WILD SOLENOIDS

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Dedicated to the memory of James T. Rogers, Jr.

Abstract. A weak solenoid is a foliated space defined as the inverse limit of finite coverings of a closed compact manifold M. The monodromy of a weak solenoid defines an equicontinuous minimal action on a Cantor space X by the fundamental group G of M. The discriminant group of this action is an obstruction to this action being homogeneous. The discriminant vanishes if the group G is abelian, but there are examples of actions of nilpotent groups for which the discriminant is non-trivial. The action is said to be stable if the discriminant group remains unchanged for the induced action on sufficiently small clopen neighborhoods in X. If the discriminant group never stabilizes as the diameter of the clopen set U tends to zero, then we say that the action is unstable, and the weak solenoid which defines it is said to be wild. In this work, we show two main results in the course of our study of the properties of the discriminant group for Cantor actions. First, the tail equivalence class of the sequence of discriminant groups obtained for the restricted action on a neighborhood basis system of a point in X defines an invariant of the return equivalence class of the action, called the asymptotic discriminant, which is consequently an invariant of the homeomorphism class of the weak solenoid. Second, we construct uncountable collections of wild solenoids with pairwise distinct asymptotic discriminant invariants for a fixed base manifold M, and hence fixed finitely-presented group G, which are thus pairwise non-homeomorphic. The study in this work is the continuation of the seminal works on homeomorphisms of weak solenoids by Rogers and Tollefson in 1971, and is dedicated to the memory of Jim Rogers.

Version date: February 10, 2017; revised June 7, 2017.

2010 *Mathematics Subject Classification.* Primary: 20E18, 37B45, 57R30; Secondary: 37B05, 57R30, 58H05.

Keywords: weak solenoids, equicontinuous foliations, classification, asymptotic discriminant, Molino theory, group chains, minimal Cantor actions, profinite groups.

1. Introduction

A matchbox manifold is a compact, connected metrizable space $\mathfrak{M}$, that is, a continuum, equipped with a decomposition $\mathcal{F}_{\mathfrak{M}}$ into leaves of constant dimension, so that the pair $(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}})$ is a foliated space in the sense of [47], for which the local transversals to the foliation are totally disconnected. In particular, the leaves of $\mathcal{F}_{\mathfrak{M}}$ are the path connected components of $\mathfrak{M}$. A matchbox manifold with 2-dimensional leaves is a lamination by surfaces in the sense of Ghys [29] and Lyubich and Minsky [40], while Sullivan called them "solenoidal spaces" in [56, 58]. The terminology "matchbox manifold" follows the usage introduced in [2, 3, 4].

A weak solenoid $\mathcal{S}_{\mathcal{P}}$, as introduced by McCord in [43], is the inverse limit space of a sequence of proper covering maps

$$\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\},$$

where M_ℓ is a compact connected manifold without boundary, and $p_{\ell+1}$ is a covering map of finite degree greater than one. The collection $\mathcal{P}$ is called a *presentation* for $\mathcal{S}_{\mathcal{P}}$. A solenoid $\mathcal{S}_{\mathcal{P}}$ is *regular* if the presentation $\mathcal{P}$ can be chosen so that for each $\ell \geq 1$, the composition $p_1 \circ \cdots \circ p_\ell \colon M_\ell \to M_0$ is a regular covering map. A weak solenoid which is not regular is said to be *irregular*, and the purpose of this work is to study the classification problem for irregular solenoids. McCord observed in [43] that a weak solenoid admits a covering by open sets which are homeomorphic to the product of a disk by a Cantor set, hence is a matchbox manifold.

The space $\mathfrak{M}$ is said to be *foliated homogeneous* if given any pair of points $x, y \in \mathfrak{M}$, then there exists a foliated homeomorphism $h \colon \mathfrak{M} \to \mathfrak{M}$ such that $h(x) = y$. A simple observation about matchbox manifolds is that a homeomorphism $\varphi \colon \mathfrak{M} \to \mathfrak{M}'$ preserves the path-connected components, hence a homeomorphism of a matchbox manifolds preserves their foliations. It follows that if $\mathfrak{M}$ is homogeneous, then it is foliated homogeneous.

The following results were obtained in [57] for the 1-dimensional case, and in [16] for matchbox manifolds with leaves of dimension 2 and higher.

THEOREM 1.1. *Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then:*

(1) The holonomy pseudogroup of $\mathcal{F}_{\mathfrak{M}}$ is equicontinuous;
(2) $\mathcal{F}_{\mathfrak{M}}$ is minimal, that is, every leaf of $\mathcal{F}_{\mathfrak{M}}$ is dense in $\mathfrak{M}$;
(3) $\mathfrak{M}$ is homeomorphic to a regular solenoid.

THEOREM 1.2. *Let $\mathfrak{M}$ be a matchbox manifold. Then the holonomy pseudogroup of $\mathcal{F}_{\mathfrak{M}}$ is equicontinuous if and only if $\mathfrak{M}$ is homeomorphic to a weak solenoid $\mathcal{S}_{\mathcal{P}}$ for some presentation $\mathcal{P}$.*

The construction of the homeomorphism between $\mathfrak{M}$ and a weak solenoid $\mathcal{S}_{\mathcal{P}}$ depends on the choice of a clopen transversal to $\mathcal{F}_{\mathfrak{M}}$. Given two such choices of clopen transversals to $\mathcal{F}_{\mathfrak{M}}$, the resulting weak solenoids are related by return equivalence [18], as discussed in Section 2.4. More generally, homeomorphic matchbox manifolds have return equivalent holonomy pseudo$\star$groups by Theorem 2.14.

Thus, in the study of the classification of matchbox manifolds up to homeomorphism, a fundamental problem is to develop invariants for the return equivalence classes of the minimal group actions on Cantor spaces, that are associated to weak solenoids. In Section 5, we define the *asymptotic discriminant* for weak solenoids, and show that it is an invariant of the homeomorphism type for equicontinuous matchbox manifolds.

The results of this work are also related to the following concept:

DEFINITION 1.3. *A* Molino space *for $(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}})$ is a homogeneous matchbox manifold $(\widehat{\mathfrak{M}}, \widehat{\mathcal{F}_{\mathfrak{M}}})$ of the same leaf dimension as $\mathfrak{M}$, and a foliated principal $\mathbf{H}$-fibration*

$$\mathbf{H} \longrightarrow \widehat{\mathfrak{M}} \xrightarrow{\widehat{q}} \mathfrak{M} \ , \tag{1}$$

where $\mathbf{H}$ is a compact topological group, $\widehat{\mathfrak{M}}$ is foliated homogeneous, and for each leaf $\widehat{L} \subset \widehat{\mathfrak{M}}$ there is a leaf $L \subset \mathfrak{M}$ such that the restriction $\widehat{q}|\widehat{L} \colon \widehat{L} \to L$ is a covering map. The fibration (1) *is called a* Molino sequence *for the foliation $\mathcal{F}_{\mathfrak{M}}$.*

Molino showed in [45, 46] (see also [34, 44]) that a Riemannian foliation $\mathcal{F}_M$ of a smooth compact manifold M always admits a Molino sequence. In this case, the fiber group $\mathbf{H}$ is a connected compact Lie group whose Lie algebra is isomorphic to the Lie algebra of transverse projectable vector fields for $\mathcal{F}_M$. Thus, $\mathbf{H}$ is locally defined, up to local isomorphism, by the transverse geometry to $\mathcal{F}_M$.

In a series of papers, [5, 6, 7, 8], Álvarez López and his coauthors formulated a *topological Molino theory* for equicontinuous foliated spaces, which is a partial generalization of the Molino theory for smooth Riemannian foliations. They formulated in [7, Definition 2.18] the notion of *strongly quasi-analytic regularity* for the holonomy pseudo$\star$group of a foliated space, or the *SQA condition*, which is discussed further in Section 4. The topological Molino theory developed in [7] applies to foliated spaces which satisfy the SQA condition.

A Molino space for a matchbox manifold $\mathfrak{M}$ is a homogeneous matchbox manifold $\widehat{\mathfrak{M}}$, which is an equicontinuous foliated space by Theorem 1.1. The quotient foliated space of an equicontinuous matchbox manifold is equicontinuous, so by Theorems 1.1 and 1.2 we have:

COROLLARY 1.4. *Let $\mathfrak{M}$ be a matchbox manifold which admits a Molino space $\widehat{\mathfrak{M}}$ with Molino sequence* (1)*. Then $\mathfrak{M}$ is homeomorphic to a weak solenoid.*

Thus, the study of Molino spaces and their properties for matchbox manifolds reduces to the study of weak solenoids. A converse to Corollary 1.4 was obtained by the authors in joint work with Dyer.

THEOREM 1.5. [22, Theorem 1.2] *Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, and let $\mathcal{P}$ be a presentation with homeomorphism $\mathfrak{M} \to \mathcal{S}_{\mathcal{P}}$ to a weak solenoid $\mathcal{S}_{\mathcal{P}}$. Then there exists a regular solenoid $\widehat{\mathcal{S}}_{\mathcal{P}}$, a compact totally disconnected group $\mathcal{D}_{\mathcal{P}}$, and a Molino sequence*

$$\mathcal{D}_{\mathcal{P}} \longrightarrow \widehat{\mathcal{S}}_{\mathcal{P}} \xrightarrow{\widehat{q}} \mathcal{S}_{\mathcal{P}} \ , \qquad (2)$$

where the spaces $\widehat{\mathcal{S}}_{\mathcal{P}}$ and $\mathcal{D}_{\mathcal{P}}$ depend on the choice of the presentation $\mathcal{P}$.

One motivation for the work in this paper is to investigate how the space $\widehat{\mathcal{S}}_{\mathcal{P}}$ depends on the choice of the presentation $\mathcal{P}$. In particular, [22] describes examples where the cardinality of the fibre group $\mathcal{D}_{\mathcal{P}}$ varies with the choice of the presentation $\mathcal{P}$. So it is important to understand when the resulting Molino sequence (2) is well-defined, up to topological conjugacy of fibrations. This is equivalent to finding conditions such that the homeomorphism type of the fiber group $\mathcal{D}_{\mathcal{P}}$ is well-defined, that is, it is independent of the choice of the presentation for $\mathcal{S}_{\mathcal{P}}$ and hence is an invariant of the homeomorphism type of $\mathfrak{M}$.

DEFINITION 1.6. *An equicontinuous matchbox manifold $\mathfrak{M}$ is said to be* stable *if its associated Molino sequence* (2) *is well-defined, up to topological conjugacy of fibrations, and otherwise is said to be* wild.

The construction of a Molino sequence for a weak solenoid $\mathcal{S}_{\mathcal{P}}$ was given in [22], using the methods of group chains, which we recall in Section 7. Moreover, a criteria in terms of group chains was given in that work for when $\mathcal{S}_{\mathcal{P}}$ is stable. The work [22] also proved stability for a few specific classes of weak solenoids. Remarkably, it turned out that the holonomy pseudo$\star$groups of solenoids in these classes satisfy the SQA condition. The first main theorem of this paper shows that there is a direct relationship between the SQA property and the stability of Molino sequence, as defined in [22]. In Definition 4.5 we introduce the LCSQA (*Locally Completely Strongly Quasi-Analytic*) property which is a localized form of the SQA property for the closure of the pseudogroup.

THEOREM 1.7. *An equicontinuous matchbox manifold $\mathfrak{M}$ is stable if and only if its holonomy* pseudo$\star$group *satisfies the LCSQA property.*

Proposition 4.7 gives the proof of this result, and Proposition 4.10 shows the intuitively clear result that the LCSQA property is an invariant of return equivalence for pseudogroup actions. We also prove in Proposition 4.9 a technical result which gives the relation between the notion of stable and the definition of the SQA condition for pseudogroups in [7, Definition 2.18].

The *Schori solenoid* was constructed in [54] as one of the first examples of non-homogeneous solenoids. The Schori solenoid is the inverse limit of 3-to-1 coverings of the genus 2 surface. It was proved in [22] that the holonomy pseudogroup of this solenoid is not LCSQA, and thus we have:

COROLLARY 1.8. *The Schori solenoid is wild.*

Another result of this paper is the introduction of an invariant which allows us to distinguish between various homeomorphism classes of wild solenoids. The *asymptotic discriminant* is defined in Section 5, where it is shown to be an invariant under return equivalence. The results of this section combine to show the following:

THEOREM 1.9. *The asymptotic discriminant of an equicontinuous matchbox manifold $\mathfrak{M}$ is well-defined, and defines an invariant of the homeomorphism class of $\mathfrak{M}$.*

Thus, the natural question is to ask whether this invariant is effective, and how does one construct examples which are distinguished by the asymptotic discriminant? The answer to this question occupies the remainder of the paper, beginning in Sections 6 and 7. Section 8 then gives a method for constructing weak solenoids which are wild and have prescribed asymptotic discriminant invariants. We apply this method in Section 9 to show the following:

THEOREM 1.10. *For $n \geq 3$, let $G \subset \mathbf{SL}(n,\mathbb{Z})$ be a torsion-free subgroup of finite index, which is thus finitely-presented. Then there exists uncountably many distinct homeomorphism types of weak solenoids which are wild, all with the same compact base manifold M_0 having fundamental group G.*

It is interesting to compare this result with some previous results about the classification problem for homogeneous solenoids. The classification of 1-dimensional solenoids up to homeomorphism was completed by Bing [12] and McCord [43], who showed that the homeomorphism type of the inverse limit $\mathcal{S}_{\mathcal{P}}$ of a 1-dimensional presentation $\mathcal{P} = \{p_{\ell+1} \colon \mathbb{S}^1 \to \mathbb{S}^1 \mid \ell \geq 0\}$ is determined by the collection of the covering degrees $\mathcal{D}_k = \{d_\ell = \deg\{p_\ell\} \geq 2 \mid \ell \geq k\}$ for k large. This classification was reformulated in terms of return equivalence for the solenoid $\mathcal{S}_{\mathcal{P}}$ by Aarts and Fokkink [1].

For a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with leaf dimension $n \geq 2$, the authors showed in joint work with Clark:

THEOREM 1.11. [18, Theorem 1.4] *Let $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}'_{\mathcal{P}'}$ be solenoids, both with base manifold the n-torus, $M_0 = \mathbb{T}^n$. Then $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}'_{\mathcal{P}'}$ are homeomorphic if and only if their global monodromy actions $(\mathfrak{X}, G, \Phi)$ and $(\mathfrak{X}', G', \Phi')$ are return equivalent, where we have $G \cong G' \cong \mathbb{Z}^n$.*

In general, for a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with leaf dimension $n \geq 2$, for which the fundamental group $G_0 = \pi_1(M_0, x_0)$ of the base manifold M_0 of the presentation $\mathcal{P}$ is not abelian, the classification problem is far from being well-understood. By Theorem 5.14 and Proposition 7.6, the asymptotic discriminant associated to an equicontinuous matchbox manifold provides a new invariant which can be used to show that a pair, or even an infinite family of solenoids, are non-homeomorphic.

We make an observation about the embedding property for weak solenoids. It follows from the definitions that if a weak solenoid $\mathcal{S}_{\mathcal{P}}$ is homeomorphic to a minimal set for a C^r-foliation $\mathcal{F}$ of a manifold, in the sense of the works [15, 37], then in the real analytic case where $r = \omega$, the holonomy pseudo$\star$group of $\mathcal{S}_{\mathcal{P}}$ satisfies the LCSQA condition. We thus obtain a non-embedding result:

COROLLARY 1.12. *The wild solenoids $\mathcal{S}_{\mathcal{P}}$ constructed in the proofs of Theorem 1.10 are not homeomorphic to the minimal sets of a real-analytic foliation on a manifold.*

A solution to the following problem would represent an extension of the ideas in [28]:

CONJECTURE 1.13. *A wild solenoid $\mathcal{S}_{\mathcal{P}}$ is not homeomorphic to the minimal set of any C^2-foliation of a finite dimensional manifold.*

Section 10 gives with some open problems about the existence of unstable actions and wild solenoids.

The authors would like to thank the anonymous referee for various comments which improved the exposition and results of this work.

2. Foliated spaces and return equivalence

In this section, we briefly recall the definitions of matchbox manifolds, which are a special class of foliated spaces as defined in [47, 14]. The construction of the pseudo⋆group associated to a matchbox manifold is recalled, and Theorem 2.14 states that a homeomorphism between minimal matchbox manifolds induces a return equivalence of their holonomy pseudo⋆groups.

2.1. **Matchbox manifolds.** Recall that a *continuum* is a compact connected metrizable space.

DEFINITION 2.1. *A* matchbox manifold of dimension n *is a continuum* $\mathfrak{M}$*, such that there exists a compact, separable, totally disconnected metric space* $\mathfrak{X}$*, and for each* $x \in \mathfrak{M}$ *there is a compact subset* $\mathfrak{T}_x \subset \mathfrak{X}$*, an open subset* $U_x \subset \mathfrak{M}$*, and a homeomorphism* $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ *defined on the closure* $\overline{U}_x$ *in* $\mathfrak{M}$*, such that* $\varphi_x(x) = (0, w_x)$ *where* $w_x \in int(\mathfrak{T}_x)$*. Moreover, it is assumed that each* φ_x *admits an extension to a homeomorphism* $\widehat{\varphi}_x \colon \widehat{U}_x \to (-2,2)^n \times \mathfrak{T}_x$ *where* $\widehat{U}_x \subset \mathfrak{M}$ *is an open subset such that* $\overline{U}_x \subset \widehat{U}_x$*. The space* $\mathfrak{T}_x$ *is called the* local transverse model *at* x*.*

The assumption that the transversals $\mathfrak{T}_x$ are totally disconnected implies that the local charts φ_x satisfy the compatibility axioms of foliation charts for a foliated space, as in [16, Section 2].

Let $\pi_x \colon \overline{U}_x \to \mathfrak{T}_x$ denote the composition of φ_x with projection onto the second factor.

For $w \in \mathfrak{T}_x$ the set $\mathcal{P}_x(w) = \pi_x^{-1}(w) \subset \overline{U}_x$ is called a *plaque* for the coordinate chart φ_x. We adopt the notation, for $z \in \overline{U}_x$, that $\mathcal{P}_x(z) = \mathcal{P}_x(\pi_x(z))$, so that $z \in \mathcal{P}_x(z)$. Note that each plaque $\mathcal{P}_x(w)$ is given the topology so that the restriction $\varphi_x \colon \mathcal{P}_x(w) \to [-1,1]^n \times \{w\}$ is a homeomorphism. Then $int(\mathcal{P}_x(w)) = \varphi_x^{-1}((-1,1)^n \times \{w\})$. Let $U_x = int(\overline{U}_x) = \varphi_x^{-1}((-1,1)^n \times int(\mathfrak{T}_x))$.

Note that if $z \in U_x \cap U_y$, then $int(\mathcal{P}_x(z)) \cap int(\mathcal{P}_y(z))$ is an open subset of both $\mathcal{P}_x(z)$ and $\mathcal{P}_y(z)$. The collection

$$\mathcal{V} = \{\varphi_x^{-1}(V \times \{w\}) \mid x \in \mathfrak{M} \ , \ w \in \mathfrak{T}_x \ , \ V \subset (-1,1)^n \text{ open}\}$$

forms the basis for the *fine topology* of $\mathfrak{M}$. The connected components of the fine topology are called leaves, and define the foliation $\mathcal{F}_{\mathfrak{M}}$ of $\mathfrak{M}$. Thus, the leaves of $\mathcal{F}_{\mathfrak{M}}$ are the path-connected components of $\mathfrak{M}$. Let $L_x \subset \mathfrak{M}$ denote the leaf of $\mathcal{F}_{\mathfrak{M}}$ containing $x \in \mathfrak{M}$.

A map $h \colon \mathfrak{M} \to \mathfrak{M}'$ is said to be a *foliated map* if the image of each leaf of $\mathcal{F}_{\mathfrak{M}}$ is contained in a leaf of $\mathcal{F}_{\mathfrak{M}'}$. As each leaf of $\mathcal{F}_{\mathfrak{M}}$ is path connected, so its image under a continuous map h is path connected, and thus the image of a leaf of $\mathcal{F}_{\mathfrak{M}}$ *must* be contained in a leaf of $\mathcal{F}_{\mathfrak{M}'}$. Thus we have:

LEMMA 2.2. *Let* $\mathfrak{M}$ *and* $\mathfrak{M}'$ *be matchbox manifolds, and* $h \colon \mathfrak{M} \to \mathfrak{M}'$ *a continuous map. Then* h *maps the leaves of* $\mathcal{F}_{\mathfrak{M}}$ *into leaves of* $\mathcal{F}_{\mathfrak{M}'}$*. In particular, any homeomorphism* $h \colon \mathfrak{M} \to \mathfrak{M}'$ *of matchbox manifolds is a foliated map.*

A matchbox manifold $\mathfrak{M}$ is said to be *smooth*, if the local charts $\varphi_x \colon \overline{U}_x \to [-1,1]^n \times \mathfrak{T}_x$ can be chosen so that for each leaf $L \subset \mathfrak{M}$ of $\mathcal{F}_{\mathfrak{M}}$, the restrictions of the charts to L defines a smooth manifold structure on L. That is, for all $x, y \in \mathfrak{M}$ with $z \in U_x \cap U_y$, there exists an open set $z \in V_z \subset U_x \cap U_y$ such that $\mathcal{P}_x(z) \cap V_z$ and $\mathcal{P}_y(z) \cap V_z$ are connected open sets, and the composition

$$\psi_{x,y;z} \equiv \varphi_y \circ \varphi_x^{-1} \colon \varphi_x(\mathcal{P}_x(z) \cap V_z) \to \varphi_y(\mathcal{P}_y(z) \cap V_z)$$

is a smooth map, where $\varphi_x(\mathcal{P}_x(z) \cap V_z) \subset \mathbb{R}^n \times \{w\} \cong \mathbb{R}^n$ and $\varphi_y(\mathcal{P}_y(z) \cap V_z) \subset \mathbb{R}^n \times \{w'\} \cong \mathbb{R}^n$. The leafwise transition maps $\psi_{x,y;z}$ are assumed to depend continuously on z in the C^∞-topology.

We recall a standard result, whose proof for foliated spaces can be found in [14, Theorem 11.4.3].

THEOREM 2.3. *Let* $\mathfrak{M}$ *be a smooth matchbox manifold. Then there exists a leafwise Riemannian metric for* $\mathcal{F}_{\mathfrak{M}}$*, such that for each* $x \in \mathfrak{M}$*, the leaf* L_x *inherits the structure of a complete Riemannian manifold with bounded geometry, and the Riemannian metric and its covariant derivatives depend continuously on* x *.*

All matchbox manifolds are assumed to be smooth with a fixed choice of metric $d_{\mathfrak{M}}$ on $\mathfrak{M}$, and a choice of a compatible leafwise Riemannian metric $d_{\mathcal{F}}$.

