

ORBIT EQUIVALENCE FOR NILPOTENT CANTOR ACTIONS

STEVEN HURDER AND OLGA LUKINA

ABSTRACT. We show that an action of a finitely-generated group G' on a Cantor space, which is continuously orbit equivalent to minimal equicontinuous action of a finitely-generated nilpotent group, must itself be a minimal equicontinuous action, and if the action of G' is effective, then G' is virtually nilpotent.

1. INTRODUCTION

Let G be a countable group, \mathfrak{X} a Cantor space, and $\Phi: G \rightarrow \text{Homeo}(\mathfrak{X})$ an action of G . We denote the action by (\mathfrak{X}, G, Φ) , and write $g \cdot x$ for $\Phi(g)(x)$ when the action is clear.

An action (\mathfrak{X}, G, Φ) is *effective* if the action homomorphism $\Phi: G \rightarrow \text{Homeo}(\mathfrak{X})$ is an injection. It is *free* if for each $g \in G$, $g \cdot x = x$ for some $x \in \mathfrak{X}$ implies that g is the identity element. The action is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x) = \{g \cdot x \mid g \in G\}$ is dense in \mathfrak{X} . The action (\mathfrak{X}, G, Φ) is *equicontinuous* (with respect to a metric $d_{\mathfrak{X}}$ on \mathfrak{X} compatible with the topology) if for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ and $g \in G$, $d_{\mathfrak{X}}(x, y) < \delta$ implies that $d_{\mathfrak{X}}(\Phi(g) \cdot x, \Phi(g) \cdot y) < \varepsilon$. This property is independent of the choice of the metric $d_{\mathfrak{X}}$ on \mathfrak{X} . The notion of a continuous orbit equivalence between Cantor actions is discussed in Section 3.

The purpose of this note is to show the following result about equicontinuous actions.

THEOREM 1.1. *Assume that G is a finitely-generated nilpotent group, and (\mathfrak{X}, G, Φ) is a minimal, equicontinuous action. Suppose that G' is also finitely-generated, and $(\mathfrak{X}', G', \Phi')$ is an action which is continuously orbit equivalent to (\mathfrak{X}, G, Φ) . Then $(\mathfrak{X}', G', \Phi')$ is a minimal equicontinuous action. Suppose also that Φ' is effective, then there exists a finite index nilpotent subgroup $H' \subset G'$.*

The conclusion that $(\mathfrak{X}', G', \Phi')$ is minimal and equicontinuous was proved for free actions by Cortez and Medynets [10, Corollary 4.4], and uses Remark 3 in [23, Section 2] that an isomorphism of full groups is realized spatially for Cantor actions. Our proof of this first conclusion of Theorem 1.1 is given in Proposition 3.2 below, and is direct though technical.

The assumption that the action Φ' is effective in the second part of Theorem 1.1 is obviously necessary, as the kernel of an action can be arbitrary, and is not reflected in the dynamics of the action. If the action is not effective, then the quotient of G' by the kernel of the action homomorphism Φ' yields an effective action to which the result applies.

The conclusion of the second part of Theorem 1.1 is a consequence of the “rigidity property” for Cantor actions as studied in the work of Cortez and Medynets [10] for free actions, the work of Li [19] for topologically free actions, and the authors [17, 18] for the general case. Theorem 1.1 can be considered as an extension of results in [10] to the non-free case, where the free hypothesis for the action is replaced by the nilpotent hypothesis on G . If G is abelian, a minimal effective G -action must be free, so the novel aspect of our result is its application to Cantor actions of non-abelian nilpotent groups, as in Example 7.2.

