LIPSHITZ MATCHBOX MANIFOLDS

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Abstract. A matchbox manifold is a connected, compact foliated space with totally disconnected transversals. It is said to be Lipshitz if there exists a metric on its transversals for which the holonomy maps are Lipshitz. Examples of Lipshitz matchbox manifolds include the exceptional minimal sets for C^1-foliations of compact manifolds, tiling spaces, the classical solenoids, and the weak solenoids of McCord and Schori, among others. We address the question: When does a Lipshitz matchbox manifold admit an embedding as a minimal set for a smooth dynamical system, or more generally for as an exceptional minimal set for a C^1-foliation of a smooth manifold? We gives examples which do embed, and examples where no Lipshitz structure can exist, so do not embed. We also discuss the classification theory for Lipshitz weak solenoids.

1. Introduction

This paper is part of the study of minimal sets for C^r-foliations, for $r \geq 1$. This is a well-studied subject for codimension-one foliations, but for higher codimension foliations, they are less well understood. The approach we take is to consider a minimal set as a foliated space, and to study in particular the cases which arise from exceptional minimal sets, where the foliated space has totally disconnected transversals, which are then each homeomorphic to the Cantor set.

This work formulates the basic notions for Lipshitz matchbox manifolds, and gives many examples of them, some of which are known to embed as an exceptional minimal set for some C^1-foliation. We also give a construction of examples of minimal matchbox manifolds which cannot be made Lipshitz for any choice of transversal metric, so do not embed in any C^1-foliation. We also formulate a variety of questions and problems relating to minimal matchbox manifolds, their possible transverse metrics, and whether they can be embedded as exceptional minimal sets for C^1-foliations.

A *matchbox manifold*, as in Definition 2.4, is a foliated space with totally disconnected transversals. They are a type of generalized lamination, and examples include the closure of leaves in compact foliated manifolds, such as exceptional minimal sets, and compact invariant sets for certain classes of dynamical systems. If $\mathfrak{M}$ is realized as the minimal set of a C^1-foliation on a smooth manifold M, then the ambient metric on M induces a metric on the transversals to $\mathfrak{M}$ for which the leafwise holonomy maps are Lipshitz. This observation is the basis for this work, in which we consider *Lipshitz matchbox manifolds*. We study some of the properties of their dynamics, and give a selection of examples. We then consider the problem of when does a Lipshitz matchbox manifold admit a homeomorphic embedding as an invariant set in a C^1-foliation of a manifold.

This work is related to the study of the following well-known question in foliation theory, as posed for example in a 1975 paper by Sondow [117]:

PROBLEM 1.1. *When is a smooth connected manifold L without boundary a leaf of a foliation of a compact smooth manifold?*

The case where L has dimension 1 is trivial, while Cantwell and Conlon showed in [25] that any surface without boundary is a leaf of a smooth codimension-1 foliation of a compact 3-manifold.

On the other hand, Ghys [51] and Inaba, Nishimori, Takamura and Tsuchiya [76], constructed 3-manifolds which are not homeomorphic to a leaf of any codimension-1 foliation of a compact

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manifold. The non-embedding examples by these authors are essentially the only known results on Problem 1.1 in this generality, and they are both for codimension-one foliations.

There is a natural variant of Problem 1.1, posed in the 1974 ICM address by Sullivan [118]:

PROBLEM 1.2. *Let L be a complete Riemannian smooth manifold without boundary. When is L* quasi-isometric *to a leaf of a C^r-foliation $\mathcal{F}$ of a compact smooth manifold M, for $r \geq 1$?*

The assumption that the embedding of L as a leaf of $\mathcal{F}$ has to preserve some metric properties forces many more restrictions on the embedding, and as a consequence, there are more obstructions to such an embedding. For example, Cantwell and Conlon studied in [23], [24] how the asymptotic behavior of the metric on L is related to the dynamics of the leaf in a codimension-one foliation.

The work of Phillips and Sullivan in [100] introduced the asymptotic Euler class of a non-compact 2-manifold with subexponential volume growth rate, and showed this may form an obstruction to embedding L as a leaf, depending on the topology of the ambient manifold M. This result was generalized by Januszkiewicz in [77] to include obstructions formed by the asymptotic Pontrjagin numbers of an open manifold with subexponential volume growth rate. The related work [75] by Meniño-Cotón and the author combines the idea behind of the Phillips and Sullivan work with Colman's notion of the tangential Lusternik-Schnirelmann category of foliations in [38].

In an alternate direction, Attie and Hurder in [7] introduced an invariant of open manifolds, its "leaf entropy", or "asymptotic leaf complexity", and gave examples of open manifolds with exponential volume growth rate that cannot be quasi-isometric to a leaf in a foliation of any codimension. Examples of surfaces with exponential growth rate that cannot be quasi-isometric to a leaf were constructed by Schweitzer in [114] and Zeghib in [127], using a variant of the approach in [7]. Also, Schweitzer [115] exhibits further examples of complete Riemannian manifolds which are not quasi-isometric to a leaf in any codimension-one foliation.

All of the non-embedding results mentioned above rely on the simple strategy, that a leaf in a compact foliated manifold M has recurrence properties, and the idea is to formulate such a property, *intrinsic* to L, which cannot be satisfied if L is homeomorphic to a leaf, or possibly quasi-isometric to a leaf. This has proven to be a difficult method to implement, as effective criteria for non-recurrence, and thus non-embeddability, have proven to be elusive to find.

A leaf L contained in a minimal set $\mathfrak{M}$ for a foliation $\mathcal{F}$ on a compact manifold M has much stronger recurrence properties. For example, Cass observed in [29] that such a leaf must be "quasi-homogeneous", and that this property is an invariant of the quasi-isometry class of a Riemannian metric on L. He consequently gave examples of complete Riemannian manifolds, including leaves of foliations, which cannot be quasi-isometric to a leaf in a minimal set. For example, Cass showed that any non-compact leaf in a Reeb foliation of $\mathbb{S}^3$ cannot be realized as a leaf of a minimal set in any codimension.

The question raised by Cass' work suggests a variant of the above questions, where we consider the closure $\mathfrak{M} = \overline{L}$ of the leaf in M, which has the structure of a *foliated space*. The notion of a foliated space $\mathfrak{M}$ is contained in the work by Ruelle [107], and a formal treatment of these spaces was given by Moore and Schochet [97, Chapter 2], as part of their development of a general formulation of the Connes measured leafwise-index theorem [39]. Candel and Conlon [22, Chapter 11] further developed the theory of foliated spaces, and gave many interesting examples. We are particularly interested in the case where the transverse model space for the foliated space $\mathfrak{M}$ is totally disconnected.

A compact connected foliated space $\mathfrak{M}$ with totally disconnected transversals is called a "matchbox manifold", in accordance with terminology introduced in continua theory [1, 2, 3]. A matchbox manifold with 2-dimensional leaves can also be considered as an "abstract" lamination by surfaces as in [53, 90]. If all leaves of $\mathfrak{M}$ are dense, then it is called a *minimal matchbox manifold.* A compact minimal set $\mathfrak{M} \subset M$ for a foliation $\mathcal{F}$ on a manifold M yields a foliated space with foliation $\mathcal{F}_{\mathfrak{M}} = \mathcal{F}|\mathfrak{M}$. If the minimal set is exceptional, then $\mathfrak{M}$ is a minimal matchbox manifold. The definition and properties of matchbox manifolds are discussed in Section 2.

The leaves of the foliation $\mathcal{F}_{\mathfrak{M}}$ of a foliated space $\mathfrak{M}$ admit a smooth Riemannian metric, which is continuous on $\mathfrak{M}$. Thus, for each leaf $L \subset \mathfrak{M}$ there is a well-defined quasi-isometry class of Riemannian metrics on L. If $\mathfrak{M}$ is homeomorphic to an invariant set for a foliation $\mathcal{F}$ of a compact manifold M, then L is quasi-isometric to a leaf.

The obstructions used in the works above, to show that a particular Riemannian manifold L cannot be quasi-isometric to a leaf of a foliation of a compact manifold M, also provide obstructions to realizing L as a leaf in a compact foliated space $\mathfrak{M}$. The study of the embedding property for foliated spaces into C^r-foliations thus introduces new criteria for showing non-embedding, criteria that depend not just on the intrinsic geometry and topology of L, but includes "extrinsic properties" of L in $\mathfrak{M}$, such as the transverse geometry and dynamics of the foliated space $\mathfrak{M}$.

The most general form of the embedding problem we consider can then be formulated as follows:

PROBLEM 1.3. *Let $\mathfrak{M}$ be a compact connected foliated space. Does there exists a homeomorphism of $\mathfrak{M}$ to an invariant set of a C^r-foliation of a manifold M, for $r \geq 1$?*

The structure of compact invariant sets for smooth flows on compact manifolds is an extremely well-studied topic, and as remarked by Kennedy and Yorke in [81], these sets can be extremely pathological. For this reason, it is reasonable to consider restricted cases of this embedding problem. The first reduction is to spaces which admit a Cantor set transversal:

PROBLEM 1.4. *Let $\mathfrak{M}$ be a minimal matchbox manifold. Does there exists a homeomorphism of $\mathfrak{M}$ to an exceptional minimal set of a C^r-foliation of a manifold M, for $r \geq 1$?*

Observe that if $\mathfrak{M}$ is an invariant set for a C^r-foliation $\mathcal{F}$ of a Riemannian manifold M, where $r \geq 1$, then the holonomy maps for the foliation $\mathcal{F}_{\mathfrak{M}}$ on $\mathfrak{M}$ are induced by the holonomy maps of $\mathcal{F}$, and there is a metric on the transversals to $\mathfrak{M}$ such that the holonomy maps of $\mathcal{F}_{\mathfrak{M}}$ are Lipshitz, as discussed in Section 4. Using this simple observation, Problem 1.4 can be considered in two ways.

PROBLEM 1.5. *Let $\mathfrak{M}$ be a Lipshitz matchbox manifold. Find obstructions to the existence of a foliated embedding $\iota \colon \mathfrak{M} \to M$, where M has a C^r-foliation $\mathcal{F}$ with $r \geq 1$.*

This problem can alternately be formulated as asking for a characterization of the Lipshitz structures which can arise for the transverse Cantor sets to exotic minimal sets in C^r-foliations. For example, in the case of a foliation obtained from the suspension of a diffeomorphism of the circle $\mathbb{S}^1$, McDuff studied in [95] the question: which Cantor sets embedded in $\mathbb{S}^1$ are the invariant sets for $C^{1+\alpha}$-diffeomorphisms of the circle?

PROBLEM 1.6. *Let $\mathfrak{M}$ be a minimal matchbox manifold. Find obstructions to the existence of a transverse Lipshitz structure for the holonomy maps of the foliation $\mathcal{F}_{\mathfrak{M}}$.*

Problems 1.5 and 1.6 have a completely different nature than the usual questions about the dynamics of smooth foliations, such as discussed in [71, 72, 124]. One aspect of this, is that while that any two Riemannian metrics on a smooth transversal $[-1,1]^n$ are quasi-isometric, and so are Lipshitz equivalent, there are many Lipshitz classes of metrics on a Cantor set. The Hausdorff dimension of a metric space gives one invariant of the Lipshitz class, for example. As discussed in Problem 8.4, "local homogeneity" provides another obstacle to metrically embedding Cantor sets. We show in Section 8 the following result.

THEOREM 1.7. *There exist compactly-generated pseudogroups $\mathcal{G}_{\mathfrak{X}}$ acting minimally on a Cantor set $\mathfrak{X}$, such that there is no metric on $\mathfrak{X}$ for which the generators of $\mathcal{G}_{\mathfrak{X}}$ satisfy a Lipshitz condition.*

COROLLARY 1.8. *There exists minimal matchbox manifolds which do not embed as an exceptional minimal set for any C^1-foliation.*

Many further questions and problems are posed throughout the text, which is organized as follows.

Sections 2 and 3 below collect together some definitions and results concerning matchbox manifolds and their dynamical properties that we use in the paper. More details can be found in the works

[22, 34, 35, 36, 97]. Section 4 defines the Lipshitz property for pseudogroup actions. The main result of this section is a proof that an embedding of a Matchbox manifold as an exceptional minimal set in a C^1-foliation yields a Lipshitz structure on it.

Section 5 discusses some examples from the literature of embeddings of matchbox manifolds as exceptional minimal sets for foliations. Then in Section 6, the notion of *McCord*, *weak* and *generalized* solenoids are introduced. These are basic examples for the study of minimal matchbox manifolds, and may represent all of the minimal ones. The construction of transverse metrics on weak solenoids for which their holonomy are Lipshitz is discussed.

Section 7 introduces an operation on minimal matchbox manifolds, called their "fusion", which amalgamates their pseudogroups. The fusion process is inspired by the method introduced by Lukina in [88], and is related to the turbularization process for codimension-one foliations. Fusion is based on properties of the Cantor set transversals, so is particular to matchbox manifolds. The fusion process is used to construct the examples in Section 8 of minimal pseudogroup Cantor actions, which cannot be Lipshitz for any metric, and thus are not homeomorphic to an exceptional minimal set in any C^1-foliation.

Finally, in Section 9, *Morita equivalence* and *Lipshitz equivalence* of minimal Lipshitz pseudogroups are introduced. The problem of the classification of matchbox manifolds up to Lipshitz equivalence is considered in detail for the the special case of weak solenoids.

2. Foliated spaces and matchbox manifolds

The notion of a foliated space is defined by Moore and Schochet [97, Chapter 2], as part of their development of a general form of the Connes measured leafwise index theorem. The text by Candel and Conlon [22, Chapter 11] further develops the theory of foliated spaces, with many examples. The author's papers with Clark [33, 34] discuss the topology and dynamics of *matchbox manifolds*.

First, recall two basic topological notions. A *continuum* Ω is a *compact, connected metrizable* space. A set $V \subset \Omega$ is *clopen* if it is non-empty, and both open and closed.

DEFINITION 2.1. *A* foliated space of dimension n *is a continuum* $\mathfrak{M}$*, such that there exists a separable metric space* $\mathfrak{X}$*, and for each* $x \in \mathfrak{M}$ *there is a compact subset* $\mathfrak{X}_x \subset \mathfrak{X}$*, an open subset* $U_x \subset \mathfrak{M}$*, and a homeomorphism defined on the closure* $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{X}_x$ *such that* $\varphi_x(x) = (0, w_x)$ *where* $w_x \in int(\mathfrak{X}_x)$*. Moreover, it is assumed that each* φ_x *admits an extension to a foliated homeomorphism* $\widehat{\varphi}_x \colon \widehat{U}_x \to (-2,2)^n \times \mathfrak{X}_x$ *where* $\overline{U}_x \subset \widehat{U}_x$*. The space* $\mathfrak{X}_x$ *is called the* local transverse model *at* x*.*

Let $\pi_x \colon \overline{U}_x \to \mathfrak{X}_x$ denote the composition of φ_x with projection onto the second factor. For $w \in \mathfrak{X}_x$ the set $\mathcal{P}_x(w) = \pi_x^{-1}(w) \subset \overline{U}_x$ is called a *plaque* for the coordinate chart φ_x. We adopt the notation, for $z \in \overline{U}_x$, that $\mathcal{P}_x(z) = \mathcal{P}_x(\pi_x(z))$, so that $z \in \mathcal{P}_x(z)$. Note that each plaque $\mathcal{P}_x(w)$ for $w \in \mathfrak{X}_x$ is given the topology so that the restriction $\varphi_x \colon \mathcal{P}_x(w) \to [-1,1]^n \times \{w\}$ is a homeomorphism. Then $int(\mathcal{P}_x(w)) = \varphi_x^{-1}((-1,1)^n \times \{w\})$. Let $U_x = int(\overline{U}_x) = \varphi_x^{-1}((-1,1)^n \times int(\mathfrak{X}_x))$. Note that if $z \in U_x \cap U_y$, then $int(\mathcal{P}_x(z)) \cap int(\mathcal{P}_y(z))$ is an open subset of both $\mathcal{P}_x(z)$ and $\mathcal{P}_y(z)$. The collection of sets

$$\mathcal{V} = \{\varphi_x^{-1}(V \times \{w\}) \mid x \in \mathfrak{M}\ ,\ w \in \mathfrak{X}_x\ ,\ V \subset (-1,1)^n \text{ open}\}$$

forms the basis for the *fine topology* of $\mathfrak{M}$. The connected components of the fine topology are called *leaves*, and define the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$. Let $L_x \subset \mathfrak{M}$ denote the leaf of $\mathcal{F}_{\mathfrak{M}}$ containing $x \in \mathfrak{M}$.

