

# ORBIT EQUIVALENCE AND CLASSIFICATION OF WEAK SOLENOIDS

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ABSTRACT. In this work, we study minimal equicontinuous actions which are locally quasi-analytic. The first main result shows that for minimal equicontinuous actions which are locally quasi-analytic, continuous orbit equivalence of the actions implies return equivalence. This generalizes results of Cortez and Medynets, and of Li. The second main result is that if  $G$  is a finitely-generated, virtually nilpotent group, then every minimal equicontinuous action by  $G$  is locally quasi-analytic. As an application, we show that the homeomorphism type of a nil-solenoid is determined by the topological full group of its monodromy action.

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## 1. INTRODUCTION AND MAIN RESULTS

The works of McCord [37] and Schori [46] in the 1960's, and the work of Fokkink and Oversteegen in [23], studied the self-homeomorphisms of weak solenoids. The authors' work in [14] studied the more general problem of the classification up to homeomorphism for weak solenoids, where it was shown that for certain classes of weak solenoids, two weak solenoids are homeomorphic if and only if their global monodromy actions are return equivalent.

The global monodromy of a weak solenoid is a minimal equicontinuous Cantor action of a finitely generated group, so the classification problem for weak solenoids motivates the study of invariants for such actions. The purpose of this work is to consider the topological full group as such an invariant, and to find conditions on the actions which imply that they are return equivalent, and hence provide a solution to the classification problem. For the class of nil-solenoids, Theorem 1.8 gives such a solution. Before stating our main results precisely, we first recall a few basic concepts.

Let  $G$  be a countably generated discrete group, and  $\Phi: G \rightarrow \text{Homeo}(\mathfrak{X})$  be an action of  $G$  on a topological space  $\mathfrak{X}$ . We also denote the action by  $(\mathfrak{X}, G, \Phi)$ , and write  $g \cdot x$  for  $\Phi(g)(x)$ . We say that  $\Phi$  is a *Cantor action* if  $\mathfrak{X}$  is a Cantor space.

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The orbit of a point  $x \in \mathfrak{X}$  is the subset  $\mathcal{O}(x) = \{g \cdot x \mid g \in G\}$ . The action is *minimal* if for all  $x \in \mathfrak{X}$ , its orbit  $\mathcal{O}(x)$  is dense in  $\mathfrak{X}$ . The *isotropy group* of  $x \in \mathfrak{X}$  is the subgroup

$$(1) \quad G_x = \{g \in G \mid g \cdot x = x\} .$$

The action  $(\mathfrak{X}, G, \Phi)$  is *effective* if the homomorphism  $\Phi$  is assumed to have trivial kernel. It is *free* if for all  $x \in \mathfrak{X}$  and  $g \in G$ , then  $g \cdot x = x$  implies that  $g = e$ , where  $e \in G$  denotes the identity of the group. Introduce the *isotropy set*

$$(2) \quad \text{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists g \in G, g \neq id, g \cdot x = x\} = \bigcup_{e \neq g \in G} \text{Fix}(g) ,$$

where  $\text{Fix}(g) = \{x \in \mathfrak{X} \mid g \cdot x = x\}$ . The action  $(\mathfrak{X}, G, \Phi)$  is said to be *topologically free* [10, 35, 44] if the set  $\text{Iso}(\Phi)$  is nowhere dense in  $\mathfrak{X}$ . Note that if  $e \neq g \in G$  and  $\Phi(g)$  acts trivially on  $\mathfrak{X}$ , then  $\text{Iso}(\Phi) = \mathfrak{X}$ , and thus a topologically free action must be effective.

The action  $(\mathfrak{X}, G, \Phi)$  is *equicontinuous* with respect to a metric  $d_{\mathfrak{X}}$  on  $\mathfrak{X}$ , if for all  $\varepsilon > 0$  there exists  $\delta > 0$ , such that for all  $x, y \in \mathfrak{X}$  and  $g \in G$  we have

$$(3) \quad d_{\mathfrak{X}}(x, y) < \delta \quad \implies \quad d_{\mathfrak{X}}(g \cdot x, g \cdot y) < \varepsilon .$$

Note that the definition is independent of the choice of the metric  $d_{\mathfrak{X}}$  on  $\mathfrak{X}$ .

For a Cantor space  $\mathfrak{X}$ , let  $\text{CO}(\mathfrak{X})$  denote the collection of all clopen (compact open) subsets of  $\mathfrak{X}$ . Note that for  $\phi \in \text{Homeo}(\mathfrak{X})$  and  $U \in \text{CO}(\mathfrak{X})$ , the image  $\phi(U) \in \text{CO}(\mathfrak{X})$ .

We say that  $U \subset \mathfrak{X}$  is *adapted* to the action  $(\mathfrak{X}, G, \Phi)$  if  $U$  is a non-empty clopen subset, and for any  $g \in G$ ,  $\Phi(g)(U) \cap U \neq \emptyset$  implies that  $\Phi(g)(U) = U$ . It follows that

$$(4) \quad G_U = \{g \in G \mid \Phi(g)(U) \cap U \neq \emptyset\}$$

is a subgroup of finite index in  $G$ , called the *stabilizer* of  $U$ .

Denote the restricted action on  $U$  by  $(U, G_U, \Phi_U)$ , and introduce the *restricted holonomy group*:

$$(5) \quad \mathcal{H}_U = \text{image} \{ \Phi_U : G_U \rightarrow \text{Homeo}(U) \} .$$

**DEFINITION 1.1.** *For  $i = 1, 2$ , let  $(\mathfrak{X}_i, G_i, \Phi_i)$  be minimal equicontinuous Cantor actions. We say the actions are return equivalent if there exists non-empty clopen subsets  $U_i \subset \mathfrak{X}_i$  such that  $U_i$  is adapted to the action  $\Phi_i$ , and there is a homeomorphism  $h : U_1 \rightarrow U_2$  whose induced action  $h_* : \text{Homeo}(U_1) \rightarrow \text{Homeo}(U_2)$  restricts to an isomorphism  $h_* : \mathcal{H}_{U_1} \rightarrow \mathcal{H}_{U_2}$ .*

Note that when  $U_i = \mathfrak{X}_i$  and both actions are effective, then this definition reduces to the usual notion of conjugacy of the actions, up to the induced group isomorphism  $h_* : G_1 \cong \mathcal{H}_{U_1} \rightarrow \mathcal{H}_{U_2} \cong G_2$ . In the terminology of [26], this is called an *isomorphism* of the actions. For the general case, though, additional assumptions on the group or the action are required to induce an isomorphism of the actions of the groups  $G_1$  and  $G_2$  from an isomorphism  $h_* : \mathcal{H}_{U_1} \rightarrow \mathcal{H}_{U_2}$ .

**PROBLEM 1.2.** *Find invariants of minimal equicontinuous Cantor actions which are sufficient to classify the actions up to return equivalence.*

In this work, we are concerned with the following conjugacy invariant of a Cantor action  $(\mathfrak{X}, G, \Phi)$ .

**DEFINITION 1.3.** *The topological full group  $[[\mathfrak{X}, G, \Phi]] \subset \text{Homeo}(\mathfrak{X})$  is the subgroup consisting of homeomorphisms  $\phi : \mathfrak{X} \rightarrow \mathfrak{X}$  such that for all  $x \in \mathfrak{X}$ , there exists  $U \in \text{CO}(\mathfrak{X})$  with  $x \in U$ , and  $g \in G$  such that their restrictions to  $U$  satisfy  $\phi|_U = \Phi(g)|_U$ .*

This definition was introduced in the work by Glasner and Weiss [27, Section 2], and has been extensively studied for its relation to the classification problem of Cantor actions, and for the algebraic properties of the group itself, as discussed for example in the Séminaire Bourbaki survey [16], and also the works [7, 8, 24, 31]. The topological full group is closely related to the notion of continuous orbit equivalence of actions, as will be discussed in Section 4. This paper was motivated by the following result of Cortez and Medynets in [15]:

**THEOREM 1.4.** *For  $i = 1, 2$ , let  $G_i$  be a finitely-generated group, and let  $(\mathfrak{X}, G_i, \Phi_i)$  be a free minimal equicontinuous Cantor action on a Cantor space  $\mathfrak{X}$ . If  $[[\mathfrak{X}, G_1, \Phi_1]] = [[\mathfrak{X}, G_2, \Phi_2]]$  then the actions  $\Phi_1$  and  $\Phi_2$  are structurally conjugate, hence are return equivalent.*

We give a generalization of Theorem 1.4 for minimal equicontinuous Cantor actions that are not free, but satisfy a property called locally quasi-analytic as given in Definition 2.1.

**THEOREM 1.5.** *For  $i = 1, 2$ , let  $G_i$  be a finitely-generated group, and  $(\mathfrak{X}, G_i, \Phi_i)$  be a minimal equicontinuous Cantor action which is locally quasi-analytic. If  $[[\mathfrak{X}, G_1, \Phi_1]] = [[\mathfrak{X}, G_2, \Phi_2]]$  then the actions  $\Phi_1$  and  $\Phi_2$  are return equivalent.*

Our second main result gives a sufficient condition for the group  $G$  which implies the LQA property.

**THEOREM 1.6.** *Let  $G$  be a Noetherian group. Then a minimal equicontinuous action  $(\mathfrak{X}, G, \Phi)$  on a Cantor space  $\mathfrak{X}$  is locally quasi-analytic.*

In particular, for  $G$  a finitely-generated nilpotent group, Corollary 3.9 shows that  $G$  is Noetherian. This yields the following application of Theorem 1.5:

**COROLLARY 1.7.** *Let  $G$  be a finitely-generated nilpotent group. Then the topological full group is a complete invariant up to return equivalence for a minimal equicontinuous action  $(\mathfrak{X}, G, \Phi)$ : if two such actions have spatially isomorphic topological full groups, then the actions are return equivalent.*

This result can be considered as a localized version of [35, Theorem 1.3], but with the Noetherian hypothesis on  $G$  in place of the homological criterion of Li in [35, Section 5].

Finally, in Remark 4.8 we introduce the asymptotic equivalence class of the collection  $[[[\mathfrak{X}, G, \Phi]]]$  of local topological full groups associated to a minimal equicontinuous action  $(\mathfrak{X}, G, \Phi)$ . In Section 5, we introduce the notion of a nil-solenoid, which is a weak solenoid whose base space is a closed nil-manifold and whose global monodromy action is effective. Precise definitions are given in Section 5, where we prove Theorem 5.4, which implies the following:

**THEOREM 1.8.** *The asymptotic local topological full group class  $[[[\mathcal{F}_P]]]$  for nil-solenoids is a complete invariant of its homeomorphism class.*

Appendix A contains a collection of examples of minimal Cantor actions which illustrate properties of actions as discussed in this work.

## 2. LOCALLY QUASI-ANALYTIC AND HAUSDORFF ACTIONS

In this section, we discuss notions of pointwise and uniform “regularity” for topological actions.

We first recall the notion of a locally quasi-analytic action. Haefliger introduced in [32] the notion of a quasi-analytic action on a topological space, in his study of pseudogroups of local isometries on locally connected spaces. The works [2, 3] by Álvarez-López, Candel and Moreira-Galicia reformulated Haefliger’s definition for the case of topological actions on Cantor spaces as follows:

**DEFINITION 2.1.** [2, Definition 9.4] *A topological action  $(\mathfrak{X}, G, \Phi)$  is locally quasi-analytic, or simply LQA, if there exists  $\varepsilon > 0$  such that for any non-empty open set  $U \subset \mathfrak{X}$  with  $\text{diam}(U) < \varepsilon$ , and for any non-empty open subset  $V \subset U$ , and elements  $g_1, g_2 \in G$*

$$(6) \quad \text{if the restrictions } \Phi(g_1)|_V = \Phi(g_2)|_V, \text{ then } \Phi(g_1)|_U = \Phi(g_2)|_U.$$

*The action is said to be quasi-analytic if (6) holds for  $U = \mathfrak{X}$ .*

Examples of equicontinuous Cantor actions  $\mathfrak{X}$  which are locally quasi-analytic, but not quasi-analytic, are given in [20, 33], and examples are given in Section A.