2.2. **Holonomy pseudogroup.** We next describe the construction of the holonomy pseudo$\star$group associated to a matchbox manifold, and the holonomy pseudogroup that it generates. We only give details sufficient for defining the notion of return equivalence in Section 2.4 below, and refer to the text [14] and the works [16, 17, 18] for details.

The first step is to choose a *regular foliated covering* of $\mathfrak{M}$, $\mathcal{U} \equiv \{\varphi_i : U_i \to (-1,1)^n \times \mathfrak{T}_i \mid 1 \leq i \leq \nu\}$ where $\mathcal{U} = \{U_1, \ldots, U_\nu\}$ is an open covering of $\mathfrak{M}$, and each chart $\varphi_i : U_i \to (-1,1)^n \times \mathfrak{T}_i$ satisfies the conditions on charts in Definition 2.1, and also satisfies the regularity conditions detailed in Proposition 2.6 of [17], which in particular includes the assumption that the plaques $\mathcal{P}_x(z)$ associated to the foliation charts are convex in the leafwise metric.

We also assume that the transverse model spaces $\mathfrak{T}_i$ form a disjoint clopen covering of $\mathfrak{X}$, so that $\mathfrak{X} = \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_\nu$ is a disjoint union. For each $1 \leq i \leq \nu$, the set $\mathcal{T}_i = \varphi_i^{-1}(\{0\} \times \mathfrak{T}_i)$ is a compact transversal to $\mathcal{F}_{\mathfrak{M}}$. Without loss of generality, we can assume that the transversals $\{\mathcal{T}_1, \ldots, \mathcal{T}_\nu\}$ are pairwise disjoint in $\mathfrak{M}$, and let $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_\nu \subset \mathfrak{M}$ denote their disjoint union. Define the maps $\tau_i : \mathfrak{T}_i \to \overline{U}_i$ given by $\tau_i(\xi) = \varphi_i^{-1}(0, \xi)$, and let $\tau : \mathfrak{X} \to \mathcal{T}$ denote the union of the maps τ_i. Each transversal $\mathcal{T}_i$ has a metric $d_{\mathcal{T}_i}$ obtained by the restriction of $d_{\mathfrak{M}}$, and we define

$$d_{\mathfrak{T}_i}(x, y) = d_{\mathfrak{M}}(\tau_i(x), \tau_i(y)) \text{ if } x, y \in \mathfrak{T}_i, \text{ for } 1 \leq i \leq \nu.$$

We may assume that $d_{\mathfrak{M}}$ is rescaled so that $\mathrm{diam}(\mathfrak{M}) < 1$. The metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$ is defined by setting

$$d_{\mathfrak{X}}(x, y) = d_{\mathfrak{T}_i}(x, y) \text{ if } x, y \in \mathfrak{T}_i \quad , \quad d_{\mathfrak{X}}(x, y) = 1 \text{ otherwise .}$$

A *leafwise path* is a continuous map $\gamma : [0,1] \to \mathfrak{M}$ such that there is a leaf L of $\mathcal{F}_{\mathfrak{M}}$ for which $\gamma(t) \in L$ for all $0 \leq t \leq 1$. Since $\mathfrak{M}$ is a matchbox manifold, and $\gamma : [0,1] \to \mathfrak{M}$ is continuous, then γ is a leafwise path by Lemma 2.2. The holonomy pseudogroup for a matchbox manifold $(\mathfrak{M}, \mathcal{F}_{\mathfrak{M}})$ is defined analogously to the construction of holonomy for foliated manifolds, and associates to a leafwise path with endpoints in the transversal space $\mathcal{T}$ a local homeomorphism of $\mathfrak{X}$. In this way, the construction generalizes the induced dynamical systems associated to a section of a flow.

For each $1 \leq i \leq \nu$, let $\pi_i : \overline{U}_i \to \mathfrak{T}_i$ be composition of the coordinate chart φ_i with the projection onto the second factor $\mathfrak{T}_i$. A pair of indices (i, j), $1 \leq i, j \leq \nu$, is said to be *admissible* if the *open* coordinate charts satisfy $U_i \cap U_j \neq \emptyset$. For (i, j) admissible, define clopen subsets $\mathfrak{D}_{i,j} = \pi_i(U_i \cap U_j) \subset \mathfrak{T}_i \subset \mathfrak{X}$. The convexity of foliation charts imply that plaques are either disjoint, or have connected intersection. This implies that there is a well-defined homeomorphism $h_{j,i} : \mathfrak{D}_{i,j} \to \mathfrak{D}_{j,i}$ with domain $D(h_{j,i}) = \mathfrak{D}_{i,j} \subset \mathfrak{T}_i$ which is a non-empty clopen subset, and range $R(h_{j,i}) = \mathfrak{D}_{j,i} \subset \mathfrak{T}_j$.

The maps $\mathcal{G}_{\mathcal{F}}^{(1)} = \{h_{j,i} \mid (i,j) \text{ admissible}\}$ are the transverse change of coordinates defined by the foliation charts. By definition, $h_{i,i}$ is the identity map on $\mathfrak{T}_i$, $h_{i,j}^{-1} = h_{j,i}$, and if $U_i \cap U_j \cap U_k \neq \emptyset$ then $h_{k,j} \circ h_{j,i} = h_{k,i}$ on their common domain of definition.

A sequence $\mathcal{I} = (i_0, i_1, \ldots, i_\alpha)$ is *admissible*, if each pair $(i_{\ell-1}, i_\ell)$ is admissible for $1 \leq \ell \leq \alpha$, and the composition

$$h_{\mathcal{I}} = h_{i_\alpha, i_{\alpha-1}} \circ \cdots \circ h_{i_1, i_0} \tag{3}$$

has non-empty domain. The domain $\mathfrak{D}_{\mathcal{I}}$ of $h_{\mathcal{I}}$ is the *maximal clopen subset* of $\mathfrak{D}_{i_0, i_1} \subset \mathfrak{T}_{i_0}$ for which the compositions are defined, and the range is $\mathfrak{R}(h_{\mathcal{I}}) \subset \mathfrak{D}_{i_\alpha, i_{\alpha-1}}$.

Given any open subset $U \subset \mathfrak{D}_{\mathcal{I}}$ define $h_{\mathcal{I}}|U \in \mathcal{G}_{\mathcal{F}}$ by restriction to U. Then the collection of maps

$$\mathcal{G}_{\mathcal{F}}^* = \{h_{\mathcal{I}}|U \mid \mathcal{I} \text{ admissible and } U \subset \mathfrak{D}(h_{\mathcal{I}})\} \subset \mathcal{G} \ . \tag{4}$$

is called the pseudo$\star$group associated to $\mathcal{G}_{\mathcal{F}}^{(1)}$ in the literature [36, 42]. The collection $\mathcal{G}_{\mathcal{F}}^*$ is closed under the operations of compositions, taking inverses, and restrictions to open sets.

The *holonomy pseudogroup* $\mathcal{G}_{\mathcal{F}}$ associated to the regular foliated covering $\mathcal{U}$ is the topological pseudogroup acting on $\mathfrak{X}$ generated by the elements of $\mathcal{G}_{\mathcal{F}}^{(1)}$. Note that $\mathcal{G}_{\mathcal{F}}^*$ contains the collection $\mathcal{G}_{\mathcal{F}}^{(1)}$, hence $\mathcal{G}_{\mathcal{F}}$ can also be considered as the pseudogroup generated by the elements of $\mathcal{G}_{\mathcal{F}}^*$.

The generating set $\mathcal{G}_{\mathcal{F}}^{(1)}$ is finite by construction, so $\mathcal{G}_{\mathcal{F}}$ is a finitely-generated pseudogroup as in [35].

Let $W \subset \mathfrak{X}$ be an open subset, then the *restriction* of $\mathcal{G}^*_{\mathcal{F}}$ to W is defined by:

$$\mathcal{G}^*_W = \{g \in \mathcal{G}^*_{\mathcal{F}} \mid \mathfrak{D}(g) \subset W \ , \ \mathfrak{R}(g) \subset W\} . \tag{5}$$

The pseudo$\star$group $\mathcal{G}^*_W$ is also referred to as the *induced* pseudo$\star$group on W.

Introduce the filtrations of $\mathcal{G}^*_{\mathcal{F}}$ by word length. For $\alpha \geq 1$, let $\mathcal{G}^{(\alpha)}_{\mathcal{F}}$ be the collection of holonomy homeomorphisms $h_{\mathcal{I}}|U \in \mathcal{G}^*_{\mathcal{F}}$ determined by admissible paths $\mathcal{I} = (i_0, \ldots, i_k)$ such that $k \leq \alpha$ and $U \subset \mathfrak{D}(h_{\mathcal{I}})$ is open. Then for each $g \in \mathcal{G}^*_{\mathcal{F}}$ there is some α such that $g \in \mathcal{G}^{(\alpha)}_{\mathcal{F}}$. Let $\|g\|$ denote the least such α, which is called the *word length* of g.

We have the following finiteness result for minimal pseudogroup actions on compact spaces, whose counterpart for group actions is a well-known folklore result:

LEMMA 2.4. [17, Lemma 4.1] *Let $U \subset \mathfrak{X}$ be an open subset. Then there exists an integer α_U such that $\mathfrak{X}$ is covered by the collection $\{h_{\mathcal{I}}(U) \mid h_{\mathcal{I}} \in \mathcal{G}^{(\alpha_U)}_{\mathcal{F}}\}$, where $h_{\mathcal{I}}(U) = h_{\mathcal{I}}(U \cap \mathfrak{D}(h_{\mathcal{I}}))$.*

2.3. **Pseudogroup dynamics.** The $\mathcal{G}_{\mathcal{F}}$-orbit of a point $w \in \mathfrak{X}$ is denoted by

$$\mathcal{O}(w) \equiv \{g(w) \mid g \in \mathcal{G}^*_{\mathcal{F}} \text{ such that } w \in \mathfrak{D}(g)\} = \{g(w) \mid g \in \mathcal{G}_{\mathcal{F}} \text{ such that } w \in \mathfrak{D}(g)\} . \tag{6}$$

The action of $\mathcal{G}_{\mathcal{F}}$ is *minimal* if for each $w \in \mathfrak{X}$ the $\mathcal{G}_{\mathcal{F}}$-orbit $\mathcal{O}(w)$ is dense in $\mathfrak{X}$.

Next, recall the definition of an *equicontinuous* pseudo$\star$group. Let $d_{\mathfrak{X}}$ denote the metric on $\mathfrak{X}$.

DEFINITION 2.5. *The* pseudo$\star$group $\mathcal{G}^*_{\mathcal{F}}$ *acting on* $\mathfrak{X}$ *is* equicontinuous *if for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $g \in \mathcal{G}^*_{\mathcal{F}}$, if $w, w' \in \mathfrak{D}(g)$ and $d_{\mathfrak{X}}(w, w') < \delta$, then $d_{\mathfrak{X}}(g(w), g(w')) < \epsilon$. That is, $\mathcal{G}^*_{\mathcal{F}}$ is equicontinuous as a family of local group actions.*

Also, we define a notion of an analytic pseudogroup as follows:

DEFINITION 2.6. *The* pseudo$\star$group $\mathcal{G}^*_{\mathcal{F}}$ *acting on* $\mathfrak{X}$ *is* analytic *if there exists an integer $q \geq 1$, an embedding $\mathfrak{X} \subset \mathbb{R}^q$, and for all admissible pairs of indices (i, j), open sets $U_{i,j} \subset \mathbb{R}^q$ such that $\mathfrak{D}_{i,j} \subset U_{i,j}$ then the map $h_{j,i} \colon \mathfrak{D}_{i,j} \to \mathfrak{D}_{j,i}$ is the restriction of a real analytic map $\widehat{h}_{j,i} \colon U_{i,j} \to U_{j,i}$.*

Finally, we recall a basic result from [16]:

THEOREM 2.7. [16, Theorem 4.12] *Let $\mathcal{G}^*_{\mathcal{F}}$ be the holonomy* pseudo$\star$group *associated to a matchbox manifold $\mathfrak{M}$. If the action of $\mathcal{G}^*_{\mathcal{F}}$ on $\mathfrak{X}$ is* equicontinuous*, then the action of $\mathcal{G}^*_{\mathcal{F}}$ on $\mathfrak{X}$ is minimal.*

The assumption that $\mathcal{G}^*_{\mathcal{F}}$ is the holonomy pseudo$\star$group derived from a connected foliated space $\mathfrak{M}$ is essential for the proof; the conclusion of Theorem 2.7 is obviously false for an arbitrary pseudo$\star$group.

2.4. **Return equivalence.** The concept of *Morita equivalence* between the holonomy groupoids associated to smooth foliated manifolds was introduced by Haefliger in [31], see also [32, 35]. We discuss next the notion of *return equivalence* between pseudo$\star$groups acting on Cantor spaces, which is the analog for matchbox manifolds of the notion of Morita equivalence. The notation "return equivalence" is borrowed from the study of dynamical systems, as the concept is based on the notion of a Poincaré section to the dynamics generated by the holonomy transport along leafwise paths.

Let $\Phi^i \colon \mathcal{G}^*_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be pseudo$\star$group actions on Cantor spaces $\mathfrak{X}_i$, for $i = 1, 2$. The primary examples of interest are the pseudo$\star$group actions associated to matchbox manifolds.

Given a clopen set $U_i \subset \mathfrak{X}_i$, for $i = 1, 2$, let $\mathcal{G}^*_{U_i}$ denote the pseudo$\star$group defined by the restriction of $\mathcal{G}^*_i$ to U_i as in (5). We say that $\mathcal{G}^*_{U_1}$ and $\mathcal{G}^*_{U_2}$ are *isomorphic* if there exists a homeomorphism $\phi \colon U_1 \to U_2$ such that the induced map $\Phi_\phi \colon \mathcal{G}^*_{U_1} \to \mathcal{G}^*_{U_2}$ is an isomorphism. That is, for all $g \in \mathcal{G}^*_{U_1}$ the map $\Phi_\phi(g) = \phi \circ g \circ \phi^{-1}$ defines an element of $\mathcal{G}^*_{U_2}$. Conversely, for $h \in \mathcal{G}^*_{U_2}$ the map $\Phi^{-1}_\phi(h) = \phi^{-1} \circ h \circ \phi$ defines an element of $\mathcal{G}^*_{U_1}$.

DEFINITION 2.8. *Let $\Phi^i \colon \mathcal{G}^*_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be minimal* pseudo$\star$group *actions, for $i = 1, 2$. Then Φ^1 and Φ^2 are* return equivalent *if there are non-empty open sets $U_i \subset \mathfrak{X}_i$ and a homeomorphism $\phi \colon U_1 \to U_2$ such that the induced map $\Phi_\phi \colon \mathcal{G}^*_{U_1} \to \mathcal{G}^*_{U_2}$ is an isomorphism.*

We first note the following result:

PROPOSITION 2.9. [18, Proposition 4.6] Return equivalence *is an equivalence relation on the class of minimal* pseudo$\star$group *actions.*

REMARK 2.10. Haefliger's definition of pseudogroup equivalence in [35, Section 1] is as follows: Let $\Phi^i\colon \mathcal{G}_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be pseudogroups, for $i = 1, 2$. Then Φ^1 and Φ^2 are equivalent if there is a pseudogroup $\Phi\colon \mathcal{G} \times \mathfrak{X} \to \mathfrak{X}$ and homeomorphisms f_i from $\mathfrak{X}_i$ onto open sets $\mathfrak{X}'_i \subset \mathfrak{X}$ such that for $i = 1, 2$, $\mathfrak{X}'_i$ meets all of the orbits of $\mathcal{G}$, and f_i induces an isomorphism from $\mathcal{G}_i$ onto the restriction $\mathcal{G}_{\mathfrak{X}'_i}$. In particular, each inclusion f_i induces an equivalence between $\mathcal{G}_i$ and $\mathcal{G}$, and Haefliger's notion of equivalence is the equivalence relation generated by such inclusions. For minimal pseudo$\star$group actions, each open set in $\mathfrak{X}$ meets every orbit of the action of $\mathcal{G}$, or equally of the action of $\mathcal{G}^*$. Thus, in the case of minimal actions, the notion of return equivalence in Definition 2.8 for pseudo$\star$group actions coincides with Haefliger's notion of equivalence for the pseudogroup actions they generate.

The notion of return equivalence is applied to matchbox manifolds as follows:

DEFINITION 2.11. *Two minimal matchbox manifolds $\mathfrak{M}_i$ for $i = 1, 2$, are* return equivalent *if there exist regular coverings of $\mathfrak{M}_i$ so that the corresponding* pseudo$\star$group*s $\mathcal{G}^*_i$ for $i = 1, 2$ are return equivalent.*

This notion is well-defined by the following two results, whose proofs are given in Section 4 of [18].

LEMMA 2.12. *Let $\mathfrak{M}$ be a minimal matchbox manifold with transversal $\mathfrak{X}$ and* pseudo$\star$group *$\mathcal{G}^*$ associated to a regular foliated covering of $\mathfrak{M}$. Let $W_1, W_2 \subset \mathfrak{X}$ be non–empty open subsets, then the restricted* pseudo$\star$group*s $\mathcal{G}^*_{W_1}$ and $\mathcal{G}^*_{W_2}$ are return equivalent.*

LEMMA 2.13. *Let $\mathfrak{M}$ be a minimal matchbox manifold, and suppose we are given regular coverings by regular foliation charts $\{\varphi_i\colon U_i \to (-1,1)^n \times \mathfrak{T}_i \mid 1 \leq i \leq \nu\}$ and $\{\varphi'_j\colon U'_j \to (-1,1)^n \times \mathfrak{T}'_j \mid 1 \leq j \leq \nu'\}$ of $\mathfrak{M}$, with transversals $\mathfrak{X}$ and $\mathfrak{X}'$ respectively. Let $W_1 \subset \mathfrak{X}$ and $W_2 \subset \mathfrak{X}'$ be non–empty open subsets. Then the restricted* pseudo$\star$group*s $\mathcal{G}^*_{W_1}$ and $\mathcal{G}^*_{W_2}$ are return equivalent.*

Finally, we have that homeomorphism implies return equivalence.

THEOREM 2.14. [18, Theorem 4.8] *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be minimal matchbox manifolds. Suppose that there exists a homeomorphism $h\colon \mathfrak{M}_1 \to \mathfrak{M}_2$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are return equivalent.*

We conclude this section with a fundamental observation about the dynamics of the pseudogroups associated to equicontinuous matchbox manifolds.

PROPOSITION 2.15. *Let $\Phi\colon \mathcal{G} \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous* pseudo$\star$group *action on a Cantor space $\mathfrak{X}$. Let $W \subset \mathfrak{X}$ be a non–empty open subset, then for any $x \in W$, there exists a clopen neighborhood $x \in U \subset W$ such that the restricted* pseudo$\star$group *$\mathcal{G}^*_U$ is induced by a group action $\Phi_U\colon \mathcal{H}_U \times U \to U$, where $\mathcal{H}_U \subset Homeo(U)$.*

REMARK 2.16. The conclusion of this result, as proved in [16, Section 6], actually includes a stronger statement. The clopen set $U \subset \mathfrak{X}$ in Proposition 2.14 is constructed using the method of "orbit coding" for the pseudo$\star$group action. Lemma 6.5 in [16] yields that if $x \in U$ and $g \in \mathcal{G}^*$ satisfies $\Phi(g)(x) \in U$ then there exists $h_{\mathcal{I}} \in \mathcal{G}^*$ defined as in (3) such that $U \subset \mathfrak{D}(h_{\mathcal{I}})$ with $\Phi(h_{\mathcal{I}})(U) = U$, and $g = h_{\mathcal{I}}|U \in \mathcal{H}_U$. Every $g \in \mathcal{H}_U$ is realized in this way. This implies, in particular, that a translate of U under the action of an element of $\mathcal{G}^*$ is either contained in U, or disjoint from U. Thus, the translates of U under the action of $\mathcal{G}^*$ form a clopen disjoint partition of $\mathfrak{X}$. The precise version of these facts is stated as Proposition 3.4 in [22].

The action $(\mathcal{H}_U, U, \Phi_U)$ obtained in Proposition 2.15 is an *equicontinuous minimal Cantor system* in the terminology of [20].

COROLLARY 2.17. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be equicontinuous matchbox manifolds, with transversals $\mathfrak{X}_1$ and $\mathfrak{X}_2$ defined by a choice of regular foliated coverings, respectively. Assume that their holonomy* pseudo⋆groups *$\mathcal{G}^*_{\mathcal{F}_1}$ and $\mathcal{G}^*_{\mathcal{F}_2}$ are return equivalent. Then there exist clopen subsets $U_i \subset \mathfrak{X}_i$ such that the restricted* pseudo⋆group *$\mathcal{G}^*_{U_i}$ is induced by a group action $\Phi_{U_i} \colon \mathcal{H}_{U_i} \times U_i \to U_i$, and there exists a homeomorphism $h \colon U_1 \to U_2$ which induces an isomorphism of the groups $\mathcal{H}^1_{U_1}$ and $\mathcal{H}^2_{U_2}$ and a conjugacy of equicontinuous minimal Cantor systems $(\mathcal{H}^1_{U_1}, U_1, \Phi^1_{U_1})$ and $(\mathcal{H}^2_{U_2}, U_2, \Phi^2_{U_2})$.*

Clopen subsets with restricted pseudo⋆group actions, induced by a group action, play an prominent role in the proofs throughout the paper, motivating the introduction of the following terminology.

DEFINITION 2.18. *Let $\Phi \colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous* pseudo⋆group *action on a Cantor space $\mathfrak{X}$. We say that a clopen subset $U \subset \mathfrak{X}$ is* adapted *to the action Φ, with group $\mathcal{H}_U \subset Homeo(U)$, if the restricted* pseudo⋆group *$\mathcal{G}^*_U$ is induced by the group action $\Phi_U \colon \mathcal{H}_U \times U \to U$.*

In the definition of return equivalence, there is no a priori restriction on the diameter of the clopen set U, so it is important to investigate properties of the associated group $\mathcal{H}_U$ which are independent of the choice of the adapted clopen subset U. For example, the condition that $\mathcal{H}_U$ is a virtually nilpotent group is such a property, which is a key idea in the work [21].