DEFINITION 2.2. *A* smooth foliated space *is a foliated space* $\mathfrak{M}$ *as above, such that there exists a choice of local charts* $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{X}_x$ *such that for all* $x, y \in \mathfrak{M}$ *with* $z \in U_x \cap U_y$*, there exists an open set* $z \in V_z \subset U_x \cap U_y$ *such that* $\mathcal{P}_x(z) \cap V_z$ *and* $\mathcal{P}_y(z) \cap V_z$ *are connected open sets, and the composition* $\psi_{x,y;z} \equiv \varphi_y \circ \varphi_x^{-1} \colon \varphi_x(\mathcal{P}_x(z) \cap V_z) \to \varphi_y(\mathcal{P}_y(z) \cap V_z)$ *is a smooth map, where* $\varphi_x(\mathcal{P}_x(z) \cap V_z) \subset \mathbb{R}^n \times \{w\} \cong \mathbb{R}^n$ *and* $\varphi_y(\mathcal{P}_y(z) \cap V_z) \subset \mathbb{R}^n \times \{w'\} \cong \mathbb{R}^n$*. The maps* $\psi_{x,y;z}$ *are assumed to depend continuously on* z *in the* C^∞*-topology on maps between subsets of* $\mathbb{R}^n$*.*

A map $f\colon \mathfrak{M} \to \mathbb{R}$ is said to be *smooth* if for each flow box $\varphi_x\colon \overline{U}_x \to [-1,1]^n \times \mathfrak{X}_x$ and $w \in \mathfrak{X}_x$ the composition $y \mapsto f \circ \varphi_x^{-1}(y,w)$ is a smooth function of $y \in (-1,1)^n$, and depends continuously on w in the C^∞-topology on maps of the plaque coordinates y. As noted in [97] and [22, Chapter 11], this allows one to define smooth partitions of unity, vector bundles, and tensors for smooth foliated spaces. In particular, one can define leafwise Riemannian metrics. We recall a standard result, whose proof for foliated spaces can be found in [22, Theorem 11.4.3].

THEOREM 2.3. *Let $\mathfrak{M}$ be a smooth foliated space. Then there exists a leafwise Riemannian metric for $\mathcal{F}_{\mathfrak{M}}$, such that for each $x \in \mathfrak{M}$, L_x inherits the structure of a complete Riemannian manifold with bounded geometry, and the Riemannian geometry of L_x depends continuously on x. In particular, each leaf L_x has the structure of a complete Riemannian manifold with bounded geometry.*

Bounded geometry implies, for example, that for each $x \in \mathfrak{M}$, there is a leafwise exponential map $\exp_x^{\mathcal{F}}\colon T_x\mathcal{F}_{\mathfrak{M}} \to L_x$ which is a surjection, and the composition $\exp_x^{\mathcal{F}}\colon T_x\mathcal{F}_{\mathfrak{M}} \to L_x \subset \mathfrak{M}$ depends continuously on x in the compact-open topology on maps.

DEFINITION 2.4. *A* matchbox manifold *is a continuum with the structure of a smooth foliated space $\mathfrak{M}$, such that the transverse model space $\mathfrak{X}$ is totally disconnected, and for each $x \in \mathfrak{M}$, the transverse model space $\mathfrak{X}_x \subset \mathfrak{X}$ is a clopen subset, hence is homeomorphic to a Cantor set.*

All matchbox manifolds are assumed to be smooth with a given leafwise Riemannian metric. The space $\mathfrak{M}$ is assumed to be metrizable, and we fix a choice for the metric $d_{\mathfrak{M}}$ on $\mathfrak{M}$. One subtlety is that the choice of $d_{\mathfrak{M}}$ then determines a metric $d_{\mathfrak{X}}$ on the transversal $\mathfrak{X}$, but the holonomy of $\mathcal{F}_{\mathfrak{M}}$ need not be Lipshitz with respect to this metric.

An important difference between a foliated matchbox manifold and a smooth foliated manifold, is that the local foliation charts for a matchbox manifold are not connected, and so must be chosen appropriately to ensure that each chart is "local". We introduce the following conventions.

For $x \in \mathfrak{M}$ and $\epsilon > 0$, let $D_{\mathfrak{M}}(x,\epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x,y) \leq \epsilon\}$ be the closed ϵ-ball about x in $\mathfrak{M}$, and $B_{\mathfrak{M}}(x,\epsilon) = \{y \in \mathfrak{M} \mid d_{\mathfrak{M}}(x,y) < \epsilon\}$ the open ϵ-ball about x.

Similarly, for $w \in \mathfrak{X}$ and $\epsilon > 0$, let $D_{\mathfrak{X}}(w,\epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w,w') \leq \epsilon\}$ be the closed ϵ-ball about w in $\mathfrak{X}$, and $B_{\mathfrak{X}}(w,\epsilon) = \{w' \in \mathfrak{X} \mid d_{\mathfrak{X}}(w,w') < \epsilon\}$ the open ϵ-ball about w.

Each leaf $L \subset \mathfrak{M}$ has a complete path-length metric, induced from the leafwise Riemannian metric:

$$d_{\mathcal{F}}(x,y) = \inf\left\{\|\gamma\| \mid \gamma\colon [0,1] \to L \text{ is piecewise C}^1 \ ,\ \gamma(0) = x \ ,\ \gamma(1) = y \ ,\ \gamma(t) \in L \quad \forall\ 0 \leq t \leq 1\right\}$$

where $\|\gamma\|$ denotes the path-length of the piecewise C^1-curve $\gamma(t)$. If $x, y \in \mathfrak{M}$ are not on the same leaf, then set $d_{\mathcal{F}}(x,y) = \infty$. For each $x \in \mathfrak{M}$ and $r > 0$, let $D_{\mathcal{F}}(x,r) = \{y \in L_x \mid d_{\mathcal{F}}(x,y) \leq r\}$.

The leafwise Riemannian metric $d_{\mathcal{F}}$ is continuous with respect to the metric $d_{\mathfrak{M}}$ on $\mathfrak{M}$, but otherwise the two metrics can be chosen independently. The metric $d_{\mathfrak{M}}$ is used to define the metric topology on $\mathfrak{M}$, while the metric $d_{\mathcal{F}}$ depends on an independent choice of the Riemannian metric on leaves.

For each $x \in \mathfrak{M}$, the Gauss Lemma implies that there exists $\lambda_x > 0$ such that $D_{\mathcal{F}_{\mathfrak{M}}}(x,\lambda_x)$ is a *strongly convex* subset for the metric $d_{\mathcal{F}}$. That is, for any pair of points $y, y' \in D_{d_{\mathcal{F}}}(x,\lambda_x)$ there is a unique shortest geodesic segment in L_x joining y and y' and contained in $D_{\mathcal{F}_{\mathfrak{M}}}(x,\lambda_x)$ (cf. [45, Chapter 3, Proposition 4.2], or [61, Theorem 9.9]). Then for all $0 < \lambda < \lambda_x$ the disk $D_{\mathcal{F}_{\mathfrak{M}}}(x,\lambda)$ is also strongly convex. The leafwise metrics have uniformly bounded geometry, so we obtain:

LEMMA 2.5. *There exists $\lambda_{\mathcal{F}} > 0$ such that for all $x \in \mathfrak{M}$, $D_{\mathcal{F}_{\mathfrak{M}}}(x,\lambda_{\mathcal{F}})$ is strongly convex.* □

The following proposition summarizes results in [34, sections 2.1 - 2.2].

PROPOSITION 2.6. *For a smooth foliated space $\mathfrak{M}$, given $\epsilon_{\mathfrak{M}} > 0$, there exist constants $\lambda_{\mathcal{F}} > 0$ and $0 < \delta_{\mathcal{U}}^{\mathcal{F}} < \lambda_{\mathcal{F}}/5$, and a covering of $\mathfrak{M}$ by foliation charts $\left\{\varphi_i\colon \overline{U}_i \to [-1,1]^n \times \mathfrak{X}_i \mid 1 \leq i \leq \nu\right\}$ with the following properties: For each $1 \leq i \leq \nu$, let $\pi_i = \pi_{x_i}\colon \overline{U}_i \to \mathfrak{X}_i$ be the projection, then*

(1) Interior: $U_i \equiv int(\overline{U}_i) = \varphi_i^{-1}\left((-1,1)^n \times B_{\mathfrak{X}}(w_i,\epsilon_i)\right)$, where $w_i \in \mathfrak{X}_i$ and $\epsilon_i > 0$.

(2) Locality: for $x_i \equiv \varphi_i^{-1}(w_i, 0) \in \mathfrak{M}$, $\overline{U}_i \subset B_{\mathfrak{M}}(x_i, \epsilon_{\mathfrak{M}})$.

For $z \in \overline{U}_i$, *the* plaque *of the chart* φ_i *through* z *is denoted by* $\mathcal{P}_i(z) = \mathcal{P}_i(\pi_i(z)) \subset \overline{U}_i$.

(3) Convexity: the plaques of φ_i *are strongly convex subsets for the leafwise metric.*
(4) Uniformity: for $w \in \mathfrak{X}_i$ *let* $x_w = \varphi_{x_i}^{-1}(0, w)$, *then*

(1)
$$D_{\mathcal{F}}(x_w, \delta^{\mathcal{F}}_{\mathcal{U}}/2) \ \subset\ \mathcal{P}_i(w) \ \subset\ D_{\mathcal{F}}(x_w, \delta^{\mathcal{F}}_{\mathcal{U}})$$

(5) The projection $\pi_i(U_i \cap U_j) = \mathfrak{X}_{i,j} \subset \mathfrak{X}_i$ *is a clopen subset for all* $1 \leq i, j \leq \nu$.

A regular foliated covering *of* $\mathfrak{M}$ *is one that satisfies the above conditions (2.6.1) to (2.6.5).*

We assume in the following that a regular foliated covering of $\mathfrak{M}$ as in Proposition 2.6 has been chosen. Let $\mathcal{U} = \{U_1, \ldots, U_\nu\}$ denote the corresponding open covering of $\mathfrak{M}$. We can assume that the spaces $\mathfrak{X}_i$ form a *disjoint clopen covering* of $\mathfrak{X}$, so that $\mathfrak{X} = \mathfrak{X}_1 \dot{\cup} \cdots \dot{\cup} \mathfrak{X}_\nu$.

Let $\epsilon_{\mathcal{U}} > 0$ be a Lebesgue number for $\mathcal{U}$. That is, given any $z \in \mathfrak{M}$ there exists some index $1 \leq i_z \leq \nu$ such that the open metric ball $B_{\mathfrak{M}}(z, \epsilon_{\mathcal{U}}) \subset U_{i_z}$.

For $1 \leq i \leq \nu$, let $\lambda_i \colon \overline{U}_i \to [-1,1]^n$ be the projection, so that for each $z \in U_i$ the restriction $\lambda_i \colon \mathcal{P}_i(z) \to [-1,1]^n$ is is a smooth coordinate system on the plaque.

For each $1 \leq i \leq \nu$ the set $\mathcal{T}_i = \varphi_i^{-1}(0, \mathfrak{X}_i)$ is a compact transversal to $\mathcal{F}$. Without loss of generality, we can assume that the transversals $\{\mathcal{T}_1, \ldots, \mathcal{T}_\nu\}$ are pairwise disjoint in $\mathfrak{M}$. Then define sections

(2)
$$\tau_i \colon \mathfrak{X}_i \to \overline{U}_i \ , \text{ defined by } \tau_i(\xi) = \varphi_i^{-1}(0, \xi) \ , \text{ so that } \pi_i(\tau_i(\xi)) = \xi.$$

Then $\mathcal{T}_i = \mathcal{T}_{x_i}$ is the image of τ_i and we let $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_\nu \subset \mathfrak{M}$ denote their disjoint union, and $\tau \colon \mathfrak{X} \to \mathcal{T}$ the union of the maps τ_i.

A map $f \colon \mathfrak{M} \to \mathfrak{M}'$ between foliated spaces is said to be a *foliated map* if the image of each leaf of $\mathcal{F}$ is contained in a leaf of $\mathcal{F}'$. If $\mathfrak{M}'$ is a matchbox manifold, then each leaf of $\mathcal{F}$ is path connected, so its image is path connected, hence must be contained in a leaf of $\mathcal{F}'$. Thus,

LEMMA 2.7. *Let* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *be matchbox manifolds, and* $f \colon \mathfrak{M}' \to \mathfrak{M}$ *a continuous map. Then* f *maps the leaves of* $\mathcal{F}'$ *to leaves of* $\mathcal{F}$. *In particular, any homeomorphism* $f \colon \mathfrak{M}' \to \mathfrak{M}$ *of matchbox manifolds is a foliated map.* □

A *leafwise path* is a continuous map $\gamma \colon [0,1] \to \mathfrak{M}$ such that there is a leaf L of $\mathcal{F}$ for which $\gamma(t) \in L$ for all $0 \leq t \leq 1$. If $\mathfrak{M}$ is a matchbox manifold, and $\gamma \colon [0,1] \to \mathfrak{M}$ is continuous, then γ is a leafwise path by Lemma 2.7. In the following, we will assume that all paths are piecewise differentiable.

The holonomy pseudogroup of a smooth foliated manifold $(M, \mathcal{F})$ generalizes the concept of a Poincaré section for a flow, which induces a discrete dynamical system associated to the flow. Associated to a leafwise path γ is a holonomy map h_γ, which is a local homeomorphism on the transversal space. For a matchbox manifold $(\mathfrak{M}, \mathcal{F})$ the holonomy along a leafwise path is defined analogously. We briefly recall below the ideas and notations of the construction of holonomy maps for matchbox manifolds; further details and proofs are given in [34, 35].

A pair of indices (i,j), $1 \leq i,j \leq \nu$, is said to be *admissible* if $U_i \cap U_j \neq \emptyset$. For (i,j) admissible, set $\mathfrak{X}_{i,j} = \pi_i(U_i \cap U_j) \subset \mathfrak{X}_i$. The regularity of foliation charts imply that plaques are either disjoint, or have connected intersection. For (i,j) admissible, there is a well-defined transverse change of coordinates homeomorphism $h_{i,j} \colon \mathfrak{X}_{i,j} \to \mathfrak{X}_{j,i}$ with domain $\mathrm{Dom}(h_{i,j}) = \mathfrak{X}_{i,j}$ and range $R(h_{i,j}) = \mathrm{Dom}(h_{j,i}) = \mathfrak{X}_{j,i}$. By definition they satisfy $h_{i,i} = Id$, $h_{i,j}^{-1} = h_{j,i}$, and if $U_i \cap U_j \cap U_k \neq \emptyset$ then $h_{k,j} \circ h_{j,i} = h_{k,i}$ on their common domain of definition. Note that the domain and range of $h_{i,j}$ are clopen subsets of $\mathfrak{X}$ by Proposition 2.6.5.

Recall that for $1 \leq i \leq \nu$, $\tau_i \colon \mathfrak{X}_i \to \mathcal{T}_i$ denotes the transverse section for the coordinate chart U_i, where $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_\nu \subset \mathfrak{M}$ denotes their disjoint union, and $\pi \colon \mathcal{T} \to \mathfrak{X}$ is the coordinate projection restricted to $\mathcal{T}$ which is a homeomorphism, with $\tau \colon \mathfrak{X} \to \mathcal{T}$ its inverse.

The *holonomy pseudogroup* $\mathcal{G}_{\mathcal{F}}$ of $\mathcal{F}$ is the topological pseudogroup modeled on $\mathfrak{X}$ generated by the elements of $\mathcal{G}_{\mathcal{F}}^{(1)} = \{h_{j,i} \mid (i,j) \text{ admissible}\}$. We also define a subgroupoid $\mathcal{G}_{\mathcal{F}}^{*} \subset \mathcal{G}_{\mathcal{F}}$ which is based on the holonomy along paths. A sequence $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ is *admissible* if each pair $(i_{\ell-1}, i_\ell)$ is admissible for $1 \leq \ell \leq \alpha$, and the composition $h_{\mathcal{I}} = h_{i_\alpha, i_{\alpha-1}} \circ \cdots \circ h_{i_1, i_0}$ has non-empty domain $\mathrm{Dom}(h_{\mathcal{I}})$, which is defined to be the maximal clopen subset of $\mathfrak{X}_{i_0}$ for which the compositions are defined. Given a open subset $U \subset \mathrm{Dom}(h_{\mathcal{I}})$ define the restriction $h_{\mathcal{I}}|U \in \mathcal{G}_{\mathcal{F}}$. Introduce

$$\mathcal{G}_{\mathcal{F}}^{*} = \{h_{\mathcal{I}}|U \mid \mathcal{I} \text{ admissible and } U \subset \mathrm{Dom}(h_{\mathcal{I}})\} \subset \mathcal{G}_{\mathcal{F}} \ . \tag{3}$$

The range of $g = h_{\mathcal{I}}|U$ is the open set $R(g) = h_{\mathcal{I}}(U) \subset \mathfrak{X}_{i_\alpha} \subset \mathfrak{X}$. Note that each map $g \in \mathcal{G}_{\mathcal{F}}^{*}$ admits a continuous extension $\overline{g} \colon \overline{\mathrm{Dom}(g)} = \overline{U} \to \mathfrak{X}_{i_\alpha}$ as $\mathrm{Dom}(h_{\mathcal{I}})$ is a clopen set for each $\mathcal{I}$.