For G abelian, Giordano, Putnam and Skau [15] give a classification of free minimal equicontinuous G -actions, up to various forms of equivalence, in terms of a locally compact group associated to

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the action using Pontrjagin duality. For the more general case where G is nilpotent, the invariant constructed in [15] can be replaced with the C^* -algebra $C^*(\mathfrak{X}, G, \Phi)$ associated to the action, which is an invariant of continuous orbit equivalence for topologically free actions (see [10, Theorem 1.2], and also [24].) The algebra $C^*(\mathfrak{X}, G, \Phi)$ is Type I, and its isomorphism invariants for G nilpotent and non-abelian are more subtle than in the case where G is abelian [3, 21, 22]. For a nilpotent Cantor action which is stable but not topologically free, the proof of Theorem 1.1 shows that the holonomy group H_U of the induced action of G_V on the adapted set U is a θ_U conjugacy invariant, so isomorphism invariants of the C^* -algebra $C^*(U, H_U, \Phi_U)$ are continuous orbit equivalence invariants of the action (\mathfrak{X}, G, Φ) . As the choice of the clopen set $U \subset \mathfrak{X}$ depends on the orbit equivalence map $h: \mathfrak{X} \rightarrow \mathfrak{X}'$, it is necessary to consider C^* -invariants which are independent of this choice to obtain invariants of the action (\mathfrak{X}, G, Φ) . One such invariant is the Type I property, but it is unclear if there are other possibilities. Regarding this, see also Remark 6.2.

The proof of Theorem 1.1 uses a combination of results from the authors' papers [17, 18], along with ideas from the work by Cortez and Medynets in [10]. Sections 2, 4 and 5 recall the basic results needed for the proof. Section 3 gives a proof of Proposition 3.2, a result of interest on its own. The proof of Theorem 1.1 is given in Section 6. Section 7 gives a selection of examples.

2. EQUICONTINUOUS ACTIONS

We recall some basic properties of minimal equicontinuous actions (\mathfrak{X}, G, Φ) on a Cantor set \mathfrak{X} . The basic reference for such actions is the book by Auslander [4].

Let $\text{CO}(\mathfrak{X})$ denote the collection of all clopen (closed and open) subsets of the Cantor space \mathfrak{X} , which forms a basis for the topology of \mathfrak{X} . For $\phi \in \text{Homeo}(\mathfrak{X})$ and $U \in \text{CO}(\mathfrak{X})$, the image $\phi(U) \in \text{CO}(\mathfrak{X})$. The following result is folklore, and a proof is given in [17, Proposition 3.1].

PROPOSITION 2.1. *A minimal Cantor action (\mathfrak{X}, G, Φ) is equicontinuous if and only if, for the induced action $\Phi_*: G \times \text{CO}(\mathfrak{X}) \rightarrow \text{CO}(\mathfrak{X})$, the G -orbit of every $U \in \text{CO}(\mathfrak{X})$ is finite.*

We say that $U \in \text{CO}(\mathfrak{X})$ is *adapted* to the action (\mathfrak{X}, G, Φ) if U is *non-empty* and for any $g \in G$, if $\Phi(g)(U) \cap U \neq \emptyset$ then $\Phi(g)(U) = U$. It follows that

$$(1) \quad G_U = \{g \in G \mid \varphi(g)(U) \cap U \neq \emptyset\}$$

is a subgroup of G , called the *stabilizer* of U . Then for $g, g' \in G$ with $g \cdot U \cap g' \cdot U \neq \emptyset$ we have $g^{-1}g' \cdot U = U$, hence $g^{-1}g' \in G_U$. Thus, the translates $\{g \cdot U \mid g \in G\}$ form a finite clopen partition of \mathfrak{X} , and are in 1-1 correspondence with the quotient space $X_U = G/G_U$. Then G acts by permutations of the finite set X_U and so the stabilizer group $G_U \subset G$ has finite index. The action of $g \in G$ on X_U is trivial precisely when g is a stabilizer of each coset $h \cdot G_U$, so $g \in C_U$ where

$$(2) \quad C_U = \bigcap_{h \in G} h G_U h^{-1} \subset G_U$$

is the largest normal subgroup of G contained in G_U . The action of the finite group $Q_U \equiv G/C_U$ on X_U by permutations is a finite approximation of the action of G on \mathfrak{X} , and the isotropy group of the identity coset $e \cdot G_U$ is $D_U \equiv G_U/C_U \subset Q_U$.