3. Ellis groups

The *Ellis (enveloping) semigroup* associated to a continuous group action $\Phi \colon G \times X \to X$ was introduced in the papers [23, 24], and is treated in the books [10, 25, 26]. We briefly recall some basic facts for the special case of equicontinuous minimal Cantor systems, and then apply these concepts to the case of equicontinuous pseudo⋆group actions.

Let X be a compact Hausdorff topological space, G be a finitely generated group, and $\Phi \colon G \times X \to X$ an equicontinuous action. Let $\Phi(G) \subset Homeo(X)$ denote the image subgroup, and consider its closure $\overline{\Phi(G)} \subset Homeo(X)$ in the *uniform topology on maps*. The group $\overline{\Phi(G)}$ is a special case of the more general construction of Ellis *semi-groups* for topological actions. Note that each $\widehat{g} \in \overline{\Phi(G)}$ is the limit of a sequence of maps in $\Phi(G)$; that is, we do not need to use ultrafilters in the definition of $\overline{\Phi(G)}$, in contrast to the more general construction of Ellis semi-groups. By a small abuse of notation, for $\widehat{g} \in Homeo(X)$, we use the notation $\widehat{g} = (g_i)$ to denote a sequence $\{g_i \mid i \geq 1\} \subset G$ such that the sequence $\{\Phi(g_i) \mid i \geq 1\} \subset Homeo(X)$ converges to $\widehat{g}$.

If the action $\Phi \colon G \times X \to X$ is equicontinuous and minimal, then $\overline{\Phi(G)}(x) = X$ for any $x \in X$. That is, the group $\overline{\Phi(G)}$ acts transitively on X. Introduce the isotropy group at x,

$$\overline{\Phi(G)}_x = \{\widehat{g} = (g_i) \in \overline{\Phi(G)} \mid \widehat{g}(x) = x\}, \tag{7}$$

then there is a natural identification $X \cong \overline{\Phi(G)}/\overline{\Phi(G)}_x$ of left G-spaces.

We next consider the application of the Ellis construction to an equicontinuous pseudo⋆group action $\Phi \colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ on a Cantor space $\mathfrak{X}$. For $x \in \mathfrak{X}$, let U with $x \in U$ be a clopen set adapted to the action Φ, with group $\mathcal{H}_U \subset Homeo(U)$. Let $\overline{\mathcal{H}_U} \subset Homeo(U)$ denote the closure of $\mathcal{H}_U$ in the uniform topology. Then, since the action of $\mathcal{H}_U$ on U is minimal, the action of $\overline{\mathcal{H}_U}$ on U is transitive, and so there is an $\overline{\mathcal{H}_U}$-equivariant homeomorphism $U \cong \overline{\mathcal{H}_U}/\overline{\mathcal{H}_U}_x$ where $\overline{\mathcal{H}_U}_x$ is the isotropy subgroup of $x \in U$.

We note the following basic observation.

LEMMA 3.1. *For $i = 1, 2$, let $\mathfrak{U}_i$ be a Cantor space, let $H_i \subset Homeo(\mathfrak{U}_i)$ be a subgroup whose action is equicontinuous, and let $\overline{H_i} \subset Homeo(\mathfrak{U}_i)$ denote the closure of H_i in the uniform topology on maps. Suppose there exists a homeomorphism $\phi \colon \mathfrak{U}_1 \to \mathfrak{U}_2$ which induces an isomorphism $\Phi_\phi \colon H_1 \to H_2$. Then ϕ induces a topological isomorphism $\overline{\Phi}_\phi \colon \overline{H_1} \to \overline{H_2}$.*

4. Strongly quasi-analytic and stable actions

The *strong quasi-analyticity* condition for a pseudo⋆group action was introduced by J. Álvarez López and A. Candel as the *quasi–effective property* Definition 9.4 in [5], and in revised form as Definition 2.18 in [7], and has a fundamental role in their construction of Molino spaces for equicontinuous foliated spaces. This condition was motivated by the search for a condition equivalent to quasi-analyticity condition introduced by Haefliger [33] for smooth foliations.

DEFINITION 4.1. *A* pseudo⋆group $\mathcal{G}^*$ *acting on a locally compact space* $\mathfrak{X}$ *is* strongly quasi-analytic*, or* SQA*, if for every* $g \in \mathcal{G}^*$ *the following condition holds: Let* $U \subset \mathfrak{D}(g)$ *be a non-empty open set in the domain of* g*, and suppose that the restriction* $g|U$ *is the identity map, then* g *is the identity map on* $\mathfrak{D}(g)$*.*

DEFINITION 4.2. *A pseudogroup* $\mathcal{G}$ *is said to be SQA if it is generated as a pseudogroup by a sub*-pseudo⋆group $\mathcal{G}^* \subset \mathcal{G}$ *which satisfies the SQA property.*

Note that Definition 4.2 does not imply that every element of $\mathcal{G}$ satisfies the SQA condition. For example, suppose there exists a non-identity map $g \in \mathcal{G}^*$ and an open set $W \subset \mathfrak{X}$ such that W is disjoint from the domain $\mathfrak{D}(g)$ and from the range $\mathfrak{R}(g)$. Define $h \in \mathcal{G}$ to be the union of the map g with the identity transformation of W, then h is a map that does not satisfy the conditions in Definition 4.1. This suggests the problem, to understand which actions are SQA:

PROBLEM 4.3. *Let* $\mathcal{G}$ *be a pseudogroup acting on a locally compact space* $\mathfrak{X}$*. Give criteria for the action such that there exists a sub*-pseudo⋆group $\mathcal{G}^*$ *of* $\mathcal{G}$ *which satisfies the SQA property and generates* $\mathcal{G}$ *as a pseudogroup.*

For example, an analytic pseudo⋆group $\mathcal{G}^*_{\mathcal{F}}$, as defined by Definition 2.6, satisfies the SQA condition.

A group action $\Phi\colon G \times \mathfrak{X} \to \mathfrak{X}$ is *free* if for any $x \in \mathfrak{X}$ and $g \in G$, then $\Phi(g)(x) = x$ implies that g is the identity element of G. The action Φ defines a pseudo⋆group $\mathcal{G}^*_{\Phi}$ obtained by considering the restrictions of the actions $\Phi(g)\colon \mathfrak{X} \to \mathfrak{X}$ to open sets in $\mathfrak{X}$, where $g \in G$. It is then immediate that $\mathcal{G}^*_{\Phi}$ satisfies the SQA condition, so the pseudogroup $\mathcal{G}_{\Phi}$ generated by the action is SQA.

We next introduce two variations on the SQA property, which will be used to give a criterion which is used in Propositions 4.7 and 4.9 to show that an equicontinuous action on a Cantor space is SQA, and also that the action of the associated Ellis group is SQA.

For the study of matchbox manifolds, the following *local* SQA property is a more natural property.

DEFINITION 4.4. *A* pseudo⋆group $\mathcal{G}^*$ *acting on a locally compact space* $\mathfrak{X}$ *is* locally strongly quasi-analytic*, or* LSQA*, if for each* $x \in \mathfrak{X}$*, there exists an open subset* $U \subset \mathfrak{X}$ *with* $x \in U$ *such that the restricted* pseudo⋆group $\mathcal{G}^*_U$*, as defined by* (5)*, is strongly quasi-analytic.*

The applications of the LSQA property to the Molino theory for equicontinuous foliated spaces requires this property for the closure of a pseudogroup action in the compact-open topology of maps for their restriction to open subsets of the transversal. We formulate this as follows.

DEFINITION 4.5. *A* pseudo⋆group $\mathcal{G}^*$ *acting on a locally compact space* $\mathfrak{X}$ *is* locally completely strongly quasi-analytic*, or* LCSQA*, if for each* $x \in \mathfrak{X}$*, there exists an open subset with* $x \in U$ *such that for any* $\widehat{g} \in \overline{\mathcal{G}^*_U}$*, suppose that* $V \subset Dom(\widehat{g}) \cap U$ *is a non-empty open set for which the restriction of* $\widehat{g}$ *to* V *is the identity map, then* $\widehat{g}$ *is the identity map on its domain* $Dom(\widehat{g})$*.*

The notion of an LCSQA pseudo⋆group action has a more straightforward formulation in the case where $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ is an equicontinuous pseudo⋆group action on a Cantor space $\mathfrak{X}$. Consider a clopen set $U \subset \mathfrak{X}$ adapted to the action Φ, with group $\mathcal{H}_U$. Then for $x \in U$, let $V \subset U$ be a clopen subset adapted to the action Φ, with group $\mathcal{H}_V$. Let $\mathcal{H}_{U,V} \subset \mathcal{H}_U \subset Homeo(U)$ denote the subgroup of elements $g \in \mathcal{H}_U$ such that $g(V) = V$. Since V is clopen in U, the closure $\overline{\mathcal{H}_{U,V}}$ consists of elements $\widehat{g} \in \overline{\mathcal{H}_U}$ such that $\widehat{g}(V) = V$. Then there exist homomorphisms induced by restriction,

$$\rho_{U,V}\colon \mathcal{H}_{U,V} \to \mathcal{H}_V \quad , \quad \overline{\rho}_{U,V}\colon \overline{\mathcal{H}_{U,V}} \to \overline{\mathcal{H}_V} \ . \tag{8}$$

DEFINITION 4.6. *Let $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous* pseudo⋆group *action on a Cantor space $\mathfrak{X}$. We say that the action is* stable *if for each $x \in \mathfrak{X}$, there exists a clopen set $U \subset \mathfrak{X}$ adapted to the action Φ, with $x \in U$ and with group $\mathcal{H}_U$, such that for any clopen set $V \subset U$ adapted to Φ with group $\mathcal{H}_V$, the restriction map $\overline{\rho}_{U,V}$* on closures *in* (8) *has trivial kernel. The action is said to be* wild *if it is not stable.*

We give two basic properties of this class of pseudo⋆group actions.

PROPOSITION 4.7. *An equicontinuous* pseudo⋆group *action $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ on a Cantor space $\mathfrak{X}$ is stable if and only if it is LCSQA.*

Proof. Suppose the action Φ is stable. Given $x \in \mathfrak{X}$, by Proposition 2.15, there is a clopen subset $x \in U \subset \mathfrak{X}$ adapted to Φ with group $\mathcal{H}_U$. We show that the action of the closure $\overline{\mathcal{H}_U}$ on U is SQA.

Let $\widehat{g} \in \overline{\mathcal{H}_U}$ be such that for some open subset $V \subset U$, the restriction $\widehat{g}|V$ is the identity map. Again by Proposition 2.15, there exists a non-empty clopen subset $W \subset V$, adapted to Φ, with group $\mathcal{H}_W$. By the assumption that the action Φ is stable, the map $\overline{\rho}_{U,W}$ has trivial kernel, and as $\widehat{g}$ restricts to the identity map on W, it follows that $\widehat{g}$ is the identity map on U, as was to be shown.

Now suppose the action of $\mathcal{G}^*$ on $\mathfrak{X}$ is LCSQA. Let $x \in \mathfrak{X}$, then there exists a clopen neighborhood $x \in W \subset \mathfrak{X}$ such that the restricted pseudogroup $\mathcal{G}^*_W$ satisfies the SQA condition for $\overline{\mathcal{G}^*_W}$. Then by Proposition 2.15, there exists a non-empty clopen subset $U \subset W$ with $x \in U$, adapted to the action Φ, with group $\mathcal{H}_U$.

Let $V \subset U$ be a non-empty clopen set, adapted to Φ. We show that the kernel of the restriction map $\overline{\rho}_{U,V}\colon \overline{\mathcal{H}_{U,V}} \to \overline{\mathcal{H}_V}$ is trivial. Let $\widehat{g} \in \overline{\mathcal{H}_{U,V}}$ be such that its restriction to V is the identity map. Then as $\mathcal{G}^*_W$ satisfies the LCSQA condition, $\widehat{g}$ must be the identity on U. □

The definition of a stable action given in [22] was formulated in terms of group chains, and we show in Proposition 7.4 below that the definition in [22] coincides with the notion in Definition 4.6. It was shown in [22], that the holonomy pseudogroup associated to the Schori solenoid is not LSQA, and so is not LCSQA. As a consequence of Proposition 4.7, we obtain that the Schori weak solenoid as described in Theorem 9.8 of [22] is an example of a non-stable equicontinuous minimal group action.

COROLLARY 4.8. *The global monodromy action of the Schori solenoid is wild.*

The techniques in the proof of Proposition 4.7 above also yield the following relation between the LCSQA and SQA properties, which gives another partial answer to Problem 4.3. We thank the referee for pointing out this application of the methods above.

PROPOSITION 4.9. *A minimal equicontinuous* pseudo⋆group *action $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ on a Cantor space $\mathfrak{X}$ is LCSQA if and only if the closure $\overline{\mathcal{G}}$ of the pseudogroup $\mathcal{G}$ it generates is SQA.*

Proof. Suppose the action of $\mathcal{G}^*$ on $\mathfrak{X}$ is LCSQA. For $x \in \mathfrak{X}$, then as in the proof of Proposition 4.7, there exists a non-empty clopen subset $U \subset \mathfrak{X}$ with $x \in U$, adapted to the action Φ, with group $\mathcal{H}_U$. Moreover, for any non-empty clopen set $V \subset U$ which is adapted to Φ, the kernel of the restriction map $\overline{\rho}_{U,V}\colon \overline{\mathcal{H}_{U,V}} \to \overline{\mathcal{H}_V}$ is trivial, so that the pseudo⋆group $\overline{\mathcal{G}^*_U}$ acting on U generated by the collection of maps in $\overline{\mathcal{H}_U}$ satisfies the SQA property.

By Remark 2.16, the translates of U under the action of $\mathcal{G}^*$ define a disjoint partition of $\mathfrak{X}$ by adapted clopen sets. Label this partition by $\{U_1, \ldots, U_k\}$ where $U_1 = U$. Then for each $1 \leq i, j \leq k$ there exists some $g_{i,j} \in \mathcal{G}^*$ such that $U_i \subset \mathfrak{D}(g_{i,j})$ and $\Phi(g_{i,j})\colon U_i \to U_j$ is a homeomorphism. As $\mathcal{G}^*$ is a pseudo⋆group, the restriction $g_{i,j}|U_i$ is in $\mathcal{G}^*$ as well, so we can assume that $\mathfrak{D}(g_{i,j}) = U_i$. Let

$$\mathcal{G}^*(U_i, U_j) = \{g \in \mathcal{G}^* \mid \mathfrak{D}(g) \subset U_i \ , \Phi(g)(U_i) \subset U_j\} \ .$$

Then for $g \in \mathcal{G}^*(U_i, U_j)$, the composition $g_{j,1} \cdot g \cdot g_{1,i} \in \mathcal{G}^*(U_1, U_1) = \mathcal{G}^*_U$. It follows that $\mathcal{G}$ is generated as a pseudogroup by the collection of maps $\{g_{i,j} \mid 1 \leq i, j \leq k\} \ \cup \ \mathcal{G}^*_U$. It remains to show that the pseudo⋆group generated by this collection, and its closure in the uniform topology on maps, satisfies the SQA property.

As U_i and U_j are disjoint for $i \neq j$, it suffices to consider maps $h \in \overline{\mathcal{G}^*(U_i, U_i)}$ for some $1 \leq i \leq k$. Then by Lemma 3.1 we have $g_{i,1} \cdot h \cdot g_{1,i} \in \overline{\mathcal{G}^*(U_1, U_1)} = \overline{\mathcal{G}^*(U, U)}$. It is given that $\overline{\mathcal{G}^*(U, U)}$ satisfies the SQA property, so the same follows for $\overline{\mathcal{G}^*(U_i, U_i)}$, as was to be shown.

Conversely, assume that the closure $\overline{\mathcal{G}}$ of the pseudogroup $\mathcal{G}$ generated by $\mathcal{G}^*$ is SQA. Then by Definition 4.2, there is some sub-pseudo⋆group $\overline{\mathcal{H}^*} \subset \overline{\mathcal{G}}$ which generates $\overline{\mathcal{G}}$, and satisfies the SQA property. Then apply the proof of Proposition 4.7 verbatim to obtain that the action of $\mathcal{H}^*$ is stable, hence the action of $\mathcal{G}^*$ is stable, and thus $\mathcal{G}^*$ satisfies the LCSQA property. □

Example 7.6 in [20] constructs a minimal equicontinuous action of a Cantor space for which the pseudo⋆group it generates is not SQA, but does satisfy the LSQA property, and so the pseudogroup generated by the action is SQA in the sense of Definition 4.2.

Finally, we show that the stable property is preserved by return equivalence. The following result is the analog of Lemma 9.5 in [5].

PROPOSITION 4.10. *The stable property is an invariant of return equivalence for equicontinuous minimal pseudogroup Cantor actions.*

Proof. For $i = 1, 2$, let $\Phi^i \colon \mathcal{G}_i \times \mathfrak{X}_i \to \mathfrak{X}_i$ be a minimal equicontinuous pseudogroup action on a Cantor space $\mathfrak{X}_i$, and assume that the actions are return equivalent. Assume that the action of $\mathcal{G}_1$ on $\mathfrak{X}_1$ is stable, then we show that the action of $\mathcal{G}_2$ on $\mathfrak{X}_2$ is also stable.

For $i = 1, 2$, by the Definition 2.8, there exist clopen subsets $W_i \subset \mathfrak{X}_i$, adapted to Φ_i, and a homeomorphism $\phi \colon W_1 \to W_2$ which induces an isomorphism $\Phi_\phi \colon \mathcal{H}^1_{W_1} \to \mathcal{H}^2_{W_2}$. Then by Lemma 3.1, there is an induced isomorphism between their closures, $\overline{\Phi}_\phi \colon \overline{\mathcal{H}^1_{W_1}} \to \overline{\mathcal{H}^2_{W_2}}$.

By Proposition 2.15, we can choose an adapted clopen subset $U_1 \subset W_1$ such that $x \in U_1$, with group $\mathcal{H}^1_{U_1}$. The image $U_2 = \phi(U_1)$ is a clopen subset of $\mathfrak{X}_2$. By Lemma 2.4, there exists an integer m such that $\mathfrak{X}_2$ is covered by the finite collection $\mathcal{U}_2 = \{h_{\mathcal{I}_1}(U_2), \ldots, h_{\mathcal{I}_m}(U_2)\}$ of clopen sets in $\mathfrak{X}_2$ where each $h_{\mathcal{I}_\ell} \in \mathcal{G}^*_2$.

Given $y \in \mathfrak{X}_2$ there exists an index $1 \leq i_y \leq m$ such that $y \in h_{\mathcal{I}_{i_y}}(U_2)$. By Proposition 2.15, we can choose a clopen set U_y, adapted to Φ_2, such that $y \in U_y \subset h_{\mathcal{I}_{i_y}}(U_2)$, with group $\mathcal{H}^2_{U_y}$.

Note that $h^{-1}_{\mathcal{I}_{i_y}}(U_y) \subset U_2$, and we set $U'_y = \phi^{-1}(h^{-1}_{\mathcal{I}_{i_y}}(U_y)) \subset U_1$. Then $h_{\mathcal{I}_\ell} \circ \phi$ induces a topological conjugacy between $\mathcal{H}^1_{U'_y}$ and $\mathcal{H}^2_{U_y}$, and by Lemma 3.1 between their closures in the uniform topologies.

Now let $V \subset U_y$ be a clopen subset, adapted to Φ_2, with group $\mathcal{H}^2_V$. Then $V' = \phi^{-1}(h^{-1}_{\mathcal{I}_{i_y}}(V)) \subset U'_y \subset U_1$ which is again a clopen subset.

Let $\widehat{g} \in \overline{\mathcal{H}^2(U_y, V)}$ be such that its restriction to V is the identity map, then it is conjugate by $h_{\mathcal{I}_\ell} \circ \phi$ to a map $\widehat{g}' \in \overline{\mathcal{H}^1_{U'_y, V'}}$ which restricts to the identity map on V'. As the action of $\mathcal{G}_1$ on $\mathfrak{X}_1$ is stable, $\widehat{g}'$ is the identity on U'_y, hence $\widehat{g}$ is the identity on U_y. Thus, the action of $\mathcal{G}_2$ on $\mathfrak{X}_2$ is stable.

Reversing the roles of $\mathcal{G}_1$ and $\mathcal{G}_2$ shows the converse implication, and completes the proof. □

5. The asymptotic discriminant invariant

In this section, we introduce the notion of tail equivalence for a sequence of group homomorphisms, and apply the results of the previous section to obtain a new invariant of the homeomorphism type for equicontinuous matchbox manifolds.