The idea of the proof for the following result appeared in the work [21] by Epstein, Millet and Tischler, in the context of topological pseudogroup actions. It has also appeared in the literature in various alternative formulations, for example as Proposition 3.6 in [44] and Lemma 2.2 in [35].

**PROPOSITION 2.2.** *An effective Cantor action  $(\mathfrak{X}, G, \Phi)$  is quasi-analytic if and only if it is topologically free.*

*Proof.* Suppose that the action  $\Phi$  is topologically free, then the isotropy set  $\text{Iso}(\Phi)$  is nowhere dense in  $\mathfrak{X}$ . Let  $V \subset \mathfrak{X}$  be a non-empty open set, and suppose that  $g_1, g_2 \in G$  satisfy  $\Phi(g_1)|V = \Phi(g_2)|V$ . Then  $\Phi(g_2^{-1}g_1)|V = \text{id}|V$  so we must have that  $g_2^{-1}g_1 = e \in G$ . Thus,  $g_1 = g_2$  and hence  $\Phi(g_1) = \Phi(g_2)$ , as was to be shown.

Conversely, suppose that the action  $\Phi$  is quasi-analytic and effective, then for each  $e \neq g \in G$  the closed set  $\text{Fix}(\Phi(g))$  has no interior. Thus,  $\text{Iso}(\Phi)$  is the countable union of closed nowhere dense sets, hence by the Baire Category Theorem,  $\text{Iso}(\Phi)$  must be nowhere dense.  $\square$

**COROLLARY 2.3.** *Let  $(\mathfrak{X}, G, \Phi)$  be a minimal topological action. If  $G$  is a finitely-generated abelian group, and the action is effective, then it is quasi-analytic, hence is topologically free.*

*Proof.* Suppose that there exists a non-empty open set  $V \subset \mathfrak{X}$  and let  $g_1, g_2 \in G$  be such that the restrictions satisfy  $\Phi(g_1)|V = \Phi(g_2)|V$ . Let  $g = g_2^{-1}g_1$  then the restriction  $\Phi(g)|V$  is the identity map. Let  $y \in \mathfrak{X}$  then there exists  $g_y \in G$  such that  $g_y \cdot y \in V$ , so  $y \in V_y \equiv g_y^{-1} \cdot V$ . Since  $\Phi(g) = \Phi(g_y^{-1}g g_y)$ , the restriction  $\Phi(g)|V_y$  is the identity map. Thus,  $\Phi(g)$  is the identity on  $\mathfrak{X}$ , and as the action is effective, we have  $g = \text{id}$ .  $\square$

The LQA property for a group action  $(\mathfrak{X}, G, \Phi)$  can be interpreted in terms of the properties of the germinal groupoid  $\mathcal{G}_\Phi$  associated to the action. This groupoid is fundamental for the study of the  $C^*$ -algebras these actions generate, as discussed for example by Renault in [43, 44]. Recall that for  $g_1, g_2 \in G$ , we say that  $\Phi(g_1)$  and  $\Phi(g_2)$  are *germinally equivalent* at  $x \in \mathfrak{X}$  if  $\Phi(g_1)(x) = \Phi(g_2)(x)$ , and there exists an open neighborhood  $x \in U \subset \mathfrak{X}$  such that the restrictions agree,  $\Phi(g_1)|U = \Phi(g_2)|U$ . We then write  $\Phi(g_1) \sim_x \Phi(g_2)$ . For  $g \in G$ , denote the equivalence class of  $\Phi(g)$  at  $x$  by  $[g]_x$ . The collection of germs  $\mathcal{G}(\mathfrak{X}, G, \Phi) = \{[g]_x \mid g \in G, x \in \mathfrak{X}\}$  is given the sheaf topology, and forms an *étale groupoid* modeled on  $\mathfrak{X}$ . We recall the following result from Winkelkemper:

**PROPOSITION 2.4.** [49, Proposition 2.1] *The germinal groupoid  $\mathcal{G}(\mathfrak{X}, G, \Phi)$  is Hausdorff at  $[g]_x$  if and only if, for all  $[g']_x \in \mathcal{G}(\mathfrak{X}, G, \Phi)$  with  $g \cdot x = g' \cdot x = y$ , if there exists a sequence  $\{x_i\} \subset \mathfrak{X}$  which converges to  $x$  such that  $[g]_{x_i} = [g']_{x_i}$  for all  $i$ , then  $[g]_x = [g']_x$ .*

Winkelkemper showed in [49, Proposition 2.3] that for a smooth foliation  $\mathcal{F}$  of a connected manifold  $M$  for which the associated holonomy pseudogroup  $\mathcal{G}_\mathcal{F}$  is generated by real analytic maps, then  $\mathcal{G}_\mathcal{F}$  is a Hausdorff space. For Cantor actions, an analogous result holds for the LQA property.

**PROPOSITION 2.5.** *If an action  $(\mathfrak{X}, G, \Phi)$  is locally quasi-analytic, then  $\mathcal{G}(\mathfrak{X}, G, \Phi)$  is Hausdorff.*

*Proof.* Assume that  $\mathcal{G}(\mathfrak{X}, G, \Phi)$  is not Hausdorff. Then there exists  $g \in G$  and  $x \in \mathfrak{X}$  such that  $\mathcal{G}(\mathfrak{X}, G, \Phi)$  is non-Hausdorff at  $[g]_x$ . By Proposition 2.4, there exists  $[g']_x \in \mathcal{G}(\mathfrak{X}, G, \Phi)$  with  $g \cdot x = g' \cdot x = y$ , and a sequence  $\{x_i\} \subset \mathfrak{X}$  which converges to  $x$  such that  $[g]_{x_i} = [g']_{x_i}$  for all  $i$ , but  $[g]_x \neq [g']_x$ . Let  $g'' = g^{-1}g' \in G$ , then  $g'' \cdot x = x$  and  $[g'']_{x_i} = [\text{id}]_{x_i}$  for all  $i$ , but  $[g'']_x \neq [\text{id}]_x$ . In particular, the action of  $\Phi(g'')$  is not the identity in any open neighborhood of  $x$ , but there exists a sequence of open sets  $x_i \in U_i \subset \mathfrak{X}$  containing  $x$  in their limit for which the restriction  $\Phi(g'')|U_i = \text{id}|U_i$ . Hence, there does not exist  $\varepsilon > 0$  such that  $\Phi(g)|U$  is quasi-analytic for all open neighborhood  $x \in U$  with  $\text{diam}(U) < \varepsilon$ . Thus, the action  $(\mathfrak{X}, G, \Phi)$  is not locally quasi-analytic.  $\square$

**REMARK 2.6.** Suppose that  $\mathcal{G}(\mathfrak{X}, G, \Phi)$  is a Cantor action such that  $\mathcal{G}(\mathfrak{X}, G, \Phi)$  is Hausdorff. Then for each  $x \in \mathfrak{X}$  and  $g, g' \in G$  with  $g \cdot x = g' \cdot x$  and  $[g]_x \neq [g']_x$ , there exists an open neighborhood  $x \in U(x, g, g') \subset \mathfrak{X}$  such that the set  $\{y \in U(x, g, g') \mid g \cdot y = g' \cdot y\}$  has no interior. If there exists  $\varepsilon > 0$  such that any open set  $U$  with  $x \in U$  and  $\text{diam}(U) < \varepsilon$  then  $U(x, g, g') = U$  for all  $g, g' \in G$ , satisfies this property, this is just saying that  $\mathcal{G}(\mathfrak{X}, G, \Phi)$  locally quasi-analytic. Thus, the locally quasi-analytic property can be viewed as a “uniform Hausdorff property” for  $\mathcal{G}(\mathfrak{X}, G, \Phi)$ .

We say that  $x \in \mathfrak{X}$  is a *non-Hausdorff point* for the action  $(\mathfrak{X}, G, \Phi)$  if there exists  $g \in G$  such that the germ  $[g]_x$  is not Hausdorff in  $\mathcal{G}(\mathfrak{X}, G, \Phi)$ . Examples of minimal equicontinuous Cantor actions with non-Hausdorff points are discussed in Section A. The existence of non-Hausdorff points has implications for the algebraic structure of the reduced  $C^*$ -algebra  $C^*(\mathfrak{X}, G, \Phi)$  associated to the action, as discussed for example in [11, 22, 44].

### 3. EQUICONTINUOUS ACTIONS

In this section, we consider some basic properties of minimal equicontinuous Cantor actions. The main results of the section are Theorem 3.6, which gives an algebraic criterion for when a group action is locally quasi-analytic, and Corollary 3.9 which applies this criterion to nilpotent groups.

First, we recall the following folklore result and give a sketch of the proof, as the ideas involved are central to the study of equicontinuous actions and used in the discussions that follow.

**PROPOSITION 3.1.** *Let  $G$  be a finitely-generated group. Then a minimal Cantor action  $(\mathfrak{X}, G, \Phi)$  is equicontinuous if and only if, for the induced action  $\Phi_*: G \times \text{CO}(\mathfrak{X}) \rightarrow \text{CO}(\mathfrak{X})$ , the  $G$ -orbit of every  $U \in \text{CO}(\mathfrak{X})$  is finite.*

*Proof.* Suppose that the action is equicontinuous, and let  $U_1 \subset \mathfrak{X}$  be a non-empty proper clopen subset. Then  $U_2 = \mathfrak{X} - U_1$  is also a non-empty proper clopen subset, and let  $\varepsilon = d_{\mathfrak{X}}(U_1, U_2) > 0$ , where  $d_{\mathfrak{X}}$  is the choice of some metric on  $\mathfrak{X}$ . Let  $\delta > 0$  be such that (3) holds for this choice of  $\varepsilon$ , for all  $g \in G$ . The iterates of the partition  $\{U_1, U_2\}$  of  $\mathfrak{X}$  define a closed partition of  $\mathfrak{X}$ ,

$$\mathcal{C} = \bigcap \{g \cdot U_i \mid i = 1, 2, g \in G\},$$

which is invariant for the action  $\Phi$  by construction. Let  $K \in \mathcal{C}$  with  $x \in K$ , and suppose that  $g \cdot x \in U_i$ . Let  $y \in \mathfrak{X}$  satisfy  $d_{\mathfrak{X}}(x, y) < \delta$  then by (3) and the choice of  $\varepsilon$  we have that  $g \cdot y \in U_i$  also. As this holds for all  $g \in G$ , the set  $K$  is open in  $\mathfrak{X}$ . Thus,  $\mathcal{C}$  is a clopen partition, and by the compactness of  $\mathfrak{X}$  it must be finite. Thus, the action  $\Phi$  permutes this finite collection, hence there is a subgroup  $G_{\mathcal{C}} \subset G$  of finite index which fixes every  $K \in \mathcal{C}$ . As  $U_1$  is the union of sets in  $\mathcal{C}$ , the subgroup  $G_{\mathcal{C}}$  also fixes  $U_1$ , and thus the  $G$ -orbit of  $U_1$  is finite.

