(q)$ of $\mathbf{SO}(q)$. Then we have

$$\begin{aligned} \mathcal{I}(\mathbf{SO}(2)) &\cong \mathbb{R}[e_m] \\ \mathcal{I}(\mathbf{SO}(2m)) &\cong \mathbb{R}[e_m, p_1, \ldots, p_{m-1}]\ ,\ q = 2m \geq 4 \\ \mathcal{I}(\mathbf{SO}(2m+1)) &\cong \mathbb{R}[p_1, \ldots, p_m]\ ,\ q = 2m+1 \geq 3 \end{aligned}$$

where the p_j are the Pontrjagin polynomials, and e_m is the Euler polynomial defined for q even.

The symmetric polynomials p_j evaluated on the curvature matrix of 2-forms associated to the connection ∇_g yields closed forms $\Delta_{\mathcal{F},g}(p_j) = p_\ell(\nabla_g) \in \Omega^{4j}(M)$. Then $\Delta_{\mathcal{F}}[p_j] = [p_j(\nabla_g)] \in H^{4j}(M;\mathbb{R})$ represents the Pontrjagin class $P_j(Q)$. The Euler form $\Delta_{\mathcal{F},g}(e_m) = e_m(\nabla_g) \in \Omega^{2m}(M)$ and the Euler class $\Delta_{\mathcal{F},g}[e_m] \in H^{2m}(M,\mathbb{R})$ are similarly defined when $q = 2m$. We thus obtain a multiplicative homomorphism

$$\Delta_{\mathcal{F},g}\colon \mathcal{I}(\mathbf{SO}(q)) \to H^*(M;\mathbb{R})$$

As noted in Theorem 3.4, Pasternack first observed that for ∇_g the adapted connection to a Riemannian foliation, the map $\Delta_{\mathcal{F},g}$ vanishes identically in degrees greater than q.

Definition 4.1. For $q = 2m$, set

$$\mathcal{I}(\mathbf{SO}(q))_{2m} \equiv \mathbb{R}[e_m, p_1, p_2, \ldots, p_{m-1}]/(\mathcal{P} \mid \deg(\mathcal{P}) > q)$$

For $q = 2m+1$, set

$$\mathcal{I}(\mathbf{SO}(q))_{2m+1} \equiv \mathbb{R}[p_1, p_2, \ldots, p_m]/(\mathcal{P} \mid \deg(\mathcal{P}) > q)$$

Corollary 4.1 (Pasternack). *Let* $(\mathcal{F}, g)$ *be a Riemannian foliation of* M *with codimension* q*. Then there is a characteristic homomorphism* $\Delta_{\mathcal{F},g} \colon \mathcal{I}(\mathbf{SO}(q))_q \to H^*(M;\mathbb{R})$*, which is functorial for transversal maps between foliated manifolds.*

Of course, if we assume that the normal bundle Q is trivial, then this map is zero in cohomology. The point of the construction of secondary classes is to obtain geometric information from the forms $p_j(\nabla_g) \in \Omega^{4j}(M)$, even if they are exact. If we do not assume that Q is trivial, then one still knows that the cohomology classes $[p_j(\nabla_g)] \in H^{4j}(M;\mathbb{R})$ lie in the image of the integral cohomology, $H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{R})$ so that one can use the construction of Cheeger-Simons differential characters as in[7,9,31,52] to define secondary invariants in the groups $H^{4j-1}(M;\mathbb{R}/\mathbb{Z})$. These classes are closely related to the Bott-Heitsch examples above, and to the secondary classes constructed below.

Given a trivialization $s \colon M \to P$, let ∇_s be the flat connection on Q for which s is parallel. Set $\nabla_t = t\nabla_g + (1-t)\nabla_s$, which we consider as a connection on the bundle Q extended as product over $M \times \mathbb{R}$. Then the Pontrjagin forms for ∇_t yield closed forms $p_j(\nabla_t) \in \Omega^{4j}(M \times \mathbb{R})$. Define the $4j-1$ degree transgression form

$$h_j = h_j(\nabla_g, s) = \int_0^1 \{\iota(\partial/\partial t) p_j(\nabla_t)\} \wedge dt \ \in \Omega^{4j-1}(M) \tag{9}$$

which satisfies the coboundary relation on forms:

$$dh_j(\nabla_g, s) = p_j(\nabla_g) - p_j(\nabla_s) = p_j(\nabla_g)$$

For $q = 2m$ we also introduce the transgression of the Euler form,

$$\chi_m = \chi_m(\nabla_g, s) = \int_0^1 \{\iota(\partial/\partial t) e_m(\nabla_t)\} \wedge dt \ \in \Omega^{q-1}(M) \tag{10}$$

which satisfies the coboundary equation $d\chi_m = e_m(\nabla_g)$. Note that if $4j > q$, then the form $p_j(\nabla_g) = 0$, so the transgression form h_j is closed. The cohomology class $[h_j] = \Delta^s_{\mathcal{F},g}(h_j) \in H^{4j-1}(M;\mathbb{R})$ is said to be a *secondary cohomology class*. In general, introduce the graded differential complexes:

$$\begin{aligned} RW_{2m} &= \Lambda\,(h_1, \ldots, h_{m-1}, \chi_m) \otimes \mathcal{I}(\mathbf{SO}(q))_{2m} \\ RW_{2m+1} &= \Lambda\,(h_1, \ldots, h_m) \otimes \mathcal{I}(\mathbf{SO}(q))_{2m+1} \end{aligned}$$

where $d_W(h_j \otimes 1) = 1 \otimes p_j$ and $d_W(\chi_m \otimes 1) = e_m \otimes 1$. For $I = (i_1 < \cdots < i_\ell)$ and $J = (j_1 \leq \cdots \leq j_k)$ set

$$h_I \otimes p_J = h_{1_1} \wedge \cdots \wedge h_{i_\ell} \otimes p_{j_1} \wedge \cdots \wedge p_{j_k} \tag{11}$$

Note that $\deg(h_I \otimes p_J) = 4(|I|+|J|)-\ell$, and that $d_W(h_I \otimes p_J) = 0$ exactly when $4i_1 + 4|J| > q$. In the following, the expression $h_I \otimes p_J$ will always assume that the indexing sets I and J are ordered as above.

Theorem 4.1 (Lazarov - Pasternack[34]). *Let $(\mathcal{F}, g)$ be a Riemannian foliation of codimension $q \geq 2$ of a manifold M without boundary, and assume that there is given a framing of the normal bundle, $s\colon M \to P$. Then the above constructions yield a map of differential graded algebras*

$$\Delta^s_{\mathcal{F},g}\colon RW_q \to \Omega^*(M) \tag{12}$$

such that the induced map on cohomology, $\Delta^s_{\mathcal{F},g}\colon H^*(RW_q) \to H^*(M;\mathbb{R})$, is independent of the choice of basic connection ∇_g, and depends only on the integrable homotopy class of $\mathcal{F}$ as a Riemannian foliation and the homotopy class of the framing s.

This construction can also be recovered from the method of truncated Weil algebras applied to the Lie algebra $\mathfrak{so}(q)$ (see Kamber and Tondeur[29,30]). The functoriality of the construction of $\Delta^s_{\mathcal{F},g}$ implies, in the usual way:[32,34]

Corollary 4.2. *There exists a universal characteristic homomorphism*

$$\Delta\colon H^*(RW_q, d_W) \to H^*(FR\Gamma_q;\mathbb{R}) \tag{13}$$

There are many natural questions about how the values of these secondary classes are related to the geometry and dynamical properties of the foliation $(\mathcal{F}, g, s)$. We discuss some known results in the following.

First, consider the role of the section $s\colon M \to P$. Given any smooth map $\varphi\colon M \to \mathbf{SO}(q)$, we obtain a new framing $s' = s \cdot \varphi\colon M \to P$ by setting $s'(x) = s(x)\cdot\varphi(x)$. Thus, φ can be thought of as a gauge transformation of the normal bundle $Q \to M$.

The cohomology of the Lie algebra $\mathfrak{so}(q)$ is isomorphic to an exterior algebra, generated by the cohomology classes of left-invariant closed forms $\tau_j \in \Lambda^{4j-1}(\mathfrak{so}(q))$ for $j < q/2$, and the Euler form $\chi_m \in \Lambda^{2m-1}(\mathfrak{so}(q))$ when $q = 2m$. The map φ pulls these back to closed forms $\varphi^*(\tau_j) \in \Omega^{4j-1}(M)$.

Theorem 4.2 (Lazarov[33,34]). *Suppose that two framings s, s' of Q are related by a gauge transformation $\varphi\colon M \to \mathbf{SO}(q)$, $s' = s\cdot\varphi$. Then on the level of forms,*

$$\Delta^{s'}_{\mathcal{F},g}(h_j) = \Delta^s_{\mathcal{F},g}(h_j) + \varphi^*(\tau_j) \tag{14}$$

In particular, for $j > q/4$, we have the relation in cohomology

$$\Delta^{s'}_{\mathcal{F},g}[h_j] = \Delta^s_{\mathcal{F},g}[h_j] + \varphi^*[\tau_j] \in H^{4j-1}(M;\mathbb{R}) \tag{15}$$

The relation (14) can be used to easily calculate exactly how the cohomology classes $\Delta^{s}_{\mathcal{F},g}[h_I \otimes p_J]$ and $\Delta^{s'}_{\mathcal{F},g}[h_I \otimes p_J]$ associated to framings s, s' are related. (See §4,[34] and[33] for details.) Here is one simple application of Theorem 4.2:

Proof of Theorem 2.5. For the product foliation of $\mathbf{SO}(q) \times \mathbb{R}^q$ we have a natural identification of the transverse orthogonal frame bundle $P = \mathbf{SO}(q) \times \mathbf{SO}(q)$. Let $s\colon \mathbf{SO}(q) \to P$ be the map $s(x) = x \times \{Id\}$, called the product framing. Then the map $\Delta^{s}_{\mathcal{F},g}\colon RW_q \to \Omega^*(M)$ is identically zero.