DEFINITION 5.1. *Let $\mathcal{A} = \{\phi_i \colon A_i \to A_{i+1} \mid i \geq 1\}$ and $\mathcal{B} = \{\psi_i \colon B_i \to B_{i+1} \mid i \geq 1\}$ be two sequences of* surjective *group homomorphisms. We say that $\mathcal{A}$ and $\mathcal{B}$ are tail equivalent, and write $\mathcal{A} \overset{t}{\sim} \mathcal{B}$, if the sequences of groups $\mathcal{A}$ and $\mathcal{B}$ are intertwined by a sequence of surjective group homomorphisms. That is, there exists:*

(1) an increasing sequence of indices $\{p_i \mid i \geq 1\ ,\ p_{i+1} > p_i \geq i \geq 1\}$;
(2) an increasing sequence of indices $\{q_i \mid i \geq 1\ ,\ q_{i+1} > q_i \geq i \geq 1\}$;
(3) a sequence $\mathcal{C} = \{\tau_i \colon C_i \to C_{i+1} \mid i \geq 1\}$ of surjective group homomorphisms;
(4) a collection of isomorphisms $\Pi_{\mathcal{AC}} \equiv \{\Pi^i_{\mathcal{AC}} \colon A_{p_i} \to C_{2i-1} \mid i \geq 1\}$;
(5) a collection of isomorphisms $\Pi_{\mathcal{BC}} \equiv \{\Pi^i_{\mathcal{BC}} \colon B_{q_i} \to C_{2i} \mid i \geq 1\}$;

such that for all $i \geq 1$, we have

$$\tau_{2i} \circ \tau_{2i-1} \circ \Pi^i_{\mathcal{AC}} = \Pi^{i+1}_{\mathcal{AC}} \circ \Phi_i\ , \tag{9}$$

$$\tau_{2i+1} \circ \tau_{2i} \circ \Pi^i_{\mathcal{BC}} = \Pi^{i+1}_{\mathcal{BC}} \circ \Psi_i\ , \tag{10}$$

where

$$\Phi_i = \phi_{p_{i+1}-1} \circ \phi_{p_{i+1}-2} \circ \cdots \circ \phi_{p_i+1} \circ \phi_{p_i} \tag{11}$$

$$\Psi_i = \psi_{q_{i+1}-1} \circ \psi_{q_{i+1}-2} \circ \cdots \circ \psi_{q_i+1} \circ \psi_{q_i}\ . \tag{12}$$

The collections of maps $\Pi_{\mathcal{AC}}$ and $\Pi_{\mathcal{BC}}$ satisfying (9) and (10) are said to *realize* $\mathcal{A} \overset{t}{\sim} \mathcal{B}$. The relations (9) and (10) give rise to a commutative diagram of maps

$$\begin{tikzcd}
\cdots & A_{p_i} \arrow[r, "\phi_{p_i}"] \arrow[d, "\Pi^i_{\mathcal{AC}}"', "\cong"] & \cdots \arrow[r, "\phi_{p_{i+1}-1}"] & A_{p_{i+1}} \arrow[r, "\phi_{p_{i+1}}"] \arrow[d, "\Pi^{i+1}_{\mathcal{AC}}"', "\cong"] & \cdots \arrow[r, "\phi_{p_{i+2}-1}"] & A_{p_{i+2}} \arrow[r, "\phi_{p_{i+2}}"] \arrow[d, "\Pi^{i+2}_{\mathcal{AC}}"', "\cong"] & \cdots & \\
\cdots & C_{2i-1} \arrow[r, "\tau_{2i-1}"'] & C_{2i} \arrow[r, "\tau_{2i}"'] & C_{2i+1} \arrow[r, "\tau_{2i+1}"'] & C_{2i+2} \arrow[r, "\tau_{2i+2}"'] & C_{2i+3} \arrow[r, "\tau_{2i+3}"'] & C_{2i+4} & \cdots \\
 & \cdots \arrow[r, "\psi_{q_i-1}"'] & B_{q_i} \arrow[r, "\psi_{q_i}"'] \arrow[u, "\cong", "\Pi^i_{\mathcal{BC}}"'] & \cdots \arrow[r, "\psi_{q_{i+1}-1}"'] & B_{q_{i+1}} \arrow[r, "\psi_{q_{i+1}}"'] \arrow[u, "\cong", "\Pi^{i+1}_{\mathcal{BC}}"'] & \cdots \arrow[r, "\psi_{q_{i+2}-1}"'] & B_{q_{i+2}} \arrow[u, "\cong", "\Pi^{i+2}_{\mathcal{BC}}"'] & \cdots
\end{tikzcd} \tag{13}$$

We omit the standard proof that "tail equivalence" is an equivalence relation. The tail equivalence class of a sequence $\mathcal{A}$ is denoted by $[\mathcal{A}]_\infty$ and is called the *asymptotic class* of $\mathcal{A}$.

We also have the usual notion of isomorphisms of sequences.

DEFINITION 5.2. *Let $\mathcal{A} = \{\phi_i \colon A_i \to A_{i+1} \mid i \geq 1\}$ and $\mathcal{B} = \{\psi_i \colon B_i \to B_{i+1} \mid i \geq 1\}$ be sequences of surjective group homomorphisms. Say that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, and write $\mathcal{A} \cong \mathcal{B}$, if there exists isomorphisms $h_i \colon A_i \to B_i$ such that $\psi_i \circ h_i = h_{i+1} \circ \phi_i$ for all $i \geq 1$.*

A sequence $\mathcal{A}$ is *constant* if each map $\phi_i \colon A_i \to A_{i+1}$ is an isomorphism for all $i \geq 1$, and $\mathcal{B}$ said to be *asymptotically constant* if it is tail equivalent to a constant sequence $\mathcal{A}$. The following result follows from the usual method of "chasing of diagrams".

LEMMA 5.3. *A sequence $\mathcal{B} = \{\psi_i \colon B_i \to B_{i+1} \mid i \geq 1\}$ of surjective homomorphisms is asymptotically constant if and only if there exists $i_0 \geq 0$ such that $ker(\psi_i)$ is trivial for all $i \geq i_0$.*

We apply the notion of tail equivalence to obtain a new invariant for the return equivalence class of an equicontinuous minimal Cantor pseudo$\star$group action $\Phi \colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ on a Cantor space $\mathfrak{X}$.

Recall that a clopen subset $U \subset \mathfrak{X}$, $x \in U$, is *adapted* to the action Φ if the restricted pseudogroup $\mathcal{G}^*_U$ is induced by a group action $\Phi_U \colon \mathcal{H}_U \times U \to U$, with image group $\mathcal{H}_U \subset Homeo(U)$. The closure $\overline{\mathcal{H}_U} \subset Homeo(U)$ of $\mathcal{H}_U$ in the uniform topology on maps is the Ellis group of the group action Φ_U as discussed in Section 3. Since Φ_U is minimal, the closure $\overline{\mathcal{H}_U}$ acts transitively on U.

We will need the following basic observation about adapted sets and their properties.

LEMMA 5.4. *Let $\Phi \colon \mathcal{G} \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal pseudogroup action on a Cantor space $\mathfrak{X}$. Let $V \subset U \subset \mathfrak{X}$ be clopen sets, and assume that both U and V are adapted to the action Φ. Then the restriction map $\overline{\rho}_{U,V} \colon \overline{\mathcal{H}_{U,V}} \to \overline{\mathcal{H}_V}$ defined in (8) is a surjection.*

Proof. Let $\widehat{h} \in \overline{\mathcal{H}_V}$ then there exists a sequence $\{h_i \in \mathcal{H}_V \mid i \geq 1\}$ with $h_i \to \widehat{h}$ in the uniform topology on maps of V. As $V \subset U$ is an open set, the maps h_i are elements in $\mathcal{G}^*_U$. As $\mathcal{G}^*_U$ is induced from the group action $\Phi_U \colon \mathcal{H}_U \times U \to U$ there exists some $g_i \in \mathcal{H}_U$ whose restriction to V satisfies $g_i|V = h_i$. The group $\overline{\mathcal{H}_U}$ is compact, so there exists a subsequence g_{i_j} which converges to $\widehat{g} \in \overline{\mathcal{H}_U}$, and also the subsequence $g_{i_j}|V$ converges to $\widehat{h}$. Thus, $\widehat{g}|V = \widehat{h}$ as was to be shown. □

DEFINITION 5.5. *Let $\Phi \colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal* pseudo⋆group *action on a Cantor space $\mathfrak{X}$. A descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ is said to be an* adapted neighborhood basis *at $x \in \mathfrak{X}$ for the action Φ if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 1$ with $\cap\, U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ.*

Proposition 2.15 implies that each $x \in \mathfrak{X}$ has an adapted neighborhood basis.

LEMMA 5.6. *Let $\Phi \colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal* pseudo⋆group *action on a Cantor space $\mathfrak{X}$. Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be an* adapted neighborhood basis *at $x \in \mathfrak{X}$. Let $\ell' > \ell > 1$, so that $U_{\ell'} \subset U_\ell$. Suppose that $\widehat{h} \in \overline{\mathcal{H}_{U_1}}$ and $\widehat{h}(U_{\ell'}) = U_{\ell'}$, then $\widehat{h}(U_\ell) = U_\ell$.*

Proof. By Remark 2.16, the translates of U_ℓ under the action of $\mathcal{H}_{U_1}$ form a disjoint clopen partition of U_1. Suppose that for $g \in \mathcal{G}$ we have $\Phi(g)(U_{\ell'}) = U_{\ell'}$ then $\Phi(g)(U_\ell) \cap U_\ell \neq \emptyset$, hence $\Phi(g)(U_\ell) = U_\ell$.

Given $\widehat{h} \in \overline{\mathcal{H}_{U_1}}$ there exists a sequence $\{g_i \mid i \geq 1\} \subset \mathcal{G}$ such that the maps $\Phi(g_i)|U_1$ converge in the uniform topology to $\widehat{h}$. As each set in the disjoint partition of U_1 by the translates of $U_{\ell'}$ is clopen, for i sufficiently large we must have $\Phi(g_i)(U_{\ell'}) = U_{\ell'}$, and hence $\Phi(g_i)(U_\ell) = U_\ell$. As U_ℓ is compact, we conclude that $\widehat{h}(U_\ell) = U_\ell$. □

In the following, we assume that $\Phi \colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ is an equicontinuous minimal pseudo⋆group action on a Cantor space $\mathfrak{X}$, and $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ is an *adapted neighborhood basis* at $x \in \mathfrak{X}$. For $\ell' > \ell \geq 1$, following the constructions in Section 4, define $\mathcal{H}_{U_\ell, U_{\ell'}} \subset \mathcal{H}_{U_\ell} \subset Homeo(U_\ell)$ to be the subgroup of elements $h \in \mathcal{H}_{U_\ell}$ such that $h(U_{\ell'}) = U_{\ell'}$, with closure $\overline{\mathcal{H}_{U_\ell, U_{\ell'}}} \subset \overline{\mathcal{H}_{U_\ell}}$. Then for $\widehat{h} \in \overline{\mathcal{H}_{U_\ell, U_{\ell'}}}$ we have $\widehat{h}(U_{\ell'}) = U_{\ell'}$, so there are well-defined restriction homomorphisms

$$\rho_{U_\ell, U_{\ell'}} \colon \mathcal{H}_{U_\ell, U_{\ell'}} \to \mathcal{H}_{U_{\ell'}} \quad , \quad \overline{\rho}_{U_\ell, U_{\ell'}} \colon \overline{\mathcal{H}_{U_\ell, U_{\ell'}}} \to \overline{\mathcal{H}_{U_{\ell'}}} \ . \tag{14}$$

Introduce the isotropy groups of the actions of the closure $\overline{\mathcal{H}_{U_1, U_\ell}}$, given by

$$\mathcal{I}(U_1, U_\ell, x) = \overline{\mathcal{H}_{U_1, U_\ell}}_x = \{\widehat{h} \in \overline{\mathcal{H}_{U_1, U_\ell}} \mid \widehat{h}(x) = x\} \ . \tag{15}$$

Then we have the following consequence of Lemma 5.6.

COROLLARY 5.7. *For $\ell' > \ell \geq 1$, $\overline{\rho}_{U_\ell, U_{\ell'}}$ induces a surjection $\sigma_{U_\ell, U_{\ell'}} \colon \mathcal{I}(U_1, U_\ell, x) \to \mathcal{I}(U_1, U_{\ell'}, x)$. Thus, there exists a sequence of surjective homomorphisms,*

$$\mathcal{I}(\mathcal{U}, x) = \{\sigma_{U_\ell, U_{\ell+1}} \colon \mathcal{I}(U_1, U_\ell, x) \to \mathcal{I}(U_1, U_{\ell+1}, x) \mid \ell \geq 1\} \ . \tag{16}$$

Proof. Let $\widehat{h} \in \mathcal{I}(U_1, U_{\ell'}, x) \subset \overline{\mathcal{H}_{U_1}}$ then $\widehat{h}(U_{\ell'}) = U_{\ell'}$. Then by Lemma 5.6 we have $\widehat{h}(U_\ell) = U_\ell$ so that $\widehat{h} \in \mathcal{I}(U_1, U_\ell, x)$ and thus $\sigma_{U_\ell, U_{\ell'}}$ is onto. □

The following results consider the tail equivalence class $[\mathcal{I}(\mathcal{U}, x)]_\infty$ of the sequence defined by (16).

LEMMA 5.8. *Let $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal* pseudo$\star$group *action on a Cantor space $\mathfrak{X}$. Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ and $\mathcal{V} = \{V_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be adapted neighborhood bases at $x \in \mathfrak{X}$. Then the sequences of surjective homomorphisms $\mathcal{I}(\mathcal{U}, x)$ and $\mathcal{I}(\mathcal{V}, x)$ are tail equivalent.*

Proof. Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ and $\mathcal{V} = \{V_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be adapted neighborhood bases at x. We introduce a third neighborhood basis $\mathcal{W} = \{W_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$, defined recursively as follows.

Set $W_1 = U_1$ and $p_1 = 1$. Then there exists $q_1 \geq 1$ such that $V_{q_1} \subset U_1$ and we set $W_2 = V_{q_1}$. Now proceed recursively, and assume that $\{p_1, p_2, \ldots, p_i\}$ and $\{q_1, q_2, \ldots, q_i\}$ have been chosen, where $W_{2j-1} = U_{p_j}$ and $W_{2j} = V_{q_j}$ for $1 \leq j \leq i$. Then choose $p_{i+1} > p_i$ such that $U_{p_{i+1}} \subset V_{q_i}$ and set $W_{2j+1} = U_{p_{i+1}}$. Choose $q_{i+1} > q_i$ such that $V_{q_{i+1}} \subset U_{p_{i+1}}$ and set $W_{2i+2} = V_{q_{i+1}}$.

Next, let $A_i = \mathcal{I}(U_1, U_i, x)$, $B_i = \mathcal{I}(V_1, V_i, x)$, and $C_i = \mathcal{I}(U_1, W_i, x)$.

Define $\phi_i = \sigma_{U_i, U_{i+1}}$, $\psi_i = \sigma_{V_i, V_{i+1}}$, and $\tau_i = \sigma_{W_i, W_{i+1}}$.

Define $\Pi^i_{\mathcal{AC}}\colon A_{p_i} \to C_{2i-1} = \mathcal{I}(U_1, W_{2i-1}, x) = \mathcal{I}(U_1, U_{p_i}, x) = A_{p_i}$ to be the identity for each $i \geq 1$.

Define $\Pi^i_{\mathcal{BC}}\colon B_{q_i} \to C_{2i} = \mathcal{I}(U_1, W_{2i}, x) = \mathcal{I}(U_1, V_{q_i}, x) = B_{q_i}$ to be the identity for each $i \geq 1$.

Then the identities (9) and (10) as in the diagram (13) are satisfied, so that $\mathcal{I}(\mathcal{U}, x)$ and $\mathcal{I}(\mathcal{V}, x)$ are tail equivalent, as was to be shown. □

LEMMA 5.9. *Let $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal* pseudo$\star$group *action on a Cantor space $\mathfrak{X}$. Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be an* adapted neighborhood basis *at $x \in \mathfrak{X}$. Let $h \in \mathcal{G}^*$ with $h(x) = y \in \mathfrak{X}$, and let $\mathcal{V} = \{V_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be an adapted neighborhood basis at y. Then the sequences of surjective homomorphisms $\mathcal{I}(\mathcal{U}, x)$ and $\mathcal{I}(\mathcal{V}, y)$ are tail equivalent.*

Proof. Recall that $\mathfrak{D}(h) \subset \mathfrak{X}$ denotes the domain of h which is an open neighborhood with $x \in \mathfrak{D}(h)$, so there exists $\ell_0 \geq 1$ such that $U_\ell \subset \mathfrak{D}(h)$ for $\ell \geq \ell_0$. Each image $h(U_\ell) \subset \mathfrak{X}$ is then a clopen neighborhood of y, so we obtain a chain of clopen neighborhoods $\mathcal{U}^h = \{h(U_\ell) \mid \ell \geq \ell_0\}$ at y. Moreover, as each U_ℓ is an adapted clopen set, the same holds for the image $h(U_\ell)$. We then have that $\mathcal{I}(\mathcal{U}, x)$ and $\mathcal{I}(\mathcal{U}^h, y)$ are tail equivalent by construction, and $\mathcal{I}(\mathcal{U}^h, y)$ is tail equivalent to $\mathcal{I}(\mathcal{V}, y)$ by Lemma 5.8. Thus, $\mathcal{I}(\mathcal{U}, x)$ and $\mathcal{I}(\mathcal{V}, y)$ are tail equivalent, as was to be shown. □

LEMMA 5.10. *Let $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal* pseudo$\star$group *action on a Cantor space $\mathfrak{X}$. Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be an* adapted neighborhood basis *at $x \in \mathfrak{X}$. Let $y \in U_1$ and $\mathcal{V} = \{V_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be an adapted neighborhood basis at y. Then the sequences of surjective homomorphisms $\mathcal{I}(\mathcal{U}, x)$ and $\mathcal{I}(\mathcal{V}, y)$ are tail equivalent.*

Proof. The action Φ is minimal, so the restricted action $\Phi|U_1$ is also minimal. As U_1 is an adapted clopen set, this implies that the action of the closure $\overline{\mathcal{H}_{U_1}}$ is transitive. Thus, there exists $\widehat{h} \in \overline{\mathcal{H}_{U_1}}$ such that $\widehat{h}(x) = y$. Introduce the sequence of clopen subsets of U_1 given by $\mathcal{U}^{\widehat{h}} = \{\widehat{h}(U_\ell) \mid \ell \geq 2\}$ which is a neighborhood basis at y. We claim that $\widehat{h}(U_\ell)$ is adapted to the action Φ, for each $\ell \geq 2$. Choose a sequence $\{h_i \in \mathcal{H}_{U_1}\}$ such that h_i converges to $\widehat{h}$ in the uniform topology on maps.

Fix $\ell \geq 2$, then U_ℓ is a clopen set and $\widehat{h} \in \mathit{Homeo}(U_1)$ implies that $\widehat{h}(U_\ell)$ is a clopen subset of U_1 and hence $\varepsilon_\ell = d_{\mathfrak{X}}\{\widehat{h}(U_\ell), U_1 - \widehat{h}(U_\ell)\} > 0$. Thus, there exists $i_\ell \geq 1$ such that $i \geq i_\ell$ implies that $h_i(U_\ell) \cap (\mathfrak{X} - \widehat{h}(U_\ell)) = \emptyset$, and hence $h_i(U_\ell) = \widehat{h}(U_\ell)$. As $h_i \in \mathcal{H}_{U_1}$ this implies that $\widehat{h}(U_\ell)$ is adapted to the action of Φ. Thus, $\mathcal{U}^{\widehat{h}}$ is an adapted neighborhood basis at y.

By Lemma 5.8 the sequences of surjective homomorphisms $\mathcal{I}(\mathcal{U}^{\widehat{h}}, y)$ and $\mathcal{I}(\mathcal{V}, y)$ are tail equivalent. We have by construction that $\mathcal{I}(\mathcal{U}, x)$ and $\mathcal{I}(\mathcal{U}^{\widehat{h}}, y)$ are tail equivalent, so $\mathcal{I}(\mathcal{U}, x)$ is tail equivalent to $\mathcal{I}(\mathcal{V}, y)$ as claimed. □

LEMMA 5.11. *Let $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal* pseudo$\star$group *action on a Cantor space $\mathfrak{X}$. Let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ and $\mathcal{V} = \{V_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be adapted neighborhood bases at $x \in \mathfrak{X}$ and $y \in \mathfrak{X}$, respectively. Then the sequences of surjective homomorphisms $\mathcal{I}(\mathcal{U}, x)$ and $\mathcal{I}(\mathcal{V}, y)$ are tail equivalent.*

Proof. By Lemma 2.4 there exists $h \in \mathcal{G}^*$ such that $y \in h(U_1 \cap \mathfrak{D}(h))$. Let $z = h^{-1}(y) \in U_1 \cap \mathfrak{D}(h)$. Then by Proposition 2.15, there exists an adapted neighborhood basis $\mathcal{W} = \{W_\ell \subset U_1 \cap \mathfrak{D}(h) \mid \ell \geq 1\}$ at z. Then by Lemma 5.10, the sequences $\mathcal{I}(\mathcal{U}, x)$ and $\mathcal{I}(\mathcal{W}, z)$ are tail equivalent.

Let $\ell_y \geq 1$ be such that $h(W_\ell) \subset V_1$ for $\ell \geq \ell_y$. Define $\mathcal{W}^h = \{h(W_\ell) \subset V_1 \mid \ell \geq \ell_y\}$. Then the sequences $\mathcal{I}(\mathcal{W}, z)$ and $\mathcal{I}(\mathcal{W}^h, y)$ are tail equivalent by construction. Then by Lemma 5.8, the sequences $\mathcal{I}(\mathcal{W}^h, y)$ and $\mathcal{I}(\mathcal{V}, y)$ are tail equivalent.

Thus, we have $\mathcal{I}(\mathcal{U}, x) \overset{t}{\sim} \mathcal{I}(\mathcal{W}, z) \overset{t}{\sim} \mathcal{I}(\mathcal{W}^h, y) \overset{t}{\sim} \mathcal{I}(\mathcal{V}, y)$ as was to be shown. □

As a consequence of Lemma 5.11 we obtain a well-defined invariant of the action Φ.

DEFINITION 5.12. *Let $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal* pseudo⋆group *action on a Cantor space $\mathfrak{X}$. The* asymptotic discriminant *of Φ is the tail equivalence class, $\mathcal{I}(\Phi) = [\mathcal{I}(\mathcal{U}, x)]_\infty$, for a choice of the basepoint $x \in \mathfrak{X}$ and a choice of an adapted neighborhood basis $\mathcal{U}$ at x.*

The explanation for the notation "asymptotic discriminant" will be made clear in Sections 7.2 and 7.3, where we give an interpretation of this class in terms of the group chain model for the action and the notion of the discriminant group for such actions, as introduced in the works [19, 20, 22].

Asymptotic discriminant defines an invariant of the return equivalence class of a pseudogroup action.

PROPOSITION 5.13. *Let $\Phi\colon \mathcal{G}^* \times \mathfrak{X} \to \mathfrak{X}$ be an equicontinuous minimal* pseudo⋆group *action on a Cantor space $\mathfrak{X}$. Then the asymptotic discriminant $\mathcal{I}(\Phi)$ depends only on the return equivalence class of the action.*

Proof. The proof is similar to that of Proposition 4.10 so we give only a sketch of the argument.

Let $\Phi^i\colon \mathcal{G}_i^* \times \mathfrak{X}_i \to \mathfrak{X}_i$ be equicontinuous minimal pseudo⋆group Cantor actions, for $i = 1, 2$, and assume that Φ^1 and Φ^2 are return equivalent. Then there are non-empty open sets $W_i \subset \mathfrak{X}_i$ and a homeomorphism $h\colon W_1 \to W_2$ such that the induced map $\Phi_h\colon \mathcal{G}^*_{W_1} \to \mathcal{G}^*_{W_2}$ is an isomorphism.

Choose $x \in \mathfrak{X}_1$ and let $\mathcal{U} = \{U_\ell \subset \mathfrak{X}_1 \mid \ell \geq 1\}$ be an adapted neighborhood basis at x for Φ^1. Let $y = \phi(x) \in \mathfrak{X}_2$ and $V_\ell = \phi(U_\ell)$ for $\ell \geq 1$. Then $\mathcal{V} = \{V_\ell \subset \mathfrak{X}_2 \mid \ell \geq 1\}$ is an adapted neighborhood basis at y for Φ^2.