Conversely, assume that the  $G$ -orbit of every  $U \in \text{CO}(\mathfrak{X})$  is finite. Fix a basepoint  $x \in \mathfrak{X}$ , and as  $\mathfrak{X}$  is a Cantor space, one can choose a descending chain of clopen sets

$$\mathfrak{X} = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_\ell \supset \cdots \supset \{x\}$$

whose intersection is  $\{x\}$ . For each  $\ell > 0$ , the intersection of the finite collection  $\{g \cdot V_\ell \mid g \in G\}$  has a clopen set containing the basepoint  $x$ , which we label  $U_\ell$ , so that  $x \in U_\ell \subset V_\ell$ . Then  $U_\ell$  is an adapted clopen subset with stabilizer group denoted by  $G_\ell \subset G$ . It then follows as in [18, Appendix A] or [15, Section 2], that the Cantor action  $(\mathfrak{X}, G, \Phi)$  is conjugate to the odometer constructed from the group chain  $\{G_\ell \mid \ell \geq 0\}$ , hence  $(\mathfrak{X}, G, \Phi)$  is a minimal equicontinuous action.  $\square$

The above proof that each  $U \in \text{CO}(\mathfrak{X})$  has finite orbit is essentially the same as what was called the ‘‘coding method’’ used to study equicontinuous pseudogroup actions in [13], and discussed for group actions in [18, Appendix A].

We discuss in more detail another key idea in the above proof. Let  $U \subset \mathfrak{X}$  be an adapted clopen set for the action  $\Phi$ , with stabilizer subgroup  $G_U \subset G$ . Then for  $g, g' \in G$  with  $g \cdot U \cap g' \cdot U \neq \emptyset$  we have  $g^{-1}g' \cdot U = U$ , hence  $g^{-1}g' \in G_U$ . Thus, the translates  $\{g \cdot U \mid g \in G\}$  form a finite clopen partition of  $\mathfrak{X}$ , and are in 1-1 correspondence with the quotient space  $X_U = G/G_U$  so the stabilizer group  $G_U \subset G$  has finite index. Note that the action of  $g \in G_U$  on  $X_U$  is trivial precisely when  $g \in C_U$ , where  $C_U \subset G_U$  is the largest normal subgroup of  $G$  contained in  $G_U$ . Thus, the action of the finite group  $G/C_U$  on  $X_U$  by permutations is a finite approximation of the action of  $G$  on  $\mathfrak{X}$ .

The restricted action of  $G_U$  on  $U$  defines a homomorphism  $\Phi_U: G_U \rightarrow \text{Homeo}(U)$ , with kernel  $\ker(\Phi_U) \subset G_U$ . Suppose that  $g \in \ker(\Phi_U)$  and the action is quasi-analytic, then  $g$  must act trivially on  $X_U$  hence  $g \in C_U$  which implies that  $g$  acts trivially on  $\mathfrak{X}$ . Thus, the quasi-analytic hypothesis is

equivalent to the assumption that  $\ker(\Phi_U)$  is a normal subgroup of  $G$ . Exactly the same reasoning yields the following criteria for an action to be locally quasi-analytic.

**PROPOSITION 3.2.** *Let  $(\mathfrak{X}, G, \Phi)$  be a minimal equicontinuous action on a Cantor space  $\mathfrak{X}$ . Suppose that the action  $\Phi$  is locally quasi-analytic. Then there exists  $\varepsilon > 0$  such that for each pair of non-empty adapted clopen sets  $V \subset U \subset \mathfrak{X}$  with  $\text{diam}(U) < \varepsilon$ , the subgroup  $\ker(\Phi_V) \subset G_V \subset G_U$  is normal in  $G_U$ .*

The action of the Grigorchuk group on the tree boundary (as discussed in Example A.6) admits pairs of clopen subsets  $V \subset U$  with arbitrarily small diameters, for which  $\ker(\Phi_V)$  is not normal in  $G_U$ , and thus its action is not locally quasi-analytic.

We next develop the idea behind the proof of Proposition 3.2 to develop an effective criteria, given in Theorem 3.6, for showing that an action must be locally quasi-analytic.

For a choice of basepoint  $x \in \mathfrak{X}$  and scale  $\varepsilon > 0$ , there exists an adapted clopen set  $U \in \text{CO}(\mathfrak{X})$  with  $x \in U$  and  $\text{diam}(U) < \varepsilon$ . Iterating this construction, for a given basepoint  $x$ , one can always construct the following:

**DEFINITION 3.3.** *Let  $(\mathfrak{X}, G, \Phi)$  be a minimal equicontinuous action on a Cantor space  $\mathfrak{X}$ . A properly descending chain of clopen sets  $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$  is said to be an adapted neighborhood basis at  $x \in \mathfrak{X}$  for the action  $\Phi$  if  $x \in U_{\ell+1} \subset U_\ell$  for all  $\ell \geq 1$  with  $\bigcap U_\ell = \{x\}$ , and each  $U_\ell$  is adapted to the action  $\Phi$ .*

For such a collection, set  $G_\ell = G_{U_\ell}$  we obtain a descending chain of finite index subgroups

$$\mathcal{G}_U = \{G = G_0 \supset G_1 \supset G_2 \supset \cdots\}.$$

The intersection  $K(\mathcal{G}_U) = \bigcap_{\ell \geq 0} G_\ell$  is called the kernel of  $\mathcal{G}_U$ .

Next, set  $X_\ell = G/G_\ell$  and note that  $G$  acts transitively on the left on  $X_\ell$ . The inclusion  $G_{\ell+1} \subset G_\ell$  induces a natural  $G$ -invariant quotient map  $p_{\ell+1}: X_{\ell+1} \rightarrow X_\ell$ . Introduce the inverse limit

$$(7) \quad X_\infty \equiv \varprojlim \{p_{\ell+1}: X_{\ell+1} \rightarrow X_\ell \mid \ell > 0\}$$

which is a Cantor space with the Tychonoff topology, and the action on the factors  $X_\ell$  induces a minimal equicontinuous action  $\Phi_x: G \times X_\infty \rightarrow X_\infty$ . Note that for  $g \in K(\mathcal{G}_U)$ , the left action of  $g$  on  $X_\ell$  fixes the coset  $e_\ell \in X_\ell$  and hence fixes the limiting point  $e_\infty \in X_\infty$ .

For each  $\ell \geq 0$ , we have the ‘‘partition coding map’’  $\sigma_\ell: \mathfrak{X} \rightarrow X_\ell$  which is  $G$ -equivariant. The maps  $\{\sigma_\ell\}$  are compatible with the quotient maps in (7), and so define a limit map  $\sigma_\infty: \mathfrak{X} \rightarrow X_\infty$ . The fact that the diameters of the clopen sets  $\{U_\ell\}$  tend to zero, implies that  $\sigma_\infty$  is a homeomorphism. Let  $\tau_x: X_\infty \rightarrow \mathfrak{X}$  denote the inverse map, which commutes with the  $G$  actions on both spaces, and satisfies  $\tau_x(e_\infty) = x$ . The minimal equicontinuous action  $(X_\infty, G, \Phi_x)$  is called the *odometer representation* centered at  $x$  for the action  $(\mathfrak{X}, G, \Phi)$ .

Suppose that  $\mathcal{U}$  and  $\mathcal{U}'$  are two choices of adapted neighborhood bases at  $x$ , then the corresponding group chains  $\mathcal{G}_U$  and  $\mathcal{G}_{U'}$  are equivalent as descending chains, as shown in [23, 18]. For two choices of basepoints  $x, x' \in \mathfrak{X}$ , with adapted neighborhood basis  $\mathcal{U}$  at  $x$  and  $\mathcal{U}'$  at  $x'$ , the corresponding group chains  $\mathcal{G}_U$  for  $\mathcal{U}$  and  $\mathcal{G}_{U'}$  for  $\mathcal{U}'$  need not be equivalent, but are *conjugate equivalent* as descending chains. The conjugacy relation on group chains, as introduced by Fokkink and Oversteegen in [23], is in essence the relation on group chains which results in conjugating them by elements of the closure of the group action. It is key for showing in [18] that the invariants of the action  $(\mathfrak{X}, G, \Phi)$  defined in [20, 33] are independent of the choices of basepoints and group chain models.

We recall a basic property of the odometer models:

**LEMMA 3.4.** *A minimal equicontinuous Cantor action  $(\mathfrak{X}, G, \Phi)$  is free if and only if for all  $x \in \mathfrak{X}$  and any adapted neighborhood basis  $\mathcal{U}$  at  $x$ , the kernel  $K(\mathcal{G}_U)$  is the trivial group.*

Let  $\mathcal{G}_U$  be the group chain associated to adapted neighborhood system at  $x$ , and suppose that  $y = g \cdot x$ , then set

$$\mathcal{G}_U^y = \{G = G_0 \supset g G_1 g^{-1} \supset g G_2 g^{-1} \supset \dots\},$$

which is the group chain associated to the adapted neighborhood basis  $\mathcal{U}^y$  at  $y$ . Clearly,  $K(\mathcal{G}_U^y) = g K(\mathcal{G}_U^x) g^{-1}$ , so the property that  $K(\mathcal{G}_U^y)$  is trivial is independent of the choice of  $y \in \mathcal{O}(x)$ . On the other hand, if  $y \notin \mathcal{O}(x)$  then whether  $K(\mathcal{G}_U^y)$  is trivial or not may depend on the choice of  $y$ .

We next prove a criterion for when a minimal equicontinuous Cantor action  $(\mathfrak{X}, G, \Phi)$  is locally quasi-analytic. Recall the following property of groups.

**DEFINITION 3.5.** *A group  $\mathfrak{G}$  is said to be Noetherian if every increasing chain of closed subgroups  $\{H_i \mid i \geq 1\}$  of  $\mathfrak{G}$  has a maximal element  $H_N$ .*

Here is the main result of this section:

**THEOREM 3.6.** *Let  $G$  be a Noetherian group. Then a minimal equicontinuous Cantor action  $(\mathfrak{X}, G, \Phi)$  is locally quasi-analytic.*

*Proof.* We assume that the action  $(\mathfrak{X}, G, \Phi)$  is not locally quasi-analytic, and construct an increasing chain of subgroups in  $G$  with no maximal element, contradicting that  $G$  is Noetherian.

Fix  $x \in \mathfrak{X}$  and let  $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$  be an adapted neighborhood basis at  $x$  for the action  $\Phi$ . For  $\ell \geq 1$ , the collection of translates  $\{g \cdot U_\ell \mid g \in G\}$  is a disjoint clopen covering of  $\mathfrak{X}$ . Let  $\lambda_\ell > 0$  be a Lebesgue number for this covering. That is, if  $U \subset \mathfrak{X}$  is an open set with  $\text{diam}(U) < \lambda_\ell$  then there exists  $g \in G$  such that  $U \subset g \cdot U_\ell$ .

The assumption that Definition 2.1 does not hold implies that for each  $\varepsilon > 0$ , there exists non-empty clopen subsets  $V \subset U \subset \mathfrak{X}$  with  $\text{diam}(U) < \varepsilon$  such that (6) fails for this pair. That is, there exists  $h_1, h_2 \in G$  such that  $\Phi(h_1)|V = \Phi(h_2)|V$  but  $\Phi(h_1)|U \neq \Phi(h_2)|U$ . Set  $h = h_2^{-1} h_1$  then  $\Phi(h)|V$  is the identity map, but  $\Phi(h)|U$  is not. Note that for this  $h$  we have  $h \cdot U \cap U \neq \emptyset$ .

We next construct an increasing chain of subgroups  $\{H_i \mid i \geq 1\}$  by induction. To begin, set  $\ell_1 = 1$ . Let  $W_1 \subset \mathfrak{X}$  be a non-empty open set with  $\text{diam}(W_1) < \lambda_1$  such that there exists a non-empty open subset  $V_1 \subset W_1$  and  $h_1 \in G$  so that  $\Phi(h_1)|V_1$  is the identity, and  $\Phi(h_1)|W_1$  is not, with  $h_1 \cdot W_1 \cap W_1 \neq \emptyset$ . Then there exists  $g_1 \in G$  such that  $W_1 \subset g_1 \cdot U_1$ , so  $g_1^{-1} \cdot W_1 \subset U_1$ . Set  $\xi_1 = g_1^{-1} h_1 g_1$  and note that  $\xi_1 \cdot U_1 \cap U_1 \neq \emptyset$ , hence  $\xi_1 \in G_{U_1}$ .