Let $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ be an admissible sequence. For each $1 \leq \ell \leq \alpha$, set $\mathcal{I}_\ell = (i_0, i_1, \ldots, i_\ell)$, and let $h_{\mathcal{I}_\ell}$ denote the corresponding holonomy map. For $\ell = 0$, let $\mathcal{I}_0 = (i_0, i_0)$. Note that $h_{\mathcal{I}_\alpha} = h_{\mathcal{I}}$ and $h_{\mathcal{I}_0} = Id \colon \mathfrak{X}_0 \to \mathfrak{X}_0$.

Given $w \in \mathrm{Dom}(h_{\mathcal{I}})$, let $x = \tau_{i_0}(w) \in L_w$. For each $0 \leq \ell \leq \alpha$, set $w_\ell = h_{\mathcal{I}_\ell}(w)$ and $x_\ell = \tau_{i_\ell}(w_\ell)$. Recall that $\mathcal{P}_{i_\ell}(x_\ell) = \mathcal{P}_{i_\ell}(w_\ell)$, where each $\mathcal{P}_{i_\ell}(w_\ell)$ is a strongly convex subset of the leaf L_w in the leafwise metric $d_{\mathcal{F}}$. Introduce the *plaque chain*

$$\mathcal{P}_{\mathcal{I}}(w) = \{\mathcal{P}_{i_0}(w_0), \mathcal{P}_{i_1}(w_1), \ldots, \mathcal{P}_{i_\alpha}(w_\alpha)\} \ . \tag{4}$$

Adopt the notation $\mathcal{P}_{\mathcal{I}}(x) \equiv \mathcal{P}_{\mathcal{I}}(w)$. Intuitively, a plaque chain $\mathcal{P}_{\mathcal{I}}(x)$ is a sequence of successively overlapping convex "tiles" in L_w starting at $x = \tau_{i_0}(w)$, ending at $y = x_\alpha = \tau_{i_\alpha}(w_\alpha)$, and with each $\mathcal{P}_{i_\ell}(x_\ell)$ "centered" on the point $x_\ell = \tau_{i_\ell}(w_\ell)$.

Let $\gamma \colon [0,1] \to \mathfrak{M}$ be a path. Set $x_0 = \gamma(0) \in U_{i_0}$, $w = \pi(x_0)$ and $x = \tau(w) \in \mathcal{T}_{i_0}$. Let $\mathcal{I}$ be an admissible sequence with $w \in \mathrm{Dom}(h_{\mathcal{I}})$. We say that $(\mathcal{I}, w)$ *covers* γ if the domain of γ admits a partition $0 = s_0 < s_1 < \cdots < s_\alpha = 1$ such that $\mathcal{P}_{\mathcal{I}}(w)$ satisfies

$$\gamma([s_\ell, s_{\ell+1}]) \subset \mathcal{P}_{i_\ell}(\xi_\ell) \ , \ 0 \leq \ell < \alpha, \text{ and } \gamma(1) \in \mathcal{P}_{i_\alpha}(\xi_\alpha). \tag{5}$$

For a path γ, we construct an admissible sequence $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ with $w \in \mathrm{Dom}(h_{\mathcal{I}})$ so that $(\mathcal{I}, w)$ covers γ, and has "uniform domains". Inductively choose a partition of the interval $[0,1]$, say $0 = s_0 < s_1 < \cdots < s_\alpha = 1$, such that for each $0 \leq \ell \leq \alpha$,

$$\gamma([s_\ell, s_{\ell+1}]) \subset D_{\mathcal{F}}(x_\ell, \epsilon_{\mathcal{U}}^{\mathcal{F}}) \quad , \quad x_\ell = \gamma(s_\ell).$$

As a notational convenience, we have let $s_{\alpha+1} = s_\alpha$, so that $\gamma([s_\alpha, s_{\alpha+1}]) = x_\alpha$. Choose $s_{\ell+1}$ to be the largest value of $s_\ell < s \leq 1$ such that $d_{\mathcal{F}}(\gamma(s_\ell), \gamma(t)) \leq \epsilon_{\mathcal{U}}^{\mathcal{F}}$ for all $s_\ell \leq t \leq s$, then $\alpha \leq \|\gamma\|/\epsilon_{\mathcal{U}}^{\mathcal{F}}$.

For each $0 \leq \ell \leq \alpha$, choose an index $1 \leq i_\ell \leq \nu$ so that $B_{\mathfrak{M}}(x_\ell, \epsilon_{\mathcal{U}}) \subset U_{i_\ell}$. Note that, for all $s_\ell \leq t \leq s_{\ell+1}$, $B_{\mathfrak{M}}(\gamma(t), \epsilon_{\mathcal{U}}/2) \subset U_{i_\ell}$, so that $x_{\ell+1} \in U_{i_\ell} \cap U_{i_{\ell+1}}$. It follows that $\mathcal{I}_\gamma = (i_0, i_1, \ldots, i_\alpha)$ is an admissible sequence. Set $h_\gamma = h_{\mathcal{I}_\gamma}$ and note that $h_\gamma(w) = w'$.

Next, consider paths $\gamma, \gamma' \colon [0,1] \to \mathfrak{M}$ with $x = \gamma(0) = \gamma'(0)$ and $y = \gamma(1) = \gamma'(1)$. Suppose that γ and γ' are homotopic relative endpoints. That is, assume there exists a continuous map $H \colon [0,1] \times [0,1] \to \mathfrak{M}$ with

$$H(0,t) = \gamma(t) \ , \ H(1,t) = \gamma'(t) \ , \ H(s,0) = x \text{ and } H(s,1) = y \quad \text{for all } 0 \leq s \leq 1$$

Then there exists partitions $0 = s_0 < s_1 < \cdots < s_\beta = 1$ and $0 = t_0 < t_1 < \cdots < t_\alpha = 1$ such that for each pair of indices $0 \leq j < \beta$ and $0 \leq k < \alpha$, there is an index $1 \leq i(j,k) \leq \nu$ such that

$$H([s_j, s_{j+1}] \times [t_k, t_{k+1}]) \subset D_{\mathcal{F}}(H(s_j, t_k), \epsilon_{\mathcal{U}}^{\mathcal{F}}) \subset U_{i(j,k)}$$

A standard argument then yields the following basic fact about holonomy maps.

LEMMA 2.8. *Let $\gamma, \gamma' \colon [0,1] \to \mathfrak{M}$ be paths with $x = \gamma(0) = \gamma'(0)$ and $y = \gamma(1) = \gamma'(1)$, and suppose they are homotopic relative endpoints. Then the induced holonomy maps h_γ and $h_{\gamma'}$ agree on an open neighborhood of $\xi_0 = \pi_{i_0}(x)$.*

Next consider the *groupoid* formed by germs of maps in $\mathcal{G}_{\mathcal{F}}$. Let $U, U', V, V' \subset \mathfrak{X}$ be open subsets with $w \in U \cap U'$. Given homeomorphisms $h \colon U \to V$ and $h' \colon U' \to V'$ with $h(w) = h'(w)$, then h and

h' have the same *germ at* w, and write $h \sim_w h'$, if there exists an open neighborhood $w \in W \subset U \cap U'$ such that $h|W = h'|W$. Note that $\sim_w$ defines an equivalence relation.

DEFINITION 2.9. *The* germ of h at w *is the equivalence class* $[h]_w$ *under the relation* $\sim_w$*. The map* $h\colon U \to V$ *is called a* representative *of* $[h]_w$*. The point* w *is called the source of* $[h]_w$ *and denoted* $s([h]_w)$*, while* $w' = h(w)$ *is called the range of* $[h]_w$ *and denoted* $r([h]_w)$*.*

The collection of all such germs $[h]_w$ for $h \in \mathcal{G}_{\mathcal{F}}$ and $w \in \mathrm{Dom}(h)$, forms the *holonomy groupoid* $\Gamma_{\mathcal{F}}$, which has the natural topology associated to sheaves of maps over $\mathcal{X}$. Let $\mathcal{R}_{\mathcal{F}} \subset \mathfrak{X} \times \mathfrak{X}$ denote the equivalence relation on $\mathfrak{X}$ induced by $\mathcal{F}$, where $(w, w') \in \mathcal{R}_{\mathcal{F}}$ if and only if w, w' correspond to points on the same leaf of $\mathcal{F}$. The product map $s \times r\colon \Gamma_{\mathcal{F}} \to \mathcal{R}_{\mathcal{F}}$ is étale; that is, the map is a local homeomorphism with discrete fibers. These notions were introduced by Haefliger for foliations [54, 55], and naturally extend to the case of matchbox manifolds.

We introduce a convenient notation for elements of $\Gamma_{\mathcal{F}}$. Let $(w, w') \in \mathcal{R}_{\mathcal{F}}$, and let γ denote a path from $x = \tau(w)$ to $y = \tau(w')$. We may assume that γ is a geodesic for the leafwise metric, and let $[h_\gamma]_w$ (or sometimes just γ_w) denote the germ at w of the holonomy map defined by γ.

It follows that there is a well-defined surjective homomorphism, the *holonomy map*,

$$h_{\mathcal{F},x}\colon \pi_1(L_x, x) \to \Gamma_w^w \equiv \{[g]_w \in \Gamma_{\mathcal{F}} \mid r([g]_w) = w\} \tag{6}$$

Moreover, if $y, z \in L$ then the homomorphism $h_{\mathcal{F},y}$ is conjugate (by an element of $\mathcal{G}_{\mathcal{F}}$) to the homomorphism $h_{\mathcal{F},z}$. A leaf L is said to have *non-trivial germinal holonomy* if for some $y \in L$, the homomorphism $h_{\mathcal{F},y}$ is non-trivial. If the homomorphism $h_{\mathcal{F},y}$ is trivial, then we say that L_y is a *leaf without holonomy*. This property depends only on L, and not the choice of $y \in L$.

LEMMA 2.10. *Given a path* $\gamma\colon [0,1] \to \mathfrak{M}$ *with* $x = \gamma(0)$ *and* $y = \gamma(1)$*. Suppose that* L_x *is a leaf without holonomy. Then there exists a leafwise geodesic segment* $\gamma'\colon [0,1] \to \mathfrak{M}$ *with* $x = \gamma'(0)$ *and* $y = \gamma'(1)$*, such that* $\|\gamma'\| = d_{\mathcal{F}}(x, y)$*, and* h_γ *and* $h_{\gamma'}$ *agree on an open neighborhood of* ξ_0*.*

Proof. The leaf L_x containing x is a complete Riemannian manifold, so there exists a geodesic segment γ' which is length minimizing between x and y. Then the holonomy maps h_γ and $h_{\gamma'}$ agree on an open neighborhood of $\xi_0 = \pi_{i_0}(x)$ by the definition of germinal holonomy. □

Next, we introduce the filtrations of $\mathcal{G}_{\mathcal{F}}^*$ by word length, and of $\Gamma_{\mathcal{F}}$ by path length, then derive estimates comparing these notions of length.

For $\alpha \geq 1$, let $\mathcal{G}_{\mathcal{F}}^{(\alpha)}$ be the collection of holonomy homeomorphisms $h_{\mathcal{I}}|U \in \mathcal{G}_{\mathcal{F}}^*$ determined by admissible paths $\mathcal{I} = (i_0, \ldots, i_k)$ such that $k \leq \alpha$ and $U \subset \mathrm{Dom}(h_{\mathcal{I}})$ is open. For each α, let $C(\alpha)$ denote the number of admissible sequences of length at most α. As there are at most ν^2 admissible pairs (i, j), we have the basic estimate that $C(\alpha) \leq \nu^{2\alpha}$. This upper bound estimate grows exponentially with α, though the exact growth rate of $C(\alpha)$ may be much less.

For each $g \in \mathcal{G}_{\mathcal{F}}^*$ there is some α such that $g \in \mathcal{G}_{\mathcal{F}}^{(\alpha)}$. Let $\|g\|$ denote the least such α, which is called the *word length* of g. Note that $\mathcal{G}_{\mathcal{F}}^{(1)}$ generates $\mathcal{G}_{\mathcal{F}}^*$.

We use the word length on $\mathcal{G}_{\mathcal{F}}^*$ to define the word length on $\Gamma_{\mathcal{F}}$, where for $\gamma_w \in \Gamma_{\mathcal{F}}$, set

$$\|\gamma_w\| \ = \ \min \ \{\|g\| \mid [g]_w = \gamma_w \text{ for } g \in \mathcal{G}_{\mathcal{F}}^*\} \, . \tag{7}$$

Introduce the *path length* of $\gamma_w \in \Gamma_{\mathcal{F}}$, by considering the infimum of the lengths $\|\gamma'\|$ for all piecewise smooth curves γ' for which $\gamma'_w = \gamma_w$. That is,

$$\ell(\gamma_w) \ = \ \inf \ \{\|\gamma'\| \mid \gamma'_w = \gamma_w\} \, . \tag{8}$$

Note that if L_w is a leaf without holonomy, set $x = \tau(w)$ and $y = \tau(w')$, then Lemma 2.10 implies that $\ell(\gamma_w) = d_{\mathcal{F}}(x, y)$. This yields a fundamental estimate, whose proof can be found in [36]:

LEMMA 2.11. *Let* $[g]_w \in \Gamma_{\mathcal{F}}$ *where* w *corresponds to a leaf without holonomy. Then*

$$d_{\mathcal{F}}(x, y)/2\delta_{\mathcal{U}}^{\mathcal{F}} \ \leq \ \|[g]_w\| \ \leq \ 1 + d_{\mathcal{F}}(x, y)/\epsilon_{\mathcal{U}}^{\mathcal{F}} \tag{9}$$

3. Dynamics

The study of the dynamics of a pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting on $\mathfrak{X}$ can be considered as a generalization of the study of continuous actions of finitely-generated groups on Cantor sets, though it differs in some fundamental ways. For a group action, each $\gamma \in \Gamma$ defines a homeomorphism $h_\gamma : \mathfrak{X} \to \mathfrak{X}$. For a pseudogroup action, given $g \in \mathcal{G}_{\mathfrak{X}}$ and $w \in \mathrm{Dom}(g)$, there is some clopen neighborhood $w \in U \subset \mathrm{Dom}(g)$ for which $g|U = h_{\mathcal{I}}|U$ where $\mathcal{I}$ is admissible sequence with $w \in \mathrm{Dom}(h_{\mathcal{I}})$. By the definition of a pseudogroup, every $g \in \mathcal{G}_{\mathfrak{X}}$ is the "union" of such maps, and the dynamical properties of the action may reflect this difference.

DEFINITION 3.1. *A pseudogroup $\mathcal{G}_{\mathfrak{X}}$ acting on a Cantor set $\mathfrak{X}$ is* compactly generated*, if there exists two collections of* clopen *subsets $\{U_1, \ldots, U_k\}$ and $\{V_1, \ldots, V_k\}$ of $\mathfrak{X}$ and homeomorphisms $\{h_i : U_i \to V_i \mid 1 \leq i \leq k\}$ which generate all elements of $\mathcal{G}_{\mathfrak{X}}$. The collection of maps $\mathcal{G}^*_{\mathfrak{X}}$ is defined to be all compositions of the generators on the maximal domains for which the composition is defined.*

We recall two definitions from topological dynamics, that of *equicontinuous* and *expansive* dynamics, as adapted to actions of compactly-generated pseudogroups.

DEFINITION 3.2. *The action of a compactly-supported pseudogroup $\mathcal{G}_{\mathfrak{X}}$ on $\mathfrak{X}$ is* expansive*, or more properly ϵ-*expansive*, if there exists $\epsilon > 0$ such that for all $w, w' \in \mathfrak{X}$, there exists $g \in \mathcal{G}^*_{\mathfrak{X}}$ with $w, w' \in D(g)$ such that $d_{\mathfrak{X}}(g(w), g(w')) \geq \epsilon$.*

DEFINITION 3.3. *The action of a compactly-supported pseudogroup $\mathcal{G}_{\mathfrak{X}}$ on $\mathfrak{X}$ is* equicontinuous *if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $g \in \mathcal{G}^*_{\mathfrak{X}}$, if $w, w' \in D(g)$ and $d_{\mathfrak{X}}(w, w') < \delta$, then $d_{\mathfrak{X}}(g(w), g(w')) < \epsilon$. Thus, $\mathcal{G}^*_{\mathfrak{X}}$ is equicontinuous as a family of local group actions.*

The *geometric entropy* for pseudogroup C^1-actions, introduced by Ghys, Langevin and Walczak [52], gives a measure of the "exponential complexity" of the orbits of the action. It has found many applications in the study of foliation dynamical systems. We give another application here, to the embedding problem for matchbox manifolds, using the extension of these ideas to the groupoid action of $\mathcal{G}_{\mathfrak{X}}$ on $\mathfrak{X}$ with the metric $d_{\mathfrak{X}}$.