The restricted homeomorphisms $h_\ell = \phi|U_\ell\colon U_\ell \to V_\ell$ induce isomorphisms $h_\ell^*\colon \mathcal{H}_{U_\ell} \to \mathcal{H}_{V_\ell}$. By Lemma 3.1 there are induced isomorphisms $\overline{h_\ell}\colon \overline{\mathcal{H}_{U_\ell}} \to \overline{\mathcal{H}_{V_\ell}}$ of their closures in the uniform topology which coincides with the compact-open topology. Restrictions of these isomorphisms to isotropy groups gives the isomorphisms $\overline{h_1}\colon \mathcal{I}(U_1, U_\ell, x) \to \mathcal{I}(V_1, V_\ell, x)$. Thus the maps $\{\overline{h_\ell} \mid \ell \geq 1\}$ give an isomorphism of the asymptotic discriminant class associated to the adapted neighborhood basis $\mathcal{U}$ with the asymptotic class of the adapted neighborhood basis $\mathcal{V}$. □

Let $\mathfrak{M}$ be an equicontinuous matchbox manifold, and choose a regular covering of $\mathfrak{M}$ as in Section 2.1, and let $\mathcal{G}_{\mathcal{F}}$ be the associated holonomy pseudogroup as defined in Section 2.2, which yields an equicontinuous minimal pseudo⋆group action $\Phi\colon \mathcal{G}^*_{\mathcal{F}} \times \mathfrak{X} \to \mathfrak{X}$ on the transversal Cantor space $\mathfrak{X}$. Let $x \in \mathfrak{X}$ and let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$ be an adapted neighborhood basis at x for Φ. Define the asymptotic discriminant of $\mathfrak{M}$ to be the asymptotic class $\mathcal{I}(\mathfrak{M}) = \mathcal{I}(\mathcal{U}, x)$ associated to the adapted neighborhood basis $\mathcal{U}$ at x. Then by Theorem 2.14 and Proposition 5.13, we have:

THEOREM 5.14. *The asymptotic discriminant $\mathcal{I}(\mathfrak{M})$ of $\mathfrak{M}$ is well-defined, and is an invariant of the homeomorphism class of $\mathfrak{M}$.*

The following result then follows from Definition 4.6 and Lemma 5.3.

COROLLARY 5.15. *An equicontinuous matchbox manifold $\mathfrak{M}$ is stable if an only if the asymptotic discriminant invariant is asymptotically constant.*

In the following sections, we show how to calculate the asymptotic discriminant for weak solenoids using the calculus of group chains, as developed in [19, 20]. The works [19, 20, 21, 22] used these methods to construct examples of various classes of weak solenoids which are stable and have non-trivial asymptotic discriminant. In fact, Theorem 10.8 in [22] shows that given any separable profinite group $\mathbf{K}$ there is a weak solenoid with constant asymptotic discriminant isomorphic to $\mathbf{K}$. This provides uncountable families of non-homeomorphic stable equicontinuous matchbox manifolds.

Our interest in this work is on the equicontinuous matchbox manifolds which are not stable, hence are wild. For such spaces, the asymptotic discriminant is not a compact subgroup of a profinite group, but a tail equivalence class of maps between such groups. In the following, we introduce the methods used to calculate the asymptotic discriminant, then in Section 9 we give a systematic construction of an uncountable collection of wild Cantor actions with distinct asymptotic discriminant groups.

6. Weak solenoids

Sections 3 and 5 formulated the stable property for equicontinuous matchbox manifolds, and defined the asymptotic discriminant invariant in terms of their holonomy pseudo$\star$groups. On the other hand, given an equicontinuous minimal pseudo$\star$group action, the works [16, 19, 20] associate to an equicontinuous minimal Cantor action a group chain model for the action, and formulate many concepts and results analogous to those in the above sections in terms of the group chain model.

In this section, we present the group chain model associated to the equicontinuous minimal holonomy Cantor actions for weak solenoids, then give in the following section an interpretation of the stable property and the asymptotic discriminant invariant in terms of group chains. These results form the basis for the construction of families of weak solenoids which are stable with non-trivial discriminant in the works [20, 21, 22], and here of families of weak solenoids which are wild.

We begin by recalling the construction of *weak solenoids*, as first introduced by McCord [43] and Schori [54], and some of their properties as developed by Rogers and Tollefson [50, 51, 52, 53] and Fokkink and Oversteegen [27].

A *presentation* for a weak solenoid is a collection

$$\mathcal{P} = \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell \mid \ell \geq 0\} \ , \tag{17}$$

where each M_ℓ is a connected compact manifold of dimension n, and each *bonding* map $p_{\ell+1}$ is a proper covering map of finite index. Associated to a presentation $\mathcal{P}$ is the weak solenoid denoted by $\mathcal{S}_\mathcal{P}$ which is defined as the inverse limit,

$$\mathcal{S}_\mathcal{P} \equiv \varprojlim \ \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \ \subset \prod_{\ell \geq 0} \ M_\ell \ . \tag{18}$$

By definition, for a sequence $\{x_\ell \in M_\ell \mid \ell \geq 0\}$, we have

$$x = (x_0, x_1, \ldots) \in \mathcal{S}_\mathcal{P} \iff p_\ell(x_\ell) = x_{\ell-1} \text{ for all } \ell \geq 1 \ . \tag{19}$$

The set $\mathcal{S}_\mathcal{P}$ is given the relative topology, induced from the product topology, so that $\mathcal{S}_\mathcal{P}$ is itself compact and connected. McCord showed in [43] that the space $\mathcal{S}_\mathcal{P}$ has a local product structure, and moreover we have:

PROPOSITION 6.1. *Let $\mathcal{P}$ be a presentation with base space M_0 and $\mathcal{S}_\mathcal{P}$ the associated weak solenoid. Then $\mathcal{S}_\mathcal{P}$ is an equicontinuous matchbox manifold of dimension n with foliation $\mathcal{F}_\mathcal{S}$ by path-connected components.*

Associated to a presentation $\mathcal{P}$ of compact manifolds is a sequence of proper surjective maps

$$q_\ell = p_1 \circ \cdots \circ p_{\ell-1} \circ p_\ell \colon M_\ell \to M_0 \ . \tag{20}$$

For each $\ell > 1$, projection onto the ℓ-th factor in the product $\displaystyle\prod_{\ell \geq 0} M_\ell$ in (18) yields a fibration map denoted by $\Pi_\ell \colon \mathcal{S}_\mathcal{P} \to M_\ell$, for which $\Pi_0 = \Pi_\ell \circ q_\ell \colon \mathcal{S}_\mathcal{P} \to M_0$. A choice of a basepoint $x_0 \in M_0$ fixes a fiber $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$, which is a Cantor set by the assumption that the fibers of each map p_ℓ have cardinality at least 2. The choice of x_0 will remain fixed throughout the following. We also then have a fixed base group $G_0 = \pi_1(M_0, x_0)$.

A choice $x \in \mathfrak{X}_0$ defines basepoints $x_\ell = \Pi_\ell(x) \in M_\ell$ for $\ell \geq 1$. For each $\ell \geq 1$, let

$$G^x_\ell = \text{image}\,\{(q_\ell)_\# \colon \pi_1(M_\ell, x_\ell) \longrightarrow G_0\} \tag{21}$$

denote the image of the induced map $(q_\ell)_\#$ on fundamental groups. Associated to the presentation $\mathcal{P}$ and basepoint $x \in \mathfrak{X}_0$ we thus obtain a descending chain of subgroups of finite index

$$\mathcal{G}^x \equiv \{G^x_\ell\}_{\ell \geq 0} = \{G_0 \supset G^x_1 \supset G^x_2 \supset \cdots \supset G^x_\ell \supset \cdots\} \ . \tag{22}$$

Note that given another choice of basepoint $y \in \mathfrak{X}_0$ with corresponding images $y_\ell = \Pi_\ell(y) \in M_\ell$, for each $\ell \geq 1$, there exists $g^y_\ell \in G_0$ such that the image group $G^y_\ell = (g^y_\ell) G^x_\ell (g^y_\ell)^{-1}$. The resulting group chain $\mathcal{G}^y = \{G^y_\ell\}_{\ell \geq 0}$ is said to be *conjugate* to $\mathcal{G}^x$.

Each quotient $X^x_\ell = G_0/G^x_\ell$ is a finite set equipped with a left G_0-action, and there are surjections $X^x_{\ell+1} \to X^x_\ell$ which commute with the action of G_0. The inverse limit

$$X^x_\infty = \varprojlim \, \{p_{\ell+1} \colon X^x_{\ell+1} \to X^x_\ell\} \ \subset \prod_{\ell \geq 0} X^x_\ell \tag{23}$$

is given the relative topology, induced from the product (Tychonoff) topology on the space $\prod_{\ell \geq 0} X^x_\ell$, so that X^x_∞ is compact. As each set X^x_ℓ is finite with cardinality at least 2 for $\ell \geq 1$, X^x_∞ is a totally disconnected perfect set, so is a Cantor space.

A sequence $(g_\ell) \subset G_0$ such that $g_\ell G^x_\ell = g_{\ell+1} G^x_\ell$ for all $\ell \geq 0$ determines a point $(g_\ell G^x_\ell) \in X^x_\infty$. Let $e \in G_0$ denote the identity element, then the sequence $e_x = (e G^x_\ell)$ is the basepoint of X^x_∞.

The action $\Phi_x \colon G_0 \times X^x_\infty \to X^x_\infty$ is given by coordinate-wise multiplication, $\Phi_0(g)(g_\ell G^x_\ell) = (g g_\ell G^x_\ell)$. We then have the standard observation:

LEMMA 6.2. *$\Phi_x \colon G_0 \times X^x_\infty \to X^x_\infty$ defines an equicontinuous Cantor minimal system $(X^x_\infty, G_0, \Phi_x)$.*

The choice of the basepoint $x \in \mathcal{S}_\mathcal{P}$ defines basepoints $x_\ell \in M_\ell$ for all $\ell \geq 1$, which gives an identification of X^x_ℓ with the fiber of the covering map $M_\ell \to M_0$. In the inverse limit, we thus obtain a homeomorphism $\tau_x \colon X^x_\infty \to \mathfrak{X}_0 = \Pi_0^{-1}(x_0)$ such that $\tau_x(e_x) = x$, which can be viewed as "coordinates" on $\mathfrak{X}_0$.

The left action of G_0 on X^x_∞ is conjugated by τ_x to an action of G_0 on $\mathfrak{X}_0$, called the *monodromy action* at x_0 for the fibration $\Pi_0 \colon \mathcal{S}_\mathcal{P} \to M_0$, where the action is defined by the holonomy transport along the leaves of the foliation $\mathcal{F}_\mathcal{S}$ on $\mathcal{S}_\mathcal{P}$.

The map $\tau_x \colon X^x_\infty \to \mathfrak{X}_0$ is used to give a "standard form" for the solenoid $\mathcal{S}_\mathcal{P}$. Let $\widetilde{M_0}$ denote the universal covering of the compact manifold M_0 and let $(X^x_\infty, G_0, \Phi_x)$ be the minimal Cantor system associated to the presentation $\mathcal{P}$ and the choice of a basepoint $x \in \mathfrak{X}_0$. Associated to the left action Φ_x of G_0 on X^x_∞ is a suspension space

$$\mathfrak{M} = \widetilde{M_0} \times X^x_\infty / (z \cdot g^{-1}, x) \sim (z, \Phi_x(g)(x)) \quad \text{for } z \in \widetilde{M_0}, \ g \in G_0 \ , \tag{24}$$

which is a minimal matchbox manifold. Moreover, $\mathfrak{M}$ has an inverse limit presentation, where all of the bonding maps between the coverings $M_\ell \to M_0$ are derived from the universal covering map $\widetilde{\pi} \colon \widetilde{M_0} \to M_0$, so are in "standard form". The following result uses the path lifting property for coverings, to show that for any presentation $\mathcal{P}$, we have:

THEOREM 6.3. [16] *Let $\mathcal{S}_\mathcal{P}$ be a weak solenoid, with base space M_0 which is a compact manifold of dimension $n \geq 1$. Then the suspension of the map τ_x yields a foliated homeomorphism $\tau^*_x \colon \mathfrak{M} \to \mathcal{S}_\mathcal{P}$.*

COROLLARY 6.4. *The homeomorphism type of a weak solenoid $\mathcal{S}_\mathcal{P}$ is completely determined by the base manifold M_0 and the associated minimal Cantor system $(X^x_\infty, G_0, \Phi_x)$.*

We conclude this discussion of this section, by introducing the following important notion:

DEFINITION 6.5. *The* kernel *of a group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ is the subgroup $K(\mathcal{G}) = \bigcap_{\ell \geq 0} G_\ell$.*

For a weak solenoid $\mathcal{S}_{\mathcal{P}}$ with choice of a basepoint $x_0 \in M_0$ and fiber $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$, the kernel subgroup $K(\mathcal{G}^x) \subset G_0$ may depend on the choice of the basepoint $x \in \mathfrak{X}_0$. The dependence of $K(\mathcal{G}^x)$ on x is a natural aspect of the dynamics of the foliation $\mathcal{F}_{\mathcal{S}}$ on $\mathcal{S}_{\mathcal{P}}$, when $K(\mathcal{G}^x)$ is interpreted in terms of the topology of the leaves of $\mathcal{F}_{\mathcal{S}}$.

The map $\tau_x^* \colon \mathfrak{M} \to \mathcal{S}_{\mathcal{P}}$ of Theorem 6.3 sends the quotient space $\widetilde{M}/K(\mathcal{G}^x)$ to the leaf $L_x \subset \mathcal{S}_{\mathcal{P}}$ through $x \in \mathfrak{X}_0$ in $\mathcal{S}_{\mathcal{P}}$ and so $K(\mathcal{G}^x)$ is naturally identified with the fundamental group $\pi_1(L_x, x)$. The holonomy homomorphism $h_x \colon \pi_1(L_x, x) \to Homeo(\mathfrak{X}_0, x)$ of the leaf L_x in the suspension foliation $\mathcal{F}_{\mathcal{S}}$ is conjugate to the left action, $\Phi_0 \colon K(\mathcal{G}^x) \to Homeo(X_\infty^x, e_x)$.

From the point of view of foliation theory, the leaves of $\mathcal{F}_{\mathcal{S}}$ with holonomy are a "small" set. There always exists leaves without holonomy, while there may exist leaves with holonomy, and so the fundamental groups of the leaves may vary accordingly. This aspect of the foliation dynamics of weak solenoids is discussed further in [22, Section 4.2].

7. Algebraic models from group chains

In this section, we develop models derived from a group chain $\mathcal{G}$ for the spaces and actions introduced in Section 3. In particular, we construct the Ellis group for the equicontinuous action on the Cantor space X_∞^x associated to a group chain in (23), and then interpret in Section 7.2 the definition of stable actions given in Definition 4.6 in terms of the group chain model of the action. In Section 7.3, we then give the group chain interpretation of the asymptotic discriminant invariant in Definition 5.12.

7.1. **Core chains.** Let $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ be a group chain in G_0. Let X_∞ be the inverse limit space defined as in (23) by the finite quotients $X_\ell = G_0/G_\ell$ with the transitive left G_0-action. This defines an equicontinuous minimal Cantor action denoted by (X_∞, G_0, Φ_0). Let $\mathcal{H} = \Phi_0(G_0) \subset Homeo(X_\infty)$ be the subgroup defined by this action, and let $\overline{\mathcal{H}} \subset Homeo(X_\infty)$ be its closure.

For each $\ell \geq 0$, the group G_0 acts transitively on the left on each quotient space X_ℓ, where G_ℓ is the isotropy group for the basepoint $x_\ell = eG_\ell \in X_\ell$. Then for $g \in G_0$ the isotropy of the point $gG_\ell \in X_\ell$ is the conjugate subgroup $gG_\ell g^{-1} \subset G_0$. The kernel of the homomorphism $\Phi_\ell \colon G_0 \to Homeo(X_\ell)$ is the intersection of the isotropy groups for the points of X_ℓ,

$$C_\ell \equiv \mathrm{core}_{G_0}\, G_\ell \equiv \bigcap_{g \in G_0} gG_\ell g^{-1} \ \subseteq \ G_\ell \ . \tag{25}$$

The subgroup C_ℓ, called the *core* of G_ℓ in G_0, is the maximal normal subgroup of G_ℓ. Note that

$$\bigcap_{\ell \geq 0} C_\ell = \bigcap_{\ell \geq 0} \bigcap_{g \in G_0} gG_\ell g^{-1} = \bigcap_{g \in G_0} \bigcap_{\ell \geq 0} gG_\ell g^{-1} = \bigcap_{g \in G_0} gK(\mathcal{G})g^{-1} \equiv N(\mathcal{G}) \ , \tag{26}$$

so that $N(\mathcal{G}) \subset K(\mathcal{G})$ is the largest normal subgroup of G_0 contained in the kernel $K(\mathcal{G})$.

Each quotient G_0/C_ℓ is a finite group, and the collection $\mathcal{C} = \{C_\ell\}_{\ell \geq 0}$ forms a descending chain of normal subgroups of G_0. The inclusions of coset spaces define bonding homomorphisms $\delta_\ell^{\ell+1}$ for the inverse sequence of quotient groups G_0/C_ℓ, and the inverse limit space

$$C_\infty = \varprojlim \left\{ \delta_\ell^{\ell+1} \colon G_0/C_{\ell+1} \to G_0/C_\ell \right\} \tag{27}$$

$$= \{(eG_0, g_1C_1, \ldots) \mid g_\ell C_\ell = g_{\ell+1}C_\ell\} \ \subset \prod_{\ell \geq 0} G_0/C_\ell \tag{28}$$

is a profinite group. Let $\widehat{\iota} \colon G_0 \to C_\infty$ be the homomorphism defined by $\widehat{\iota}(g) = (gC_\ell)$ for $g \in G_0$. Note that $N(\mathcal{G}) = ker(\widehat{\iota})$, so that $\widehat{\iota}$ is injective if and only if the kernel $K(\mathcal{G})$ has trivial core.

Let $\widehat{\Phi}_0$ be the induced left action of G_0 on C_∞, where $\widehat{\Phi}_0(g)(g_\ell C_\ell) = (gg_\ell C_\ell)$ for $g \in G_0$. Let $(C_\infty, G_0, \widehat{\Phi}_0)$ denote the minimal Cantor system defined by this action. As C_∞ is a group, the isotropy subgroup of the identity $\widehat{e}_\infty = (eC_\ell) \in C_\infty$ is the subgroup $N(\mathcal{G}) \subset G_0$ which is the kernel of the homomorphism $\Phi_0 \colon G_0 \to Homeo(C_\infty)$.

Introduce the descending chain of clopen neighborhoods of the identity $\widehat{e}_\infty$ in C_∞:

$$C_{n,\infty} = \lim_{\longleftarrow} \left\{\delta_\ell^{\ell+1} \colon G_n/C_{\ell+1} \to G_n/C_\ell \mid \ell \geq n\right\} = \{(g_\ell C_\ell) \in C_\infty \mid g_\ell \in G_n\} \ . \tag{29}$$

Note that $C_{n,\infty}$ can also be identified with the closure in C_∞ of the image of $\widehat{\iota}(G_n)$.

The collection $\{C_{n,\infty} \mid n \geq 1\}$ of clopen subsets of C_∞ defines a neighborhood system of $\widehat{e}_\infty$.

Observe that for each $\ell \geq 0$, the quotient group $D_\ell = G_\ell/C_\ell \subset G_0/C_\ell$ and so the inverse limit

$$\mathcal{D} = \lim_{\longleftarrow} \left\{\delta_\ell^{\ell+1} \colon D_{\ell+1} \to D_\ell\right\} \tag{30}$$

is a closed subgroup of C_∞. The group $\mathcal{D}$ is called the *discriminant group* of the action (V_0, G_0, Φ_0).

The relationship between C_∞ and the Ellis group of (X_∞, G_0, Φ_0) is given by the following result.

THEOREM 7.1 (Theorem 4.4, [20])**.** *Let (X_∞, G_0, Φ_0) be the equicontinuous minimal Cantor action associated to the group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$. Then the map $\widehat{\iota} \colon G_0 \to C_\infty$ induces an isomorphism of topological groups $\Upsilon_0 \colon \overline{\Phi_0(G_0)} \to C_\infty$ such that the restriction gives an isomorphism $\Upsilon_{\widehat{e}} \colon \overline{\Phi_0(G_0)}_{\widehat{e}} \cong \mathcal{D}$.*

Proof. We give the essential idea of the proof. For $\ell \geq 1$, we have $N(\mathcal{G}) \subset C_\ell$, so the image $\Phi_0(G_0) \cong G_0/N(\mathcal{G})$ maps onto the finite quotient group G_0/C_ℓ. Moreover, the subgroup G_ℓ/C_ℓ is the isotropy group of the action of G_0/C_ℓ on X_ℓ. Passing to inverse limits, we obtain the isomorphisms $\Upsilon_0 \colon \overline{\Phi_0(G_0)} \to C_\infty$ and $\Upsilon_{\widehat{e}} \colon \overline{\Phi_0(G_0)}_{\widehat{e}} \to \mathcal{D}$. □

The following is a key observation about the discriminant subgroup defined by (30).

PROPOSITION 7.2 (Proposition 5.3, [20])**.** *Let (X_∞, G_0, Φ_0) be the equicontinuous minimal Cantor action associated to the group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$. Then*

$$\mathrm{core}_{G_0}\, \mathcal{D} \ = \ \bigcap_{g \in G_0} g\ \mathcal{D}\ g^{-1} \ = \ \{\widehat{e}\} \ . \tag{31}$$

Moreover, the maximal normal subgroup in C_∞ of $\mathcal{D}$ is trivial.

7.2. **Stable group chains.** We next consider the stability properties of the discriminant group for the minimal Cantor system (X_∞, G_0, Φ_0) associated to the group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$. We use the notations of Section 4. For $g \in G_0$ and $(g_\ell G_\ell) \in X_\infty$ let $g \cdot (g_\ell G_\ell) = \Phi_0(g)(g_\ell G_\ell)$.

For $n \geq 1$, define the clopen neighborhood of $\widehat{x}_0 = (eG_\ell) \in X_\infty$,

$$U_n = \{(g_\ell G_\ell) \in X_\infty \mid g_\ell \in G_n\} \subset X_\infty \ . \tag{32}$$

Note that for each $n \geq 1$, the G_0-translates of the set U_n form a partition of X_∞, so the collection $\{g \cdot U_n \mid n \geq 1\ ,\ g \in G_0\}$ forms a basis of clopen sets for the topology of X_∞. In particular, the restriction of the G_0-action on X_∞ to the set U_n is given by the action of G_n on U_n.