On the other hand, the connecting map $\delta\colon \mathbf{SO}(q) \to FR\Gamma_q$ in (1) classifies the Riemannian foliation $\mathcal{F}_\delta$ of $\mathbf{SO}(q) \times \mathbb{R}^q$, obtained via the pull-back of the standard product foliation of $\mathbf{SO}(q) \times \mathbb{R}^q$ via the action of $\mathbf{SO}(q)$ on $\mathbb{R}^q$. However, the normal framing of $\mathcal{F}_\delta$ is the product framing on $\mathbf{SO}(q) \times \mathbb{R}^q$. Let $\varphi\colon \mathbf{SO}(q) \to \mathbf{SO}(q)$ be defined by $\varphi(x) = x^{-1}$ for $x \in \mathbf{SO}(q)$. Then $\mathcal{F}_\delta$ is diffeomorphic to the product foliation of $\mathbf{SO}(q) \times \mathbb{R}^q$ with the framing defined by the gauge action of φ.

It follows from Theorem 4.2 that for $j > q/4$, $\Delta^{s'}_{\mathcal{F},g}[h_j] = \varphi^*[\tau_j] = \pm\tau_j \in H^{4j-1}(\mathbf{SO}(q); \mathbb{R})$ is a generator. Hence, the connecting map $\delta\colon \mathbf{SO}(q) \to FR\Gamma_q$ cannot be homotopic to the identity if there exists $j > q/4$ such that $\tau_j \in H^{4j-1}(\mathbf{SO}(q); \mathbb{R})$ is non-zero. This is the case for all $q > 2$. □

The original Chern-Simons invariants of 3-manifolds[9] can be considered as examples of the above constructions. Let M be a closed oriented, connected 3-manifold with Riemannian metric g. Consider M as foliated by points, then we obtain a Riemannian foliation of codimension 3. Choose an oriented framing $s\colon M \times \mathbb{R}^3 \to TM$, then the transgression form $\Delta^{s}_{\mathcal{F},g}(h_1) \in H^3(M; \mathbb{R}) \cong \mathbb{R}$ is well-defined. Note that by formula (15), the mod $\mathbb{Z}$-reduction $\overline{\Delta^{s}_{\mathcal{F},g}(h_1)} \in H^3(M; \mathbb{R}/\mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}$ is then independent of the choice of framing. This invariant of the metric is just the Chern-Simons invariant.[9] for (M, g). On the other hand, Atiyah showed[1] that for a 3-manifold, there is a "canonical" choice of framing s_0 for TM, so that there is a canonical $\mathbb{R}$-valued Chern-Simons invariant, $\Delta^{s_0}_{\mathcal{F},g}(h_1) \in \mathbb{R}$.

Chern and Simons[9] also show that the values of $\overline{\Delta^{s}_{\mathcal{F},g}(h_1)} \in \mathbb{R}/\mathbb{Z}$ can vary non-trivially with the choice of Riemannian metric.

One of the standard problems in foliation theory, is to determine whether the universal characteristic map is injective. For the classifying space $B\Gamma_q$ of smooth foliations, this remains one of the outstanding open problems.[24] In contrast, for Riemannian foliations, the universal map (13) is injective. We present here a new proof of this, based on Theorem 3.5.

Theorem 4.3 (Hurder[19]). *There exists a compact manifold M and a Riemannian foliation $\mathcal{F}$ of M with trivial normal bundle, such that $\mathcal{F}$ is defined by a fibration over a compact manifold of dimension q, and the characteristic map $\Delta^s_{\mathcal{F},g}\colon H^*(RW_q) \to H^*(M)$ is injective. Moreover, if q is odd, then M can be chosen to be connected.*

Proof. Let B be the compact, oriented Riemannian manifold defined in the proof Theorem 3.5. Let M be the bundle of oriented orthonormal frames for TB. The basepoint map $\pi\colon M \to B$ defines a fibration $\mathbf{SO}(q) \to M \to B$, whose fiber $L_x = \pi^{-1}(x)$ over $x \in B$ is the group $\mathbf{SO}(q)$ of oriented orthonormal frames in T_xB. Let $\mathcal{F}$ be the foliation defined by the fibration. The Riemannian metric on B lifts to the transverse metric on the normal bundle $Q = \pi^*TB$. The bundle Q has a canonical framing s, where for $b \in B$ and $A \in \mathbf{SO}(q)$ the framing of $Q_{x,A}$ is that defined by the matrix A.

The normal bundle restricted to L_x is trivial, as it is just the constant lift of T_xB. That is, $Q|L_x \cong \pi^*(T_xB) \cong L_x \times \mathbb{R}^q$. The basic connection ∇_g restricted to $Q|L_x$ is the connection associated to the product bundle $L_x \times \mathbb{R}^q$. However, the canonical framing of $Q \to M$ restricted to $Q|L_x$ is twisted by $\mathbf{SO}(q)$. Thus, the connection ∇_s on Q for which the canonical framing is parallel, restricts to the Maurer-Cartan form on $\mathbf{SO}(q) \times \mathbb{R}^q$ along each fiber L_x.