Note that  $g_1^{-1} \cdot V_1 \subset U_1$ . As the action of  $G_{U_1}$  on  $U_1$  is minimal, there exists  $\gamma_1 \in G_{U_1}$  such that  $\gamma_1 \cdot x \in g_1^{-1} \cdot V_1$ . Set  $V'_1 = \gamma_1^{-1} g_1^{-1} \cdot V_1$  so that  $x \in V'_1 \subset U_1$ . Then for

$$k_1 = \gamma_1^{-1} \xi_1 \gamma_1 = \gamma_1^{-1} g_1^{-1} h_1 g_1 \gamma_1,$$

we have that  $\Phi(k_1)|U_1$  is not the identity, but  $\Phi(k_1)|V'_1$  is the identity. Choose  $\ell_2 > \ell_1 = 1$  such that  $U_{\ell_2} \subset V'_1$  and thus  $\Phi(k_1)|U_{\ell_2}$  is also the identity.

Assume that for  $i > 1$ , and increasing sequence of integers  $1 = \ell_1 < \ell_2 < \dots < \ell_i$  have been chosen as above. Let  $W_i \subset \mathfrak{X}$  be a non-empty open set with  $\text{diam}(W_i) < \lambda_{\ell_i}$  such that there exists a non-empty open subset  $V_i \subset W_i$  and  $h_i \in G$  so that  $\Phi(h_i)|V_i$  is the identity, and  $\Phi(h_i)|W_i$  is not the identity, with  $h_i \cdot W_i \cap W_i \neq \emptyset$ . Then there exists  $g_i \in G$  such that  $W_i \subset g_i \cdot U_{\ell_i}$  and so  $g_i^{-1} \cdot W_i \subset U_{\ell_i}$ . Set  $\xi_i = g_i^{-1} h_i g_i$  and note that  $\xi_i \cdot U_{\ell_i} \cap U_{\ell_i} \neq \emptyset$ , hence  $\xi_i \in G_{U_{\ell_i}}$ .

Note that  $g_i^{-1} \cdot V_i \subset U_{\ell_i}$ . As the action of  $G_{U_{\ell_i}}$  on  $U_{\ell_i}$  is minimal, there exists  $\gamma_i \in G_{U_{\ell_i}}$  such that  $\gamma_i \cdot x \in g_i^{-1} \cdot V_i$ . Set  $V'_i = \gamma_i^{-1} g_i^{-1} \cdot V_i$  so that  $x \in V'_i \subset U_{\ell_i}$ . Then for

$$k_i = \gamma_i^{-1} \xi_i \gamma_i = \gamma_i^{-1} g_i^{-1} h_i g_i \gamma_i,$$

we have that  $\Phi(k_i)|U_{\ell_i}$  is not the identity, but  $\Phi(k_i)|V'_i$  is the identity. Choose  $\ell_{i+1} > \ell_i$  such that  $U_{\ell_{i+1}} \subset V'_i$  and thus  $\Phi(k_i)|U_{\ell_{i+1}}$  is also the identity.

Now assume that indices  $\{\ell_i \mid i \geq 1\}$  and elements  $\{k_i \mid i \geq 1\}$  have been chose as above. Define:

$$(8) \quad H_i = \{g \in G_{\ell_i} \mid \Phi(g)|U_{\ell_i} = id\}.$$

Observe that for each  $g \in H_i$  and  $j > i$  we have  $U_{\ell_j} \subset U_{\ell_i}$  hence  $g \cdot U_{\ell_j} = U_{\ell_j}$  thus  $H_i \subset G_{\ell_j}$ . Moreover,  $\Phi(g)|_{U_{\ell_j}}$  is the identity, hence  $H_i \subset H_j$ .

Now let  $i \geq 1$ , then by choice,  $\Phi(h_i)|_{W_i}$  is not the identity, hence  $\Phi(k_i)|_{U_{\ell_i}}$  is not the identity, so  $k_i \notin H_i$ . On the other hand, for  $j > i$  the restriction  $\Phi(k_i)|_{U_{\ell_j}}$  is the identity, hence  $k_i \in H_j$ .

It follows that the collection  $\{H_i \mid i \geq 1\}$  forms a strictly increasing chain of subgroups in  $G$ , which therefore is not Noetherian.  $\square$

The proof of Theorem 3.6 yields the following interesting result.

**COROLLARY 3.7.** *Let  $(\mathfrak{X}, G, \Phi)$  be a minimal equicontinuous Cantor action which is not locally quasi-analytic. Then for any  $x \in \mathfrak{X}$ , the isotropy subgroup  $G_x \subset G$  contains an infinite strictly increasing chain of subgroups.*

*Proof.* For given  $x$ , choose an adapted neighborhood basis at  $x$ . Then for  $H_i$  as constructed in the proof above, the action of each  $h \in H_i$  fixes the set  $U_{\ell_{i+1}}$  so in particular  $h \in G_x$ . Thus  $H_i \subset G_x$ .  $\square$

It is remarkable that the first construction of a non-homogenous weak solenoid by Schori in [46], see also [12], has this ascending chain property for the isotropy subgroups of its monodromy action. The Grigorchuk groups [28], and more generally branch groups [5, 29], provide a large class of examples of groups acting on trees which give rise to Cantor actions that are not locally quasi-analytic. This is discussed further in Example A.6.

It is easy to see that a finite group is Noetherian, and that the infinite cyclic group  $\mathbb{Z}$  is Noetherian. This observation is the basis for the next result.

Recall that a group  $\mathfrak{G}$  is *polycyclic* if there exists a chain of subgroups

$$(9) \quad \{e\} = \mathfrak{G}_{k+1} \subset \mathfrak{G}_k \subset \cdots \subset \mathfrak{G}_0 = \mathfrak{G}$$

such that each  $\mathfrak{G}_{\ell+1}$  is normal in  $\mathfrak{G}_\ell$  and the quotient  $\mathfrak{G}_\ell/\mathfrak{G}_{\ell+1}$  is a cyclic group. For example, a finitely-generated nilpotent group is polycyclic. A group is *virtually polycyclic* if there exists a finite group  $Q_0$  and a surjective homomorphism  $\pi_0: \mathfrak{G} \rightarrow Q_0$  such that the kernel  $\mathfrak{G}_0 = \ker(\pi_0)$  is polycyclic. We then have the folklore result, whose proof we include for completeness:

**PROPOSITION 3.8.** *Let  $\mathfrak{G}$  be a virtually polycyclic group, then  $\mathfrak{G}$  is Noetherian.*

*Proof.* Let  $\{H_i \mid i \geq 1\}$  be subgroups of  $\mathfrak{G}$  with  $H_i \subset H_{i+1}$ . Let  $\pi_0: \mathfrak{G} \rightarrow Q_0$  be a surjection onto a finite group  $Q_0$  such that  $\mathfrak{G}_0 \equiv \ker(\pi_0)$  is a nilpotent group. Then there exists  $i_0 \geq 1$  such that  $\pi_0(H_i) = \pi_0(H_{i_0}) \subset Q_0$  for all  $i \geq i_0$ . Set  $H_{0,i} = \ker\{\pi_0: H_i \rightarrow Q_0\}$  for  $i \geq i_0$ . Then  $\{H_{0,i} \mid i \geq i_0\}$  is an increasing chain of subgroups of  $\mathfrak{G}_0$ .

Let  $\mathfrak{G}_1 \subset \mathfrak{G}_0$  be a normal subgroup such that  $Q_1 = \mathfrak{G}_0/\mathfrak{G}_1$  is cyclic, and let  $\pi_1: \mathfrak{G}_0 \rightarrow Q_1$  be the quotient map. Then  $\{\pi_1(H_{0,i}) \mid i \geq i_0\}$  is an increasing chain of subgroups of the cyclic group  $Q_1$ , so there exists  $i_1 \geq i_0$  such that  $\pi_1(H_{0,i}) = \pi_1(H_{0,i_1}) \subset Q_1$  for all  $i \geq i_1$ . Set  $H_{1,i} = \ker\{\pi_1: H_{0,i} \rightarrow Q_1\}$  for  $i \geq i_1$ . Then  $\{H_{1,i} \mid i \geq i_1\}$  is an increasing chain of subgroups of  $\mathfrak{G}_1$ .

Continue recursively in this way, until we obtain a cyclic normal subgroup  $\mathfrak{G}_k \subset \mathfrak{G}_{k-1}$  as in (9), and an increasing chain of subgroups  $\{H_{k,i} \mid i \geq i_k\}$  of  $\mathfrak{G}_k$ . Then there exists  $N \geq i_k$  such that  $H_{k,i} = H_{k,N}$  for all  $k \geq N$ .

We claim that  $H_N$  is a maximal element for the chain  $\{H_i \mid i \geq 1\}$ . Let  $n \geq N$ . Then  $n \geq N \geq i_0$  implies that  $\pi_0(H_n) = \pi_0(H_N)$ , and so  $H_{1,n} = H_{1,N}$ . Proceeding recursively for  $1 \leq \ell \leq k$ , we have that  $n \geq N \geq i_\ell$  implies that  $\pi_{\ell+1}(H_{\ell,n}) = \pi_{\ell+1}(H_{\ell,N})$  and so  $H_{\ell+1,n} = H_{\ell+1,N}$ . Thus,  $H_n = H_N$  so  $H_N$  is maximal.  $\square$

Thus, we have the following extension of Corollary 2.3.

**COROLLARY 3.9.** *Let  $G$  be a finitely generated nilpotent group. Then a minimal equicontinuous Cantor action  $(\mathfrak{X}, G, \Phi)$  is locally quasi-analytic.*

The authors along with Jessica Dyer, gave in [18, Example 8.5] examples of minimal equicontinuous Cantor actions  $(\mathfrak{X}, G, \Phi)$  where  $G$  is a torsion-free 2-step nilpotent group, its discriminant group  $\mathcal{D}_x$  is a Cantor space, and the action is locally quasi-analytic. It was asked in [20] whether it is possible to construct examples of minimal equicontinuous Cantor actions  $(\mathfrak{X}, G, \Phi)$  where  $G$  is a finitely-generated nilpotent group, and the action is not locally quasi-analytic. Corollary 3.9 shows this to be impossible.

We conclude this section with a discussion of the *stable* property for actions, which was introduced in the authors' works [20, 33]. Let  $(\mathfrak{X}, G, \Phi)$  be a minimal equicontinuous Cantor action, and let  $\Phi: G \rightarrow \text{Homeo}(\mathfrak{X})$  be the homomorphism defined by the action. Then the closure  $\overline{\mathcal{H}_\Phi} \subset \text{Homeo}(X)$  in the uniform topology of the subgroup  $\mathcal{H}_\Phi = \Phi(G)$  is a compact totally disconnected group which acts transitively on  $\mathfrak{X}$ . For  $x \in \mathfrak{X}$ , the isotropy subgroup  $\mathcal{D}_x = \{h \in \overline{\mathcal{H}_\Phi} \mid h \cdot x = x\}$  is a closed subgroup, which is thus either finite or a Cantor group. This subgroup is called the *discriminant group* of the action at  $x$  in the work [18], and its conjugacy class in  $\overline{\mathcal{H}_\Phi}$  is independent of  $x$ .

Note that the isotropy subgroup satisfies  $G_x \subset \mathcal{D}_x$ . Fokkink and Oversteegen gave in [23, Section 8] the first examples of Cantor actions for which  $\mathcal{D}_x$  is a Cantor group but  $G_x$  the trivial group. Further examples with this property are constructed in [20, Section 10].