We begin with the basic notion of ϵ-separated sets, due to Bowen [19] for diffeomorphisms, and extended to groupoids in [52]. Let $\epsilon > 0$ and $\ell > 0$. A subset $\mathcal{E} \subset \mathfrak{X}$ is said to be $(d_{\mathfrak{X}}, \epsilon, \ell)$-separated if for all $w, w' \in \mathcal{E} \cap \mathfrak{X}_i$ there exists $g \in \mathcal{G}^*_{\mathfrak{X}}$ with $w, w' \in \mathrm{Dom}(g) \subset \mathfrak{X}_i$, and $\|g\|_w \leq \ell$ so that $d_{\mathfrak{X}}(g(w), g(w')) \geq \epsilon$. If $w \in \mathfrak{X}_i$ and $w' \in \mathfrak{X}_j$ for $i \neq j$ then they are (ϵ, ℓ)-separated by default. The "expansion growth function" counts the maximum of this quantity:

$$h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon, \ell) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset \mathfrak{X} \text{ is } (d_{\mathfrak{X}}, \epsilon, \ell)\text{-separated}\}$$

The entropy is then defined to be the exponential growth type of the expansion growth function:

$$h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon) = \limsup_{\ell \to \infty} \ \ln \{h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon, \ell)\} / \ell \quad , \quad h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) = \lim_{\epsilon \to 0} \ h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}, \epsilon)$$

Note that the quantity $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) \geq 0$, and it may take the value $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) = \infty$.

Recall that any two metrics compatible with the topology on the compact space $\mathfrak{X}$ are comparable, in that given metrics $d_{\mathfrak{X}}$ and $d'_{\mathfrak{X}}$ on $\mathfrak{X}$, given $\epsilon > 0$ there exists $\epsilon' > 0$ so that for any $x, y \in \mathfrak{X}$ with $d_{\mathfrak{X}}(x, y) \geq \epsilon$, then $d'_{\mathfrak{X}}(x, y) \geq \epsilon'$. It then follows, as in [19, 52]:

THEOREM 3.4. *Given an action of a compactly-supported pseudogroup $\mathcal{G}_{\mathfrak{X}}$ on $\mathfrak{X}$, let $d_{\mathfrak{X}}$ and $d'_{\mathfrak{X}}$ be compatible transverse metrics on $\mathfrak{X}$. Then $0 < h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) < \infty$ if and only if $0 < h(\mathcal{G}_{\mathfrak{X}}, d'_{\mathfrak{X}}) < \infty$.*

4. Lipshitz foliations and geometry

In this section, we define the Lipshitz property for a matchbox manifold $\mathfrak{M}$, and then consider its relation to embeddings of $\mathfrak{M}$ in a Riemannian manifold M. Note that our definition of Lipshitz for homeomorphisms in Definition 4.1 is equivalent to the bi-Lipshitz condition as used in some works. The basic result is that if $\mathfrak{M}$ is homeomorphic to an exceptional minimal set in a C^1-foliation, then its transversal space $\mathfrak{X}$ has a metric for which the pseudogroup $\mathcal{G}_{\mathfrak{M}}$ is Lipshitz.

It is a standard fact that there is a *unique* Cantor set, up to homeomorphism. More precisely, any two compact, perfect totally disconnected non-empty sets are homeomorphic. See [96, Chapter 12] for a proof and discussion of this result. In particular, for a given Cantor set $\mathfrak{X}$, any non-empty clopen subset $U \subset \mathfrak{X}$ is homeomorphic to $\mathfrak{X}$.

On the other hand, there are many possible metrics on $\mathfrak{X}$ which are compatible with the topology. Two metrics $d_{\mathfrak{X}}$ and $d'_{\mathfrak{X}}$ are *L-equivalent*, if they satisfy a Lipshitz condition for some $C \geq 1$,

$$C^{-1} \cdot d_{\mathfrak{X}}(x,y) \ \leq \ d'_{\mathfrak{X}}(x,y) \ \leq \ C \cdot d_{\mathfrak{X}}(x,y) \quad \text{for all } x,y \in \mathfrak{X} \tag{10}$$

The study of the *Lipshitz geometry* of the pair $(\mathfrak{X}, d_{\mathfrak{X}})$ investigates the geometric properties common to all metrics in the Lipshitz class of the given metric $d_{\mathfrak{X}}$. For example, the Hausdorff dimension of a Cantor transversal to an exceptional minimal set for a C^1-foliation of codimension-one has been studied in the works by Cantwell and Conlon [26], Matsumoto [93], Gelfert and Rams [50], and Biś and Urbanski [16]. The Markov minimal sets for codimension-one foliations, as studied in these works, are related to the Cantor sets defined by contracting Iterated Function Systems, and the more general class of self-similar fractals. These are naturally embedded in the Riemannian manifold on which the collection of embeddings are defined , so have a given Lipshitz class of metrics. This is a very-well studied subject, and we mention only works of Rams and his coauthors in [42, 50], and the works of Rao, Ruan, Wang and Xi as in [103, 104] which are more closely related to our theme.

There are also a variety of classes of metrics on a Cantor set, which are not necessarily Lipshitz invariant, but arise naturally from dynamical considerations. For example, recall that an *ultrametric* on $\mathfrak{X}$ is a metric $d_{\mathfrak{X}}$ which satisfies the *strong triangle inequality*,

$$d_{\mathfrak{X}}(x,z) \leq \max\{d_{\mathfrak{X}}(x,y), d_{\mathfrak{X}}(y,z)\} \quad \text{for all } x,y,z \in \mathfrak{X}.$$

The endset of a tree is endowed with a natural ultrametric, and if the tree is "bushy" then this endset is a Cantor set. The Lipshitz geometry of boundary ultrametrics is fundamental in the study of the quasi-isometric classification of trees; see for example, the survey text [20], and also Hughes [66, 67, 68]. Ultrametrics also arise naturally in the study of minimal Cantor systems via Bratteli diagrams. The work [13, 78] by Bellisard, Julien and Savinien studies the Lipshitz embedding and L-equivalence properties for Cantor sets defined by Bratteli diagrams, especially those associated to substitution tiling spaces.

DEFINITION 4.1. *The action of a compactly-supported pseudogroup $\mathcal{G}_{\mathfrak{X}}$ is* Lipshitz *with respect to $d_{\mathfrak{X}}$ if there exists $C \geq 1$ such that for each (i,j) admissible with corresponding transition map $h_{i,j}$, and for all $w, w' \in \mathrm{Dom}(h_{i,j})$ we have*

$$C^{-1} \cdot d_{\mathfrak{X}}(w,w') \leq d_{\mathfrak{X}}(h_{i,j}(w), h_{i,j}(w')) \leq C \cdot d_{\mathfrak{X}}(w,w') \ . \tag{11}$$

We then say that $\mathcal{G}^*_{\mathfrak{X}}$ is C-Lipshitz with respect to $d_{\mathfrak{X}}$.

One can also define finer metric conditions on the action os a pseudogroup $\mathcal{G}_{\mathfrak{X}}$, such as the Zygmund condition used in [69] which can be used to define "quasi-conformal" properties of homeomorphisms, as in [49, 91, 99, 121, 123]. This topic will not be pursued here, though is related to examples of matchbox manifolds obtained via the action of hyperbolic groups on their "boundary at infinity". The study of the Lipshitz properties of Gromov hyperbolic groups acting on their boundaries is a massive subject, as discussed in the survey works [20, 79].

The following is an immediate consequence of the definitions.

LEMMA 4.2. *Suppose that the action of $\mathcal{G}^*_{\mathfrak{X}}$ on $\mathfrak{X}$ is C-Lipshitz with respect to $d_{\mathfrak{X}}$. Then for all $g \in \mathcal{G}^*_{\mathfrak{X}}$ with word length $\|g\| \leq \alpha$, and $w, w' \in \mathrm{Dom}(g)$ we have*

$$C^{-\alpha} \cdot d_{\mathfrak{X}}(w,w') \ \leq \ d_{\mathfrak{X}}(g(w), g(w')) \ \leq \ C^{\alpha} \cdot d_{\mathfrak{X}}(w,w') \ . \tag{12}$$

The claim of the following result is intuitively clear, but requires some care, and illustrates some of the subtleties of dealing with foliation charts which have totally disconnected transversals.

Recall that $\tau \colon \mathfrak{X} \to \mathcal{T} \subset \mathfrak{M}$ is the transversal to $\mathcal{F}$ associated to the chosen good covering. Let $d_{\mathfrak{X}}$ be the metric induced on $\mathfrak{X}$ by the restriction of $d_{\mathfrak{M}}$ on $\mathfrak{M}$ to the image of $\tau \colon \mathfrak{X} \to \mathcal{T}$.

PROPOSITION 4.3. *Let $\mathfrak{M}$ be a minimal matchbox manifold, and M a smooth Riemannian manifold with a C^1-foliation $\mathcal{F}$, and $\mathcal{Z} \subset M$ an exceptional minimal set for $\mathcal{F}$. Suppose there exists a homeomorphism $f \colon \mathfrak{M} \to M$ onto $\mathcal{Z}$, then there exists a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the action on $\mathfrak{X}$ of the holonomy pseudogroup $\mathcal{G}^*_{\mathfrak{M}}$ is Lipshitz.*

Proof. Lemma 2.7 implies that the restriction of f to a leaf L of $\mathcal{F}_{\mathfrak{M}}$ is a homeomorphism onto a leaf $\mathcal{L}$ of $\mathcal{F}$, in the restricted topology on $\mathcal{Z}$.

Choose a good covering $\{\phi_\alpha \colon V_\alpha \to (-1,1)^n \times (-1,1)^q \mid 1 \leq \alpha \leq k\}$ for the foliation $\mathcal{F}$ of M, as in [22], where n is the leaf dimension of $\mathcal{F}$, and q is the codimension of $\mathcal{F}$ in M. Set $T_\alpha = \phi_\alpha^{-1}(\{0\} \times (-1,1)^q)$, then the union $T = T_1 \cup \cdots \cup T_k$ is a complete transversal for $\mathcal{F}$. We can assume without loss of generality that the closures of the transversals are disjoint. The Riemannian metric on TM restricts to a Riemannian metric on each T_α and thus defines a path-length metric denoted by d_{T_α} on each submanifold $T_\alpha \subset M$. Extend the metrics on each T_α to a metric on T, by declaring $d_T(u,v) = 1$ if $u \in T_\alpha$ and $v \in T_\beta$ for $\alpha \neq \beta$.

A pair (α, β) is said to be *admissible* if $V_\alpha \cap V_\beta \neq \emptyset$. For (α,β) admissible, the overlap of plaques in these charts defines a holonomy map $g_{\alpha,\beta}$. The assumption that $\mathcal{F}$ is a C^1-foliation implies that $g_{\alpha,\beta}$ is a C^1-map from an open subset of T_α to an open set of T_β. For each $u \in \mathrm{Dom}(g_{\alpha,\beta})$ let $D_u(g_{\alpha,\beta})$ denote the matrix of differentials for $g_{\alpha,\beta}$ at $u \in \mathrm{Dom}(g_{\alpha,\beta})$, with respect to the framing of the tangent spaces to the sections T_α induced by the coordinate charts. Let $\|D_u(g_{\alpha,\beta})\|$ denote the matrix sup-norm of $D_u(g_{\alpha,\beta})$ with respect to the Riemannian metric induced on the sections. The assumption that we have a good covering implies that the maps $g_{\alpha,\beta}$ admit continuous C^1-extensions, so the norms $\|D_u(g_{\alpha,\beta})\|$ have uniform upper bounds for all admissible pairs (α,β) and all $u \in \mathrm{Dom}(g_{\alpha,\beta})$. Define:

$$C'_{\mathcal{F}} = \max \{\|D_u(g_{\alpha,\beta})\| \mid (\alpha,\beta) \text{ admissible }, \ u \in \mathrm{Dom}(g_{\alpha,\beta})\} \ < \ \infty \tag{13}$$

It follows that the pseudogroup for $\mathcal{F}$ defined by the maps $\{g_{\alpha,\beta} \mid (\alpha,\beta) \text{ admissible}\}$ is $C'_{\mathcal{F}}$-Lipshitz.

Recall that $\mathcal{T}_i \subset \mathfrak{M}$, for $1 \leq i \leq \nu$, are the Cantor transversals to $\mathfrak{M}$ defined by a good covering for $\mathfrak{M}$ as in Definition 2.1. For each $x \in \mathcal{T}_i$ there exists $1 \leq \alpha \leq k$ with $f(x) \in V_\alpha$, and thus a clopen neighborhood $W(i,x,\alpha) \subset \mathcal{T}_i$ for which $f(W(i,x,\alpha)) \subset V_\alpha$. If $W(i,x,\alpha)$ is sufficiently small, then the plaque projection of the image $f(W(i,x,\alpha))$ into T_α is a homeomorphism onto its image, and so the metric d_{T_α} on T_α induces a metric on $W(i,x,\alpha)$. As each $\mathcal{T}_i$ is compact, we can choose a finite covering $\{U_k\}$ of the union $\mathcal{T} = \mathcal{T}_1 \cup \cdots \mathcal{T}_\nu$ where each $W_k = W(i,x,\alpha)$ for appropriate (i,x,α). It may happen that for $x,y \in U_k$ there is an admissible pair (i,j) for the covering of $\mathfrak{M}$ such that $f(h_{i,j}(x))$ and $f(h_{i,j}(y))$ are not contained in the same foliation chart V_ℓ. However, as there are only a finite number of admissible pairs (i,j) for the covering of $\mathfrak{M}$ by foliation charts, we can refine the finite clopen covering $\{U_i\}$ of $\mathcal{T}$, so that this condition is satisfied.

We then obtain a metric $d_{\mathcal{T}}$ on $\mathcal{T}$ by declaring the distance between points $x, y \in \mathcal{T}$ equal to the induced metric if x,y belong to the same clopen set in $\{U_k\}$, and to have distance 1 otherwise. The metric $d_{\mathcal{T}}$ induces a metric denoted by $d_{\mathfrak{X}}$ on $\mathfrak{X}$. We claim that the action of $\mathcal{G}^*_{\mathfrak{X}}$ on $\mathfrak{X}$ is $C_{\mathcal{F}}$-Lipshitz for $d_{\mathfrak{X}}$ and an appropriate $C_{\mathcal{F}} \geq 1$.

Suppose that $x,y \in U_k$ for some clopen set in the covering of $\mathcal{T}$. Then $f(h_{i,j}(x))$ and $f(h_{i,j}(y))$ are contained in the same foliation chart V_ℓ by construction. Note that x and $h_{i,j}(x)$ are contained in the same leaf of $\mathcal{F}_{\mathfrak{M}}$ so their images $f(x)$ and $f(h_{i,j}(x))$ are contained in the same leaf of $\mathcal{F}$. Thus, there is a plaque chain of length at most $\lambda_{f,x}$ between these two points. The same holds for the point y, so there is a plaque-chain of length $\lambda_{f,y}$ between $f(y)$ and $f(h_{i,j}(y)$. By the compactness of $\mathcal{T}$ there is a uniform upper bound λ_f for all such pairs. (Curiously, the author's paper [60] uses a similar argument, and discusses its significance.) Thus, by Lemma 4.2 we have the estimate for $x,y \in U_k$ with projections $w = \pi(x), w' = \pi(y) \in \mathfrak{X}_i$, and $C''_{\mathcal{F}} = (C'_{\mathcal{F}})^{\lambda_f}$,

$$(C''_{\mathcal{F}})^{-1} \cdot d_{\mathfrak{X}}(w,w') \leq d_{\mathfrak{X}}(h_{i,j}(w), h_{i,j}(w')) \leq C''_{\mathcal{F}} \cdot d_{\mathfrak{X}}(w,w') \ . \tag{14}$$

If x,y do not belong to the same set U_k then $d_{\mathfrak{X}}(w,w') = 1$ by definition, so their exists $C'''_{\mathcal{F}} \geq 1$ such that (14) holds for such pairs. Set $C_{\mathcal{F}} = \max\{C''_{\mathcal{F}}, C'''_{\mathcal{F}}\}$, and the claim follows. □

COROLLARY 4.4. *Let $\mathfrak{M}$ be a matchbox manifold which embeds as an exceptional minimal set for C^1-foliation $\mathcal{F}$ on a compact smooth manifold M, as in Proposition 4.3. Then there is a transverse metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$ such that $h(\mathcal{G}_{\mathfrak{X}}, d_{\mathfrak{X}}) < \infty$.*

Proof. Let $d_{\mathfrak{X}}$ be the metric on $\mathfrak{X}$ constructed in the proof of Proposition 4.3. Then $\mathfrak{X}$ is covered by disjoint clopen sets for which $d_{\mathfrak{X}}$ is the pull-back of the metric on transversals to the foliation $\mathcal{F}$, so each of these sets has Hausdorff dimension bounded above by the codimension q of $\mathcal{F}$. It follows that the Hausdorff dimension satisfies $HD(\mathfrak{X}, d_{\mathfrak{X}}) \leq q$. As the holonomy maps of $\mathcal{G}^*_{\mathfrak{X}}$ are Lipshitz for this metric, the techniques used in the proof of [52, Proposition 2.7] yield the result. □

5. Examples from foliations

In this section, we recall some examples of embedding minimal matchbox manifolds as exceptional minimal sets in C^r-foliations, for $r \geq 1$. We first consider the case for foliations of codimension-one. The prototypical example is the well-known construction by Denjoy:

THEOREM 5.1 (Denjoy [43])**.** *There exist a C^1-diffeomorphism f of the circle $\mathbb{S}^1$ with no fixed points, and with a non-empty wandering set W so that the complement $\mathbf{K} = \mathbb{S}^1 - W$ is an invariant Cantor set.*

The induced action $f \colon \mathbb{Z} \times \mathbf{K} \to \mathbf{K}$ of $\mathbb{Z}$ on the invariant set $\mathbf{K}$ is called a *Denjoy minimal system.*

The C^1-hypotheses on the diffeomorphism f is far from optimal. The celebrated *Denjoy Theorem* states that if the diffeomorphism f is C^2, or even C^{1+bv} where this means that its derivative has bounded variation, then no Cantor minimal set exist. Various optimal conditions on the derivative of f such that it admits a Cantor minimal set are discussed in Hu and Sullivan [65]. McDuff [95] formulated a set of necessary and sufficient conditions on an embedded Cantor set $\mathbf{K} \subset \mathbb{S}^1$ so that it is an invariant set of a $C^{1+\alpha}$-diffeomorphism, for $0 < \alpha < 1$.