By Chern-Weil theory, the forms $\Delta^s_{\mathcal{F},g}(h_j) = h_j(\nabla_g, s)$ restricted to $L_x = \mathbf{SO}(q)$ are closed, and their classes in cohomology define the free exterior generators for the cohomology $H^*(\mathbf{SO}(q); \mathbb{R})$. (In the even case $q = 2m$, one must include the Euler class χ_m as well.)

Give the algebra RW_q the basic filtration by the degree in $\mathcal{I}(\mathbf{SO}(q))_q$, and the forms in $\Omega^*(M)$ the basic filtration by their degree in $\pi^*\Omega^*(B)$. (See[30] for example.) The characteristic map $\Delta^s_{\mathcal{F},g}$ preserves the filtrations, hence induces a map of the associated Leray-Hirsch spectral sequences,

$$\Delta^{*,*}_r\colon E^{*,*}_r(RW_q, d_W) \to E^{*,*}_r(M, d_r)$$

For $r = 2$, we then have

$$\Delta^{*,*}_2\colon E^{*,*}_2(RW_q) \cong (RW_q, d_W) \to E^{*,*}_r(M, d_2) \cong H^*(\mathbf{SO}(q); \mathbb{R}) \otimes H^*(B; \mathbb{R})$$

which is injective by the remark above. Pass to the E_∞–limit to obtain that $\Delta^s_{\mathcal{F},g}\colon H^*(RW_q) \to H^*(M)$ induces an injective map of associated graded algebras, hence is injective. □

It seems to be an artifact of the proof that for $q \geq 4$ even, the manifold M we obtain is not connected.

Problem 4.1. *For $q \geq 4$ even, does there exists a closed, connected manifold M and a Riemannian foliation $\mathcal{F}$ of M of codimension-q and trivial normal bundle, such that the secondary characteristic map $\Delta^s_{\mathcal{F},g} \colon H^*(RW_q) \to H^*(M)$ injects? Is there a cohomological obstruction to the existence of such an example?*

Note that in the examples constructed in the proof of Theorem 4.3, the image of the monomials $h_I \otimes p_J$ for $4i_1 + 4|J| > q$ (and hence $h_I \otimes p_J$ is d_W-closed) are integral:

$$\Delta^s_{\mathcal{F},g}[h_I \otimes p_J] \in \text{Image}\,\{H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{R})\}$$

This follows since the restriction of the forms $\Delta^s_{\mathcal{F},g}(h_I)$ to the leaves of $\mathcal{F}$ are integral cohomology classes. In general, one cannot expect a similar integrality result to hold for examples with all leaves compact, as is shown by the Chern-Simons example previously mentioned. However, a more restricted statement holds.

Definition 4.2. A foliation $\mathcal{F}$ of a manifold M is *compact Hausdorff* if every leaf of $\mathcal{F}$ is a compact manifold, and the leaf space $M/\mathcal{F}$ is a Hausdorff space.

Theorem 4.4 (Epstein,[12] Millett[37]). *A compact Hausdorff foliation $\mathcal{F}$ admits a holonomy-invariant Riemannian metric on its normal bundle Q.*

In the next section, we discuss the division of the secondary classes into "rigid" and "variable" classes. One can show the following:

Theorem 4.5. *Let $\mathcal{F}$ be a compact Hausdorff foliation of codimension q of M with trivial normal bundle. If $h_I \otimes p_J$ is a rigid class, then*

$$\Delta^s_{\mathcal{F},g}[h_I \otimes p_J] \in \text{Image}\,\{H^*(M, \mathbb{Q}) \to H^*(M, \mathbb{R})\}$$

This follows for the case when the leaf space $M/\mathcal{F}$ is a smooth manifold from[18] whose methods extend to this more general situation.

It is an interesting problem to find geometric conditions on a Riemannian foliation which imply the rationality of the secondary classes.[11] Rationality should be associated to rigidity properties for the global holonomy of the leaf closures, one of the fundamental geometric concepts in the Molino Structure theory discussed in §6. One expects rationality results for the secondary classes analogous to the celebrated results of Reznikov,[50,51] possibly with some additional assumptions on the geometry of the leaves.

5. Variation of secondary classes

The secondary classes of a foliation are divided into two types, the "rigid" and the "variable" classes. Examples show that the variable classes are sensitive to both the geometry and dynamical properties of the foliation, while the rigid classes seem to be topological in nature.

A monomial $h_I \otimes p_J \in RW_q$ is said to be *rigid* if $\deg(p_{i_1} \wedge p_J) > q + 2$. Note that if $4i_1 + 4|J| > q$, then this condition is automatically satisfied when $q = 4k$ or $q = 4k + 1$. Here is the key property of the rigid classes:

Theorem 5.1 (Lazarov and Pasternack, Theorem 5.5[34]). *Let $(\mathcal{F}_t, g_t, s_t)$ be a smooth 1-parameter family of framed Riemannian foliations. Let $h_I \otimes p_J \in RW_q$ be a rigid class. Then*

$$\Delta^{s_0}_{\mathcal{F}_0, g_0}[h_I \otimes p_J] = \Delta^{s_1}_{\mathcal{F}_1, g_1}[h_I \otimes p_J] \in H^*(M; \mathbb{R})$$

Note that the family $\{(\mathcal{F}_t, g_t) \mid 0 \leq t \leq 1\}$ need not be a Riemannian foliation of codimension-q of $M \times [0, 1]$.