Given an adapted clopen neighborhood  $x \in U$  for the action  $\Phi$ , with stabilizer subgroup  $G_U \subset G$ , the restricted action of  $G_U$  on  $U$  is again minimal, and there is a homomorphism  $\Phi_U: U \rightarrow \text{Homeo}(U)$  with image denoted by  $\mathcal{H}_U \subset \text{Homeo}(U)$ , which acts minimally on  $U$ . Let  $\overline{\mathcal{H}_U} \subset \text{Homeo}(U)$  denote the closure of  $\mathcal{H}_U$  in the uniform topology on maps. Then  $\overline{\mathcal{H}_U}$  is a profinite group which acts transitively on  $U$ , and let  $(\overline{\mathcal{H}_U})_x$  denote the isotropy group for the action at  $x$ . The inclusion  $U \subset \mathfrak{X}$  induces via restriction a continuous surjective homomorphism  $\sigma_U: \mathcal{D}_x \rightarrow (\overline{\mathcal{H}_U})_x$ . This restriction map need not be an isomorphism. The action  $(\mathfrak{X}, G, \Phi)$  is said to be *stable* if there exists  $\varepsilon > 0$  such that  $\text{diam}(U) < \varepsilon$  implies that the kernel of  $\sigma_U$  is independent (up to conjugacy) of the choice of such  $U$ . This construction is discussed in detail in [20, 33]. In particular, it is shown in [33, Proposition 4.7] that a stable action is locally quasi-analytic.

#### 4. CONTINUOUS ORBIT EQUIVALENCE AND RIGIDITY

The concept of *continuous orbit equivalence* between Cantor actions was introduced by Mike Boyle in his thesis [9], and has played a fundamental role in the classification of Cantor actions in many subsequent works [8, 24, 25]. The related notion of the topological full group of an action in Definition 1.3 has provided a rich source of examples of finitely generated groups with exceptional properties, as discussed for example in [16]. In this section, we show how these notions can be used to show return equivalence for minimal equicontinuous Cantor actions.

**DEFINITION 4.1.** *For  $i = 1, 2$ , let  $(\mathfrak{X}_i, G_i, \Phi_i)$  be an action on a Cantor space  $\mathfrak{X}_i$ . The actions are said to be continuously orbit equivalent if there exists a homeomorphism  $h: \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$  and continuous functions*

- (1)  $\alpha: G_1 \times \mathfrak{X}_1 \rightarrow G_2$ ,  $h(\Phi_1(g_1)(x_1)) = \Phi_2(\alpha(g_1, x_1), h(x_1))$  for all  $g_1 \in G_1$  and  $x_1 \in \mathfrak{X}_1$ ;
- (2)  $\beta: G_2 \times \mathfrak{X}_2 \rightarrow G_1$ ,  $h^{-1}(\Phi_2(g_2, x_2)) = \Phi_1(\beta(g_2, x_2), h^{-1}(x_2))$  for all  $g_2 \in G_2$  and  $x_2 \in \mathfrak{X}_2$ .

*The actions are said to be virtually continuously orbit equivalent if there exists adapted clopen subsets  $U_i \subset \mathfrak{X}_i$  such that the restricted actions  $(U_i, G_{U_i}, \Phi_i)$  are continuously orbit equivalent.*

The homeomorphism  $h$  is called a *continuous orbit equivalence* between the two actions. Note that there is no assumption made that the functions  $\alpha$  and  $\beta$  satisfy the cocycle property.

Given an action  $(\mathfrak{X}_2, G_2, \Phi_2)$  and a homeomorphism  $h: \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$ , define the conjugate action  $(\mathfrak{X}_1, G_2, \Phi_2^h)$  by setting

$$\Phi_2^h(g_2, x_1) = h^{-1}(\Phi_2(g_2, h(x_1))) \text{ for } g_2 \in G_2, x_1 \in \mathfrak{X}_1.$$

We recall an observation from [27, Section 2] about the topological full group:

**PROPOSITION 4.2.** *For  $i = 1, 2$ , let  $(\mathfrak{X}_i, G_i, \Phi_i)$  be an action on a Cantor space  $\mathfrak{X}_i$ . Then a homeomorphism  $h: \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$  is a continuous orbit equivalence between the actions, if and only if the following two conditions hold:*

- (1)  $\Phi_1(g_1, \cdot) \in [[\mathfrak{X}_1, G_2, \Phi_2^h]]$  for all  $g_1 \in G_1$ ;
- (2)  $\Phi_2^h(g_2, \cdot) \in [[\mathfrak{X}_1, G_1, \Phi_1]]$  for all  $g_2 \in G_2$ .

*That is, the actions are continuously orbit equivalent if and only if the images  $\Phi_1(G_1) \subset \text{Homeo}(\mathfrak{X}_1)$  and  $\Phi_2^h(G_2) = h^{-1} \circ \Phi_2(G_2) \circ h \subset \text{Homeo}(\mathfrak{X}_1)$  generate the same topological full group.*

In the works [15, 35], the following is called the “rigidity problem” for Cantor actions.

**PROBLEM 4.3.** *For  $i = 1, 2$ , let  $(\mathfrak{X}, G_i, \Phi_i)$  be an action on a fixed Cantor space  $\mathfrak{X}$ , and suppose that  $[[\mathfrak{X}, G_1, \Phi_1]] = [[\mathfrak{X}, G_2, \Phi_2]]$ . Give conditions on the actions  $\Phi_1$  and  $\Phi_2$  which are sufficient to imply that they are return equivalent.*

Given a Cantor action  $(\mathfrak{X}, G_1, \Phi_1)$ , one can construct a new Cantor action  $(\mathfrak{X}, G_2, \Phi_2)$  with the same topological full group by simply adjoining elements of  $[[\mathfrak{X}_1, G_1, \Phi_1]]$  to the action  $\Phi_1(G_1)$ . We discuss this construction further in Example A.4. The take-away of the construction of such examples, is that one requirement for a solution of the rigidity problem is that both actions must have a property which rules out such constructions.

Let  $(\mathfrak{X}_i, G_i, \Phi_i)$  be a minimal equicontinuous action on a Cantor space  $\mathfrak{X}_i$  for  $i = 1, 2$ . Assume there exists a continuous orbit equivalence  $h: \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$  between two actions, then by the above remarks, we can assume without loss of generality that  $\mathfrak{X}_1 = \mathfrak{X}_2 = \mathfrak{X}$ , and there exists functions  $\alpha, \beta$  as in Definition 4.1. If the both actions are free, then the functions  $\alpha$  and  $\beta$  are uniquely determined by the actions and satisfy a cocycle property. This is a key point in the works [15, 26]. It was observed in [10, 35], and also implicitly in [44], that if the actions are topologically free on  $\mathfrak{X}$ , then the orbit functions  $\alpha$  and  $\beta$  still must satisfy the cocycle identities. Li showed that if, in addition, the actions satisfy the *continuous cocycle rigidity* property of [35, Section 4], then the actions are conjugate.

The work by Cortez and Medynets [15] showed that for free minimal equicontinuous Cantor actions, the orbit cocycles  $\alpha$  and  $\beta$  induce a return equivalence between the two actions. The observation behind the proof of Theorem 1.5 is that if both actions are assumed to be locally quasi-analytic, then the method of proof for [15, Theorem 3.3] can be used to show the actions are return equivalent.

We now begin the proof of Theorem 1.5. Assume that  $(\mathfrak{X}, G, \Phi)$  and  $(\mathfrak{X}, H, \Psi)$  are a minimal equicontinuous actions on a Cantor space  $\mathfrak{X}$ , and  $\varepsilon > 0$  is chosen so that both actions are locally quasi-analytic for clopen sets  $U \subset \mathfrak{X}$  satisfying  $\text{diam}(U) < \varepsilon$ . Choose a basepoint  $x \in \mathfrak{X}$ .

Let  $V \subset \mathfrak{X}$  be an adapted clopen set for the action  $\Psi$  with  $x \in V$  and  $\text{diam}(V) < \varepsilon$ , with stabilizer group  $H_V \subset H$ . Then by Proposition 2.2, the action of  $\mathcal{H}_V^\Psi = \Psi_V(H_V) \subset \text{Homeo}(V)$  on  $V$  is topologically free.

Let  $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 1\}$  be an adapted neighborhood basis at  $x$  for the action  $(\mathfrak{X}, G, \Phi)$ , with  $U_1 \subset V$ . Let  $\mathcal{G}_\mathcal{U} = \{G_\ell \mid \ell \geq 0\}$  be the associated group chain with  $G_0 = G$  and  $G_1 = G_{U_1}$ . Then the action of  $\mathcal{H}_{U_1}^\Phi = \Phi_{U_1}(G_1) \subset \text{Homeo}(U_1)$  on  $U_1$  is also topologically free.

Let  $\alpha: G \times \mathfrak{X} \rightarrow H$  be the continuous function in Definition 4.1. The subgroup  $G_1 \subset G$  has finite index, and  $G$  is finitely-generated, so there exists a finite generating set  $\{g_1, \dots, g_k\} \subset G_1$ . By the continuity of  $\alpha$ , there exists  $\delta_1 > 0$  such that

$$(10) \quad \alpha(g_i, x) = \alpha(g_i, y) \text{ for } 1 \leq i \leq k, \ x, y \in \mathfrak{X} \text{ with } d_{\mathfrak{X}}(x, y) < \delta_1 .$$

Let  $\ell_2 \geq 2$  be such that  $\text{diam}(g \cdot U_{\ell_2}) < \delta_1$  for all  $g \in G_1$ . The collection  $\{g \cdot U_{\ell_2} \mid g \in G_1\}$  is a finite clopen partition of  $U_1$  so there exists  $0 < \delta_2 < \delta_1$  such that for any  $y \in U_1$  there exists  $g \in G_1$  such that  $B_{d_{\mathfrak{X}}}(y, \delta_2) \subset g \cdot U_{\ell_2}$ . That is,  $\delta_2$  is a Lebesgue number for the open covering of  $U_1$ .

By the uniform continuity of the action  $\Phi$ , there exists  $0 < \delta_3 \leq \delta_2$  such that, for all  $g \in G$ ,

$$(11) \quad d_{\mathfrak{X}}(\Phi(g)(x), \Phi(g)(y)) < \delta_2 \text{ for all } x, y \in \mathfrak{X} \text{ with } d_{\mathfrak{X}}(x, y) < \delta_3 .$$

Let  $\ell_3 > \ell_2$  be such that  $\text{diam}(g \cdot U_{\ell_3}) < \delta_3$  for all  $g \in G_1$ . Note that  $U_{\ell_3} \subset U_{\ell_2} \subset U_1$ .

Now consider the restriction  $\alpha: G_1 \times U_1 \rightarrow H$ . For each  $g \in G_1$  and  $x \in U_1$  we have  $\Phi(g)(x) \in U_1$ . Let  $h \in H$  be such that  $\Psi(h)(x) = \Phi(g)(x)$ , then  $U_1 \subset V$  implies that  $\Psi(h)(V) \cap V \neq \emptyset$  hence  $h \in H_V$ . Recall that  $\mathcal{H}_V^\Psi = \Psi_V(H_V) \subset \text{Homeo}(V)$ , thus the restriction of  $\alpha$  to  $G_1 \times U_1$  induces a map  $\hat{\alpha}_1 = \Psi_V \circ \alpha: G_1 \times U_1 \rightarrow \mathcal{H}_V^\Psi$ .

**LEMMA 4.4.** *The map  $\hat{\alpha}_1: G_1 \times U_1 \rightarrow \mathcal{H}_V^\Psi$  is a cocycle over the action of  $\Phi_{U_1}$  on  $U_1$ .*

*Proof.* Let  $g, g' \in G_1$  and let  $x \in U_1$ . Then  $\Phi_{U_1}(g'g)(x) = \Phi_{U_1}(g')(\Phi_{U_1}(g)(x))$ .