The Denjoy example played a fundamental role in the construction of counter-examples to the Seifert Conjecture, which enabled Schweitzer in [113] to construct the first C^1-examples of flows on 3-manifolds without periodic orbits. Schweitzer's construction embedded a suspension of the Denjoy minimal set as an isolated minimal set for a flow contained in a plug embedded in $\mathbb{R}^3$, and motivated Harrison's construction [56, 57] of a $C^{2+\alpha}$-flow in $\mathbb{R}^3$ with an *isolated* minimal limit set homeomorphic to a suspension of the Denjoy set, for $\alpha < 1$. On the other hand, Knill constructed in [83] a smooth diffeomorphism in the 2-dimensional annulus with a minimal set homeomorphic to the Denjoy set, so the suspension of this diffeomorphism yields a codimension-2 smooth foliation defined by a flow, with a minimal set homeomorphic to the Denjoy minimal set in $\mathbb{T}^2$. Note that the periodic orbits for the Knill diffeomorphism contain the Denjoy set in its closure, so this example is not sufficient for constructing smooth counter-examples to the Seifert Conjecture. The Knill example illustrates that the degree of differentiability r for a C^r-embedding of a Cantor minimal system may depend on the codimension, as well as the dynamical behavior of the action in open neighborhoods.

In some cases, there are analogs of the above results for the case of a finitely-generated group acting minimally on a Cantor set. For example, Pixton gave a generalization of the Denjoy construction:

THEOREM 5.2 (Pixton [101])**.** *Suppose that $0 < \alpha < 1/(n+1)$, then there exist a $C^{1+\alpha}$-action of $\mathbb{Z}^n$ on the circle $\mathbb{S}^1$ with no fixed points and with a non-empty wandering set W so that the complement $\mathbf{K} = \mathbb{S}^1 - W$ is an invariant Cantor set.*

The suspension of a C^1-action of $\mathbb{Z}^n$ on $\mathbb{S}^1$ yields a foliation with leaves covered by $\mathbb{R}^n$, and with an exceptional minimal set. This yields an embedded minimal matchbox manifold, whose holonomy is conjugate to the given action. The Pixton-type examples have been further studied by Deroin, Kleptsyn and Navas in [44], and Kleptsyn and Navas in [82]. Note that all of these Pixton-type examples, of codimension-one foliations with contractible leaves, their minimal sets have the remarkable property that such a minimal set is a generalized solenoid. That is, the minimal set is homeomorphic to an inverse limit of a system of maps between compact simplicial complexes, as shown by Theorem 6.4 below.

An exceptional minimal set for a codimension-one C^2-foliation of a compact manifold cannot be of "Denjoy type" by Sacksteder's Theorem [108], which implies that the exceptional minimal set must have an element of holonomy which is a transverse contraction along a leaf of the minimal set. A special class of these examples, the *Markov minimal sets*, were studied by Hector [58, 59], Cantwell and Conlon [26], and Matsumoto [93] for example. It remains an open problem to characterize the embeddings of Cantor minimal systems in C^r-foliations of codimension-one, for $r \geq 1$ (see [70]).

For foliations with codimension $q \geq 2$, we recall some of the constructions of exceptional minimal sets for foliations with leaves of higher dimension $n \geq 2$. As remarked previously, if we allow foliations defined by a flow, then the possible minimal sets are so varied as to be unclassifiable. Thus, we consider examples which are "essentially" higher dimensional, and are not the obvious result of taking a product of 1-dimensional foliations.

The Markov minimal sets for codimension-one foliations are a special case of minimal sets defined by an *Iterated Function System* (or *IFS* for short), acting on $\mathbb{R}^q$, or embedded in a manifold such as $\mathbb{S}^q$. Such examples can be realized by a foliation with 2-dimensional leaves, using the suspension construction. The properties of minimal sets defined by an IFS is extremely well-studied, especially the Hausdorff dimension of invariant Cantor sets defined by the IFS. For example, the work of Julien and Savinien in [78] estimates the Hausdorff dimension for a self-similar Cantor set with an ultrametric, and they derive estimates for its Lipshitz embedding dimension.

Given a finitely-generated group Γ and a C^r-action $\varphi \colon \Gamma \times N \to N$ on a compact manifold N of dimension q, the suspension of the action (see [21]) yields a C^r-foliation of codimension-q. In general, it is impossible to determine if such an action φ has an invariant Cantor set on which the action is minimal, except in very special cases. For example, consider a lattice subgroup $\Gamma \subset G$ of the rank one connected Lie group $G = SO(q,1)$. The boundary at infinity for the associated symmetric space $\mathbb{H}^q = SO(q,1)/O(q)$ is diffeomorphic to $\mathbb{S}^q$. If the group Γ is a non-uniform lattice, then the action of Γ on its limit set in $\mathbb{S}^q$ defines a minimal Cantor action, and the suspension of this action is a minimal matchbox manifold embedded in the smooth foliation associated to the action of Γ on $\mathbb{S}^q$.

The Williams solenoids were introduced in the papers [125, 126]. Williams proved that for an Axiom A diffeomorphism $f \colon M \to M$ of a compact manifold M with an expanding attractor $\Omega \subset M$, then Ω admits a stationary presentation, as defined in the next section, and so is homeomorphic to a generalized solenoid. The unstable manifolds for f restricted to an open neighborhood U of Ω form a $C^{0,\infty}$-foliation of U. That is, the foliation has C^0-pseudogroup maps, with smoothly embedded leaves, and Ω is the unique minimal set.

PROBLEM 5.3. *Does there exists a C^r-foliation of a compact manifold M with exceptional minimal set homeomorphic to a Williams solenoid?*

6. Solenoids

In this section, we introduce the notions of McCrod, weak and generalized solenoids, and recall some of their properties, including the construction of metrics on the transverse Cantor sets for which the action is Lipshitz. There are many open questions about when such examples can be realized as exceptional minimal sets for C^r-foliations.

The Denjoy minimal set $\mathfrak{M}$ for a C^1-flow on $\mathbb{T}^2$ has the remarkable property that it is homeomorphic to an inverse limit space defined by iterations of a map between the pointed wedge of two circles, $\mathbb{S}^1 \vee_{x_0} \mathbb{S}^1$, as explained in [31]. Also, the construction of the Hirsch examples in [62] (see also [15]), which yield real analytic foliations of codimension-one with exceptional minimal sets, the exceptional minimal set is homeomorphic to the inverse limit space defined by iterations of a 2-fold covering map of the 2-torus $\mathbb{T}^2$. For both of these classes of examples, their exceptional minimal sets are homeomorphic to generalized solenoids.

A *presentation* is a collection $\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\}$, where each M_ℓ is a connected compact simplicial complex of dimension n, and each "bonding map" $p_{\ell+1}$ is a proper surjective map of simplicial complexes with discrete fibers. For $\ell \geq 0$ and $x \in M_\ell$, the set $\{p_{\ell+1}^{-1}(x)\} \subset M_{\ell+1}$ is compact and discrete, so the cardinality $\#\{p_{\ell+1}^{-1}(x)\} < \infty$, though it need not be constant in either ℓ or x. For our applications, we assume that $\#\{p_{\ell+1}^{-1}(x)\} > 1$ for all $x \in M_\ell$.

Associated to $\mathcal{P}$ is the inverse limit space, called a *generalized solenoid*,

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \ \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \ \subset \prod_{\ell \geq 0} \ M_\ell, \tag{15}$$

where the set $\mathcal{S}_{\mathcal{P}}$ is given the product topology. By definition we have, for a sequence $\{x_\ell \in M_\ell\}$,

$$x = (x_0, x_1, \ldots) \in \mathcal{S}_{\mathcal{P}} \iff p_\ell(x_\ell) = x_{\ell-1} \text{ for all } \ell \geq 1 \ . \tag{16}$$

We say the presentation $\mathcal{P}$ is *stationary* if $M_\ell = M_0$ for all $\ell \geq 0$, and the bonding maps $p_\ell = p_1$ for all $\ell \geq 1$. A solenoid $\mathcal{S}_{\mathcal{P}}$ obtained from a stationary presentation $\mathcal{P}$ has a self-map σ defined by the shift, $\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$. The map σ can be considered as a type of expanding map on $\mathcal{S}_{\mathcal{P}}$, though in fact it may be expanding only in some directions, as discussed in Section 3 of [15].

For example, if $M_\ell = \mathbb{S}^1$ for each $\ell \geq 0$, and the map p_ℓ is a proper covering map of degree $m_\ell > 1$ for $\ell \geq 1$, then $\mathcal{S}_{\mathcal{P}}$ is a classical *Vietoris solenoid*, introduced in [122]. The exceptional minimal set for a Hirsch foliation is homeomorphic to the space $\mathcal{S}_{\mathcal{P}}$ defined by a presentation $\mathcal{P}$ which is defined by 2-fold covering maps of $\mathbb{T}^2$ to itself. Thus, this minimal set for the Hirsch foliation is homeomorphic to the product of the classical Vietoris solenoid with $\mathbb{S}^1$.

If M_ℓ is a compact manifold without boundary for each $\ell \geq 0$, and the map p_ℓ is a proper covering map of degree $m_\ell > 1$ for $\ell \geq 1$, then $\mathcal{S}_{\mathcal{P}}$ is said to be a *weak solenoid.* This generalization of the classical solenoids was originally considered in the papers by McCord [94] and Schori [112].

Associated to a presentation is a sequence of proper surjective maps

$$q_\ell = p_1 \circ \cdots \circ p_{\ell-1} \circ p_\ell \colon M_\ell \to M_0.$$

as well as a fibration map $\Pi_\ell \colon \mathcal{S}_{\mathcal{P}} \to M_\ell$ obtained by projection onto the ℓ-th factor. Then it follows from the definitions that $\Pi_0 = \Pi_\ell \circ q_\ell \colon \mathcal{S}_{\mathcal{P}} \to M_0$ for all $\ell \geq 1$. A choice of a basepoint $x \in \mathcal{S}_{\mathcal{P}}$ gives basepoints $x_\ell = \Pi_\ell(x) \in M_\ell$, and we define $\mathcal{H}^x_\ell = \pi_1(M_\ell, x_\ell)$. Let $\mathfrak{X}_x = \Pi_0^{-1}(x)$ denote the fiber of x, which is Cantor set by the assumption on the cardinality of the fibers of each map p_ℓ.

A weak solenoid $\mathcal{S}_{\mathcal{P}}$ is said to be a *normal* (or *McCord*) *solenoid* if the tower of coverings in the presentation is normal. That is, given a basepoint $x \in \mathcal{S}_{\mathcal{P}}$ then for each $\ell \geq 1$, the image $(q_\ell)_\# \colon \mathcal{H}^x_\ell \longrightarrow \mathcal{H}^x_0$ is a normal subgroup. Then each quotient $G^x_\ell = \mathcal{H}^x_0 / \mathcal{H}^x_\ell$ is finite group, and there are surjections $G^x_{\ell+1} \to G^x_\ell$. The fiber $\mathfrak{X}_x$ is then naturally identified with the inverse limit group

$$G^x_\infty = \varprojlim \ \{p_{\ell+1} \colon G^x_{\ell+1} \to G^x_\ell\} \ \subset \prod_{\ell \geq 0} \ G^x_\ell. \tag{17}$$

By assumption, all covering maps are proper, so have degree larger than 1, which implies that G^x_∞ with the product topology is a *Cantor group.* Moreover, the action of the fundamental group $\mathcal{H}^x_0$ on the fiber G^x_∞ is induced from the coordinate-wise multiplication on the product in (17). In the case of the Vietoris solenoid, where each map $p_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1$ is a double cover, the fiber G^x_∞ is the dyadic group. The fiber of a higher-dimensional McCord solenoid is a generalization of this example.

In general, for a weak solenoid $\mathcal{S}_{\mathcal{P}}$, the quotient $X_\ell = \mathcal{H}^x_0 / \mathcal{H}^x_\ell$ is a finite set with a left action of the fundamental group $\mathcal{H}^x_0$. Choose a metric d_ℓ on X_ℓ which is invariant under this action, and for definiteness assume that X_ℓ has diameter equal to 1. In the case of a McCord solenoid, then d_ℓ is unique. Choose a positive series $\{a_\ell \mid a_\ell > 0\}$ with total sum 1, then define a metric on $\mathfrak{X}_x$ by setting, for $u, v \in \mathfrak{X}_x$ so $u = (x_0, u_1, u_2, \ldots)$ and $v = (x_0, v_1, v_2, \ldots)$,

$$d_{\mathfrak{X}}(u, v) = a_1 d_1(u_1, v_1) + a_2 d_1(u_2, v_2) + \cdots \tag{18}$$

Observe that $d_{\mathfrak{X}}$ is invariant under the action of $\mathcal{H}^x_0$, so the holonomy for the fibration $\Pi_0 \colon \mathcal{S}_{\mathcal{P}} \to M_0$ acts by isometries. It may happen that we have two presentations $\mathcal{P}$ and $\mathcal{P}'$ over the same base manifold M_0 such that their inverse limits are homeomorphic as fibrations, $h \colon \mathcal{S}_{\mathcal{P}} \cong \mathcal{S}_{\mathcal{P}'}$. However, the map h need not be Lipshitz on fibers for the metrics associated to the presentations as above.

LEMMA 6.1. *Let $\mathcal{P}$ be a presentation of a weak solenoid $\mathcal{S}_{\mathcal{P}}$, choose a basepoint $x \in \mathcal{S}_{\mathcal{P}}$ and set $\mathfrak{X}_x = \Pi_0^{-1}(x)$, and recall that $\mathcal{H}_0^x = \pi_1(M_0, x_0)$. Then the left action of $\mathcal{H}_0^x$ on $\mathfrak{X}_x$ is minimal.*

Proof. The left action of $\mathcal{H}_0^x$ on each quotient space $X_\ell = \mathcal{H}_0^x / \mathcal{H}_\ell^x$ is transitive, so the orbits are dense in the product topology for $\mathfrak{X}_x$. □

Let $\widetilde{M}_0$ denote the universal covering of M_0. Associated to the action of $\mathcal{H}_0^x$ on $\mathfrak{X}_x$ is a suspension minimal matchbox manifold

$$\mathfrak{M} = \widetilde{M}_0 \times \mathfrak{X}_x / (y_0 \cdot g^{-1}, x) \sim (y_0, g \cdot x) \quad \text{for } y_0 \in \widetilde{M}_0,\ g \in \mathcal{H}_0^x \tag{19}$$

It is shown in [34] that:

PROPOSITION 6.2. *Let $\mathcal{S}_{\mathcal{P}}$ be a weak solenoid with base space M_0 which is a compact manifold of dimension $n \geq 1$. Then there is a foliated homeomorphism $\mathcal{S}_{\mathcal{P}} \cong \mathfrak{M}$, and so $\mathcal{S}_{\mathcal{P}}$ is an equicontinuous minimal matchbox manifold of dimension n.*

The McCord solenoids have a nice characterization among the matchbox manifolds. A continuum Ω is *homogeneous* if its group of homeomorphisms is point-transitive. It is also shown in [34] that:

THEOREM 6.3. *Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a McCord solenoid $\mathcal{S}_{\mathcal{P}}$ as foliated spaces.*

The McCord solenoids can be thought of as the transitive models in codimension-zero foliation theory, for the for *TP* spaces in a topological Molino's Theory. See [4] for a discussion of Molino's Theory in the context of equicontinuous foliations. Note that all leaves in a McCord solenoid are homeomorphic as the spaces are homogeneous. If the images of the maps $(q_\ell)_\# \colon \mathcal{H}_\ell^x \longrightarrow \mathcal{H}_0^x$ have trivial intersection, then all leaves of the foliation $\mathcal{F}_{\mathfrak{M}}$ for $\mathfrak{M} = \mathcal{S}_{\mathcal{P}}$ are isometric to the universal covering of the base manifold M_0. In the case of weak solenoids, the leaves of $\mathcal{F}_{\mathfrak{M}}$ need not be homeomorphic, and the work [32] gives examples where the leaves even have differing numbers of ends. There is no analog of this behavior in the context of Riemannian foliations on manifolds.