For the special case where $q = 4k - 2 \geq 6$, a stronger form of the above result is true:

Theorem 5.2 (Lazarov and Pasternack, Theorem 5.6[34]). *Let $(\mathcal{F}, g_t, s_t)$ be a smooth 1-parameter family, where $\mathcal{F}$ is a fixed foliation of codimension q, each g_t is a holonomy invariant Riemannian metric on Q, and s_t is a smooth family of framings on Q. Let $h_I \otimes p_J \in RW_q$ satisfy* $\deg(p_{i_1} \wedge p_J) > q + 1$. *Then*

$$\Delta^{s_0}_{\mathcal{F}, g_0}[h_I \otimes p_J] = \Delta^{s_1}_{\mathcal{F}, g_1}[h_I \otimes p_J] \in H^*(M; \mathbb{R})$$

We say that these classes are metric rigid*. Thus, the classes $[h_I \otimes p_J] \in H^*(RW_q)$ are metric rigid when* $\deg(h_{i_1} \otimes p_J) > q$*, and rigid under all deformations when* $\deg(h_{i_1} \otimes p_J) > q + 1$.

A closed monomial $h_I \otimes p_J$ which is not rigid, is said to be *variable*. In the special case $q = 2$, the class $[\chi_1 \otimes e_1] \in H^3(RW_2)$ is variable. For $q > 2$, neither the Euler class e_m or its transgression χ_m can occur in a variable class, so for $q = 4k - 2$ or $q = 4k - 1$, the variable classes are spanned by the closed monomials

$$\mathcal{V}_q = \{h_I \otimes p_J \mid 4i_1 + 4|J| = 4k\} \tag{16}$$

Let v^k_q denote the dimension of the subspace of $H^k(RW_q)$ spanned by the variable monomials.

Theorem 5.2 implies that for codimension $q = 4k - 2 \geq 6$, in order to continuously vary the value of a variable class $h_I \otimes p_J$ it is necessary to deform the underlying foliation. For $q = 4k - 1$, the value of variable class may (possibly) be continuously varied by simply changing the transverse metric for the foliation. We illustrate this with two examples.

Example 5.1 (Chern-Simons, Example 2 in §6[9]). *Consider* $\mathbb{S}^3$ *as the Lie group* $\mathbf{SU}(2)$ *with Lie algebra spanned by*

$$X = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, Y = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, Z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

which gives a framing s *of* $T\mathbb{S}^3$*. Let* g_u *be the Riemannian metric on* $\mathbb{S}^3$ *for which the parallel Lie vector fields* $\{u \cdot X, Y, Z\}$ *are an orthonormal basis. Let* $\mathcal{F}$ *denote the point-foliation of* $\mathbb{S}^3$*. Then* $[h_1] \in H^3(RW_3)$ *and for each* $u > 0$*, we have* $\Delta^s_{\mathcal{F},g_u}[h_1] \in H^3(\mathbb{S}^3;\mathbb{R}) \cong \mathbb{R}$*.*

Theorem 5.3 (Theorem 6.9[9]). $\dfrac{d}{du}|_{u=1} \left(\Delta^s_{\mathcal{F},g_u}[h_1]\right) \neq 0$

One expects similar results also hold for other compact Lie groups of dimension $4k - 1 \geq 7$, although the author does not know of a published calculation of this.

Chern and Simons also prove a fundamental fact about the conformal rigidity of the transgression classes, and as their calculations are all local, the result carries over to Riemannian foliations:

Theorem 5.4 (Theorem 4.5[9]). *The rigid secondary classes in codimension* $q = 4k-1$ *are conformal invariants. That is, let* $(\mathcal{F}, g)$ *be a Riemannian foliation of codimension* $q = 4k - 1$ *of the closed manifold* M*. Let* s *be a framing of the normal bundle* Q*. Let* $\mu\colon M \to \mathbb{R}$ *be a smooth function, which is constant along the leaves of* $\mathcal{F}$*. Define a conformal deformation of* g *by setting* $g_t = \exp(\mu(t)) \cdot g$*. Then for all* $[h_I \otimes p_J] \in H^*(RW_q, d_W)$ *with* $4i_1 + 4|J| = q + 1$*,*

$$\Delta^s_{\mathcal{F},g_t}[h_I \otimes p_J] = \Delta^s_{\mathcal{F},g}[h_I \otimes p_J] \in H^*(M;\mathbb{R})$$

Combining Theorems 5.1, 5.2 and 5.4 we obtain:

Corollary 5.1. *The secondary classes of Riemannian foliations are conformal invariants.*

A modification of the original examples of Bott[3] and Baum-Cheeger[2] show that all of the variable secondary classes vary independently, by a suitable variation of foliations.