The action of  $\mathcal{H}_V^\Psi$  is topologically free, so there exists a dense  $\mathcal{H}_V^\Psi$ -invariant subset  $Z \subset V$  such that the action of  $\mathcal{H}_V^\Psi$  on  $Z$  is free. That is, for  $\psi, \psi' \in \mathcal{H}_V^\Psi$  and  $z \in Z$ , if  $\psi(z) = \psi'(z)$  then  $\psi = \psi' \in \text{Homeo}(V)$ .

Let  $x \in Z$  and  $h = \alpha_1(g, x)$ , then  $\psi = \Psi_V(h)$  is the unique map in  $\mathcal{H}_V^\Psi$  such that  $\psi(x) = x'$ . Note that  $x' \in Z$  so there is a unique  $\psi' \in \mathcal{H}_V^\Psi$  such that  $x'' = \psi'(x') = \Phi_{U_1}(g')(x')$ . Then  $x'' = \psi' \circ \psi(x)$ .

By the defining identity (1) in Definition 4.1 with  $h$  the identity map, for  $x \in Z \cap U_1$  we have that

$$\hat{\alpha}_1(g, x) = \psi, \hat{\alpha}_1(g', x') = \psi', \hat{\alpha}_1(g'g, x) = \psi' \psi.$$

The continuity of the actions  $\Phi$  and  $\Psi$  imply that this holds for all  $x \in U_1$  as  $Z \cap U_1$  is dense in  $U_1$ . Thus, we have the cocycle identity

$$(12) \quad \hat{\alpha}_1(g'g, x) = \hat{\alpha}_1(g', \Phi_{U_1}(x)) \hat{\alpha}_1(g, x)$$

for all  $g', g \in G_{U_1}$  and  $x \in U_1$ , as was to be shown.  $\square$

Next, we show that the cocycle  $\hat{\alpha}_1$  is defined by a group homomorphism when restricted to the clopen subset  $U_{\ell_3} \subset U_1$ . The method of proof is the same as for the proof of [15, Theorem 3.3].

Recall that  $\{g_1, \dots, g_k\} \subset G_1$  is a set of generators, then for each  $1 \leq i \leq k$ , by the choice of  $U_{\ell_2}$  and (10), for any  $g' \in G_1$  the value of  $\hat{\alpha}_1(g_i, x)$  is constant for  $x \in g' \cdot U_{\ell_2}$ .

By the choice of  $\delta_1$ , for  $d_{\mathbb{X}}(x, y) < \delta_1$  and  $1 \leq i \leq k$ , we have  $\hat{\alpha}_1(g_i, x) = \hat{\alpha}_1(g_i, y)$ . Then by the choice of  $\delta_3$  and  $U_{\ell_3}$ , for any  $1 \leq i \leq k$ ,  $g' \in G_{U_1}$  and  $x, y \in U_{\ell_3}$  then

$$\hat{\alpha}_1(g_i, \Phi_{U_1}(g')(x)) = \hat{\alpha}_1(g_i, \Phi_{U_1}(g')(y)).$$

We apply this to the case where  $g = g_{i_1} \cdots g_{i_\nu} \in G_{U_{\ell_3}}$ . Set  $g' = g_{i_2} \cdots g_{i_\nu}$  then for  $x, y \in Z \cap U_{\ell_3}$ ,

$$(13) \quad \begin{aligned} \hat{\alpha}_1(g, x) &= \hat{\alpha}_1(g_{i_1} g', x) = \hat{\alpha}_1(g_{i_1}, \Phi_{U_1}(g')(x)) \circ \hat{\alpha}_1(g', x) \\ &= \hat{\alpha}_1(g_{i_1}, \Phi_{U_1}(g')(y)) \circ \hat{\alpha}_1(g', y) = \hat{\alpha}_1(g_{i_1} g', y) = \hat{\alpha}_1(g, y). \end{aligned}$$

As the values of  $\hat{\alpha}_1(g, x), \hat{\alpha}_1(g, y) \in \mathcal{H}_V^\Psi$  are defined using the identity (1) in Definition 4.1, the identity (13) holds on the closure of  $Z \cap U_{\ell_3}$  which is all of  $U_{\ell_3}$ . Thus, the calculation (13) shows that the restricted cocycle  $\hat{\alpha}_3: G_{U_{\ell_3}} \times U_{\ell_3} \rightarrow \mathcal{H}_V^\Psi$  is independent of the second factor  $U_{\ell_3}$ , hence induces a group homomorphism denoted by  $\tilde{\alpha}_3: G_{U_{\ell_3}} \rightarrow \mathcal{H}_V^\Psi$ .

Suppose that  $g, g' \in G_{U_{\ell_3}}$  satisfy  $\Phi_{U_{\ell_3}}(g) = \Phi_{U_{\ell_3}}(g') \in \text{Homeo}(U_{\ell_3})$ , then by the defining identity (1) in Definition 4.1 with  $h$  the identity map, for  $x \in Z \cap U_1$  we have  $\hat{\alpha}_1(g, x) = \hat{\alpha}_1(g', x) \in \mathcal{H}_V^\Psi$ . It follows that  $\hat{\alpha}_3$  induces a homomorphism  $\hat{\alpha}_3: \mathcal{H}_{U_{\ell_3}}^\Phi \rightarrow \mathcal{H}_V^\Psi$ . The homomorphism  $\hat{\alpha}_3$  is in fact an isomorphism. The proof is omitted, as it follows by the same arguments as in the works [35, 15].

Set  $\mathbf{G} = \mathcal{H}_{U_{\ell_3}}^\Phi$  and  $\mathbf{H} = \hat{\alpha}_3(\mathbf{G}) \subset \mathcal{H}_V^\Psi$ , and denote by  $\mathbf{A} \equiv \hat{\alpha}_3: \mathbf{G} \rightarrow \mathbf{H}$ .

Set  $W = U_{\ell_3}$ . Note that the group  $\mathbf{G}$  is the stabilizer of  $W$ , and the orbits of  $\mathbf{H}$  on points in  $V$  equal the orbits of  $\mathbf{G}$ , hence  $\mathbf{H}$  also stabilizes  $W$ . Thus,  $\mathbf{A}$  conjugates the action of  $\mathbf{G}$  and  $\mathbf{H}$  on  $W$ . This completes the proof of Theorem 1.5.

**REMARK 4.5.** The conclusion of Theorem 1.5 cannot be improved to obtain a conjugacy of the actions  $\Phi$  and  $\Psi$ , as illustrated in Examples A.2 and A.3. However, one can show that  $\mathbf{G}$  and  $\mathbf{H}$  determine subgroups of the same index in  $G$  and  $H$ , respectively, using the same methods as in the proof of [15, Theorem 3.3], so the actions  $\Phi$  and  $\Psi$  are structurally conjugate in the sense of [15].

**REMARK 4.6.** The proof of Theorem 1.5 in this section applies equally well for the case where the actions  $(\mathfrak{X}_i, G_i, \Phi_i)$  for  $i = 1, 2$  are virtually continuously orbit equivalent, simply by replacing the spaces  $\mathfrak{X}_i$  with the clopen subsets  $U_i$  and considering the actions of the subgroups  $G_{U_i} \subset G_i$  of finite index. The conclusion is again that the actions  $\Phi$  and  $\Psi$  are return equivalent. In this case, it is not assumed that the subgroup index  $[G_1 : G_{U_1}]$  equals the index  $[G_2 : G_{U_2}]$ , so this notion is somewhat more general than the notion of structural orbit equivalence in [15]. This more general situation is relevant to the classification problem for weak solenoids, as discussed in the next Section 5.

**REMARK 4.7.** Medynets states in [38, Remark 3] (and also in Theorem 4.2 in [15]) that if the topological full groups of two minimal Cantor actions  $(\mathfrak{X}_i, G_i, \Phi_i)$ ,  $i = 1, 2$ , are isomorphic as abstract groups, then there is a homeomorphism  $h: \mathfrak{X}_1 \rightarrow \mathfrak{X}_2$  which induces the isomorphism. That is, every such abstract group isomorphism is implemented by a spatial isomorphism. As pointed out in [15], this statement implies that the group isomorphism class of the full group  $[[\mathfrak{X}, G, \Phi]]$  determines the return equivalence class of the minimal equicontinuous action  $(\mathfrak{X}, G, \Phi)$ . Moreover, the group structure of topological full groups for Cantor actions has been studied in many recent works, as discussed for example in the Séminaire Bourbaki lecture by de Cornulier [16]. It thus seems an interesting problem, to determine “asymptotic invariants” of return equivalence directly from the group structure of the full groups  $[[U, G_U, \Phi_U]]$ , where  $U \subset \mathfrak{X}$  is an adapted clopen set for the action. For example, one can ask how to determine the *asymptotic discriminant invariant* of a minimal equicontinuous Cantor action, as introduced in [33], from the abstract algebraic properties of the partially ordered collection of groups

$$(14) \quad [[[\mathfrak{X}, G, \Phi]]] \equiv \{[[U, G_U, \Phi_U]] \mid U \subset \mathfrak{X} \text{ is an adapted clopen subset}\} .$$

**REMARK 4.8.** Introduce the notion of an asymptotic equivalence relation on the collections of local topological full groups  $[[[\mathfrak{X}, G, \Phi]]]$  by requiring that there exists  $\varepsilon > 0$ , such that there are local continuous orbit equivalences between the adapted clopen sets in the collection  $[[[\mathfrak{X}, G, \Phi]]]$ , where we assume that all such  $U$  have  $\text{diam}(U) < \varepsilon$ . Then consider the pro-object obtained by letting  $\varepsilon$  tend to 0. This approach is used to define the asymptotic discriminant in [33], which distinguishes large classes of minimal equicontinuous actions.

**QUESTION 4.9.** *What dynamical information about a minimal equicontinuous action  $(\mathfrak{X}, G, \Phi)$  is captured by the asymptotic equivalence class of  $[[[\mathfrak{X}, G, \Phi]]]$  ?*

## 5. CLASSIFICATION OF NIL-SOLENOIDS

In this section, we recall the construction and a few basic properties of weak solenoids, which are a special class continua introduced by McCord [37]. Schori subsequently gave an example in [46] of a weak solenoid whose Cantor fiber was not a Cantor group, which began the study of the self-homeomorphism group of weak solenoids. This was further investigated by Fokkink and Oversteegen in [23], who introduced the methods that were further developed in a sequence of works by the authors with Dyer in [18, 19, 20].

The classification of weak solenoids was studied in the works [1, 14, 33], where it is shown in particular that a homeomorphism between weak solenoids induces a return equivalence between their global monodromy Cantor actions. This result is recalled as Theorem 5.2 below. We introduce the class of nil-solenoids, then give the proof of Theorem 1.8, which states that the topological full groups for nil-solenoids characterize their homeomorphism types. This represents a broad generalization of the classification result for 1-dimensional solenoids by Aarts and Fokkink given in [1].

We first recall some basic concepts of weak solenoids. Let  $M_0$  be a closed connected manifold, and  $x_0 \in M_0$  a choice of basepoint. Let  $G = \pi_1(M_0, x_0)$  denote its fundamental group. Let  $\mathcal{G}$  be a

properly descending chain of finite index subgroups,

$$(15) \quad \mathcal{G} = \{G = G_0 \supset G_1 \supset G_2 \supset \dots\}.$$

For  $\ell \geq 0$ , each subgroup  $G_\ell$  determines a finite covering  $\pi_\ell: M_\ell \rightarrow M_0$ , where  $M_\ell$  is a closed manifold. The inclusions  $G_{\ell+1} \subset G_\ell$  induce non-trivial proper covering maps  $p_{\ell+1}: M_{\ell+1} \rightarrow M_\ell$ . The collection of these maps,  $\mathcal{P} = \{p_{\ell+1}: M_{\ell+1} \rightarrow M_\ell \mid \ell \geq 0\}$ , is called a presentation.