DEFINITION 2.2. *Let (\mathfrak{X}, G, Φ) be a minimal equicontinuous action. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ is said to be an adapted neighborhood basis at $x \in \mathfrak{X}$ for the action Φ if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 0$ with $\bigcap U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ .*

PROPOSITION 2.3. *Let (\mathfrak{X}, G, Φ) be a minimal equicontinuous action. Given $x \in \mathfrak{X}$, there exists an adapted neighborhood basis \mathcal{U} at x for the action Φ .*

Proof. Given $x \in \mathfrak{X}$ and $\varepsilon > 0$, Proposition 2.1 implies there exists an adapted clopen set $U \in \text{CO}(\mathfrak{X})$ with $x \in U$ and $\text{diam}(U) < \varepsilon$. Thus, one can choose a descending chain \mathcal{U} of adapted sets in $\text{CO}(\mathfrak{X})$ whose intersection is x . \square

It follows immediately from Proposition 2.3 that a minimal equicontinuous action is conjugate to a generalized odometer, as shown in [12, 10].

3. CONTINUOUS ORBIT EQUIVALENCE

A *conjugacy* between two actions (\mathfrak{X}, G, Φ) and $(\mathfrak{X}', G, \Phi')$ is a homeomorphism $\phi: \mathfrak{X} \rightarrow \mathfrak{X}'$ such that $\Phi(g) = \phi^{-1} \circ \Phi'(g) \circ \phi$ for all $g \in G$. We are concerned with the following weaker notion of equivalence, which was introduced by Boyle in his thesis [6]:

DEFINITION 3.1. *Let (\mathfrak{X}, G, Φ) and $(\mathfrak{X}', H, \Phi')$ be actions. A continuous orbit equivalence is a homeomorphism $\phi: \mathfrak{X} \rightarrow \mathfrak{X}'$ which maps orbits of the action Φ to orbits of the action Φ' . Moreover, the homeomorphism ϕ and its inverse ϕ^{-1} are required to be “locally constant”:*

(1) *for each $x \in \mathfrak{X}$ and $g \in G$, there exists $\alpha(g, x) \in H$ and an open set $x \in U_{g,x} \subset \mathfrak{X}$ such that $\Phi'(\alpha(g, x)) \circ \phi|_{U_{g,x}} = \phi \circ \Phi(g)|_{U_{g,x}}$;*

(2) *for each $y \in \mathfrak{X}'$ and $h \in H$, there exists $\beta(h, y) \in G$ and an open set $y \in V_{h,y} \subset \mathfrak{X}'$ such that $\phi \circ \Phi(\beta(h, y))|_{V_{h,y}} = \Phi'(h) \circ \phi|_{V_{h,y}}$.*

If (\mathfrak{X}, G, Φ) is a minimal action, then a continuous orbit equivalence $\phi: \mathfrak{X} \rightarrow \mathfrak{X}'$ with an action $(\mathfrak{X}', H, \Phi')$ maps the dense orbits of Φ to the orbits of Φ' which must therefore also be dense. That is, $(\mathfrak{X}', H, \Phi')$ is also a minimal action.

Suppose that (\mathfrak{X}, G, Φ) and $(\mathfrak{X}', H, \Phi')$ are orbit equivalent by a homeomorphism $\phi: \mathfrak{X} \rightarrow \mathfrak{X}'$, then form the conjugate action $\Psi: H \times \mathfrak{X} \rightarrow \mathfrak{X}$ where $\Psi = \phi^{-1} \circ \Phi' \circ \phi$. It then follows that the identity map is an orbit equivalence between the actions (\mathfrak{X}, G, Φ) and (\mathfrak{X}, H, Ψ) . Thus, we can always reduce to the case where ϕ is the identity map, and if (\mathfrak{X}, G, Φ) is minimal then (\mathfrak{X}, H, Ψ) is also minimal.