Now consider a matchbox manifold $\mathfrak{M}$ of dimension n, but whose pseudogroup is not necessarily equicontinuous. The holonomy pseudogroup also need not be Lipshitz for the given metric. The main result of [36] states:

THEOREM 6.4. *Let $\mathfrak{M}$ be a minimal matchbox manifold without holonomy. Then there exists a presentation $\mathcal{P}$ by simplicial maps between compact branched manifolds such that $\mathfrak{M}$ is homeomorphic to $\mathcal{S}_{\mathcal{P}}$ as foliated spaces.*

The exceptional minimal sets for the Denjoy and Pixton examples discussed in Section 5 have the property that all of their leaves are diffeomorphic to $\mathbb{R}^n$, and so are without holonomy. It follows from Theorem 6.4 each such minimal set admits a presentation, and by Proposition 4.3 the inverse limit $\mathcal{S}_{\mathcal{P}}$ inherits a transverse metric $d_{\mathfrak{X}}$ such that the holonomy maps are Lipshitz. Thus we have:

COROLLARY 6.5. *Let $\mathfrak{M}$ be an exceptional minimal set for a C^1-foliation $\mathcal{F}$ of a compact manifold M. If all leaves of $\mathcal{F}|\mathfrak{M}$ are simply connected, then there is a Lipshitz homeomorphism of $\mathfrak{M}$ with the inverse limit space $\mathcal{S}_{\mathcal{P}}$ defined by a presentation $\mathcal{P}$, given by simplicial maps between compact branched manifolds.*

In the case of the Denjoy and Pixton examples, the presentation $\mathcal{P}$ one obtains is stationary.

PROBLEM 6.6. *Let $\mathfrak{M}$ be an exceptional minimal set for a C^r-foliation $\mathcal{F}$ of a compact manifold M, where $r \geq 1$, and assume that $\mathfrak{M}$ is without holonomy. Find conditions on the holonomy pseudogroup $\mathcal{G}_{\mathfrak{M}}$ for $\mathcal{F}_{\mathfrak{M}}$ which are sufficient to imply that $\mathfrak{M}$ admits a stationary presentation.*

One approach to this problem, is to ask is the existence of approximations to the foliation $\mathcal{F}_{\mathfrak{M}}$ on $\mathfrak{M}$ by the compact branched manifolds $M_\ell = M_0$ in a stationary presentation $\mathcal{P}$, implies some form of "finiteness" for the holonomy maps of the pseudogroup $\mathcal{G}_{\mathfrak{M}}$. Such finiteness conditions may be derived, for example, from the induced action of the associated shift map σ on the direct limit of cohomology groups associated to the tower of maps in the presentation.

Theorem 6.4 is a generalization of a celebrated result by Anderson and Putnam in [5] for tiling spaces. Given a repetitive, aperiodic tiling of the Euclidean space $\mathbb{R}^n$ with finite local complexity, the associated tiling space Ω is defined as the closure of the set of translations by $\mathbb{R}^n$ of the given tiling, in an appropriate Gromov-Hausdorff topology on the space of tilings on $\mathbb{R}^n$. The space Ω is a matchbox manifold in our sense, whose leaves are defined by a free action of $\mathbb{R}^n$ on Ω (for example, see [102, 109, 111].) One of the remarkable results of the theory of tilings of $\mathbb{R}^n$ is the theorem of Anderson and Putnam and its extensions, that the tiling space Ω admits a presentation as an inverse limit of a tower of branched flat manifolds [5, 110, 111], where the branched manifolds are the union of finite collections of tiles. While Theorem 6.4 is a generalization of these results from tiling space theory, it is rarely the case that the branched manifolds in the presentation $\mathcal{P}$ for a general matchbox manifold have a nice description, or any description at all.

Other generalizations of the Anderson-Putnam theorem for more general cases have been given. For example, the work of Benedetti and Gambaudo in [14] discusses constructing towers for special classes of matchbox manifolds with holonomy, where the leaves are defined by a locally-free action of a connected Lie group G. Their work suggests what appears to be a difficult problem:

PROBLEM 6.7. *Let $\mathfrak{M}$ be a minimal matchbox manifold with leaves having non-trivial holonomy. Must $\mathfrak{M}$ be homeomorphic to an inverse limit $\mathcal{S}_{\mathcal{P}}$ for some presentation $\mathcal{P}$ where the spaces M_ℓ are branched manifolds?*

Note that a solution to this problem would yield a presentation for an exceptional minimal set in a C^2-foliation of codimension-one, which must always have leaves with holonomy. The existence of such a presentation would provide an alternate approach to the celebrated result of Duminy on the ends of leaves in exceptional minimal sets [27].

Theorem 5.8 in the paper [86] states a solution to Problem 6.7, though it seems that the claimed result conflicts with the results of [14] for a model of generalized tiling spaces defined by G-actions with non-trivial holonomy. Also, the results of Section 6 of the same paper conflict with other established results concerning weak solenoids, so a further understanding of Problem 6.7 is required.

PROBLEM 6.8. *Given a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with presentation $\mathcal{P}$ and associated transverse metric given by* (18)*, does there exists a Lipshitz embedding of $\mathcal{S}_{\mathcal{P}}$ as an exceptional minimal set for a C^r-foliation of a smooth manifold M.*

The problem is of interest whether M is assumed compact, or open without boundary, and for any $r \geq 1$. All known results are for the case where the base $M_0 = \mathbb{T}^n$ is a torus. The case of Vietoris solenoids where $M_0 = \mathbb{S}^1$ was proposed for study by Smale in [116], and smooth embeddings for 1-dimensional solenoids have been constructed for flow dynamical systems, as in the works by Gambaudo and Tresser [48], Gambaudo, Sullivan and Tresser [48], and Markus and Meyer [92].

The case when the base manifold $M_0 = \mathbb{T}^n$ for $n \geq 2$ was studied by the author with Clark in [33], where it was shown that for every presentation $\mathcal{P}$ there exists a refinement $\mathcal{P}'$ which can be realized in a C^r-foliation. That is, every topological type can be realized, though the metric induced on the inverse limit depends on the presentation $\mathcal{P}$.

All of the known examples of weak solenoids which embed as exceptional minimal sets for C^2-foliations have abelian fundamental group $\mathcal{H}_x$ and so are consequently McCord solenoids. It seems plausible, based on the proofs in [33], to suggest that if a weak solenoid admits an embedding in a C^2-foliation, then it must be a McCord solenoid with nilpotent covering groups, or possibly a stronger conclusion holds, that the covering groups must be abelian.

7. Fusion of Cantor minimal systems

There is a well-known method of "amalgamating" two foliations $\mathcal{F}_1, \mathcal{F}_2$ of the same codimension along compact transversals. We recall this method, and then introduce the analogue of this technique for minimal matchbox manifolds, to obtain their "fusion".

Assume there are given two foliations of codimension-one, say $\mathcal{F}_1$ and $\mathcal{F}_2$ on manifolds M_1 and M_2. We may assume that M_1 and M_2 have the same dimension, $n+1$. Otherwise, if say $\dim M_1 = m_1 > m_2 = \dim M_2$, then we replace M_2 with the product $M_2 \times \mathbb{S}^k$ where $k = m_1 - m_2$, and the leaves of $\mathcal{F}_2$ with their products with $\mathbb{S}^k$ as well. Then choose transversals $\iota_i \colon \mathbb{S}^1 \to M_i$ for $i = 1, 2$, which always exists for codimension-one foliations [21, 84]. For $\epsilon > 0$ sufficiently small, the ϵ-disk neighborhoods of the images of ι_i have induced foliations whose leaves are ϵ-disks. Identify the boundary of the disk bundle over the image of ι_1 with the same for ι_2, and denote the resulting surgered manifold by $M = M_1 \#_\iota M_2$. Then M has a foliation of codimension-one, and whose foliation pseudogroup is the amalgamation, or "pseudogroup free product", of the pseudogroups for $\mathcal{F}_1$ and $\mathcal{F}_2$. This very useful construction is used, for example, to construct the Reeb foliation on $\mathbb{S}^3$, among other applications [21]. For codimension $q > 1$, the existence of a compact manifold N and embeddings $\iota_i \colon N \to M_i$ transverse to the foliations is not guaranteed, and in general, one does not expect that two such transversals will exist.

We next consider the amalgamation of two minimal actions $\varphi_i \colon \Gamma_i \times \mathbf{K}_i \to \mathbf{K}_i$ for $i = 1, 2$ of finitely generated groups Γ_i on Cantor sets $\mathbf{K}_i$. Choose clopen subsets $V_i \subset \mathbf{K}_i$ and a homeomorphism $h \colon V_1 \to V_2$. Define the Cantor set $\mathbf{K} = \mathbf{K}_1 \#_h \mathbf{K}_2$ obtained from the disjoint union $\mathbf{K}_1 \cup \mathbf{K}_2$ by identifying the clopen subsets V_1 and V_2 using the map h. The action of each element $\gamma \in \Gamma_1$ on $\mathbf{K}$ is defined to act via $\varphi_1(\gamma)$ on $\mathbf{K}_1$, and acts as the identity on the complement $\mathbf{K}_2 - V_2$. Analogously, the action of φ_2 extends to an action of the elements of Γ_2 on $\mathbf{K}$. This produces an action φ of the free product $\Gamma_1 * \Gamma_2$ on $\mathbf{K}$. Note that if the actions φ_i are minimal, then the action of φ on $\mathbf{K}$ is also minimal.

If the action φ_i is realized as the holonomy of a suspension matchbox manifold $\mathfrak{M}_i$ as in (19), then the action of φ is realized as the holonomy of a surgered matchbox manifold $\mathfrak{M} = \mathfrak{M}_1 \#_h \mathfrak{M}_2$ constructed analogously to the method described above for codimension-one foliations. If the leaf dimensions do not agree, then this is corrected by multiplying by an appropriate factor of $\mathbb{S}^k$.

There are many variations which are possible in this construction. The holonomy pseudogroups of minimal matchbox manifolds $\mathfrak{M}_1$ and $\mathfrak{M}_2$ can be amalgamated, using the choices of transversal clopen sets $V_1 \subset \mathfrak{X}_1$ for $\mathfrak{M}_1$ and $V_2 \subset \mathfrak{X}_2$ for $\mathfrak{M}_2$, and the choice of a homeomorphism $h \colon V_1 \to V_2$. The resulting holonomy pseudogroup $\mathcal{G}_{\mathfrak{M}}$ for $\mathfrak{M}$ depends on the pseudogroups for the factors, but also depends on the choices of the clopen sets V_i and the homeomorphism h which identifies them. This construction is analogous to the construction of a new graph matchbox manifold, from two given graph matchbox manifolds, which was introduced Lukina in [88] as part of her study of the dynamics of examples using the Ghys-Kenyon construction. Lukina called this process "fusion", and we adopt the same notation for the process described here.

DEFINITION 7.1. *Let $\mathfrak{M}_i$ be minimal matchbox manifolds with transversals $\mathfrak{X}_i$ for $i = 1, 2$. Choose clopen subsets $V_i \subset \mathfrak{X}_i$ and a homeomorphism $h \colon V_1 \to V_2$. Then the minimal matchbox manifold $\mathfrak{M} = \mathfrak{M}_1 \#_h \mathfrak{M}_2$ is said to be the* fusion *of $\mathfrak{M}_1$ with $\mathfrak{M}_2$ over h.*

The concept of fusion for matchbox manifolds illustrates some of their fundamental differences with smooth foliations. A clopen transversal for a smooth foliation must be a compact submanifold without boundary, which does not always exist. Here is an interesting basic question:

PROBLEM 7.2. *How are the dynamical properties of a fusion $\mathfrak{M} = \mathfrak{M}_1 \#_h \mathfrak{M}_2$ related to the dynamical properties of the factors $\mathfrak{M}_1$ and $\mathfrak{M}_2$? In particular, describe the geometric structure of the leaves in $\mathfrak{M}$, in terms of the structure of the leaves of the factors $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and the fusion map $h \colon V_1 \to V_2$ between transversals. Show that the theory of hierarchies for the leaves of graph matchbox manifolds in Lukina* [88] *also apply for fusion in the context of matchbox manifolds.*

8. A non-embeddable example

In this section, we construct a minimal Cantor system defined by the action of a compactly-generated pseudogroup $\mathcal{G}_{\mathfrak{X}}$, such that there does not exists a metric on $\mathfrak{X}$ for which that action is Lipshitz. This pseudogroup can be realized as the holonomy of a minimal matchbox manifold $\mathfrak{M}$, and thus $\mathfrak{M}$ does not embed as an exceptional minimal set for any C^1-foliation. There are many variations on the construction, and thus there is a wide variety of non-embeddable minimal matchbox manifolds.

The first step in our construction is to choose a minimal action of a group on a Cantor set $\mathfrak{X}$, so that the action contains an expanding map. There are many possibilities for such an action, and as described below, we use the shift space model as it is one of the simplest such examples. Other examples are provided by the holonomy pseudogroups of tiling spaces defined by a substitution, which are well-studied in the tiling space literature. The existence of the expanding map forces a type of Lipshitz rigidity for the action, as shown in the proof of Lemma 8.2 below. We "fuse" to the standard shift action, a "hyper-contracting" map φ on a wandering domain in $\mathfrak{X}$, which violates the Lipshitz condition for the metric determined by the expanding map. The construction of the hyper-contraction φ uses in a fundamental way the property of the Cantor set, that any two clopen subsets of $\mathfrak{X}$ are homeomorphic.

We begin by constructing the model for the Cantor set $\mathfrak{X}$. Let $G_\ell = \mathbb{Z}/(2^\ell\,\mathbb{Z})$ be the cyclic group of order 2^ℓ, and let $p_{\ell+1}$ be the natural quotient map. Set:

$$\mathfrak{X} \;=\; \varprojlim\, \{p_{\ell+1}\colon G_{\ell+1} \to G_\ell\} \;\subset \prod_{\ell\geq 1}\, \mathbb{Z}/(2^\ell\,\mathbb{Z})\ . \tag{20}$$

Observe that $\mathfrak{X}$ is homeomorphic to the fiber over a point $\theta_0 \in \mathbb{S}^1$ of the Vietoris solenoid defined by the 2-times map of $\mathbb{S}^1$. The holonomy of this solenoid is given by an action $A\colon \mathbb{Z}\times\mathfrak{X}\to\mathfrak{X}$, where $\mathbb{Z}$ acts on each factor $\mathbb{Z}/(2^\ell\,\mathbb{Z})$ by translation. This A action of $\mathbb{Z}$ on $\mathfrak{X}$ is minimal.

Let $\sigma\colon \mathfrak{X}\to\mathfrak{X}$ be the shift map, given by $\sigma(x_1,\ldots) = (x_2, x_3, \ldots)$. Note that σ is a $2-1$ map, and so is not invertible. Since $G_1 = \{0,1\}$, we obtain a partition of $\mathfrak{X}$ into clopen subsets, for $i = 0, 1$,

$$U_1(i) = \{(i, x_2, x_3, \ldots) \mid 0 \leq x_j < 2^j\ ,\ p_{j+1}(x_{j+1}) = x_j\ ,\ j > 1\}.$$

Observe that $\mathrm{diam}_{\mathfrak{X}}(U_1(0)) = \mathrm{diam}_{\mathfrak{X}}(U_1(1)) = d_{\mathfrak{X}}(U_1(0), U_1(1)) = 1/3$.

The restriction $\sigma_i = \sigma|U_1(i)\colon U_1(i)\to\mathfrak{X}$ is $1-1$ and onto, with inverse map $\tau_i = \sigma_i^{-1}\colon \mathfrak{X}\to U_1(i)$ given by the usual formula for the section, $\tau_i(x_1,x_2,x_3,\ldots) = (i, x_1, x_2, x_3, \ldots)$.

The metric on $\mathfrak{X}$ is defined by, for $\overline{x} = (x_1, x_2, x_3, \ldots)$ and $\overline{y} = (y_1, y_2, y_3, \ldots)$, then

$$d_{\mathfrak{X}}\,(\overline{x}, \overline{y}) = \sum_{\ell=1}^{\infty}\ 3^{-\ell}\delta(x_\ell, y_\ell), \tag{21}$$

where $\delta(x_\ell, y_\ell) = 0$ if $x_\ell = y_\ell$, and is equal to 1 otherwise. Then $\mathrm{diam}_{\mathfrak{X}}(\mathfrak{X}) = 1/2$.

Note that A acts via isometries for this metric, while σ is a 3-times expanding map, where $d_{\mathfrak{X}}(\sigma(\overline{x}), \sigma(\overline{y})) = 3d_{\mathfrak{X}}(\overline{x}, \overline{y})$ for $\overline{x}, \overline{y}\in U_1(i)$. It follows that the maps τ_i are 1/3-times contracting maps.