Associated to  $\mathcal{P}$  is the *weak solenoid*  $\mathcal{S}_\mathcal{P}$  which is the inverse limit space,

$$(16) \quad \mathcal{S}_\mathcal{P} \equiv \varprojlim \{p_{\ell+1}: M_{\ell+1} \rightarrow M_\ell\} \subset \prod_{\ell \geq 0} M_\ell.$$

By definition, for a sequence  $\{x_\ell \in M_\ell \mid \ell \geq 0\}$ , we have

$$(17) \quad x = (x_0, x_1, \dots) \in \mathcal{S}_\mathcal{P} \iff p_\ell(x_\ell) = x_{\ell-1} \text{ for all } \ell \geq 1.$$

The set  $\mathcal{S}_\mathcal{P}$  is given the relative topology, induced from the product topology, so that  $\mathcal{S}_\mathcal{P}$  is compact and connected. There is a canonical map  $\Pi_0: \mathcal{S}_\mathcal{P} \rightarrow M_0$  given by projection onto the first component.

For example, if  $M_\ell = \mathbb{S}^1$  for each  $\ell \geq 0$ , and the map  $p_\ell$  is a proper covering map of degree  $m_\ell > 1$  for  $\ell \geq 1$ , then  $\mathcal{S}_\mathcal{P}$  is an example of a classic solenoid. The generalization of 1-dimensional solenoids to the class of weak solenoids was introduced in the papers by McCord [37]. In particular, McCord showed that  $\mathcal{S}_\mathcal{P}$  has a uniform local product structure.

**PROPOSITION 5.1.** [37, 13] *Let  $\mathcal{S}_\mathcal{P}$  be a weak solenoid, whose base space  $M_0$  is a compact manifold of dimension  $n \geq 1$ . Then  $\mathcal{S}_\mathcal{P}$  is a foliated space, with foliation  $\mathcal{F}_\mathcal{P}$  whose leaves have dimension  $n$ . That is, for each  $x \in \mathcal{S}_\mathcal{P}$  there is an open neighborhood  $x \in U_x \subset \mathcal{S}_\mathcal{P}$  homeomorphic to the product space  $(-1, 1)^n \times K_x$  where  $K_x$  is a Cantor space, which is a foliation chart for  $\mathcal{F}_\mathcal{P}$ .*

A weak solenoid is a *matchbox manifold* of dimension  $n$  in the terminology of [13], or a *solenoidal manifold* in the terminology of [47, 48].

Let  $\mathfrak{X}_0 = \Pi_0^{-1}(x_0)$  denote the fiber over the basepoint  $x_0 \in M_0$ . The global monodromy of the foliation  $\mathcal{F}_\mathcal{P}$  on  $\mathcal{S}_\mathcal{P}$  is the action  $\Phi_0: G \times \mathfrak{X}_0 \rightarrow \mathfrak{X}_0$  defined by the holonomy transport along the leaves of the foliation  $\mathcal{F}_\mathcal{S}$ . This action is minimal and equicontinuous. A choice of basepoint  $x = (x_0, x_1, x_2, \dots) \in \mathfrak{X}_0$  defines basepoints in each covering  $M_\ell$ .

Let  $X_\infty$  be the inverse limit sequence associated to  $\mathcal{G}$  as defined by (7), and let  $\Phi: G \times X_\infty \rightarrow X_\infty$  be the associated minimal equivariant Cantor action. Then the homeomorphism  $\tau_x: X_\infty \rightarrow \mathfrak{X}_0$  defined in Section 3 conjugates the actions  $(\mathfrak{X}_0, G, \Phi_0)$  and  $(X_\infty, G, \Phi)$ . Moreover, the kernel  $K(\mathcal{G})$  is isomorphic to the fundamental group  $\pi_1(L_x, x)$  of the leaf  $L_x \subset \mathcal{S}_\mathcal{P}$  containing  $x$ .

We next recall a result implicit in the work of Fokkink and Oversteegen [23].

**THEOREM 5.2.** [14, Theorem 1.1] *Let  $\mathcal{P}$  be a presentation of a weak solenoid  $\mathcal{S}_\mathcal{P}$  over a closed manifold  $M_0$  of dimension  $n$  with fundamental group  $G$ , and  $\mathcal{P}'$  be a presentation of a weak solenoid  $\mathcal{S}_{\mathcal{P}'}$  over a closed manifold  $M'_0$  of dimension  $n'$  with fundamental group  $G'$ . Suppose that  $\mathcal{S}_\mathcal{P}$  is homeomorphic to  $\mathcal{S}_{\mathcal{P}'}$ , then the global holonomy action  $(X_\infty, G, \Phi)$  of  $\mathcal{F}_\mathcal{P}$  is return equivalent to the global holonomy action  $(X'_\infty, G', \Phi')$  of  $\mathcal{F}_{\mathcal{P}'}$ .*

It follows from the techniques used in the proof of Theorem 5.2, that associated to a weak solenoid  $\mathcal{S}_\mathcal{P}$ , there is a well-defined asymptotic equivalence class of local topological full groups as discussed in Remark 4.7. We denote this asymptotic class by  $[[[\mathcal{F}_\mathcal{P}]]]$ . Recall that in Remark 4.8 the notion of asymptotic equivalence between of local topological full groups was introduced.

**COROLLARY 5.3.** *Let  $\mathcal{S}_\mathcal{P}$  and  $\mathcal{S}_{\mathcal{P}'}$  be homeomorphic weak solenoids. Then the asymptotic local topological full group classes for  $\mathcal{F}_{\mathcal{P}'}$  and  $\mathcal{F}_\mathcal{P}$  are equivalent. That is,  $[[[\mathcal{F}_{\mathcal{P}'}]]] \sim [[[\mathcal{F}_\mathcal{P}]]]$ .*

We can now give an application of Theorem 1.5 which gives a converse to Corollary 5.3 for a subclass of weak solenoids, and thus a partial answer to Question 4.9.

A closed manifold  $M_0$  is said to be a *nilmanifold* if its fundamental group  $G = \pi_1(M_0, x_0)$  is a nilpotent group, for some choice of basepoint  $x_0 \in M_0$ , and its universal covering  $\widetilde{M}_0$  is a contractible space. In particular, this implies that the fundamental group  $G$  is torsion free. Let  $\mathcal{G}$  be a group chain as in (15), such that the kernel  $K(\mathcal{G})$  is the trivial group. Then the inverse limit space  $\mathcal{S}_{\mathcal{P}}$  is called a *nil-solenoid*.

**THEOREM 5.4.** *Let  $\mathcal{S}_{\mathcal{P}}$  and  $\mathcal{S}_{\mathcal{P}'}$  be nil-solenoids, and suppose that the asymptotic local topological full group classes for  $\mathcal{F}_{\mathcal{P}}$  and  $\mathcal{F}_{\mathcal{P}'}$  are equivalent. Then  $\mathcal{S}_{\mathcal{P}}$  and  $\mathcal{S}_{\mathcal{P}'}$  are homeomorphic continua.*

*Proof.* Let  $G$  denote the fundamental group for the closed manifold  $M_0$  in the presentation  $\mathcal{P}$ , and  $G'_0$  the fundamental group for the closed manifold  $M'_0$  in the presentation  $\mathcal{P}'$ .

The assumption that  $[[[\mathcal{F}_{\mathcal{P}}]]] \sim [[[\mathcal{F}_{\mathcal{P}'}]]]$  implies there exists adapted clopen sets  $U \subset X_{\infty}$  and  $U' \subset X'_{\infty}$  and a homeomorphism  $h: U \rightarrow U'$  that induces an isomorphism between the topological full groups  $[U, G_U, \Phi_U]$  and  $[U', G'_{U'}, \Phi'_{U'}]$  for the minimal equicontinuous Cantor actions  $(U, G_U, \Phi_U)$  and  $(U', G'_{U'}, \Phi'_{U'})$ .

As  $G_0$  is torsion-free nilpotent and finitely generated, the same also holds for the subgroup  $G_U$  of finite index, and similarly for  $G'_{U'}$ . It then follows from Corollary 3.9 that both restricted actions are locally quasi-analytic. It then follows from Theorem 1.5 that the restricted actions are return equivalent, and thus the actions  $(X_{\infty}, G, \Phi)$  and  $(X'_{\infty}, G', \Phi')$  are return equivalent.

Note that the assumption in the definition of a nil-solenoid  $\mathcal{S}_{\mathcal{P}}$  that the kernel  $K(\mathcal{G})$  for its defining group chain  $\mathcal{G}$  is trivial, implies that the foliation  $\mathcal{F}_{\mathcal{P}}$  has a simply connected leaf, which is homeomorphic to the universal covering of the base manifold  $M_0$  hence is contractible. Analogous comments hold for the nil-solenoid  $\mathcal{S}_{\mathcal{P}'}$ .

Return equivalence implies that the fundamental group  $G_0 = \pi_1(M, x_0)$  contains a subgroup of finite-index which is isomorphic to a subgroup of finite index in  $G'_0 = \pi_1(M', x'_0)$ . For a nil-manifold, its dimension is determined by the cohomological dimension of its fundamental group, so  $M_0$  and  $M'_0$  have finite coverings of the same dimension, hence must have equal dimensions.

Then by Theorem 1.5 of [14], the continua  $\mathcal{S}_{\mathcal{P}}$  and  $\mathcal{S}_{\mathcal{P}'}$  are homeomorphic.  $\square$

## APPENDIX A. EXAMPLES

In this appendix, we give a collection of examples of minimal Cantor actions, which illustrate the complexities that these actions can exhibit. The first examples are elementary, then increase in the subtleness of their properties. The presentations of the examples are brief, with references to their detailed constructions and study in the cited literature, as the interest is in showing the possibilities.

**EXAMPLE A.1** (Abelian actions). By Corollary 2.3, for  $G$  a finitely-generated abelian group, every effective minimal action of  $G$  on a Cantor space  $\mathfrak{X}$  is quasi-analytic, hence topologically free. The work of Li in [35] studies the relation between continuous orbit equivalence and conjugation of minimal equicontinuous Cantor actions. Example 3.5 in [35] constructs two abelian actions of  $\mathbb{Z}^n$  which are continuously orbit equivalent but not conjugate. By Theorem 1.2 in [15], or Theorem 1.5 above, the actions must be return equivalent. The results of Giordano, Putman and Skau in [26] classifies such actions in terms of their dual invariants.

**EXAMPLE A.2** (Direct products). Consider first a “toy” example. Let  $G$  and  $G'$  be non-isomorphic finite groups of the same order,  $N = |G| = |G'| \geq 4$ . Let  $X = \{1, 2, \dots, N\}$  be a finite set, so obviously not a Cantor space. Choose an isomorphism  $\varphi: G \rightarrow X$  and define the action  $(X, G, \Phi)$  by using  $\varphi$  to conjugate the left action of  $G$  on itself to an action on  $X$ . Likewise, choose an isomorphism  $\varphi': G' \rightarrow X$  and define the action  $(X, G', \Phi')$  by using  $\varphi'$  to conjugate the left action of  $G'$  on itself to an action on  $X$ . Note that for both of these actions, there is only one orbit of the action. Then both actions are minimal and equicontinuous, and the identity map  $X \rightarrow X$  is a continuous orbit equivalence. However, the actions  $\Phi$  and  $\Phi'$  cannot be conjugate as there is no group isomorphism  $h_*: G \rightarrow G'$ .

This simple example can then be modified to obtain examples which show that the conclusion of Theorem 1.5 cannot be improved to obtain a conjugacy of the actions. Let  $(\mathfrak{X}, H, \Psi)$  be any minimal equicontinuous Cantor action of a free abelian group  $H$ . Set  $\mathfrak{X} = X \times \mathfrak{Y}$  which is a Cantor space. Then the product actions  $(\mathfrak{X}, G \times H, \Phi \times \Psi)$  and  $(\mathfrak{X}, G' \times H, \Phi' \times \Psi)$  are locally quasi-analytic and continuously orbit equivalent, so satisfy the hypotheses of Theorem 1.5, but cannot be conjugate.