Let $\mathcal{H}_{U_n} = \Phi_0(G_n) \subset Homeo(U_n)$, then the Ellis group for the restricted action to U_n is the closure $\overline{\mathcal{H}_{U_n}} = \overline{\Phi_0(G_n)} \subset Homeo(U_n)$. For $m > n \geq 0$, note that $\mathcal{H}_{U_n,U_m} = \Phi_0(G_m) \subset \mathcal{H}_{U_n}$, and so $\overline{\mathcal{H}_{U_n,U_m}} = \overline{\Phi_0(G_m)} \subset \overline{\mathcal{H}_{U_n}}$. We next express the restriction map $\overline{\rho}_{U_n,U_m} \colon \overline{\mathcal{H}_{U_n,U_m}} \to \overline{\mathcal{H}_{U_m}}$ and its kernel in terms of group chains and their properties.

We first derive an inverse limit presentation for $\overline{\mathcal{H}_{U_n}}$. Introduce the truncated group chain associated to the restricted action of G_n on U_n,

$$\mathcal{G}_n = \{G_\ell\}_{\ell \geq n} = \{G_n \supset G_{n+1} \supset G_{n+2} \supset \cdots\} \ . \tag{33}$$

For each $\ell \geq n \geq 0$, we have the core subgroup for the truncated chain $\mathcal{G}_n$ defined by

$$C_\ell^n \equiv \mathrm{core}_{G_n}\, G_\ell \equiv \bigcap_{g \in G_n} g G_\ell g^{-1} \subset G_n \ . \tag{34}$$

Observe that C^n_ℓ is the kernel of the action of G_n on the quotient set G_n/G_ℓ, and $C^0_\ell = C_\ell$. Moreover, for all $\ell > m \geq n \geq 0$, we have $C^n_\ell \subset C^m_\ell \subset G_\ell \subset G_m \subset G_n$, and C^n_ℓ is a normal subgroup of G_m. Define the profinite group

$$\mathfrak{C}^n_m \cong \lim_{\longleftarrow} \left\{ \delta^{\ell+1}_\ell \colon G_m/C^n_\ell \to G_m/C^n_{\ell+1} \mid \ell \geq m \right\} \tag{35}$$

$$= \left\{ (g_\ell C^n_\ell) \mid \ell \geq m \ , \ g_m \in G_m \ , \ g_{\ell+1} C^n_\ell = g_\ell C^n_\ell \right\} \ . \tag{36}$$

Then $\mathfrak{C}^n_n$ is the Ellis group for the truncated group chain $\mathcal{G}_n$. In particular, $\mathfrak{C}^0_0 = C_\infty$.

By definition we have that $\mathfrak{C}^n_m \subset \mathfrak{C}^n_n$, and so $\mathfrak{C}^n_m$ is a clopen neighborhood of the identity in $\mathfrak{C}^n_n$. For example, $\mathfrak{C}^0_m$ is the clopen neighborhood of the identity $\widehat{e}_\infty$ in C_∞ introduced in (29).

For $n \geq 0$, the homomorphism $\iota_n \colon G_n \to \mathfrak{C}^n_n$ has dense image, hence induces an isomorphism $\Upsilon_n \colon \overline{\mathcal{H}_{U_n}} \to \mathfrak{C}^n_n$, which follows by analogous reasons as that of Theorem 7.1. That is, $\mathfrak{C}^n_n$ is isomorphic to the Ellis group of the action of G_n on U_n.

Next, for $m \geq n \geq 0$, the subgroup $\iota_n(G_m) \subset \mathfrak{C}^n_m$ is dense, hence induces an isomorphism $\Upsilon_{n,m} \colon \overline{\mathcal{H}_{U_n,U_m}} \to \mathfrak{C}^n_m$. That is, $\mathfrak{C}^n_m \subset \mathfrak{C}^n_n$ is a group chain model for $\overline{\mathcal{H}_{U_n,U_m}} \subset \overline{\mathcal{H}_{U_n}}$.

For each $\ell \geq k \geq m \geq n$, the inclusion $C^n_\ell \subset C^m_\ell$ induces group surjections denoted by

$$\phi^{k,\ell}_{n,m} \colon G_k/C^n_\ell \longrightarrow G_k/C^m_\ell \ . \tag{37}$$

For $k \geq m \geq n \geq 0$, standard methods as in [49], show that the maps in (37) yield surjective homomorphisms of profinite groups $\widehat{\phi}^k_{n,m} \colon \mathfrak{C}^n_k \to \mathfrak{C}^m_k$ which commute with the left action of G_0. In particular, for $k = m$, we obtain a surjective homomorphism

$$\widehat{\phi}^m_{n,m} \colon \mathfrak{C}^n_m \to \mathfrak{C}^m_m \tag{38}$$

which is a chain representation of the restriction map $\overline{\rho}_{U_n,U_m}$. We express its kernel in terms of the discriminant groups of the truncated group chains $\mathcal{G}_n$.

For $m > n \geq 0$, the discriminant group associated to the group chain $\mathcal{G}_n$ is given by

$$\mathcal{D}_n = \lim_{\longleftarrow} \left\{ \delta^n_{\ell,\ell+1} \colon G_{\ell+1}/C^n_{\ell+1} \to G_\ell/C^n_\ell \mid \ell \geq n \right\} \subset \mathfrak{C}^n_n \tag{39}$$

$$\cong \lim_{\longleftarrow} \left\{ \delta^n_{\ell,\ell+1} \colon G_{\ell+1}/C^n_{\ell+1} \to G_\ell/C^n_\ell \mid \ell \geq m \right\} \subset \mathfrak{C}^n_m \ . \tag{40}$$

Let $\mathcal{D}_{n,m} \subset \mathfrak{C}^n_m$ denote the image of $\mathcal{D}_n$ under the map (40). It then follows from (37) that for $m > n \geq 0$, there are surjective homomorphisms,

$$\mathcal{D} \ \stackrel{\psi_{0,n}}{\longrightarrow} \ \mathcal{D}_n \cong \mathcal{D}_{n,m} \ \stackrel{\psi_{n,m}}{\longrightarrow} \ \mathcal{D}_m \ , \tag{41}$$

where the map $\psi_{n,m}$ is the restriction of the surjection $\widehat{\phi}^m_{n,m} \colon \mathfrak{C}^n_m \to \mathfrak{C}^m_m$ in (38).

We can now formulate the notion of a stable group chain, as introduced in the work [22].

DEFINITION 7.3. *A group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ is said to be* stable *if there exists $n_0 \geq 0$ such that for all $m > n \geq n_0$, the map $\psi_{n,m} \colon \mathcal{D}_n \to \mathcal{D}_m$ in* (41) *is an isomorphism. Otherwise, the group chain is said to be* wild.

We then have the following fundamental result:

PROPOSITION 7.4. *Let $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ be a group chain. Then the equicontinuous minimal Cantor action (X_∞, G_0, Φ_0) associated to $\mathcal{G}$ is stable in the sense of Definition 4.6 if and only if the group chain $\mathcal{G}$ is stable in the sense of Definition 7.3.*

Proof. Recall that for $U_m \subset X_\infty$ defined by (32) we have $G_m = \{g \in G_0 \mid \Phi_0(g)(U_m) = U_m\}$.

The group $\mathfrak{C}^m_m$ is the inverse limit group formed from the quotients G_m/C^m_ℓ, where for $\ell \geq m$, the subgroup $C^m_\ell \subset G_\ell \subset G_m$ is normal in G_m. As noted above, the group $\mathfrak{C}^m_m$ acts transitively on U_m with isotropy group $\mathcal{D}_m$.

Analogously, the group $\mathfrak{C}^n_m$ is the inverse limit group formed from the quotients G_m/C^n_ℓ, where for $\ell \geq m \geq n$, the subgroup $C^n_\ell \subset G_\ell \subset G_m$ is normal in G_n, hence is also normal in $G_m \subset G_n$. As $\widehat{\phi}^m_{n,m} \colon \mathfrak{C}^n_m \to \mathfrak{C}^m_m$ in (38) is surjective, the group $\mathfrak{C}^n_m$ also acts transitively on U_m, with isotropy $\mathcal{D}_{n,m}$.

Thus, we have the commutative diagram of fibrations,

$$\begin{tikzcd} \mathcal{D}_{n,m} \arrow[r, "\psi_{n,m}"] \arrow[d] & \mathcal{D}_m \arrow[d] \\ \mathfrak{C}_{n,m} \arrow[r, "\widehat{\phi}^m_{n,m}"] \arrow[d] & \mathfrak{C}_{m,m} \arrow[d] \\ U_m \arrow[r, "="] & U_m \end{tikzcd} \tag{42}$$

Assume first that the group chain $\mathcal{G}$ is stable in the sense of Definition 7.3, and let $n_0 \geq 0$ be such that $\psi_{n,m} \colon \mathcal{D}_{n,m} \to \mathcal{D}_m$ is an isomorphism for all $m > n \geq n_0$. Then the center map $\widehat{\phi}^m_{n,m}$ in (42) is an isomorphism. Thus, for $\widehat{g} \in \mathfrak{C}^n_m$ if $\widehat{\phi}^m_{n,m}(\widehat{g}) \in \mathfrak{C}^m_m$ acts trivially on U_m, then $\widehat{g}$ also acts trivially on U_m. That is, the restriction map $\overline{\rho}_{U_n,U_m}$ is injective. As this holds for all $m > n \geq n_0$, by Proposition 4.10 this implies that the action Φ_0 is stable in the sense of Definition 4.6.

Now suppose that the action $\Phi_0 \colon G_0 \times X_\infty \to X_\infty$ is stable in the sense of Definition 4.6. The collection of clopen sets $\{U_\ell \mid \ell \geq 0\}$ is a neighborhood basis around $x_0 \in X_\infty$, so there exists $n_0 > 0$ such that the restriction map $\overline{\rho}_{U_n,U_m}$ on closures in (8) is an isomorphism for all $m > n \geq n_0$. We claim that $\psi_{n,m} \colon \mathcal{D}_n \cong \mathcal{D}_{n,m} \to \mathcal{D}_m$ is an isomorphism.

Let $\widehat{g} \in \overline{\mathcal{H}_{U_n,U_m}}$, and suppose that $\overline{\rho}_{U_n,U_m}(\widehat{g})$ is the identity. Then we can write $\widehat{g} = (g_\ell C^n_\ell) \in \mathfrak{C}^n_m$ where $g_\ell \in G_m$ for $\ell \geq m$, and the element $\widehat{\phi}^m_{n,m}(g_\ell C^n_\ell) = (g_\ell C^m_\ell) \in \mathfrak{C}^m_m$ acts as the identity by multiplication on the left on $U_m \cong \mathfrak{C}^m_m/\mathcal{D}_m$. That is, for all $(h_\ell C^m_\ell) \in \mathfrak{C}^m_m$, we have the coset identity

$$(g_\ell C^m_\ell) \cdot (h_\ell C^m_\ell)\ \mathcal{D}_m = (h_\ell C^m_\ell)\ \mathcal{D}_m\ ,$$

which implies that $(g_\ell C^m_\ell) \in (h_\ell C^m_\ell)\mathcal{D}_m(h_\ell C^m_\ell)^{-1}$ for all $(h_\ell C^m_\ell) \in \mathfrak{C}^m_m$. We have that

$$\mathrm{core}_{G_m}\mathcal{D}_m = \bigcap_{h \in G_m} h\ \mathcal{D}_m\ h^{-1} = \{\widehat{e}_m\}\ ,\ \widehat{e}_m = (eC^m_\ell) \in \mathfrak{C}^m_m\ , \tag{43}$$

which implies that $(g_\ell C^m_\ell) = \widehat{e}_m$, or $g_\ell \in C^m_\ell$ for all $\ell \geq m$. Thus we may suppose that $\widehat{g} = (g_\ell C^n_\ell)$ with $g_\ell \in C^m_\ell$ for all $\ell \geq m$.

By the assumption that the action is stable, we have that $\widehat{g}$ acts trivially on U_n. In particular, for all $(h_\ell C^n_\ell) \in \mathfrak{C}^n_m$, we have the identity of cosets,

$$(g_\ell C^n_\ell) \cdot (h_\ell C^n_\ell)\ \mathcal{D}_{n,m} = (h_\ell C^n_\ell)\ \mathcal{D}_{n,m}\ , \tag{44}$$

which implies by an argument as before that $(g_\ell C^n_\ell) = \widehat{e}_n$.

This shows that if $\widehat{g} \in \mathfrak{C}_{n,m}$ is mapped onto $\widehat{e}_m \in \mathfrak{C}^m_m$, then $\widehat{g} = \widehat{e}_n$, hence the kernel of the map $\widehat{\phi}^m_{n,m}$ is trivial. Since $\psi_{n,m}$ is the restriction of $\widehat{\phi}^m_{n,m}$, it also has a trivial kernel. This shows that the group chain $\mathcal{G}$ is stable in the sense of Definition 7.3. □

7.3. **Asymptotic discriminant for group chains.** We consider the asymptotic discriminant invariant for a minimal Cantor system (X_∞, G_0, Φ_0) associated to the group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$. We use the notations of Sections 4 and 5.

Recall the definition of the clopen sets U_n in (32). It was observed that the restriction of the G_0-action on X_∞ to the set U_n is given by the action of G_n on U_n, so that the collection of clopen sets $\mathcal{U} = \{U_\ell \mid \ell \geq 0\}$ is an adapted neighborhood basis at $x_0 \in X_\infty$ for the action Φ_0.

The discriminant for the action of G_n on U_n is given by $\mathcal{D}_n$ as in (39), which either by Theorem 7.1 or direct calculation, is identified with the isotropy group at x_0 of the action of $\mathfrak{C}_{n,n}$ on U_n. Thus, in the notation of the definition (15), we have the identification $\mathcal{D}_n = \mathcal{I}(U_1, U_n, x_0)$. Moreover, the surjective maps in Corollary 5.7 are identified with the surjective maps in (41), so for $m > n \geq 1$ there is a commutative diagram

$$\begin{array}{ccccc} \sigma_{U_n,U_m} & : & \mathcal{I}(U_1, U_n, x) & \longrightarrow & \mathcal{I}(U_1, U_m, x) \\ & & \cong\big\downarrow & & \big\downarrow\cong \\ \psi_{n,m} & : & \mathcal{D}_n \cong \mathcal{D}_{n,m} & \longrightarrow & \mathcal{D}_m \cong \mathcal{D}_{m,m} \end{array} \tag{45}$$

Then in analogy with Definition 5.12, we introduce the following invariant for $\mathcal{G}$.

DEFINITION 7.5. *Let $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$ be a group chain. Then the* asymptotic discriminant *for $\mathcal{G}$ is the tail equivalence class $[\mathcal{D}(\mathcal{G})]_\infty$ of the sequence of surjective group homomorphisms*

$$\mathcal{D}(\mathcal{G}) = \{\psi_{n,n+1} \colon \mathcal{D}_n \to \mathcal{D}_{n+1} \mid n \geq 1\} \tag{46}$$

defined by the discriminant groups for the restricted actions of G_n on the clopen sets $U_n \subset \mathfrak{X}_\infty$.

The definition of $\mathcal{D}(\mathcal{G})$ above involves a minor abuse of notation, or at least a change in usage for the notation. In the works [19, 20, 21] the discriminant group of a group chain was defined to be $\mathcal{D} = \mathcal{D}_1$ while the work [22] studied the discriminant groups $\mathcal{D}_n$ as a function of n, and was concerned mainly with the case when the group chain is stable, so that $\mathcal{D}_n \cong \mathcal{D}_m$ for some $n_0 \geq 1$ and $m > n \geq n_0$. By Lemma 5.3, in the stable case the sequence of surjective group homomorphisms $\mathcal{D}(\mathcal{G})$ as defined above is asymptotically constant, so the two usages are thus compatible. For the case when $\mathcal{D}(\mathcal{G})$ is unstable, or wild, then the notion of the discriminant invariant for the group chain $\mathcal{G}$ depends on the section, so can be considered as indeterminately defined. Thus, the usage in Definition 7.5 is more precise.

The discussion above shows the following:

PROPOSITION 7.6. *Let (X_∞, G_0, Φ_0) be the equicontinuous minimal Cantor system associated to the group chain $\mathcal{G} = \{G_\ell\}_{\ell \geq 0}$. Then the asymptotic class $[\mathcal{I}(\Phi_0)]_\infty$ for the sequence of surjective homomorphisms defined by* (16) *equals the asymptotic class $[\mathcal{D}(\mathcal{G})]_\infty$ for the sequence of surjective homomorphisms $\mathcal{D}(\mathcal{G})$ defined by* (46).

It remains to show that there exists an abundance of examples of wild group chains for which the asymptotic discriminant invariants are distinct, so that this is an effective invariant of weak solenoids. We construct such examples in the next two sections.

8. General constructions of solenoids

In this section, we recall two general constructions which are used to produce examples of weak solenoids with prescribed properties. The first is a well-known construction which yields a foliated space from a group action, called the *suspension construction*, as discussed in [14, Chapter 3] for example, or Chapter 6 of [13] in the case of smooth foliations. In Section 8.1 we describe this construction in the context of Cantor actions.

The second construction produces group chains whose associated equicontinuous minimal Cantor systems (X_∞, G_0, Φ_0) have prescribed discriminant invariants. This is described in Section 8.2, and is inspired by a construction introduced by Fokkink and Oversteegen [27, Lemma 37] and attributed to Hendrik Lenstra. This method was used in [22, Section 10] to construct examples of stable but not homogeneous solenoids. In Section 9, the methods of Section 8.2 are used to construct examples of wild actions, and thus by the methods of Section 8.1 we obtain examples of wild solenoids.

8.1. **The suspension construction.** Let X be a Cantor space, and H a *finitely-generated* group, though not necessarily finitely-presented, and assume there is given an action $\varphi \colon H \to Homeo(X)$. Suppose that H admits a generating set $\{g_1, \ldots, g_k\}$, let $\alpha_k \colon \mathbb{Z} * \cdots * \mathbb{Z} \twoheadrightarrow H$ be the surjective homomorphism from the free group on k generators to H, given by mapping generators to generators. Next, let Σ_k be a compact surface without boundary of genus k. For a choice of basepoint $x_0 \in \Sigma_k$, set $G_0 = \pi_1(\Sigma_k, x_0)$, then there is a homomorphism $\beta_k : G_0 \to \mathbb{Z} * \cdots * \mathbb{Z}$ onto the free group of k generators. The composition of these maps yields a homomorphism

$$\Phi_0 = \varphi \circ \alpha_k \circ \beta_k : G_0 = \pi_1(\Sigma_k, x_0) \to \mathbb{Z} * \cdots * \mathbb{Z} \to H \to Homeo(X) \ . \tag{47}$$

Next, let $\widetilde{\Sigma}_k$ denote the universal covering space of Σ_k, equipped with the right action of G_0 by covering transformations. Form the product space $\widetilde{\Sigma}_k \times X$ which has a foliation $\widetilde{\mathcal{F}}$ whose leaves are the slices $\widetilde{\Sigma}_k \times \{x\}$ for each $x \in X$. Define a right action of G_0 on $\widetilde{\Sigma}_k \times X$, which for $g \in G_0$ is given by $(y, x) \cdot g = (y \cdot g, \Phi_0(g^{-1})(x))$. For each g, this action preserves the foliation $\widetilde{\mathcal{F}}$, so we obtain a foliation $\mathcal{F}_{\mathfrak{M}}$ on the quotient space $\mathfrak{M} = (\widetilde{\Sigma}_k \times X)/G_0$. Note that all leaves of $\mathcal{F}_{\mathfrak{M}}$ are surfaces, which are in general non-compact.

The space $\mathfrak{M}$ is a foliated Cantor bundle over Σ_k, and the holonomy of this bundle $\pi : \mathfrak{M} \to \Sigma_k$ acting on the fiber $V_0 = \pi^{-1}(x_0)$ is canonically identified with the action $\Phi_0 : G_0 \to Homeo(X)$. Consequently, if the action Φ_0 is minimal then the foliation $\mathcal{F}_{\mathfrak{M}}$ is minimal. If the action Φ_0 is equicontinuous then $\mathcal{F}_{\mathfrak{M}}$ is an equicontinuous foliation in the sense of Definition 2.5.

There is a variation of the above construction, where we assume that G_0 is a *finitely-presented* group, and there is given a homomorphism $\Phi_0 : G_0 \to Homeo(X)$. Then there is a folklore result (for example, see [41]) that there exists a closed connected 4-manifold M such that for a choice of basepoint $b_0 \in M$, the fundamental group $\pi_1(M, b_0)$ is isomorphic to G_0. Then the suspension construction can be applied to the homomorphism $\Phi_0 : \pi_1(M, b_0) \to Homeo(X)$, where we replace Σ_k above with M, and the space $\widetilde{\Sigma}_k$ with the universal covering $\widetilde{M}$ of M.

8.2. **The Lenstra construction.** We give a general construction of a group chain with prescribed core and discriminant groups. See [22, Section 10] for further discussion of this construction, and the proof of Lemma 37 in [27] for the motivation behind this approach to constructing group chains.

Let $\widehat{H}$ be a given profinite group, with a finitely-generated dense subgroup $H \subset \widehat{H}$. Let $\mathcal{D} \subset \widehat{H}$ be a closed subgroup of infinite index, and so $\mathcal{D}$ is either a finite group or is a profinite group.

The quotient $X = \widehat{H}/\mathcal{D}$ is a Cantor space. As H is dense in $\widehat{H}$, the left action of H on X is minimal. The *rational core* in $\widehat{H}$ of $\mathcal{D}$ is defined to be the subgroup

$$\mathrm{core}_H \ \mathcal{D} = \bigcap_{h \in H} h\mathcal{D}h^{-1} \subset \widehat{H} \ . \tag{48}$$

LEMMA 8.1. *The left action of $\widehat{H}$ on X is effective if and only if* $\mathrm{core}_H \ \mathcal{D} = \{\widehat{e}\}$.

Proof. Suppose there exists $\widehat{e} \neq \widehat{k} \in \mathrm{core}_H \ \mathcal{D}$, then $h^{-1}\widehat{k}h \in \mathcal{D}$ for all $h \in H$, and so $\widehat{k} \cdot h\mathcal{D} = h\mathcal{D}$ for all $h \in H$. Thus, the continuous action of $\widehat{k}$ fixes the dense subset $\{h\mathcal{D} \mid h \in H\} \subset X$, hence must act trivially on all of X. That is, the action of $\widehat{H}$ on X is not effective. Thus, if the action is effective, then $\mathrm{core}_H \ \mathcal{D} = \{\widehat{e}\}$.