We use the techniques from the works [10, 17, 18] and Proposition 2.3 below to show the following result, which yields the first part of Theorem 1.1:

PROPOSITION 3.2. *Suppose that G and H are finitely-generated groups, and the identity map is a continuous orbit equivalence between the equicontinuous action (\mathfrak{X}, G, Φ) and the action (\mathfrak{X}, H, Ψ) , then (\mathfrak{X}, H, Ψ) is also equicontinuous.*

Proof. We first establish some technical preliminaries. Recall that $d_{\mathfrak{X}}$ is a metric on \mathfrak{X} compatible with the topology. Let α and β be the maps in Definition 3.1 for h the identity map. That is, we have continuous maps $\alpha: G \times \mathfrak{X} \rightarrow H$ and $\beta: H \times \mathfrak{X} \rightarrow G$ so that for $y \in \mathfrak{X}$ and $g \in G$, there exist a clopen set $y \in U_{g,y} \subset \mathfrak{X}$ with

$$(3) \quad \Psi(\alpha(g, y)) \cdot z = \Phi(g) \cdot z \text{ for } z \in U_{g,y} ,$$

and for $h \in H$, there exists a clopen set $y \in V_{h,y} \subset \mathfrak{X}$ so that

$$(4) \quad \Phi(\beta(h, y)) \cdot z = \Psi(h) \cdot z \text{ for } z \in V_{h,y} .$$

Let $\Delta(H) \equiv \{h_1, \dots, h_\mu\} \subset H$ be a set of generators which satisfy $h_i^{-1} \in \Delta(H)$ for all $1 \leq i \leq \mu$. That is, $\Delta(H)$ is a symmetric generating set for H .

For each $1 \leq j \leq \mu$ we have an open covering of \mathfrak{X} by the sets $\{V_{h_j, y} \mid y \in \mathfrak{X}\}$. As \mathfrak{X} is compact there exists a Lebesgue number $\varepsilon_j > 0$ for the covering. Then $\varepsilon' = \min\{\varepsilon_1, \dots, \varepsilon_\mu\} > 0$ is a Lebesgue number for all of these coverings.

Let $U' \subset \mathfrak{X}$ be an adapted set such that for all $g \in G$, we have $\text{diam}_{d_{\mathfrak{X}}}(g \cdot U') < \varepsilon'$. Then the translates $\mathcal{U}' = \{g \cdot U' \mid g \in G\}$ form a finite covering of \mathfrak{X} by disjoint clopen sets, hence we have

$$\varepsilon'' = \min \{ \text{dist}_{d_{\mathfrak{X}}}(g \cdot U', g' \cdot U') \mid g \cdot U' \neq g' \cdot U' \} > 0 .$$

Then for $0 < \lambda < \varepsilon''$ and $y \in g \cdot U'$, the ball of radius λ about y satisfies $B_{d_{\mathfrak{X}}}(y, \lambda) \subset g \cdot U'$.

Choose $\varepsilon > 0$ with $\varepsilon < \min \{\varepsilon', \varepsilon''\}$. As the action (\mathfrak{X}, G', Ψ) is equicontinuous, there exists $\delta > 0$ such that for all $g \in G$ and $x, y \in \mathfrak{X}$ with $d_{\mathfrak{X}}(x, y) < \delta$, then $d_{d_{\mathfrak{X}}}(\Phi(g) \cdot x, \Phi(g) \cdot y) < \varepsilon$. Note that $\delta \leq \varepsilon$ follows if we let g be the identity element.

Let $U \subset \mathfrak{X}$ be an adapted set with $\text{diam}_{d_{\mathfrak{X}}}(g \cdot U) < \delta$ for all $g \in G$.

We can now complete the proof of the proposition. Let $h \in G'$ and $x, y \in \mathfrak{X}$ satisfy $d_{\mathfrak{X}}(x, y) < \delta$. Express h in terms of the generators $\Delta(H)$, so $h = h_{j_m} \cdots h_{j_1}$ for $1 \leq j_\ell \leq \mu$. We proceed using an induction argument.

Let $g_{x,0} \in G$ so that $x \in g_{x,0} \cdot U$, then we also have $y \in g_{x,0} \cdot U$ by the choice of λ . Set $x_0 = x$ and $y_0 = y$. Then by the choice of ε with $\varepsilon < \varepsilon''$, and the choice of U , there exists $z_0 \in \mathfrak{X}$ such that $B_{d_{\mathfrak{X}}}(x_0, \varepsilon) \subset V_{h_{j_1}, z_0}$ and so also $y_0 \in V_{h_{j_1}, z_0}$.