Example 5.2 (Lazarov-Pasternack[35]). *Let* $\alpha = (\alpha_1, \ldots, \alpha_{2k}) \in \mathbb{R}^{2k}$*. Let* $(x_1, y_1, x_2, y_2, \ldots, x_{2k}, y_{2k})$ *denote coordinates on* $\mathbb{R}^{4k}$*, and define a Killing vector field* X_α *on* $\mathbb{R}^{4k}$ *by*

$$X_\alpha = \sum_{i=1}^{k} \alpha_i \{x_i \partial/\partial y_i - y_i \partial/\partial x_i\}$$

Let $\phi_t^\alpha \colon \mathbb{R}^{4k} \to \mathbb{R}^{4k}$ *be the isometric flow of* X_α*, which restricts to an isometric flow on the unit sphere* $\mathbb{S}^{4k-1}$*, so defines a Riemannian foliation* $\mathcal{F}_\alpha$ *of codimension* $q = 4k-2$ *of* $\mathbb{S}^{4k-1}$*.*

Let $h_i \otimes p_J$ satisfy $4i + 4|J| = 4k$. Associated to $p_i \wedge p_J$ is an Ad-invariant polynomial $\varphi_{i,J}$ on $\mathfrak{so}(4k)$ of degree $2k$. Let $M \to \mathbb{S}^{4k-1}$ denote the bundle of orthonormal frames for the normal bundle to $\mathcal{F}_\alpha$, for α near $0 \in \mathbb{R}^{2k}$. The spectral sequence for $\mathbf{SO}(4k-2) \to M \to \mathbb{S}^{4k-1}$ collapses at the $E_2^{r,s}$-term, hence $H^*(M;\mathbb{R}) \cong H^*(\mathbb{S}^{4k-1}, \mathbb{R}) \otimes H^*(\mathbf{SO}(4k-2);\mathbb{R})$. Let $[C] \in H^{4k-1}(M, \mathbb{R})$ correspond to the fundamental class of the base.

Theorem 5.5 (§§2 & 3[35]). *There exists* $\lambda \neq 0$ *independent of the choice of* $p_i \wedge p_J$ *such that*

$$\langle \Delta^s_{\mathcal{F}_\alpha, g}[h_i \otimes p_J], [C] \rangle = \lambda \cdot \frac{\varphi_{i,J}(\alpha_1, \ldots, \alpha_{2k})}{\alpha_1 \cdots \alpha_{2k}} \qquad (17)$$

These examples are for $q = 4k-2$. Multiplying by a factor of $\mathbb{S}^1$ in the transverse direction yields examples with codimension $4k-1$, and the same secondary invariants. Hence, we have the following corollary, due to Lazarov and Pasternack:

Corollary 5.2 (Theorem 3.6[35]). *Let* $q = 4k-2$ *or* $4k-1$*. Evaluation on a basis of* $H^{4k-1}(RW_q; d_W)$ *defines a surjective map*

$$\pi_{4k-1}(BR\Gamma_q^+) \to \mathbf{R}^{v_q^{4k-1}} \qquad (18)$$

In particular, all of the variable secondary classes in degree $4k-1$ *vary independently.*

Although not stated by Lazarov and Pasternack,[35] these examples also imply that all of the variable secondary classes for Riemannian foliations vary independently, as stated in Theorem 4.[20]

The papers[19,30,34–36,53,54] contain a more extensive collection of examples of the calculation of the secondary classes for Riemannian foliations.

We mention also the very interesting work of Morita[43] which shows there is an extended set of secondary invariants, beyond those described above. This paper uses the Chern-Weil approach of Kamber and Tondeur to extend the construction of secondary invariants for Riemannian foliations to include an affine factor in its transverse holonomy group. Moreover, Morita gives examples to show these additional classes are non-zero for a natural sets of examples, and hence for $q > 2$, give further non-triviality results for the homotopy type of $FR\Gamma_q$.

6. Molino Structure Theory

The values of certain of the secondary classes for Riemannian foliations can vary under an appropriate deformation of the underlying Riemannian foliation. This raises the question, exactly what aspects of the dynamics of $\mathcal{F}$ contributes to this variation? Molino's Structure Theory for Riemannian foliations provides a framework for studying this problem, as highlighted in Molino's survey.[42] We recall below some of the main results of this theory, in order to formulate some of the open questions. The reader can consult Molino,[40,41] Haefliger,[16,17] or Moerdijk and Mrčun[39] for further details.

Recall that we assume M is a closed, connected smooth manifold, $(\mathcal{F}, g)$ is a smooth Riemannian foliation of codimension q with tangential distribution $F = T\mathcal{F}$, and that the normal bundle $Q \to M$ to $\mathcal{F}$ is oriented.

Let $\pi : \widehat{M} \to M$ be the bundle of oriented orthonormal frames for Q. For $x \in M$, the fiber $\pi^{-1}(x) = \mathbf{Fr}^+(Q_x)$ is the space of orthogonal frames of Q_x with positive orientation. The manifold $\widehat{M}$ is a principal right $\mathbf{SO}(q)$-bundle. Set $\widehat{x} = (x, e) \in \widehat{M}$ for $e \in \mathbf{Fr}^+(Q_x)$.