To show the converse, assume that $\mathrm{core}_H \ \mathcal{D} = \{\widehat{e}\}$, then observe that

$$\mathrm{core}_{\widehat{H}} \ \mathcal{D} = \bigcap_{\widehat{h} \in \widehat{H}} \widehat{h}\mathcal{D}\widehat{h}^{-1} \subset \bigcap_{h \in H} h\mathcal{D}h^{-1} = \mathrm{core}_H \ \mathcal{D} = \{\widehat{e}\} \ .$$

Thus, the intersection of all isotropy groups $\{\widehat{h}\mathcal{D}\widehat{h}^{-1} \mid \widehat{h} \in \widehat{H}\}$ for the action of $\widehat{H}$ on X is trivial, hence the action is effective. □

Now assume that the closed subgroup $\mathcal{D} \subset \widehat{H}$ is chosen with infinite index and $\mathrm{core}_H \ \mathcal{D} = \{\widehat{e}\}$. Thus, by Lemma 8.1 the left action of $\widehat{H}$ on X is effective. We give the construction of a group chain $\mathcal{H} = \{H_\ell\}_{\ell \geq 0}$ with $H_0 = H$, and discriminant group isomorphic to $\mathcal{D}$.

By the assumption that $\widehat{H}$ is a profinite group, there exists a clopen neighborhood system $\{\widehat{U}_\ell \mid \ell \geq 1\}$ about the identity in $\widehat{H}$, so that:

(1) each $\widehat{U}_\ell$ is a normal subgroup of $\widehat{H}$;
(2) for each $\ell \geq 0$ there is a proper inclusion $\widehat{U}_{\ell+1} \subset \widehat{U}_\ell$;
(3) the intersection $\bigcap_{\ell \geq 1} \widehat{U}_\ell = \{\widehat{e}\}$.

In particular, each quotient $\widehat{H}/\widehat{U}_\ell$ is a finite group. Let $\iota_\ell^{\ell+1} \colon \widehat{H}/\widehat{U}_{\ell+1} \to \widehat{H}/\widehat{U}_\ell$ be the map induced by the inclusions of cosets. The sequence of quotient maps

$$\{q_\ell \colon \widehat{H} \to \widehat{H}/\widehat{U}_\ell \mid \ell \geq 1\} \tag{49}$$

induces an isomorphism ι_∞ of $\widehat{H}$ with the inverse limit group

$$\iota_\infty \colon \widehat{H} \cong H_\infty \equiv \varprojlim \left\{\iota_\ell^{\ell+1} \colon \widehat{H}/\widehat{U}_{\ell+1} \to \widehat{H}/\widehat{U}_\ell\right\} \ . \tag{50}$$

Next, for each $\ell \geq 1$, set $\widehat{W}_\ell = \widehat{U}_\ell \cdot \mathcal{D}$ which is a subgroup of $\widehat{H}$, as $\widehat{U}_\ell$ is a normal subgroup. Moreover, the assumption that $\mathcal{D}$ is compact implies that each $\widehat{W}_\ell$ is a clopen subset of $\widehat{H}$. Set $H_\ell = H \cap \widehat{W}_\ell$ which is a subgroup of finite index in H, and so $\mathcal{H} = \{H_\ell\}_{\ell \geq 0}$ is a group chain in H.

Next, for each $\ell \geq 0$, set $X_\ell = H/H_\ell$ which is a finite set with a transitive left action of H. There are quotient maps $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$ which commute with the G-action, and introduce the inverse limit space

$$X_\infty = \varprojlim \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell\} \subset \prod_{\ell \geq 0} X_\ell \ , \tag{51}$$

which is a totally disconnected compact perfect set, so is a Cantor set. The diagonal left action of H on X_∞ is equicontinuous and minimal, so we obtain an equicontinuous minimal Cantor system (X_∞, H, Φ). Note that H is dense in $\widehat{H}$ implies that

$$X_\ell = H/H_\ell \cong \widehat{H}/\widehat{W}_\ell. \tag{52}$$

Thus, there are surjections

$$\tau_\ell \colon X \equiv \widehat{H}/\mathcal{D} \to \widehat{H}/\widehat{W}_\ell \cong H/H_\ell = X_\ell \tag{53}$$

such that $p_{\ell+1} \circ \tau_{\ell+1} = \tau_\ell$. The collection of maps $\{\tau_\ell \colon X \to X_\ell \mid \ell \geq 1\}$ then induces an H-equivariant homeomorphism $\tau_\infty \colon X \cong X_\infty$. In particular, for each $y = (g_\ell H_\ell) \in X_\infty$ there is a sequence $(h_\ell) \in \widehat{H}$ such that for the sequence of cosets $(h_\ell \mathcal{D}) \in \widehat{H}/\mathcal{D}$ we have $\tau_\infty(h_\ell \mathcal{D}) = (g_\ell H_\ell)$.

We next relate the closed group $\mathcal{D} \subset \widehat{H}$ to the discriminant group $\mathcal{D}_\infty$ for the group chain $\mathcal{H} = \{H_\ell\}_{\ell \geq 0}$. Recall that $C_\ell = \mathrm{core}_H \ H_\ell = \bigcap_{g \in H} g H_\ell g^{-1} \subset H$ is the core in H of H_ℓ, and

$$\mathcal{D}_\infty \equiv \varprojlim \left\{\delta_\ell^{\ell+1} \colon H_{\ell+1}/C_{\ell+1} \to H_\ell/C_\ell\right\} \ . \tag{54}$$

LEMMA 8.2. *There is an isomorphism $\mathcal{D}_\infty \to \mathcal{D}$.*

Proof. Consider the quotient maps (49), that is, for $\ell \geq 1$, we have $q_\ell \colon \widehat{H} \to \widehat{H}/\widehat{U}_\ell$, where $\widehat{H}/\widehat{U}_\ell$ is a finite group. Since H is dense in $\widehat{H}$, it has non-trivial intersection with each clopen set $g\widehat{U}_\ell$. Then

$$\mathcal{D}^\ell \equiv q_\ell(\mathcal{D}) = q_\ell(\widehat{W}_\ell) = q_\ell(H \cap \widehat{W}_\ell) = q_\ell(H_\ell) \ , \tag{55}$$

where the first equality holds because $\widehat{W}_\ell = \widehat{U}_\ell \cdot \mathcal{D}$. As q_ℓ are homomorphisms, and the rational core of $\mathcal{D}$ is trivial by Lemma 8.1, we have

$$\begin{aligned} q_\ell(C_\ell) &= q_\ell(\bigcap_{g \in H} g \ H_\ell \ g^{-1}) = \bigcap_{g \in H} q_\ell(g) \ q_\ell(H_\ell) \ q_\ell(g)^{-1} \\ &= \bigcap_{g \in H} q_\ell(g) \ q_\ell(\mathcal{D}) \ q_\ell(g)^{-1} = q_\ell(\bigcap_{g \in H} g\mathcal{D}g^{-1}) = \{e_\ell\} \subset \widehat{H}/\widehat{U}_\ell \ . \end{aligned} \tag{56}$$

Since by definition C_ℓ is a subgroup of $H_\ell \subset H$, it follows that $C_\ell \subseteq H \cap \widehat{U}_\ell$. The subgroup $\widehat{U}_\ell$ is normal in $\widehat{H}$, so $H \cap \widehat{U}_\ell$ is normal in H and contained in $H_\ell = H \cap \widehat{W}_\ell$, hence $H \cap \widehat{U}_\ell \subseteq C_\ell$, as C_ℓ is maximal. We then obtain $C_\ell = H \cap \widehat{U}_\ell = H_\ell \cap \widehat{U}_\ell$. In particular, for all $\ell \geq 0$, we obtain induced isomorphisms on the quotients, $\overline{q}_\ell \colon H/C_\ell \to \widehat{H}/\widehat{U}_\ell$, and so by (55) we have that $\overline{q}_\ell(H_\ell/C_\ell) = q_\ell(\mathcal{D}) = \mathcal{D}^\ell$.

The map $\iota_\ell^{\ell+1} \colon H_{\ell+1} \to H_\ell$ induces a map $\iota_\ell^{\ell+1} \colon \mathcal{D}^{\ell+1} \to \mathcal{D}^\ell$, and so for the inverse limit we have

$$\mathcal{D}_\infty = \varprojlim \left\{\delta_\ell^{\ell+1} \colon H_{\ell+1}/C_{\ell+1} \to H_\ell/C_\ell\right\} \cong \varprojlim \left\{\iota_\ell^{\ell+1} \colon \mathcal{D}^{\ell+1} \to \mathcal{D}^\ell\right\} \cong \mathcal{D} \ , \tag{57}$$

where the isomorphism on the right hand side of (57) follows as $\{\widehat{U}_\ell \mid \ell \geq 1\}$ is a clopen neighborhood system about the identity in $\widehat{H}$. □

8.3. **Wild criteria.** We next formulate the stability property in Definition 7.3 in terms of the subgroup $\mathcal{D}$, and obtain criteria which are sufficient to imply that the equicontinuous minimal Cantor system (X_∞, H, Φ) is wild. First, note that for $n \geq 1$, we have $\mathcal{D} \subset \widehat{W}_n = \mathcal{D} \cdot \widehat{U}_n = \widehat{U}_n \cdot \mathcal{D}$, and define $U_n \equiv \widehat{W}_n/\mathcal{D} \subset \widehat{H}/\mathcal{D} = X$, which is a clopen neighborhood of the basepoint $x_0 = \widehat{e}\mathcal{D} \in X$. The proof of the following is immediate:

LEMMA 8.3. *For $\ell \geq 1$, $g \in H$ satisfies $g \cdot U_\ell = U_\ell$ if and only if $g \in H_\ell$.*

Lemma 8.3 implies that we can identify U_n with the inverse limit in (32). We now consider the Ellis group and the discriminant group of the truncated group chain $\mathcal{H}_n = \{H_\ell\}_{\ell \geq n}$ as in (39). For each $n \geq 1$, let $\rho_n \colon \widehat{W}_n \to Aut(\widehat{W}_n)$ which for $\widehat{h} \in \widehat{W}_n$ and $\widehat{g} \in \widehat{W}_n$ is defined by $\rho_n(\widehat{h})(\widehat{g}) = \widehat{h}\ \widehat{g}\ \widehat{h}^{-1}$, and let $\widehat{\rho}_n = \rho_n | \mathcal{D} \colon \mathcal{D} \to Aut(\widehat{W}_n)$ be its restriction to $\mathcal{D}$.

LEMMA 8.4. *Let $\widehat{h} \in \mathcal{D}$ then $\rho_n(\widehat{h}) \in Aut(\widehat{W}_n)$ is the identity implies that the left action of $\widehat{h}$ on U_n is the identity.*

Proof. Assume that $\rho_n(\widehat{h})$ is the identity, then $\widehat{h}\ \widehat{g}\ \widehat{h}^{-1} = \widehat{g}$ for every $\widehat{g} \in \widehat{W}_n$. Then for the action on cosets of $\mathcal{D}$ we have $\widehat{h} \cdot \widehat{g}\mathcal{D} = \widehat{h}\widehat{g}\mathcal{D} = \widehat{g}\widehat{h}\mathcal{D} = \widehat{g}\mathcal{D}$ as $\widehat{h} \in \mathcal{D}$. □

Let $\mathcal{K}_n = ker(\widehat{\rho}_n) \subset \mathcal{D}$ denote the kernel of the representation $\widehat{\rho}_n$. Then $m > n$ implies $\mathcal{K}_n \subseteq \mathcal{K}_m$.

We would like to obtain representations of the Ellis group of the restricted action on U_n and its discriminant group, in terms of $\widehat{W}_n$, $\mathcal{D}$ and $\mathcal{K}_n$. Note that the converse of Lemma 8.4 does not hold, as the identity

$$\widehat{d} \cdot \widehat{g}\mathcal{D} = \widehat{d}\widehat{g}\widehat{d}^{-1}\mathcal{D} = \widehat{g}\mathcal{D}$$

only implies that the adjoint action of $\widehat{d}$ permutes points within cosets of $\mathcal{D}$ in $\widehat{W}_n$ but need not act as the identity on $\mathcal{D}$. Thus the group $\mathcal{D}/\mathcal{K}_n$ may act on U_n non-effectively and cannot be identified with the isotropy group of the Ellis group action, and in particular, this means that $\mathcal{D}/\mathcal{K}_n$ may have non-trivial core in $\widehat{W}_n/\mathcal{K}_n$.

We want to find a condition which eliminates this situation. Consider the sequence

$$\widehat{W}_0 = \widehat{H} \supset \widehat{W}_1 \supset \cdots \supset \widehat{W}_n \supset \cdots \ .$$

The discriminant group $\mathcal{D}$ of the action of $\widehat{W}_0$ on $X = \widehat{W}_0/\mathcal{D}$ has trivial core by Proposition 7.2. Consider the action of $\widehat{W}_n$ on $U_n = \widehat{W}_n/\mathcal{D}$. Note that the action restricted to $\mathcal{D}$ induces an action of $\mathcal{D}/\mathcal{K}_n$ by Lemma 8.4. Suppose there exists $\widehat{g} \in \widehat{W}_n$ such that the action of $\widehat{g}$ on U_n is trivial, but $\widehat{g} \notin \mathcal{K}_n$. Then for each $\widehat{h} \in \widehat{W}_n$ we have $\widehat{g}\widehat{h}\mathcal{D} = \widehat{h}\mathcal{D}$, so $\widehat{g}\widehat{h}\mathcal{D}\widehat{h}^{-1} = \widehat{h}\mathcal{D}\widehat{h}^{-1}$, that is, $\widehat{g} \in \bigcap_{\widehat{h} \in \widehat{W}_n} \widehat{h}\mathcal{D}\widehat{h}^{-1}$.

Thus, the discriminant group of the restricted action is isomorphic to $\mathcal{D}/\mathcal{K}_n$ if the following condition is satisfied:

$$\mathrm{core}_{\widehat{W}_n}\ \mathcal{D}/\mathcal{K}_n = \bigcap_{\widehat{h} \in \widehat{W}_n} \widehat{h}\,(\mathcal{D}/\mathcal{K}_n)\ \widehat{h}^{-1} = \{\widehat{e}_n\} \ . \tag{58}$$

Recall that there are quotient maps $q_n \colon \widehat{H} \to \widehat{H}/\widehat{U}_n$, and denote $H_{n,n} = q_n(\widehat{W}_n) = q_n(H_n) = \mathcal{D}^n$, where the equalities follow from (55). Consider the truncated chain $\mathcal{H}_n = \{H_\ell\}_{\ell \geq n}$, and denote

$H_{n,\ell} = q_\ell(H_n)$. Also, we have $\mathcal{D}^\ell = q_\ell(\mathcal{D}) = q_\ell(H_\ell)$. A computation similar to (56) shows that

$$q_\ell(\bigcap_{g \in H_n} g \ H_\ell \ g^{-1}) = \bigcap_{g \in H_{n,\ell}} k \ \mathcal{D}^\ell \ k^{-1} = \mathrm{core}_{H_{n,\ell}} \ \mathcal{D}^\ell \ ,$$

and then (58) implies that for all $\ell > n > 1$ we must have

$$\mathrm{core}_{H_{n,\ell}} \ \mathcal{D}^\ell = \bigcap_{k \in H_{n,\ell}} k \mathcal{D}^\ell k^{-1} = \{e_\ell\} \subset H_{n,\ell} \ . \tag{59}$$

We then have the following sufficient criteria for an action to be wild:

PROPOSITION 8.5. *Suppose that the $H_{n,\ell}$–core of $\mathcal{D}^\ell$ is trivial in $\widehat{W}_\ell/\widehat{U}_\ell$ for all $\ell > n \geq 1$, and that the chain $\{\mathcal{K}_n \mid n \geq 1\}$ is not bounded above. Then the equicontinuous minimal Cantor system (X_∞, G, Φ) is wild.*

Proof. By assumption, there exists an increasing sequence of integers $\{n_\ell \mid \ell \geq 1\}$ such that for all $\ell \geq 1$, the inclusion $\mathcal{K}_{n_\ell} \subset \mathcal{K}_{n_{\ell+1}}$ is proper. Thus, there exists a sequence of elements $\widehat{d}_{\ell+1} \in \widehat{W}_{n_\ell}$ such that $\widehat{d}_{\ell+1} \in \mathcal{K}_{n_{\ell+1}}$ but $\widehat{d}_{\ell+1} \notin \mathcal{K}_{n_\ell}$. The assumption on the triviality of $H_{n,\ell}$–cores implies that the discriminant group of the restricted action is isomorphic to $\mathcal{D}/\mathcal{K}_n$. By assumption on the kernels $\mathcal{K}_n$ of the maps $\psi_{n,n+1} : \mathcal{D}/\mathcal{K}_n \to \mathcal{D}/\mathcal{K}_{n+1}$ of the discriminant groups, defined in Section 7.3, these maps are not injective for any $n \geq 0$, and it follows that the action of G on X is wild. □

REMARK 8.6. We note that the actions satisfying the conditions of Proposition 8.5 are not LCSQA. Indeed, consider the collection of elements $\{\widehat{d}_\ell\}$ chosen as in the proof of the proposition. Since $\widehat{d}_{\ell+1} \in \mathcal{K}_{n_{\ell+1}}$, by Lemma 8.4 $\widehat{d}_{\ell+1}$ acts trivially on the clopen set $U_{n_{\ell+1}} = \widehat{W}_{\ell+1}/\mathcal{D}$, and non-trivially on the larger set $U_{n_\ell} = \widehat{W}_\ell/\mathcal{D}$. Since $\bigcap U_{n_\ell}$ is a singleton, the action is not LCSQA.

We conclude with some remarks about wild actions. Recall that the kernel for the group chain $\mathcal{H}$ is

$$K(\mathcal{H}) = \bigcap_{\ell \geq 0} H_\ell = \bigcap_{\ell \geq 0} \left(H \cap \widehat{W}_\ell\right) = H \ \cap \ \bigcap_{\ell \geq 0} \widehat{W}_\ell = H \cap \mathcal{D} \ . \tag{60}$$

LEMMA 8.7. *Let $\mathcal{D} \subset \widehat{H}$ be a closed subgroup, and let $\mathcal{H} = \{H_\ell\}_{\ell \geq 0}$ with $H_0 = H \subset \widehat{H}$ be the group chain associated with the choice of a clopen neighborhood system $\{U_\ell \mid \ell \geq 1\}$ about the identity in $\widehat{H}$. Given $y = (g_\ell H_\ell) \in X \cong \widehat{H}/\mathcal{D}$, let $\mathcal{H}^y = \{g_\ell H_\ell g_\ell^{-1}\}_{\ell \geq 0}$ be the conjugate group chain to $\mathcal{H}$. Then for $\widehat{h} = (h_\ell) \in \widehat{H}$ such that $\tau(\widehat{h}\mathcal{D}) = y$, we have*

$$K(\mathcal{H}^y) = H \ \cap \ (\widehat{h} \ \mathcal{D} \ \widehat{h}^{-1}) \ . \tag{61}$$

In particular, if $\mathcal{D}$ satisfies $H \cap \widehat{g}\mathcal{D}\widehat{g}^{-1} = \{e\}$ for all $\widehat{g} \in \widehat{H}$, then the kernel of each group chain $\mathcal{H}^y$ centered at $y \in X$ is trivial.

If the kernel $K(\mathcal{H}^y)$ is trivial for all $y \in X$ then the action of H on X satisfies the SQA property. It may still happen that the action of the Ellis group $\widehat{H}$ of the action is not LCSQA, the hypotheses of Proposition 8.5 are satisfied, and so the action of H on X is wild.

9. Non-homeomorphic wild solenoids

In this section, we give a construction of weak solenoids whose monodromy actions are wild, inspired by the constructions of stable solenoids in Section 10.5 of [22]. More precisely, we construct equicontinuous minimal Cantor actions which are wild, then use the suspension construction described in Section 8.1 to realize these actions as the global monodromy actions of weak solenoids.

The following examples are inspired by the constructions of Lubotzky in [39] of compact torsion subgroups in the profinite completion of finitely generated, torsion free groups. We do not require the full strength of Lubotzky's methods, though the reader may find the discussion in Section 10.5 of [22] helpful for establishing the context.

9.1. **Wild actions.** We first recall some basic facts as used in [39]. For $N \geq 3$, let $\Gamma_N = \mathbf{SL}_N(\mathbb{Z})$ denote the $N \times N$ matrices with integer entries and determinant 1. The group Γ_N is finitely-generated and residually finite, and hence so are all finite index subgroups of Γ_N. Moreover, each subgroup of finite index $G \subset \Gamma_N$ is also *finitely-presented*, as discussed in the proof of [48, Theorem 4.7.10].

For an integer $M \geq 2$, let $\Gamma_N(M)$ denote the congruence subgroup

$$\Gamma_N(M) \equiv \mathrm{Ker}\,\{\varphi_M \colon \mathbf{SL}_M(\mathbb{Z}) \to \mathbf{SL}_N(\mathbb{Z}/M\mathbb{Z})\} \ .$$

For $M \geq 3$, $\Gamma_N(M)$ is torsion-free. Moreover, by the congruence subgroup property, every finite index subgroup of Γ_N contains $\Gamma_N(M)$ for some non-zero M; that is, congruence subgroups form a neighborhood basis about the identity for the profinite topology of $\mathbf{SL}_N(\mathbb{Z})$. This implies that

$$\widehat{\mathbf{SL}_N(\mathbb{Z})} \equiv \lim_{\longleftarrow} \ \mathbf{SL}_N(\mathbb{Z}/M\mathbb{Z}) \cong \mathbf{SL}_N(\widehat{\mathbb{Z}}) \cong \prod_{p \in \mathcal{P}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \ , \tag{62}$$

where $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, and $\widehat{\mathbb{Z}}_p$ is the p-adic completion of $\mathbb{Z}$. Here, $\mathcal{P} = \{p_i \mid i \geq 1\}$ is the set of primes listed in increasing order, where $p_1 = 2$ and $p_2 = 3$. The second isomorphism uses the observation that $\widehat{\mathbb{Z}} \cong \prod_{p \in \mathcal{P}} \widehat{\mathbb{Z}}_p$. Note that the matrix-group factors in the Cartesian product on the right hand side of (62) commute with each other.