Example 7.6 in [18] gives another variation on a product action, where the finite group  $G$  is assumed to be simple, and one chooses a non-trivial proper subgroup  $K \subset G$ . The space  $X \cong G/K$  is given the left action by  $G$ , then using a product action as above, we obtain a minimal equicontinuous Cantor action which is locally quasi-analytic and has discriminant  $\mathcal{D}_x \cong K$ . If  $K, K' \subset G$  are non-isomorphic proper subgroups with the same order, then for product actions with the actions of  $G$  and  $G/K$  and  $G/K'$  as above, we obtain minimal equicontinuous Cantor actions which are locally quasi-analytic and continuously orbit equivalent, but cannot be conjugate as their discriminants are not isomorphic as their discriminants are not equal.

**EXAMPLE A.3** (Semi-direct products). The direct product construction above can be made somewhat more interesting by using a semi-direct product construction, as in:

- Example 7.5 in [18], where  $G$  is the infinite dihedral group, and the discriminant  $\mathcal{D}_x \cong \mathbb{Z}/2\mathbb{Z}$ . This action is non-homogeneous but is stable.
- Examples 8.8 and 8.9 in [18], where  $G$  is a generalized dihedral group, and the discriminant of the action is a Cantor subgroup.
- Example 7.1 in [19], where  $G$  is a semi-direct product of  $\mathbb{Z}^n$  with a finite subgroup  $H \subset \Sigma(n)$  of the permutation group on  $n$  letters.

Note that in all of these cited examples, the semi-direct product construction results in an extension of an infinite group by a finite normal subgroup.

**EXAMPLE A.4** (Full group modifications). Let  $(\mathfrak{X}, G, \Phi)$  be a minimal equicontinuous Cantor action. Assume the action  $\Phi$  is topologically free, so that we can identify  $G$  with its image  $\Phi(G) \subset \text{Homeo}(\mathfrak{X})$ . The idea of this construction is to modify the group  $G$  by the addition of a finite set of elements from its topological full group  $[[\mathfrak{X}, G, \Phi]]$ .

Let  $U \subset \mathfrak{X}$  be a proper adapted clopen subset, with stabilizer group  $G_U \subset G$ . Let  $A: G_U \rightarrow G_U$  be a non-trivial automorphism. Define a new action  $\Phi_U^A: G_U \rightarrow \text{Homeo}(\mathfrak{X})$  by setting, for  $g \in G_U$ :

$$(18) \quad \Phi_U^A(g)(x) = A(g) \cdot x \text{ for } x \in U, \text{ and } \Phi_U^A(g)(x) = x \text{ for } x \in \mathfrak{X} - U .$$

Let  $H \subset \text{Homeo}(\mathfrak{X})$  denote the group generated by the images  $\Phi(G)$  and  $\Phi_U^A(G_U)$ . Then  $H$  is finitely generated, and by construction, the action  $\Psi: H \times \mathfrak{X} \rightarrow \mathfrak{X}$  is continuously orbit equivalent to the given action  $\Phi$ . Note that the action of  $H$  is locally quasi-analytic but not topologically free.

The clopen set  $U$  is adapted to both actions, and the restriction of  $\Psi$  to  $U$  is just the conjugated action  $\Phi_U^A: G_U \rightarrow \text{Homeo}(\mathfrak{X})$ . Thus, the images  $\Phi_U(G_U) \subset \text{Homeo}(U)$  and  $\Phi_U^A(G_U) \subset \text{Homeo}(U)$  are equal. It follows that the actions are return equivalent. However,  $H$  is almost surely not isomorphic to  $G$ , so the actions  $(\mathfrak{X}, G, \Phi)$  and  $(\mathfrak{X}, H, \Psi)$  cannot be conjugate.

**EXAMPLE A.5** (Heisenberg). In her thesis [17], Dyer constructed examples of subgroup chains in the 2-dimensional Heisenberg group, so that the discriminant of the Cantor action constructed from the chain is a Cantor group. These are probably the simplest examples of Cantor actions by torsion-free finitely generated nilpotent groups. We recall Example 8.5 from [18].

Let  $\mathcal{H}$  be the discrete Heisenberg group, presented in the form  $\mathcal{H} = (\mathbb{Z}^3, *)$  with the group operation  $*$  given by  $(x, y, z) * (x', y', z') = (x + x', y + y', z + z' + xy')$ . Let  $A_n = \begin{pmatrix} p^n & 0 \\ 0 & q^n \end{pmatrix}$ , where  $p$  and  $q$  are distinct primes, and consider the action represented by a group chain

$$G_0 = \mathcal{H} , \{G_n \mid n \geq 1\} \text{ where } G_n \equiv A_n \mathbb{Z}^2 \times p^n \mathbb{Z} .$$

Then the discriminant  $\mathcal{D}_x$  is a Cantor group, by explicit calculation [17, Example 5.14]. It would be interesting to have a procedure for constructing nilpotent actions of higher dimension whose discriminants are Cantor groups. All such examples must be locally quasi-analytic by Corollary 3.9. It is unknown whether it is possible to construct a Cantor action of a finitely generated nilpotent group which is not stable. This seems to be a subtle question.

**EXAMPLE A.6** (Actions on rooted trees). Every minimal equicontinuous Cantor action can be interpreted as an action on a rooted spherically homogeneous tree. For actions as discussed in the previous examples, this fact is just one facet of their study, but for other actions it is a part of their definition. We briefly recall some basic facts about trees, then describe two classes of “arboreal actions” currently under investigation.

Let  $T$  be a tree. That is,  $T$  consists of the set of vertices  $V = \cup_{\ell \geq 0} V_\ell$ , where each  $V_\ell$  is a finite set, and of the set of edges  $E$ , where an edge  $t = [a, b]$  can join two vertices  $a$  and  $b$  only if  $a \in V_\ell$  and  $b \in V_{\ell+1}$ , and every vertex  $b \in V_{\ell+1}$  is joined to precisely one vertex in  $V_\ell$ . We assume that  $V_0$  is a singleton, so  $T$  is *rooted* with root vertex  $V_0 = \{v_0\}$ . The tree  $T$  is *spherically homogeneous*, if there is a sequence of positive integers  $(n_1, n_2, \dots)$  such that for every  $\ell \geq 1$ , every vertex  $v \in V_{\ell-1}$  is joined by an edge to precisely  $n_\ell$  vertices in  $V_\ell$ . We assume that  $n_\ell > 1$  for all  $\ell \geq 1$ .

An automorphism  $g \in \text{Aut}(T)$  of the rooted tree  $T$  permutes the vertices within each level  $V_\ell$ , while preserving the connectedness of the tree. That is, for all  $\ell \geq 1$  the vertices  $v_{\ell-1} \in V_{\ell-1}$  and  $v_\ell \in V_\ell$  are joined by an edge if and only if  $g \cdot v_{\ell-1} \in V_{\ell-1}$  and  $g \cdot v_\ell \in V_\ell$  are joined by an edge. Thus  $\text{Aut}(T)$  acts on infinite paths in the tree. Here, a path in  $T$  is an ordered infinite sequence of vertices  $(v_\ell)_{\ell \geq 0}$  such that for all  $\ell \geq 0$  the vertex  $v_\ell$  is at level  $V_\ell$ , and the consecutive vertices  $v_\ell$  and  $v_{\ell+1}$  are joined by an edge. Denote by  $\mathcal{P}$  the space of all infinite paths in  $T$  with cylinder set topology. As  $n_\ell > 1$  for  $\ell \geq 1$ , the space of paths  $\mathcal{P}$  is a Cantor space.

Let  $(\mathfrak{X}, G, \Phi)$  be an effective minimal equicontinuous Cantor actions. The “tree model” for this action is constructed as follows. Choose a basepoint  $x \in \mathfrak{X}$ , and choose an adapted neighborhood system  $\mathcal{U} = \{U_\ell \mid \ell \geq 1\}$  at  $x$ , with stabilizer groups  $G_\ell = G_{U_\ell}$ . Set the root vertex  $v_0 = \{x\}$ , and let the vertices at level  $\ell \geq 1$  be the collection of clopen sets  $V_\ell \equiv \{g \cdot U_\ell \mid G \in G/G_\ell\}$ . For  $g, g' \in G$ , there is an edge  $t = [g \cdot U_\ell, g' \cdot U_{\ell+1}]$  between  $g \cdot U_\ell$  and  $g' \cdot U_{\ell+1}$  exactly when  $g' \cdot U_{\ell+1} \subset g \cdot U_\ell$ . This defines a spherically homogeneous tree  $T_{\mathcal{U}}$  where the degree  $n_\ell = [G_{\ell-1} : G_\ell]$ . The “partition coding map”  $\tau_\infty : X_\infty \rightarrow \mathfrak{X}$  introduced in Section 3 can be interpreted as defining a homeomorphism  $\tau_{\mathcal{U}} : \mathcal{P}_{\mathcal{U}} \rightarrow \mathfrak{X}$  which commutes with the action of  $G$ .

Now let  $T$  be a rooted spherically homogeneous tree. It was shown in the work [6] that  $\text{Aut}(T)$  is the infinite wreath product of the symmetric groups  $\mathfrak{S}(n_\ell)$ , and so has a dense countably generated subgroup  $G_0 \subset \text{Aut}(T)$ . This is described in [36, Section 2.5], and see also Nekrashevych [39]. Moreover, it is shown in [36] that the action of  $G_0$  on the path space  $\mathcal{P}$  is not locally quasi-analytic. Here is a basic question:

**PROBLEM A.7.** *Let  $T$  be a rooted spherically homogeneous tree. Construct finitely-generated subgroups  $G_0 \subset \text{Aut}(T)$  such that the induced action on the Cantor space of paths  $\mathcal{P}$  is not locally quasi-analytic.*

One interest in constructing finitely-generated groups acting on Cantor spaces is that each such action can be realized as the global monodromy action of a suspension foliation as described in Section 8.1 of [33]. If the action is a minimal equicontinuous Cantor action, then the resulting foliated space is homeomorphic to a weak solenoid as described in Section 5 above.

The non-Hausdorff examples of Grigorchuk and Nekrashevych provide one class of solutions to Problem A.7. The construction by Grigorchuk of his celebrated groups with intermediate growth in [28] introduced the notion of groups acting on rooted spherically homogeneous trees which are generated by automata. The branch groups of [5, 29] are a far reaching extension of Grigorchuk’s original construction. It is easy to see that the action of the standard Grigorchuk group on the Cantor set of paths is not locally quasi-analytic. Nekrashevych studied in [40, 41] the dynamical properties of these actions, and showed that they often contain non-Hausdorff elements in their

action groupoids. An action containing a non-Hausdorff element cannot be locally quasi-analytic by Proposition 2.5. We can thus ask a more specific version of Problem A.7.

**PROBLEM A.8.** *Characterize the branch groups for which the action on the Cantor space of infinite paths is not locally quasi-analytic, or not stable.*

The work by Grigorchuk and Savchuk [30] solves an analogous problem in the measurable category.

Odoni initiated in [42] the study of arboreal representations of the absolute Galois group. Such representations are given by the action of a profinite group on a spherically homogeneous rooted tree. This field of study is surveyed by Jones in [34]. The second author developed in [36] a method of associating to an arboreal representation an infinite chain of discrete groups, and gave examples of stable and not stable arboreal representations. The examples of not stable actions given in [36] are by infinitely-generated groups, and it is an open problem to construct arboreal representations which are finitely generated and not locally quasi-analytic, or not stable.

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