For $\overline{x}\in\mathfrak{X}$, set $\overline{x}_\ell = (x_1, \ldots, x_\ell)$. Then for $\ell\geq 1$, define the clopen neighborhood of $\overline{x}$,

$$U_\ell(\overline{x}) = \{(x_1, \ldots, x_\ell, \xi_{\ell+1}, \xi_{\ell+2}, \ldots) \mid 0 \leq \xi_j < 2^j\ ,\ p_{j+1}(\xi_{j+1}) = \xi_j\ ,\ j > \ell\}. \tag{22}$$

Observe that for all such sets, the restriction $\sigma^\ell\colon U_\ell(\overline{x})\to\mathfrak{X}$ is $1-1$ and onto, and is 3^ℓ-expansive. In particular, $\mathrm{diam}_{\mathfrak{X}}(U_\ell(\overline{x})) = 3^{-\ell}/2$.

Next, we construct an action $\varphi\colon \mathbb{Z}\times\mathfrak{X}\to\mathfrak{X}$. Choose two distinct points $\overline{y}, \overline{z}\in\mathfrak{X}$, and choose a sequence $\{\overline{x}_k \mid -\infty < k < \infty\}\subset\mathfrak{X} - \{\overline{y}, \overline{z}\}$ of distinct points with $\lim\limits_{k\to\infty}\overline{x}_k = \overline{y}$ and $\lim\limits_{k\to-\infty}\overline{x}_k = \overline{z}$.

Now choose disjoint clopen neighborhoods $V_k\subset\mathfrak{X}$ of the points $\overline{x}_k$ recursively as follows. Let

$$\rho_0 = \inf\{d_{\mathfrak{X}}(\overline{x}_0, \overline{x}_j) \mid |j| > 0\}\ .$$

Note that this implies $d_{\mathfrak{X}}(\overline{x}_0, \overline{y}) \geq \rho_0$ and $d_{\mathfrak{X}}(\overline{x}_0, \overline{z}) \geq \rho_0$. Choose $\ell_0 > 0$ such that $3^{-\ell_0}/2 < \rho_0/3$. Let V_0 be the clopen set $V_0 = U_{\ell_0}(\overline{x}_0)$, and then $\mathrm{diam}_{\mathfrak{X}}(V_0) = 3^{-\ell_0}/2 < \rho_0/3$.

Assume that for $-k < i < k$, constants $\rho_i > 0$ and clopen sets V_i have been defined. Then set

$$\rho_k = \inf\{d_{\mathfrak{X}}(\overline{x}_i, \overline{x}_j) \mid -k \leq i \leq k \ , |j| > |k|\}.$$

For $k > 0$, choose $\ell_k > \ell_{k-1}$ such that $3^{-\ell_k}/2 < \rho_k/(3\,\ell_k!)$. Let V_k be the clopen set $V_k = U_{\ell_k}(\overline{x}_k)$, and $V_{-k} = U_{\ell_k}(\overline{x}_{-k})$. Note that $\mathrm{diam}_{\mathfrak{X}}(V_k) = \mathrm{diam}_{\mathfrak{X}}(V_{-k}) < \rho_k/(3\,\ell_k!)$.

Choose a homeomorphism $\varphi \colon \mathfrak{X} \to \mathfrak{X}$ such that for all $-\infty < k < \infty$, the restriction $\varphi_k \colon V_k \to V_{k+1}$ is a homeomorphism onto, which we may assume maps $\overline{x}_k$ to $\overline{x}_{k+1}$. The map φ is defined to be the identity on the complement of the union $V = \cup\{V_k \mid -\infty < k < \infty\}$. In particular, note that $\varphi(\overline{y}) = \overline{y}$ and $\varphi(\overline{z}) = \overline{z}$, so the map φ is continuous.

Let $\mathcal{G}_{\mathfrak{X}} = \langle A, \tau_1, \tau_2, \varphi\rangle$ be the pseudogroup generated by these maps.

PROPOSITION 8.1. *There does not exists a metric $d'_{\mathfrak{X}}$ on $\mathfrak{X}$ such that the generators $\{A, \tau_1, \tau_2, \varphi\}$ of $\mathcal{G}_{\mathfrak{X}}$ satisfy the Lipshitz condition* (11) *for any $C > 1$.*

Proof. Assume that such a metric $d'_{\mathfrak{X}}$ exists, with Lipshitz constant $C > 1$ for the the generators $\{A, \tau_1, \tau_2, , \varphi\}$. Let $d'_0 = \mathrm{diam}'_{\mathfrak{X}}(\mathfrak{X}) < \infty$.

The metric $d'_{\mathfrak{X}}$ defines the topology on $\mathfrak{X}$, so there exists some $\delta_0 > 0$ such that

$$B_{d'_{\mathfrak{X}}}(\overline{x}_0, \delta_0) = \{\xi \in \mathfrak{X} \mid d'_{\mathfrak{X}}(\overline{x}_0, \xi) < \delta_0\} \subset V_0.$$

The C-Lipshitz condition for φ implies that for each $k > 0$, we have $B_{d'_{\mathfrak{X}}}(\overline{x}_k, \delta_0/C^k) \subset \varphi^k(V_0)$.

We next use the Lipshitz condition to get an estimate on the rate of contraction for the maps τ_i with respect to the metric $d'_{\mathfrak{X}}$, formulated in terms of uniform open neighborhoods of the diagonal $\Delta \subset \mathfrak{X} \times \mathfrak{X}$. For $\epsilon > 0$, define

$$\mathcal{U}(\epsilon) = \{(\overline{x}, \overline{y}) \in \mathfrak{X} \times \mathfrak{X} \mid d_{\mathfrak{X}}(\overline{x}, \overline{y}) < \epsilon\} \quad , \quad \mathcal{U}'(\epsilon) = \{(\overline{x}, \overline{y}) \in \mathfrak{X} \times \mathfrak{X} \mid d'_{\mathfrak{X}}(\overline{x}, \overline{y}) < \epsilon\} \tag{23}$$

LEMMA 8.2. *There exists an integer $b > 0$ and $r_0 > 0$ so that for all $m \geq 1$ and $\xi \in \mathfrak{X}$ and $\zeta = \sigma^{(mb+1)}(\xi)$, we have the inclusions*

$$B_{d_{\mathfrak{X}}}(\xi, 3^{-(mb+1)}/4)) \subset \sigma^{-(mb+1)}(B_{d'_{\mathfrak{X}}}(\zeta, r_0)) \subset B_{d'_{\mathfrak{X}}}(\zeta, r_0/C^{(m-1)b+1}). \tag{24}$$

Proof. By the definition of the metric $d_{\mathfrak{X}}$ so we have

$$\mathcal{U}(1/4) \subset \{U_1(0) \times U_1(0)\} \cup \{U_1(1) \times U_1(1)\}\,.$$

Note that for all $\epsilon > 0$, we have $(\sigma \times \sigma)^{-1}(\mathcal{U}(\epsilon)) = \mathcal{U}(\epsilon/3)$. On the other hand, the Lipshitz condition for σ with respect to the metric $d'_{\mathfrak{X}}$ implies $\mathcal{U}'(\epsilon/C) \subset (\sigma \times \sigma)^{-1}(\mathcal{U}'(\epsilon))$.

The continuity of the metric $d'_{\mathfrak{X}}$ with respect to $d_{\mathfrak{X}}$ and compactness of $\mathfrak{X}$ imply there exists $r_0 > 0$ such that $\mathcal{U}'(r_0) \subset \mathcal{U}(1/4)$. Thus we have,

$$\mathcal{U}'(r_0/C) \subset (\sigma \times \sigma)^{-1}(\mathcal{U}'(r_0)) \subset (\sigma \times \sigma)^{-1}(\mathcal{U}(1/4)) = \mathcal{U}(3^{-1}/4)\ . \tag{25}$$

By the continuity of the metric $d'_{\mathfrak{X}}$, there exists an integer $b > 0$ such that $\mathcal{U}(3^{-(b+1)}/4) \subset \mathcal{U}'(r_0/C)$, so that we have

$$\mathcal{U}(3^{-(b+1)}/4) \subset \mathcal{U}'(r_0/C) \subset (\sigma \times \sigma)^{-1}(\mathcal{U}'(r_0)) \subset (\sigma \times \sigma)^{-1}(\mathcal{U}(1/4)) = \mathcal{U}(3^{-1}/4)\ . \tag{26}$$

Apply $(\sigma \times \sigma)^{-b}$ to the terms in (26) to obtain

$$\mathcal{U}(3^{-(2b+1)}\epsilon/4) \subset \mathcal{U}'(r_0/C^{b+1}) \subset (\sigma \times \sigma)^{-(b+1)}(\mathcal{U}'(r_0)) \subset \mathcal{U}(3^{-(b+1)}/4)\ . \tag{27}$$

Continuing in this manner, applying (26) recursively, we obtain for $m \geq 1$,

$$\mathcal{U}(3^{-(mb+1)}/4) \subset \mathcal{U}'(r_0/C^{mb+1}) \subset (\sigma \times \sigma)^{-(mb+1)}(\mathcal{U}'(r_0)) \subset \mathcal{U}(3^{-((m-1)b+1)}/4) \subset \mathcal{U}'(r_0/C^{(m-1)b+1})$$

Then take ball-slices through the diagonal in $\mathfrak{X} \times \mathfrak{X}$ and let $\xi \in \mathfrak{X}$ with $\zeta = \sigma^{(mb+1)}(\xi)$ then the inclusions in (24) follow. □

The proof of Proposition 8.1 now follows. Now for each $k > 0$, let m_k be the least integer so that $m_k b > b + k - 1 + \ln(r_0/\delta_0)/\ln(C)$. Then we have $r_0/C^{(m_k-1)b+1} < \delta_0/C^k$. Recall that

$$B_{d'_{\mathfrak{X}}}(\overline{x}_k, \delta_0/C^k) \subset \varphi^k(V_0) = V_k = U_{\ell_k}(\overline{x}_k). \tag{28}$$

Combine (28) with (24) of Lemma 8.2 for the points $\xi = \overline{x}_k$ and $\zeta = \sigma^{(m_k b+1)}(\overline{x}_k)$ to obtain:

$$B_{d_{\mathfrak{X}}}(\xi, 3^{-(m_k b+1)}/4) \subset \sigma^{-(m_k b+1)}(B_{d'_{\mathfrak{X}}}(\zeta, r_0)) \subset B_{d'_{\mathfrak{X}}}(\xi, r_0/C^{(m_k-1)b+1}) \subset B_{d'_{\mathfrak{X}}}(\xi, \delta_0/C^k). \tag{29}$$

Apply the map $\sigma^{(m_k b+1)}$ to (29) to obtain

$$B_{d_{\mathfrak{X}}}(\zeta, 1/4) = \sigma^{(m_k b+1)}(B_{d_{\mathfrak{X}}}(\xi, 3^{-(m_k b+1)}/4)) \subset \sigma^{(m_k b+1)}(B_{d'_{\mathfrak{X}}}(\xi, \delta_0/C^k)). \tag{30}$$

Also, apply the map $\sigma^{(m_k b+1)}$ to (28) to obtain

$$\sigma^{(m_k b+1)}(B_{d'_{\mathfrak{X}}}(\xi, \delta_0/C^k)) \subset \sigma^{(m_k b+1)}(\varphi^k(V_0)) = \sigma^{(m_k b+1)}(U_{\ell_k}(\xi)) = U_{\ell_k-(m_k b+1)}(\zeta). \tag{31}$$

Combine (30) and (31) to obtain

$$B_{d_{\mathfrak{X}}}(\zeta, 1/4) \subset \sigma^{(m_k b+1)}(B_{d'_{\mathfrak{X}}}(\xi, \delta_0/C^k)) \subset U_{\ell_k-(m_k b+1)}(\zeta). \tag{32}$$

However, $\lim_{k\to\infty} \{\ell_k - (m_k b + 1)\} = \infty$ by the choice of the indices ℓ_k, so this is impossible. □

REMARK 8.3. Note that in the proof of Proposition 8.1, two key properties were used. One is the existence of the uniformly expanding map $\sigma \colon \mathfrak{X} \to \mathfrak{X}$. The proof of Lemma 8.2 will also work for a map σ which does not have constant scale factor, which is 3 in the case of the shift map σ above for the particular choice of metric $d_{\mathfrak{X}}$, though it will involve more tedious estimates. The other ingredient required, is a hypercontraction map φ which violates the "local Lipshitz symmetry" of the metric near the fixed point for the map.

Thus, one can augment any Lipshitz pseudogroup action on a Cantor set which contains a uniform contraction, by adding one or more maps which are hypercontractions, to obtain a pseudogroup which admits no metric for which the amalgamated action is Lipshitz. The role of the generator A is simply to make the action minimal. Thus, there are many examples possible of minimal Cantor actions which do not admit a metric for which the action is Lipshitz.

There is a geometric interpretation of the result of Proposition 8.1. The pseudogroup $\mathcal{G}_{\mathfrak{X}}$ can be realized as the holonomy pseudogroup of a matchbox manifold $\mathfrak{M}$ with surface leaves, using a modified suspension construction as described by Lozano-Rojo and Lukina in [87]. The leaves of $\mathcal{F}_{\mathfrak{M}}$ are modeled by the Cayley graphs [85] of the orbits of $\mathcal{G}_{\mathfrak{X}}$. The conclusion of Proposition 8.1 can be viewed as the transversal consequence of a lack of homogeneity for the leaves of $\mathcal{F}_{\mathfrak{M}}$. We explain this further.

Let $\Gamma = \langle A, \tau_1, \tau_2 \rangle$ denote the pseudogroup generated by the shift and translation maps acting on $\mathfrak{X}$ as in (20). The suspension construction of [87] yields a matchbox manifold $\mathfrak{N}$ with foliation $\mathcal{F}_{\mathfrak{N}}$. The leaves of $\mathcal{F}_{\mathfrak{N}}$ are modeled by the Cayley graphs of the orbits of Γ, each of which is a copy of the usual rooted binary tree, trivalent except at the root point.

The addition of the hypercontraction φ to the generating set of Γ yields the pseudogroup $\mathcal{G}_{\mathfrak{X}}$, which can also be realized using the construction in [87] to obtain a matchbox manifold $\mathfrak{M}$. The addition of this generator can be interpreted as gluing copies of the Cayley graphs of leaves of $\mathcal{F}_{\mathfrak{N}}$ to themselves, along an edge which reaches further and further towards infinity on the graphs. This is a version of the fusion process described more generally by Lukina in [89]. The choice of the sets V_ℓ above implies these attached copies of the tree to itself are spaced increasingly further out along the branches of the tree, which destroys the homogeneity of the Cayley graphs of the orbits Γ. Correspondingly, the leaves of $\mathcal{F}_{\mathfrak{M}}$ are obtained by attaching cylinder sets to the leaves of $\mathcal{F}_{\mathfrak{N}}$ at increasing distances.

The work of Lukina [89] studies the Hausdorff dimensions of graph manifolds, and various examples are calculated. It would be interesting to know whether the Hausdorff dimension of $\mathcal{G}_{\mathfrak{X}}$ with respect to the natural metric $d_{\mathfrak{X}}$ is infinite, which would give another interpretation of the non-embedding of $\mathfrak{M}$ into C^1-foliations, as such an imbedding would imply that the Hausdorff dimension is finite.

We conclude this section with another remark, and a question. Recall that Problem 1.5 asks for obstructions to the existence of an embedding $\iota\colon \mathfrak{M} \to M$ of a Lipshitz matchbox manifold as an exceptional minimal set for a C^1-foliation $\mathcal{F}$ on M. Such an embedding implies in particular that the transverse Cantor set $\mathfrak{X}$ admits a Lipshitz embedding into the Euclidean space $\mathbb{R}^q$. The question of when a metric space admits a Lipshitz embedding in $\mathbb{R}^q$ dates from the 1928 paper [18], and is certainly well-studied. For example, the doubling property of Assouad [6], and the weakening of this condition by Olson and Robinson [98], prove embedding criteria for metrics. These are types of "asymptotic small-scale homogeneity" properties of the metric $d_{\mathfrak{X}}$, which suggests an alternate approach to the Lipshitz embedding problem for minimal pseudogroups.

PROBLEM 8.4. *Let $\mathfrak{X}$ be a Cantor space with metric $d_{\mathfrak{X}}$. Let $\mathcal{G}_{\mathfrak{M}}$ be a compactly-generated pseudogroup acting minimally on $\mathfrak{X}$, and which is Lipshitz with respect to $d_{\mathfrak{X}}$. Does this imply that the metric $d_{\mathfrak{X}}$ satisfies some version of the doubling condition? Or, is it possible to construct compactly-generated pseudogroup actions on Cantor sets for which no Lipshitz metric is doubling?*

For a Cantor set $\mathfrak{X}$ with an ultrametric $d_{\mathfrak{X}}$, the Lipshitz embedding problem for $(\mathfrak{X}, d_{\mathfrak{X}})$ has been solved for various special cases. The work of Julien and Savinien in [78] estimates the Hausdorff dimension for a self-similar Cantor set with an ultrametric, and they derive estimates for its Lipshitz embedding dimension. The embedding properties of ultrametrics on Cantor sets which are the boundary of a hyperbolic group are discussed by Buyalo and Schroeder in [20, Chapter 8].