Fix an integer $N \geq 3$. Let $G \subset \Gamma_N$ be a finite index, torsion-free subgroup, which is thus finitely presented, and its profinite completion $\widehat{G}$ is a clopen subgroup of finite index in $\widehat{\mathbf{SL}_N(\mathbb{Z})}$. Then there exists an index $k_G \geq 2$ such that for $\mathcal{P}(G) = \{p_i \mid i \geq k_G\}$ we have

$$\prod_{i \geq k_G} \mathbf{SL}_N(\widehat{\mathbb{Z}}_{p_i}) \subset \widehat{G} \ , \tag{63}$$

where the inclusion is via the isomorphism in (62). In particular, this means that the diagonal embedding of G is dense in the product of groups on the left-hand-side in (63).

Let $\widehat{H} = \prod_{i \geq k_G} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$ be the infinite product of finite groups, with the Tychonoff product topology, which is thus a Cantor group. The group $\mathbf{SL}_N(\mathbb{Z})$ maps onto each factor $\mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$, and let $Q \colon \mathbf{SL}_N(\mathbb{Z}) \to \widehat{H}$ denote the product of all these factor maps.

Let $H = Q(G) \subset \widehat{H}$ denote the image of G. Then H is a finitely-generated subgroup of $\widehat{H}$.

LEMMA 9.1. *The subgroup $H \subset \widehat{H}$ is dense in $\widehat{H}$.*

Proof. Let $\mathcal{I} \subset \mathcal{P}(G)$ be a finite subset of primes, and introduce the projection map

$$\theta_{\mathcal{I}} \colon \prod_{p \in \mathcal{P}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \longrightarrow \prod_{p \in \mathcal{I}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \ .$$

Also, for each $p \in \mathcal{I}$, there is a surjective map $\widehat{\mathbb{Z}}_p \to \mathbb{Z}/p\mathbb{Z}$, which induces a surjective map

$$\alpha_{\mathcal{I}} \colon \prod_{p \in \mathcal{I}} \mathbf{SL}_N(\widehat{\mathbb{Z}}_p) \longrightarrow \prod_{p \in \mathcal{I}} \mathbf{SL}_N(\mathbb{Z}/p\mathbb{Z}) \equiv \widehat{H}_{\mathcal{I}} \ ,$$

where $\widehat{H}_{\mathcal{I}}$ is a finite subgroup of $\widehat{H}$. Introduce the composition map $\kappa_{\mathcal{I}} = \alpha_{\mathcal{I}} \circ \theta_{\mathcal{I}} \colon \widehat{\mathbf{SL}_N(\mathbb{Z})} \to \widehat{H}_{\mathcal{I}}$.

Since $G \subset \mathbf{SL}_N(\mathbb{Z})$ is dense in the product (63), the restriction $\theta_{\mathcal{I}}|G$ has dense image, and thus the restriction $\kappa_{\mathcal{I}} \colon G \to \widehat{H}_{\mathcal{I}}$ is surjective, as $\widehat{H}_{\mathcal{I}}$ is finite so has the discrete topology.

The subgroups $\widehat{H}_{\mathcal{I}} \subset \widehat{H}$, for $\mathcal{I} \subset \mathcal{P}(G)$ a finite subset, are dense in the product Tychonoff topology on $\widehat{H}$, so it follows that $H = Q(G)$ is dense in $\widehat{H}$ as claimed. □

Note that it follows from the above construction that the kernel of the restricted map $Q \colon G \to \widehat{H}$ is not a finitely-generated subgroup of G, so the image group $H = Q(G)$ will not be finitely presented.

For each $p_i \in \mathcal{P}(G)$, choose a non-trivial subgroup $A_{p_i} \subset \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$ with trivial core. That is, A_{p_i} is chosen so that the trivial group is the largest normal subgroup of $\mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$ contained in A_{p_i}. For example, the alternating group on N symbols embeds into $\mathbf{SL}_N(\mathbb{Z})$, and so it embeds into $\mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$. For $N = 3$ or $N \geq 5$, the alternating group on N symbols is simple, so A_{p_i} may be chosen to be any non-trivial proper subgroup of the image of the alternating group in $\mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$. In the examples constructed in Section 9.2, the subgroups A_{p_i} are non-trivial cyclic subgroups.

Since A_{p_i} has trivial core, it follows that each $g \in \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$ with $g \notin A_{p_i}$ acts non-trivially on the left on $Y_{p_i} = \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})/A_{p_i}$. Then define

$$\mathcal{D} = \prod_{i \geq k_G} A_{p_i} \subset \widehat{H} \ ,$$

and introduce the quotient space

$$X = \widehat{H}/\mathcal{D} = \prod_{i \geq k_G} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})/A_{p_i} \cong \prod_{i \geq k_G} Y_{p_i} \ .$$

Define the action of G on X by setting, for $g \in G$ and $\widehat{x} \in \widehat{H}/\mathcal{D}$, $\Phi_0(g)(\widehat{x}) = Q(g) \cdot \widehat{x}$. Then we obtain an equicontinuous minimal Cantor action (X, G, Φ_0). Note that the kernel of the homomorphism $\Phi_0 \colon G \to Homeo(X)$ is an infinitely generated subgroup.

We next calculate the asymptotic discriminant groups $\mathcal{D}_n$ for (X, G, Φ_0), using the criteria given in Proposition 8.5. In order to simplify the calculations, we also require that each subgroup A_{p_i} is assumed to be abelian. This condition is satisfied for the examples given in Section 9.2.

For $\ell > k_G$ define

$$\widehat{U}_\ell = \prod_{k_G \leq i < \ell} \{e_{p_i}\} \ \times \ \prod_{i \geq \ell} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z}) \quad , \quad U_\ell = \prod_{k_G \leq i < \ell} \{e_{p_i}\} \ \times \ \prod_{i \geq \ell} Y_{p_i} \ , \tag{64}$$

where $e_{p_i} \in A_{p_i} \subset \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$ is the identity element. Each $\widehat{U}_\ell$ is a clopen subset of $\widehat{H}$, and U_ℓ is a clopen neighborhood of $x_0 \in X$. Thus, we obtain a descending chain of clopen neighborhoods $\{\widehat{U}_\ell \mid \ell > k_G\}$ of the identity in $\widehat{H}$. Observe that for $\ell > k_G$

$$\widehat{W}_\ell = \widehat{U}_\ell \cdot \mathcal{D} = \prod_{k_G \leq i < \ell} A_{p_i} \ \times \ \prod_{i \geq \ell} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z}) \ .$$

Set $H_\ell = H \cap \widehat{W}_\ell$, so that by (52) the finite set X_ℓ has the form

$$X_\ell = \widehat{H}/\widehat{W}_\ell \cong \prod_{k_G \leq i < \ell} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})/A_{p_i} \cong \prod_{k_G \leq i < \ell} Y_{p_i} \ .$$

Observe that since the groups A_{p_i} are abelian and have trivial core in $\mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z})$, for each $\ell \geq k_G$, the kernel $\mathcal{K}_\ell$ of the adjoint representation $\widehat{\rho}_\ell \colon \mathcal{D} \to Aut(\widehat{W}_\ell)$ is given by

$$\mathcal{K}_\ell = \prod_{k_G \leq i < \ell} A_{p_i} \ \times \ \prod_{i \geq \ell} \{e_{p_i}\} \ , \tag{65}$$

so that $\{\mathcal{K}_\ell \mid \ell > k_G\} \subset \mathcal{D}$ form an increasing chain of subgroups of $\mathcal{D}$ whose union is dense in $\mathcal{D}$.

Next, for $\ell > k_G$, we have the projection onto the quotient group, as in (49),

$$q_\ell \colon \widehat{H} \to \widehat{H}/\widehat{U}_\ell \cong \prod_{k_G \leq i < \ell} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z}) \ .$$

For $\ell > n > k_G$, recall that $H_n = H \cap \widehat{W}_n$ and $\widehat{W}_n = \widehat{U}_n \cdot \mathcal{D}$ where $\widehat{U}_n$ is defined by (64), then set

$$H_{n,\ell} = q_\ell(H_n) = q_\ell(\widehat{W}_n) \cong \prod_{k_G \leq i < n} A_{p_i} \ \times \ \prod_{n \leq i < \ell} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z}) \subset \prod_{k_G \leq i < \ell} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z}) \ .$$

Finally, the image under the quotient map $q_n \colon \widehat{H} \to \widehat{H}/\widehat{U}_n$ of the group $\mathcal{D}$ is given by

$$q_\ell(\mathcal{D}) = \prod_{k_G \leq i < \ell} A_{p_i} \subset \prod_{k_G \leq i < \ell} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z}) \ ,$$

and by (65)

$$q_\ell(\mathcal{D}/\mathcal{K}_n) = \prod_{n \leq i < \ell} A_{p_i} \subset \prod_{k_G \leq i < \ell} \mathbf{SL}_N(\mathbb{Z}/p_i\mathbb{Z}) \ . \tag{66}$$

By the choice of each subgroup A_{p_i} we have that the $H_{n,\ell}$–core of $q_\ell(\mathcal{D}/\mathcal{K}_n)$ is trivial, and so by Proposition 8.5 the discriminant group $\mathcal{D}_{n,n}$ of the action, restricted to $U_n = \widehat{W}_n/\mathcal{D}$, is isomorphic to $\mathcal{D}/\mathcal{K}_n$. Then, taking the inverse limit of groups in (66), we obtain that

$$\mathcal{D}_n = \prod_{i \geq n} A_{p_i} \subset \widehat{H} \ , \tag{67}$$

and the surjective homomorphisms $\psi_n \colon \mathcal{D}_n \to \mathcal{D}_{n+1}$ in (46) are given by

$$\psi_{n,n+1} \colon \prod_{i \geq n} A_{p_i} \to \prod_{i \geq n+1} A_{p_i} \tag{68}$$

In particular, we have $ker(\psi_{n,n+1}) = A_{p_n}$ for all $n \geq 1$. We have thus shown:

PROPOSITION 9.2. *The action (X, G, Φ_0) is wild, with asymptotic discriminant given by the asymptotic class of the sequence of surjective homomorphisms $\{\psi_{n,n+1} \colon \mathcal{D}_n \to \mathcal{D}_{n+1} \mid n \geq k_G\}$.*

9.2. **Proof of Theorem 1.10.** We will prove Theorem 1.10 for $N = 3$, and the proof for $N > 3$ is verbatim the same. Let $G \subset \mathbf{SL}_3(\mathbb{Z})$ be a torsion-free subgroup of finite index in the 3×3 integer matrices, then G is necessarily finitely presented as discussed above. Wild actions of G are constructed by the method in Section 9.1, and we now have to show that there is an uncountable number of distinct homeomorphism types of the suspensions of such actions.

Let $k_G \geq 2$ be chosen so that for $\mathcal{P}(G) = \{p_i \mid i \geq k_G\}$ we have

$$\prod_{i \geq k_G} \mathbf{SL}_3(\widehat{\mathbb{Z}}_{p_i}) \subset \widehat{G} \ , \ \text{and set } \widehat{H} = \prod_{i \geq k_G} \mathbf{SL}_3(\mathbb{Z}/p_i\mathbb{Z}) \ . \tag{69}$$

For each $i \geq k_G$ we next choose a non-trivial abelian subgroup $A_{p_i} \subset \mathbf{SL}_3(\mathbb{Z}/p_i\mathbb{Z})$ with trivial core. The idea is to choose, for each p_i two subgroups of $\mathbf{SL}_3(\mathbb{Z}/p_i\mathbb{Z})$ which have orders p_i and p_i^2, and hence cannot be isomorphic. Introduce the abelian matrix subgroups

$$A^1_{p_i} = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \mid a \in \mathbb{Z}/p_i \right\} \quad , \quad A^2_{p_i} = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} \mid a, b \in \mathbb{Z}/p_i \right\} \ . \tag{70}$$

Then both $A^1_{p_i}$ and $A^2_{p_i}$ have trivial cores in $\mathbf{SL}_3(\mathbb{Z}/p_i\mathbb{Z})$.

Let $\mathcal{I} \in \Sigma_{k_G} \equiv \prod_{i \geq k_G} \{1, 2\}$, so $\mathcal{I} = (s_{k_G}, s_{k_G+1}, \ldots)$ where each $s_i \in \{1, 2\}$. Now define

$$\mathcal{D}^{\mathcal{I}} = \prod_{i \geq k_G} A^{s_i}_{p_i} \subset \widehat{H} \ . \tag{71}$$

Then set $X^{\mathcal{I}} = \widehat{H}/\mathcal{D}^{\mathcal{I}}$ and let $(X^{\mathcal{I}}, G, \Phi_0^{\mathcal{I}})$ be the associated minimal Cantor action, constructed as in Section 9.1.

LEMMA 9.3. *Let $\mathcal{I}_1 = (s_{k_G}, s_{k_G+1}, \ldots)$ and $\mathcal{I}_2 = (t_{k_G}, t_{k_G+1}, \ldots)$ be sequences in Σ_{k_G}. Then the sequences of surjective homomorphisms $\{\psi^1_{n,n+1} : \mathcal{D}^{\mathcal{I}_1}_n \to \mathcal{D}^{\mathcal{I}_1}_{n+1}\}$ and $\{\psi^2_{n,n+1} : \mathcal{D}^{\mathcal{I}_2}_n \to \mathcal{D}^{\mathcal{I}_2}_{n+1}\}$ are tail equivalent if and only if there exists $n \geq k_G$ such that for all $i \geq n$ we have $s_i = t_i$. In particular, the number of such tail equivalence classes in Σ_2 is uncountable.*

Proof. By (67), we have $\mathcal{D}^{\mathcal{I}_1}_n \cong \prod_{i \geq n} A^{s_i}_{p_i} \quad , \quad \mathcal{D}^{\mathcal{I}_2}_n \cong \prod_{i \geq n} A^{t_i}_{p_i}$.

Thus, if $\mathcal{I}_1$ and $\mathcal{I}_2$ satisfy the conditions of the lemma for some $n \geq k_G$, then $\mathcal{D}^{\mathcal{I}_1}_i$ and $\mathcal{D}^{\mathcal{I}_2}_i$ are isomorphic for all $i \geq n$, and so the sequences of surjective homomorphisms of discriminant groups are tail equivalent.

Now assume that the sequences associated to $\mathcal{I}_1$ and $\mathcal{I}_2$ are tail equivalent. Then there exists sequences $\{n_i\}_{i\geq k_G}$ and $\{m_i\}_{i\geq k_G}$ and a sequence of surjective homomorphisms τ_{n_i,m_i} and $\kappa_{m_i,n_{i+1}}$,

$$\mathcal{D}^{\mathcal{I}_1}_{n_1} \xrightarrow[\tau_{n_1,m_1}]{} \mathcal{D}^{\mathcal{I}_2}_{m_1} \xrightarrow[\kappa_{m_1,n_2}]{} \mathcal{D}^{\mathcal{I}_1}_{n_2} \xrightarrow[\tau_{n_2,m_2}]{} \mathcal{D}^{\mathcal{I}_2}_{m_2} \longrightarrow \cdots$$

such that

$$\psi^1_{n_i,n_{i+1}} = \tau_{n_i,m_i} \circ \kappa_{m_i,n_{i+1}} \quad , \quad \psi^2_{m_i,m_{i+1}} = \kappa_{m_i,n_{i+1}} \circ \tau_{n_{i+1},m_{i+1}} \ .$$

Then set $n = \max\{m_i, n_{i+1}\}$. For $j \geq n$, consider a subgroup of $\mathcal{D}^{\mathcal{I}_2}_{m_i}$ isomorphic to $A^{t_j}_{p_j}$. Since p_j is a prime, its image under $\kappa_{m_i,n_{i+1}}$ is the subgroup of $\mathcal{D}^{\mathcal{I}_1}_{n_{i+1}}$, isomorphic to $A^{s_j}_{p_j}$, and its preimage under τ_{n_i,m_i} is a subgroup of $\mathcal{D}^{\mathcal{I}_1}_{n_i}$, isomorphic to $A^{s_j}_{p_j}$.

If $t_j = 1$, then $s_j = 1$, as a subgroup of order p_j cannot be mapped surjectively on a subgroup of order p_j^2. If $t_j = 2$, then $s_j = 2$ as the preimage of $A^{t_j}_{p_j}$ in $\mathcal{D}^{\mathcal{I}_1}_{n_i}$ must contain at least the same number of elements as $A^{t_j}_{p_j}$. This completes the proof. □

To complete the proof of Theorem 1.10, we use the second method of suspension given in Section 8.1. Choose a compact 4-manifold M without boundary whose fundamental group is isomorphic to the finitely-presented group G chosen above. Let $\widehat{H}$ be defined as in (69), and let $Q\colon \mathbf{SL}_N(\mathbb{Z}) \to \widehat{H}$ be the homomorphism constructed in Section 9.1, which restricts to a map $Q_G\colon G \to \widehat{H}$, whose image $H = Q(G)$ is dense by Lemma 9.1.

Note that while G is finitely presented, the group H is obtained by adding to a presentation for G the relations given by the generators of the kernel of $Q\colon G \to H$, which is infinitely generated. Thus, the presentation for H is finitely generated but not finitely presented.

For each index $\mathcal{I}$ as in Lemma 9.3, define the quotient space $X^{\mathcal{I}} = \widehat{H}/\mathcal{D}^{\mathcal{I}}$ which is a Cantor set, then the map Q_G induces an equicontinuous action $\Phi^{\mathcal{I}}_0\colon \pi_1(M, b_0) \cong G \to Homeo(X^{\mathcal{I}})$. Let $H^{\mathcal{I}} = \Phi^{\mathcal{I}}_0(G)$ denote the image of this action. Then by the suspension construction, we obtain a weak solenoid, denoted by $\mathcal{S}_{\mathcal{I}}$, which admits a presentation with base manifold M and global holonomy group $H^{\mathcal{I}}$.

Suppose that $\mathcal{S}_{\mathcal{I}_1}$ and $\mathcal{S}_{\mathcal{I}_2}$ are homeomorphic, then by Theorem 2.14 and Corollary 2.17 their induced holonomy actions are return equivalent. Then by Proposition 5.13, the asymptotic discriminant classes of $\Phi^{\mathcal{I}_1}_0$ and $\Phi^{\mathcal{I}_2}_0$ are equal. It thus follows by Lemma 9.3 that there are uncountably many distinct homeomorphism classes of weak solenoids $\mathcal{S}_{\mathcal{I}}$ which are wild, all with the same compact base manifold M having fundamental group G.

10. Concluding Remarks

We conclude with some remarks about the construction of examples of wild equicontinuous minimal Cantor actions, and formulate some open questions and problems.

First, in the construction of the examples $(X^{\mathcal{I}}, G, \Phi^{\mathcal{I}}_0)$, the choice of the space $\widehat{H}$ and the closed subgroups $\mathcal{D}^{\mathcal{I}}$ with trivial core subgroups is straightforward, given the fact that there exists a cofinite subset $\mathcal{P}(G)$ of primes so that (63) holds for any choice of a subgroup of finite index $G \subset \mathbf{SL}_N(\mathbb{Z})$, when $N \geq 3$. This latter fact relies on the deep result, the solution of the Congruence Subgroup Problem by Bass, Milnor and Serre [11], as explained by Lubotzky in the proof of [39, Proposition 2]. There are many variations of this choice of families of discriminant groups $\mathcal{D}^{\mathcal{I}}$, especially as $N \geq 3$ increases. The remarkable fact is that there is a finitely presented group G which then acts minimally on the product space $\widehat{H}$. Note that in this construction, the global holonomy groups $H^{\mathcal{I}}$ are finitely generated, but are not finitely presented.

Let $\widehat{Q}\colon \widehat{\mathbf{SL}_N(\mathbb{Z})} \to \widehat{H}$ be the extension of the map Q defined in Section 9.1. Then for the choice of $G \subset \mathbf{SL}_N(\mathbb{Z})$ as above, the restriction of $\widehat{Q}\colon \widehat{G} \to \widehat{H}$ is also surjective by Lemma 9.1. Given a non-trivial subgroup $\mathcal{D} \subset \widehat{H}$ satisfying the condition $\mathrm{core}_H\ \mathcal{D} = \{\widehat{e}\}$ in Lemma 8.1, let $\mathcal{E} = \widehat{Q}^{-1}(\mathcal{D}) \subset \widehat{G}$. Then the action of G on the quotient $\widehat{G}/\mathcal{E}$ is wild if the action via Q on $\widehat{H}/\mathcal{D}$ is wild. This suggests the general formulation of the existence problem for wild actions:

PROBLEM 10.1. *Let G be a finitely presented, residually finite group, and let $\widehat{G}$ denote its profinite completion. Find conditions on G such that there exists a closed subgroup $\mathcal{E} \subset \widehat{G}$ such that the induced action on $\mathfrak{X} = \widehat{G}/\mathcal{E}$ is wild.*

Here is an alternative method to construct further examples of wild actions on Cantor spaces. Choose a countably infinite collection of finite simple groups, $\{H_i \mid i \geq 1\}$, which are pairwise non-isomorphic. For example, one can let $H_i = Alt(i)$ be the alternating group on i symbols, for $i \geq 5$. Then for each such finite group H_i, we can then choose any proper, non-trivial subgroup $A_i \subset H_i$ and it will have trivial core. Moreover, one consequence of the Classification Theorem for finite simple groups is that each H_i can be generated by two elements, as shown in [9, 30]. Set $\widehat{H} = \prod_{i \geq 1} H_i$ and $\mathcal{D} = \prod_{i \geq 1} A_i$, then set $\mathfrak{X} = \widehat{H}/\mathcal{D}$. To complete the construction, it is necessary to show there is a finitely generated subgroup $H \subset \widehat{H}$ which is dense in the Tychonoff topology.

PROBLEM 10.2. *Characterize the countably infinite collections of finite simple non-abelian groups $\{H_i \mid i \geq 1\}$ which are pairwise non-isomorphic, such that their product $\widehat{H}$ contains a finitely generated dense subgroup H for the Tychonoff topology on $\widehat{H}$?*

For example, the works by Kassabov and Nikolov [38] and Segal [55] discuss aspects of the finite generation problem for profinite groups which are related to Problem 10.2.

Finally, we observe that in all of the known examples of equicontinuous minimal Cantor actions (X, G, Φ) which are wild, and where G is finitely-presented, then the action $\Phi \colon G \to Homeo(X)$ has infinitely-generated kernel.

PROBLEM 10.3. *Show that an equicontinuous minimal Cantor action (X, G, Φ) of a finitely-presented group G, for which the kernel of $\Phi \colon G \to Homeo(X)$ is finitely-generated, must be tame.*

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