It follows that $\beta(h_{j_1}, x_0) = \beta(h_{j_1}, y_0) \in G$. Then set $g_{j_1} = \beta(h_{j_1}, x_0)$ and by (4) we have

$$x_1 = \Psi(h_{j_1}) \cdot x_0 = \Phi(g_{j_1}) \cdot x_0 \quad , \quad y_1 = \Psi(h_{j_1}) \cdot y_0 = \Phi(g_{j_1}) \cdot y_0 \quad .$$

Note that $d_{\mathfrak{X}}(x_1, y_1) < \varepsilon$ by the equicontinuity hypothesis, and the choice of δ .

Now let $1 \leq \ell < \mu$, and assume that $\{x_0, x_1, \dots, x_\ell\}$, $\{y_0, y_1, \dots, y_\ell\}$, and $\{g_{j_1}, g_{j_2}, \dots, g_{j_\ell}\} \subset G$ have been chosen so that for $1 \leq i \leq \ell$ we have

$$x_i = \Psi(h_{j_i}) \cdot x_{i-1} = \Phi(g_{j_i}) \cdot x_{i-1} \quad , \quad y_i = \Psi(h_{j_i}) \cdot y_{i-1} = \Phi(g_{j_i}) \cdot y_{i-1} \quad .$$

and we have $d_{\mathfrak{X}}(x_i, y_i) < \varepsilon$. Then there exists $z_\ell \in \mathfrak{X}$ such that $B_{d_{\mathfrak{X}}}(x_\ell, \varepsilon) \subset V_{h_{j_{\ell+1}}, z_\ell}$ and so also $y_\ell \in V_{h_{j_{\ell+1}}, z_\ell}$. It follows that $\beta(h_{j_{\ell+1}}, x_\ell) = \beta(h_{j_{\ell+1}}, y_\ell) \in G$. Then set $g_{j_{\ell+1}} = \beta(h_{j_{\ell+1}}, x_\ell)$ and define $\widehat{g}_{\ell+1} \equiv g_{j_{\ell+1}} \cdot g_{j_\ell} \cdots g_{j_1}$. Then by (4) and the previous choices, we have

$$\begin{aligned} x_{\ell+1} &= \Psi(h_{j_{\ell+1}}) \cdot x_\ell = \Phi(g_{j_{\ell+1}}) \cdot x_\ell = \Phi(\widehat{g}_{\ell+1}) \cdot x_0 \\ y_{\ell+1} &= \Psi(h_{j_{\ell+1}}) \cdot y_\ell = \Phi(g_{j_{\ell+1}}) \cdot y_\ell = \Phi(\widehat{g}_{\ell+1}) \cdot y_0 \quad . \end{aligned}$$

Then the equicontinuity hypothesis applied to $\Phi(\widehat{g}_{\ell+1})$ and the assumption that $d_{\mathfrak{X}}(x_0, y_0) < \delta$ yields $d_{\mathfrak{X}}(x_{\ell+1}, y_{\ell+1}) < \varepsilon$. Thus, for $\ell = \mu - 1$ we obtain the estimate

$$d_{\mathfrak{X}}(\Psi(h) \cdot x, \Psi(h) \cdot y) = d_{\mathfrak{X}}(\Phi(\widehat{g}_\mu) \cdot x_0, \Phi(\widehat{g}_\mu) \cdot y_0) < \varepsilon$$

as was to be shown. \square

REMARK 3.3. There is an unusual analogy between the proof of Proposition 3.2 above, and a key aspect of the method used in the proof of Theorem 1.5 in [9]. The latter uses a ‘‘path lifting’’ argument along the leaves of a foliated space to show that the holonomy action of a lamination which fibers over a manifold with arbitrarily small fibers is equicontinuous. This assumes a strong hypotheses about the shape properties of the manifold, which is used to prove the required path lifting property. On the other hand, the proof above uses the map β , which is given as part of the assumption that two actions are continuously orbit equivalent, to ‘‘lift’’ the maps $\Psi(h_{j_i})$ connecting the points in the chain $\{x_0, x_1, \dots, x_\mu\}$ to maps $\Phi(g_{j_i})$ which also connect the points in the chain.