Finally, recall that every Cantor set embeds homeomorphically to a Cantor set in $\mathbb{R}^2$, and any two such are homeomorphic by a homeomorphism of $\mathbb{R}^2$ restricted to the set. This classical fact, due to Brouwer, is proved in detail by Moise in Chapter 12 of [96]. It has been used to construct topological embeddings of solenoids in codimension-two foliations, as in the work of Clark and Fokkink [30].

On the other hand, the tameness property of Cantor sets in $\mathbb{R}^2$ does not hold for all Cantor sets embedded in $\mathbb{R}^3$. The *Antoine's Necklace* is the classical example of this, as discussed in Chapter 18 of [96], and in Section 4.6 of [63]. It seems natural to ask the naive question:

PROBLEM 8.5. *Let $\mathfrak{A}$ denote the Antoine Cantor set embedded in $\mathbb{R}^3$, with the metric $d_{\mathfrak{A}}$ on $\mathfrak{A}$ induced by the restriction of the Euclidean metric. Does there is some exceptional minimal set for a C^1-foliation of codimension thrtee, whose transverse model space is Lipshitz equivalent to $(\mathfrak{A}, d_{\mathfrak{A}})$?*

9. Classification of Lipshitz solenoids

In this section, we define *Morita equivalence* and *Lipshitz equivalence* of minimal pseudogroups, and consider the problem of Lipshitz classification for the special case of McCord solenoids. While the condition of Morita equivalence is well-known and studied, Lipshitz equivalence seems less commonly studied, except possibly for group and semi-group actions on their boundaries.

Let $\mathcal{G}_{\mathfrak{X}}$ be a minimal pseudogroup acting on a Cantor space $\mathfrak{X}$, and let $V \subset \mathfrak{X}$ be a clopen subset. The induced pseudogroup $\mathcal{G}_{\mathfrak{X}}|V$ is defined as the restrictions of all maps in $\mathcal{G}_{\mathfrak{X}}$ with domain and range in V. The following is then the adaptation of the notion of Morita equivalence of groupoids, as in Haefliger [55], to the context of minimal Cantor actions.

DEFINITION 9.1. *Let $\mathcal{G}_{\mathfrak{X}}$ be a minimal pseudogroup action on the Cantor set $\mathfrak{X}$ via Lipshitz homeomorphisms with respect to the metric $d_{\mathfrak{X}}$. Likewise, let $\mathcal{G}_{\mathfrak{Y}}$ be a minimal pseudogroup action on the Cantor set $\mathfrak{Y}$ via Lipshitz homeomorphisms with respect to the metric $d_{\mathfrak{Y}}$. Then*

1. *$(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}, d_{\mathfrak{X}})$ is* Morita equivalent *to $(\mathcal{G}_{\mathfrak{Y}}, \mathfrak{Y}, d_{\mathfrak{Y}})$ if there exist clopen subsets $V \subset \mathfrak{X}$ and $W \subset \mathfrak{Y}$, and a homeomorphism $h\colon V \to W$ which conjugates $\mathcal{G}_{\mathfrak{X}}|V$ to $\mathcal{G}_{\mathfrak{Y}}|W$.*
2. *$(\mathcal{G}_{\mathfrak{X}}, \mathfrak{X}, d_{\mathfrak{X}})$ is* Lipshitz equivalent *to $(\mathcal{G}_{\mathfrak{Y}}, \mathfrak{Y}, d_{\mathfrak{Y}})$ if the conjugation h is Lipshitz.*

Morita equivalence is sometimes called *return equivalence* in the literature [2, 46, 37].

Morita equivalence is a basic notion for the study of C^*-algebra invariants for foliation groupoids, as discussed by Renault [105] and Connes [39]. Lipshitz equivalence is a basic notion for the study of *metric non-commutative geometry.*

The strongest results for classification, up to Morita equivalence, have been obtained for 1-dimensional minimal matchbox manifolds. Fokkink showed in his thesis [46] (see also Barge and Williams [9]) that if f_1, f_2 are C^1-actions on $\mathbb{S}^1$ which admit a Cantor minimal set, then the induced minimal Cantor actions are Morita equivalent if and only if they have rotation numbers which are conjugate under the linear fractional action of $SL(2,\mathbb{Z})$ on $\mathbb{R}$. This implies there are uncountably many non-homeomorphic minimal matchbox manifolds which embed as minimal sets for C^1-foliations of $\mathbb{T}^2$. There is a higher-dimensional version of this result for torus-like matchbox manifolds, proved in [37]. See the papers [10, 12, 12] for the classification of 1-dimensional minimal matchbox manifolds embedded in compact surfaces.

In general, the classification problem modulo orbit equivalence, is unsolvable for the pseudogroups associated to minimal matchbox manifolds of dimension $n \geq 2$, as this is the case already for the McCord solenoids with base manifold $\mathbb{T}^n$ where $n \geq 2$. See [64, 80, 119, 120] for discussions of the undecidability of the Borel classification problem up to orbit equivalence.

The advantage of considering Lipshitz equivalence of groupoid actions, is that while the equivalence is more refined, it can also be more practical to determine when two actions are not Lipshitz equivalent. We discuss the difference between Morita and Lipshitz classification in the case of the weak solenoids, where there are a well-known criteria for Morita equivalence.

First, we recall the criteria for when two weak solenoids are homeomorphic, as given in [34, Section 9]. Assume that we are given two presentations, where all spaces $\{M_\ell \mid \ell \geq 0\}$ and $\{N_\ell \mid \ell \geq 0\}$ are compact oriented manifolds, and all bonding maps are orientation-preserving coverings,

$$\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\} \quad , \quad \mathcal{Q} = \{q_{\ell+1} \colon N_{\ell+1} \to N_\ell \mid \ell \geq 0\} \tag{33}$$

which define weak solenoids $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{Q}$ as in (15), respectively. Choose basepoints $\overline{x} \in \mathcal{S}_\mathcal{P}$ and $\overline{y} \in \mathcal{S}_\mathcal{Q}$. We consider the special case where $M_0 = N_0$, as the more general case easily reduces to this one, and the key issues are more evident in this special case. Let $\Pi_\ell^\mathcal{P} \colon \mathcal{S}_\mathcal{P} \to M_\ell$ denote the fibration map onto the factor M_ℓ for $\mathcal{S}_\mathcal{P}$, and $\Pi_\ell^\mathcal{Q} \colon \mathcal{S}_\mathcal{Q} \to N_\ell$ that for $\mathcal{S}_\mathcal{Q}$.

We can assume that $x_0 = y_0$ in M_0, where $x_0 = \Pi_0^\mathcal{P}(\overline{x})$ and $y_0 = \Pi_0^\mathcal{Q}(\overline{y})$, then set $\mathcal{H}_0 = \pi_1(M_0, x_0)$, where we suppress the dependence on basepoints. Define the subgroups $\mathcal{H}_\ell \subset \mathcal{H}_0$ which are the images of the groups $\pi_1(M_\ell, x_\ell)$ under the maps $(q_\ell)_\#$ associated to $\mathcal{P}$, and let $\mathcal{G}_\ell \subset \mathcal{H}_0$ be the corresponding images of the groups $\pi_1(N_\ell, y_\ell)$. Then we obtain two nested sequences of subgroups

$$\begin{array}{ccccccccc}
\subset & \mathcal{H}_{\ell+1} & \subset & \mathcal{H}_\ell & \subset & \cdots & \subset & \mathcal{H}_1 & \subset & \mathcal{H}_0 \\
 & & & & & & & & & \| \\
\subset & \mathcal{G}_{\ell+1} & \subset & \mathcal{G}_\ell & \subset & \cdots & \subset & \mathcal{G}_1 & \subset & \mathcal{G}_0
\end{array}$$

The proof of the following result can be found in the papers [47, 94, 106, 112].

THEOREM 9.2. *The weak solenoids $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{Q}$ are basepoint homeomorphic if and only if there exists $\ell_0 \geq 0$ and $\nu_0 \geq 0$, such that for every $\ell \geq \ell_0$ there exists $\nu_\ell \geq \nu_0$ with $\mathcal{G}_{\nu_\ell} \subset \mathcal{H}_\ell$, and for every $\nu \geq \nu_0$ there exists $\ell_\nu \geq \ell_0$ with $\mathcal{H}_{\ell_\nu} \subset \mathcal{G}_\nu$.*

The condition on bonding maps in Theorem 9.2 is called *tower equivalence* of the subgroup chains.

Let $\mathfrak{X}$ denote the fiber of $\Pi_0^\mathcal{P}$ over $\overline{x}$, and $\mathfrak{Y}$ the fiber of $\Pi_0^\mathcal{Q}$ over $\overline{y}$. Then the monodromy of the fibration $\Pi_0^\mathcal{P}$ defines the actions of $\mathcal{H}_0$ on $\mathfrak{X}$, and the action of $\mathcal{H}_0 = \mathcal{G}_0$ on $\mathfrak{Y}$ is defined by the monodromy of $\Pi_0^\mathcal{Q}$. Then results of Clark, Lukina and the author yield:

THEOREM 9.3 ([34]). *If the weak solenoids $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{Q}$ are basepoint homeomorphic, with $M_0 = N_0$, then the holonomy actions of $\mathcal{H}_0$ on $\mathfrak{X}$ and on $\mathfrak{Y}$ are Morita equivalent.*

THEOREM 9.4 ([37]). *If the weak solenoids $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{Q}$ have base manifold $M_0 = N_0 = \mathbb{T}^n$, and the holonomy actions of $\mathcal{H}_0$ on $\mathfrak{X}$ and on $\mathfrak{Y}$ are Morita equivalent, then $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_\mathcal{Q}$ are basepoint homeomorphic.*

It follows that the classification problem for matchbox manifolds which are homeomorphic to a McCord solenoid with base $\mathbb{T}^n$, reduces to the study of the Morita equivalence class of their holonomy pseudogroups, which by Theorem 9.2 reduces to a problem concerning the tower equivalence of subgroup chains in $\mathbb{Z}^n$. This equivalence problem is undecidable for $n \geq 2$.

In the case of classical Vietoris solenoids, where $M_0 = \mathbb{S}^1$ and $\mathcal{H}_0 = \mathbb{Z}$, the classification is much more straightforward. For each $\ell > 0$ there exists integers $m_\ell > 1$ and $n_\ell > 1$, defined recursively, so that $\mathcal{H}_\ell = \langle m_1 m_2 \cdots m_\ell \rangle \subset \mathbb{Z}$, and $\mathcal{G}_\ell = \langle n_1 n_2 \cdots n_\ell \rangle \subset \mathbb{Z}$. Let P be the set of all prime factors of the integers $\{m_\ell \mid \ell > 0\}$, included with multiplicity, and let Q be the same for the integers $\{n_\ell \mid \ell > 0\}$. For example, for the dyadic solenoid, the set $P = \{2, 2, 2, \ldots\}$ is an infinite collection of copies of the prime 2. These infinite sets of primes P and Q are ordered by the sequence in which they appear in the factorizations of the covering degrees m_ℓ and n_ℓ.

If the two sets P and Q are in *bijective* correspondence, then it is an exercise to show that the tower equivalence condition of Theorem 9.2 is satisfied for the presentations $\mathcal{P}$ and $\mathcal{Q}$, which yields the classification of Vietoris solenoids up to homeomorphism by Baer [8], and also the classification up to Morita equivalence of the associated minimal $\mathbb{Z}$-actions on the Cantor set fibers.

However, for the metrics on the Cantor sections $\mathfrak{X} \subset \mathfrak{M} = \mathcal{S}_{\mathcal{P}}$ and $\mathfrak{Y} \subset \mathfrak{N} = \mathcal{S}_{\mathcal{P}}$ as defined by the formula in (21), it is evident that if the bijection $\sigma \colon P \leftrightarrow Q$ permutes the elements by increasing large degrees, with respect to their ordering, then the induced map between the fibers, $h_\sigma \colon \mathfrak{X} \cong \mathfrak{Y}$, will not be Lipshitz. This motivates introducing the following invariant of a tower of equivalences.

Let $\mathcal{P}$ and $\mathcal{Q}$ be presentations with common base manifold M_0, and suppose there exists a tower equivalence between them. That is, there exists $\ell_0 \geq 0$ and $\nu_0 \geq 0$, such that for every $\ell \geq \ell_0$ there exists $\nu_\ell \geq \nu_0$ with $\mathcal{G}_{\nu_\ell} \subset \mathcal{H}_\ell$, and for every $\nu \geq \nu_0$ there exists $\ell_\nu \geq \ell_0$ with $\mathcal{H}_{\ell_\nu} \subset \mathcal{G}_\nu$. Define the *displacement* of these indexing functions $\ell \mapsto \nu_\ell$ and $\nu \mapsto \ell_\nu$ to be

$$\mathrm{Disp}(\ell_\nu, \nu_\ell) = \max\left\{\sup\left\{|\ell_\nu - \nu| \ \mid \nu \geq \nu_0\right\} \ , \ \sup\left\{|\nu_\ell - \ell| \ \mid \ell \geq \ell_0\right\}\right\} \tag{34}$$

If $\mathrm{Disp}(\ell_\nu, \nu_\ell) < \infty$, then we say that $\mathcal{P}$ and $\mathcal{Q}$ are *bounded tower equivalent*.

THEOREM 9.5. *Let $\mathcal{P}$ and $\mathcal{Q}$ be presentations with common base manifold M_0, and suppose there exists a tower equivalence between them, defined by maps $\ell \mapsto \nu_\ell$ and $\nu \mapsto \ell_\nu$. Let the fiber metrics be defined by the formula* (18) *with $a_\ell = 3^{-\ell}$. Then the action of $\mathcal{H}_0$ on the fiber $\mathfrak{X}$ of $\Pi_0^{\mathcal{P}}$ is Lipshitz equivalent to the action of $\mathcal{H}_0$ on the fiber $\mathfrak{Y}$ of $\Pi_0^{\mathcal{Q}}$ if and only if $\mathcal{P}$ and $\mathcal{Q}$ are bounded tower equivalent.*

The proof that $\mathrm{Disp}(\ell_\nu, \nu_\ell) < \infty$ implies Lipshitz equivalence for the metrics defined by (18) with $a_\ell = 3^{-\ell}$ is an exercise in the definitions, using the expression (17) for the metric on the fibers. The converse direction, that Lipshitz equivalence implies bounded tower equivalence, follows from a careful consideration of metrics used in the proof of the main theorem in [34], and will be presented elsewhere.

We give a simple example of Theorem 9.5, in the case of Vietoris solenoids. With the notation as above, suppose the the covering degrees m_ℓ for the presentation $\mathcal{P}$ with base $M_0 = \mathbb{S}^1$ are given by $m_\ell = 2$ for ℓ odd, and $m_\ell = 3$ for ℓ even. Let the covering degrees for the presentation $\mathcal{Q}$ be given by the sequence $\{n_1, n_2, n_3, \ldots\} = \{2, 3, 2, 2, 3, 2, 2, 2, 2, 3, \ldots\}$. In general, the ℓ-th cover of degree 3 is followed by 2^ℓ covers of degree 2. Then these two sequences are clearly tower equivalent, but their displacement is infinite. It follows that the matchbox manifolds $\mathfrak{M} = \mathcal{S}_{\mathcal{P}}$ and $\mathfrak{N} = \mathcal{S}_{\mathcal{P}}$ are homeomorphic, but are not Lipshitz equivalent.

We conclude with some remarks about the Lipshitz classification problem for solenoids.

Note that Theorem 9.5 applies to all equicontinuous matchbox manifolds, as [34, Theorem 1.4] implies that such spaces are always homeomorphic to a weak solenoid.

If the weights a_ℓ in formula (17) for the fiber metrics are said to have *bounded ratios*, if there exists $0 < \lambda_1 < \lambda_2$ such that $\lambda_1 < |a_{\ell+1}/a_\ell| < \lambda_2$ for all $\ell \geq 1$. Theorem 9.5 holds in the generality of fiber metrics with bounded ratios, the proof of which is a straightforward though more tedious exercise. If the weights do not have bounded ratios, then the conclusion of the theorem is no longer valid.

The work of Julien and Savinien in [78] considers the relation between the boundedness of the weights in the metric (17), and the Hausdorff dimension of the resulting metric, as applied to the transversal to a substitution tiling space. This is closely related to the constructions given above.

Finally, if $\mathfrak{M}$ is a minimal matchbox manifold without holonomy, then Theorem 6.4 implies that $\mathfrak{M}$ is foliated homeomorphic to an inverse limit space $\mathcal{S}_{\mathcal{P}}$ where $\mathcal{P}$ is a presentation by bonding maps between branched n-manifolds.

PROBLEM 9.6. *Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are minimal Lipshitz matchbox manifolds without holonomy, and having leaves of the same dimension. If their holonomy pseudogroups $\mathcal{G}_{\mathfrak{M}}$ and $\mathcal{G}_{\mathfrak{N}}$ are Lipshitz equivalent, what can be said about the relation between presentations for $\mathfrak{M}$ and $\mathfrak{N}$?*

The proof of Theorem 6.4 in [36] does not suggest an immediate relationship, unlike the case for the equicontinuous matchbox manifolds as for Theorem 9.5.

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