The manifold $\widehat{M}$ has a Riemannian foliation $\widehat{\mathcal{F}}$, whose leaves are the holonomy coverings of the leaves of $\mathcal{F}$. The definition of $\widehat{\mathcal{F}}$ can be found in the sources cited above, but there is an easy intuitive definition. Let X denote a vector field on M which is everywhere tangent to the leaves of $\mathcal{F}$, so that its flow $\varphi_t \colon M \to M$ defines $\mathcal{F}$-preserving diffeomorphisms. For each $x \in M$, $t \mapsto \varphi_t(x)$ defines a path in the leaf L_x^h through x. The differential of these maps induce transverse isometries $D_x\varphi_t \colon Q_x \to Q_{\varphi_t(x)}$ which act on the oriented frames of Q, hence define paths in $\widehat{M}$. Given $\widehat{x} = (x, e) \in \widehat{M}$, the leaf $\widehat{L}_{\widehat{x}}^h$ is defined by declaring that the path $t \mapsto D_x\varphi_t(e)$ is tangent to $\widehat{L}_{\widehat{x}}^h$. It follows from the construction that the restriction $\pi : \widehat{L}_{\widehat{x}}^h \to L_x^h$ of the projection π to each leaf of $\widehat{\mathcal{F}}$ is a covering map.

There is an $\mathbf{SO}(q)$-invariant Riemannian metric $\widehat{g}$ on $T\widehat{M}$ such that $\widehat{\mathcal{F}}$ is Riemannian. The metric $\widehat{g}$ satisfies $d\pi : T\widehat{\mathcal{F}} \to T\mathcal{F}$ is an isometry, and the restriction of $\widehat{g}$ to the tangent space $T\pi$ of the fibers of π is induced from the natural bi-invariant metric on $\mathbf{SO}(q)$. Then $d\pi$ restricted to the orthogonal complement $(T\widehat{\mathcal{F}} \oplus T\pi)^{\perp}$ is a Riemannian submersion to Q.

A fundamental observation is that $\widehat{\mathcal{F}}$ is *Transversally Parallelizable* (TP). Let $\mathbf{Diff}(\widehat{M}, \widehat{\mathcal{F}})$ denote the subgroup of diffeomorphisms of $\widehat{M}$ which map leaves to leaves for $\widehat{\mathcal{F}}$, not necessarily taking a leaf to itself. The TP condition is that $\mathbf{Diff}(\widehat{M}, \widehat{\mathcal{F}})$ acts transitively on $\widehat{M}$.

Given $\widehat{x} = (x, e) \in \widehat{M}$, let $\overline{L_x^h}$ denote the closure of the leaf L_x in M, and let $\overline{L_{\widehat{x}}^h}$ denote the closure of the leaf $\widehat{L}_{\widehat{x}}^h$ in $\widehat{M}$. For notational convenience, we set $N_x = \overline{L_x^h}$ and $N_{\widehat{x}} = \overline{L_{\widehat{x}}^h}$. Note that the distinction between $N_x \subset M$ and $N_{\widehat{x}} \subset \widehat{M}$ is indicated by the basepoint.

Theorem 6.1. *Given any pair of points $\widehat{x}, \widehat{y} \in \widehat{M}$, there is a diffeomorphism $\Phi \in \mathbf{Diff}(\widehat{M}, \widehat{\mathcal{F}})$ which restricts to a foliated diffeomorphism, $\Phi\colon N_{\widehat{x}} \to N_{\widehat{y}}$. Hence, given any pair of points $x, y \in M$, the universal coverings of the leaves L_x^h and L_y^h of $\mathcal{F}$ are diffeomorphic and quasi-isometric.*

This is a key property of Riemannian foliations, and is used to establish the general Molino Structure Theory, which gives a description of the closures of the leaves of $\mathcal{F}$ and $\widehat{\mathcal{F}}$.

Theorem 6.2 (Molino[40,41]). *Let M be a closed, connected smooth manifold, and $(\mathcal{F}, g)$ a smooth Riemannian foliation of codimension q of M. Let $W = M/\overline{\mathcal{F}}$ be the quotient of M by the closures of the leaves of $\mathcal{F}$, and $\Upsilon : M \to W$ the quotient map.*

(1) For each $\widehat{x} \in \widehat{M}$, the closure $N_{\widehat{x}}$ of $\widehat{L}_{\widehat{x}}^h$ is a submanifold of $\widehat{M}$.
(2) The set of all leaf closures $N_{\widehat{x}}$ defines a foliation $\widehat{\mathcal{E}}$ of $\widehat{M}$ with all leaves compact without holonomy.
(3) The quotient leaf space $\widehat{W}$ is a closed manifold with an induced right $\mathbf{SO}(q)$-action, and the induced fibration $\widehat{\Upsilon} : \widehat{M} \to \widehat{W}$ is $\mathbf{SO}(q)$–equivariant.
(4) W is a Hausdorff space, and there is an $\mathbf{SO}(q)$-equivariant commutative diagram:

$$\begin{array}{ccccc} & \mathbf{SO}(q) & = & \mathbf{SO}(q) & \\ & \downarrow & & \downarrow & \\ & \widehat{M} & \xrightarrow{\widehat{\Upsilon}} & \widehat{W} & \\ \pi & \downarrow & & \downarrow & \widehat{\pi} \\ & M & \xrightarrow{\Upsilon} & W & \end{array}$$

The second main result of the structure theory provides a description of the closures of the leaves of $\mathcal{F}$ and $\widehat{\mathcal{F}}$, and the structure of $\widehat{\mathcal{F}}|N_{\widehat{x}}$.

Theorem 6.3 (Molino[40,41]). *Let M be a closed, connected smooth manifold, and $(\mathcal{F}, g)$ a smooth Riemannian foliation of codimension q of M.*

(1) There exists a simply connected Lie group G, whose Lie algebra $\mathfrak{g}$ is spanned by the holonomy-invariant vector fields on $N_{\widehat{x}}$ transverse to $\widehat{\mathcal{F}}$, such that the restricted foliation $\widehat{\mathcal{F}}$ of $N_{\widehat{x}}$ is a Lie G-foliation with all leaves dense, defined by a Maurer-Cartan connection 1-form $\omega_{\mathfrak{g}}^{\widehat{x}} : TN_{\widehat{x}} \longrightarrow \mathfrak{g}$.

(2) Let $\rho_{\widehat{x}} \colon \pi_1(N_{\widehat{x}}, \widehat{x}) \to G$ be the global holonomy map of the flat connection $\omega_{\mathfrak{g}}^{\widehat{x}}$. Then the image $\widehat{\mathcal{N}}_{\widehat{x}} \subset G$ of $\rho_{\widehat{x}}$ is dense in G.

7. Some open problems

Theorems 6.2 and 6.3 suggest a number of questions about the secondary classes of Riemannian foliations. It is worth recalling that for the example constructed in the proof of Theorem 4.3 of a Riemannian foliation for which the characteristic map is injective, all of its leaves are compact, and so the structural Lie group G of Theorem 6.3 reduces to the trivial group. For this example, all of the secondary classes are integral.

The first two problems invoke the structure of the quotient manifold $\widehat{W} = \widehat{M}/\mathcal{E}$ and space $W = M/\overline{\mathcal{F}}$.

Problem 7.1. *Suppose that foliation $\overline{\mathcal{F}}$ of M by the leaf closures of $\mathcal{F}$ is a* non-singular *foliation. Show that all secondary classes of $\mathcal{F}$ are rational. In the case where every leaf of $\mathcal{F}$ is dense in M, so W reduces to a point, are the secondary classes necessarily integral?*

In all examples where there exists a family of foliations for which the secondary classes vary non-trivially, the quotient space W is singular, hence the action of $\mathbf{SO}(q)$ on $\widehat{W}$ has singular orbits. The action of $\mathbf{SO}(q)$ thus defines a stratification of $\widehat{W}$. (See[28] for a discussion of the stratifications associated to a Riemannian foliation, and some of their properties.)

Problem 7.2. *How do the values of the secondary classes for a Riemannian foliation depend upon the $\mathbf{SO}(q)$-stratification of $\widehat{W}$? Are there conditions on the structure of the stratification which are sufficient to imply that the secondary classes are rational?*

The next problems concern the role of the structural Lie group G of a Riemannian foliation $\mathcal{F}$.

Problem 7.3. *Suppose the structural Lie group G is nilpotent. For example, if all leaves of $\mathcal{F}$ have polynomial growth, the G must be nilpotent.*[6,56] *Show that all rigid secondary classes of $\mathcal{F}$ are rational.*

All of the known examples of families of Riemannian foliations for which the secondary classes vary non-trivially are obtained by the action of an abelian group $\mathbb{R}^p$, and so the structural Lie group G is necessarily abelian. In contrast, one can ask whether there is a generalization to the secondary classes of Riemannian foliations of the results of Reznikov that the rigid secondary classes of flat bundles must be rational.[50,51]

Problem 7.4. *Suppose the structural Lie group G is semi-simple with real rank at least 2, without any factors of $\mathbb{R}$. Must the values of the secondary classes be rigid under deformation? Are all of the characteristic classes of $\mathcal{F}$ are rational?*

Problem 7.5. *Assume the leaves of $\mathcal{F}$ admit a Riemannian metric for which they are Riemannian locally symmetric spaces of higher rank.*[55,57] *Must all of the characteristic classes of $\mathcal{F}$ be rational?*

The final question is more global in nature, as it asks how the topology of the ambient manifold M influences the values of the secondary classes for a Riemannian foliation $\mathcal{F}$ of M. Of course, one influence might be that the cohomology group $H^\ell(M;\mathbb{R}) = \{0\}$ where $\ell = \deg(h_I \otimes p_J)$, and then $\Delta_{\mathcal{F}}(h_I \otimes p_J) = 0$ is rather immediate. Are there more subtle influences, such as whether particular restrictions on the fundamental group $\pi_1(M)$ restrict the values of the secondary classes for Riemannian foliations of M?

Problem 7.6. *How does the topology of a compact manifold M influence the secondary classes for a Riemannian foliation $(\mathcal{F}, g)$ with normal framing s of M?*

There are various partial results for Problem 7.6 in the literature,[34,36,54] but no systematic treatment. It seems likely that an analysis such as in Ghys[13] for Riemannian foliations of simply connected manifolds would yield new results in the direction of this question.

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