4. STABLE ACTIONS

The ‘‘topologically free’’ property for an action (\mathfrak{X}, G, Φ) is a weakening of the notion of a free action. For $g \in G$ let $\text{Fix}(g) = \{x \in \mathfrak{X} \mid \Phi(g) \cdot x = x\}$, then introduce the *isotropy set*

$$(5) \quad \text{Iso}(\mathfrak{X}, G, \Phi) = \{x \in \mathfrak{X} \mid \exists g \in G, g \neq id, \Phi(g) \cdot x = x\} = \bigcup_{e \neq g \in G} \text{Fix}(g) \quad .$$

DEFINITION 4.1. (\mathfrak{X}, G, Φ) is topologically free if the set $\text{Iso}(\mathfrak{X}, G, \Phi)$ is nowhere dense in \mathfrak{X} .

The notion of a topologically free action on a Cantor set first appeared in the work of Boyle and Tomiyama [7] in their study of flip-conjugacy. Renault showed in [24, Section 3] that an action is topologically free if and only if the associated action groupoid is *essentially principal*. Topological freeness and related ideas are discussed in more detail in [19, Section 2].

For equicontinuous actions on Cantor spaces, there is a related notion of a quasi-analytic action, introduced by Álvarez López and Candel in [1, Definition 9.4], and also Álvarez López and Moreira Galicia in [2, Definition 2.18].

DEFINITION 4.2. *An equicontinuous action (\mathfrak{X}, G, Φ) is quasi-analytic if for any $g \in G$ and adapted set $U \subset \mathfrak{X}$ with $\Phi(g)(U) = U$, if $\Phi(g)|U$ is the identity, then $\Phi(g)$ is the identity map.*

This notion admits a local formulation first given in [14, 17], which is fundamental for this work.

DEFINITION 4.3. *An equicontinuous action (\mathfrak{X}, G, Φ) is locally quasi-analytic, or LQA, if there exists $\varepsilon > 0$ so that for any adapted set $U \subset \mathfrak{X}$ with $\text{diam}_{d_{\mathfrak{X}}}(U) < \varepsilon$, then for $g \in G$ with $\Phi(g)(U) = U$, and any adapted set $V \subset U$, if $\Phi(g)|V$ is the identity map on V , then $\Phi(g)|U$ is the identity map on U .*

In order to introduce the notion of a stable action, we briefly recall another construction. Let (\mathfrak{X}, G, Φ) be an equicontinuous action, and let $\Phi(G) \subset \text{Homeo}(\mathfrak{X})$ denote the image subgroup. Introduce the closure $\mathfrak{G}(\Phi) \equiv \overline{\Phi(G)} \subset \text{Homeo}(X)$ in the uniform topology of maps. That is, each element $\hat{g} \in \mathfrak{G}(\Phi)$ is the uniform limit of a sequence of maps $\{\Phi(g_i) \mid i \geq 1\} \subset \Phi(G)$. If the action Φ is minimal, then the action $\hat{\Phi}$ of $\mathfrak{G}(\Phi)$ on \mathfrak{X} is transitive. The notion of a stable action was introduced in the authors' works [14, 17], and is equivalent to the following definition by Theorem 1.3 in [18].

DEFINITION 4.4. *Let (\mathfrak{X}, G, Φ) be a minimal equicontinuous action on a Cantor space \mathfrak{X} , and let $\mathfrak{G}(\Phi)$ be the closure of $\Phi(G)$. The action Φ is said to be stable if the action of the closure $\mathfrak{G}(\Phi)$ on \mathfrak{X} is locally quasi-analytic. In particular, a stable action is locally quasi-analytic.*

An action which is not stable is said to be *wild*. A wild action admits a decreasing chain of adapted sets $\{\mathfrak{X} \supset U_1 \supset U_2 \supset \dots\}$ such that for $K_\ell = \{\hat{g} \in \mathfrak{G}(\Phi) \mid \hat{\Phi}(\hat{g})|U_\ell = id\} \subset \mathfrak{G}(\Phi)$, the chain of closed subgroups $\{K_1 \subset K_2 \subset \dots\}$ is increasing without bound in $\mathfrak{G}(\Phi)$. Theorem 1.4 in [18] shows that the existence of such a chain for a minimal equicontinuous action is preserved by a continuous orbit equivalence, which yields the following result:

THEOREM 4.5. *Let $h: \mathfrak{X} \rightarrow \mathfrak{X}'$ be a continuous orbit equivalence between minimal equicontinuous actions (\mathfrak{X}, G, Φ) and $(\mathfrak{X}', G', \Psi)$. If G is finitely generated, and $(\mathfrak{X}', G', \Psi)$ is stable, then (\mathfrak{X}, G, Φ) is stable.*

Finally, the stable property raises the question of when a Cantor action must be stable. The following property of a group provides such a criterion.

DEFINITION 4.6. [5] *A group \mathfrak{G} is said to be Noetherian if every increasing chain of closed subgroups $\{K_\ell \mid \ell \geq 1\}$ of \mathfrak{G} has a maximal element K_N .*

The interpretation of the wild condition for a Cantor action in terms of an increasing chain of subgroups, yields the following result:

THEOREM 4.7. [16, Theorem 1.6] *Let G be a Noetherian group. Then a minimal equicontinuous action (\mathfrak{X}, G, Φ) on a Cantor space \mathfrak{X} is locally quasi-analytic.*

Recall that a group Γ is *polycyclic* if there exists a chain of subgroups

$$(6) \quad \{e\} = \Gamma_{k+1} \subset \Gamma_k \subset \dots \subset \Gamma_0 = \Gamma$$

such that each $\Gamma_{\ell+1}$ is normal in Γ_ℓ and the quotient $\Gamma_\ell/\Gamma_{\ell+1}$ is a cyclic group. For example, a finitely-generated nilpotent group is polycyclic. A group Γ is *virtually polycyclic* if there exists a subgroup $\Gamma_0 \subset \Gamma$ of finite index such that Γ_0 is polycyclic. The following result is folklore.

PROPOSITION 4.8. *Let Γ be a virtually polycyclic group, then Γ is Noetherian.*

COROLLARY 4.9. *Let G be a finitely-generated nilpotent group. Then a minimal equicontinuous Cantor action (\mathfrak{X}, G, Φ) is locally quasi-analytic.*

It is elementary that an effective minimal action of an abelian group must be free, and Corollary 4.9 can be viewed as a generalization of this fact.

5. RETURN EQUIVALENCE

Recall from Section 3 that a conjugacy between two actions (\mathfrak{X}, G, Φ) and $(\mathfrak{X}', G, \Phi')$ is a homeomorphism $\phi: \mathfrak{X} \rightarrow \mathfrak{X}'$ such that $\Phi(g) = \phi^{-1} \circ \Psi(g) \circ \phi$ for all $g \in G$. We require a slightly more general form of conjugacy, as studied by Li in [19] and Cortez and Medynets, who called it θ -conjugacy.

DEFINITION 5.1. *The actions (\mathfrak{X}, G, Φ) and (\mathfrak{X}, H, Ψ) are said to be θ -conjugate if there exists a group isomorphism $\theta: G \rightarrow H$ such that*

$$(7) \quad \Phi(g)(x) = \Psi^\theta(x) \equiv \Psi(\theta(g))(x) \quad , \text{ for all } g \in G \text{ , } x \in \mathfrak{X} \text{ .}$$

For $G = H$ and θ the identity map, then θ -conjugacy reduces to the usual notion of conjugacy.

For the case where $G = H = \mathbb{Z}$, the involution $\theta(n) = -n$ is the only non-trivial isomorphism, and in this case θ -conjugacy is the same as flip-conjugacy as studied by Boyle and Tamiyama [7].

The notion of return equivalence for minimal equicontinuous actions was introduced in [8] for the study of the homeomorphism types of weak solenoids. For the geometric applications in [14, 16], the holonomy action on a transversal is the fundamental concept. Accordingly, return equivalence for actions is formulated in terms of the image group H_U for an adapted subset $U \subset \mathfrak{X}$.

Let $U \subset \mathfrak{X}$ be adapted for the Cantor action (\mathfrak{X}, G, Φ) , and let $\Phi_U: G_U \times U \rightarrow U$ denote the restricted action of G_U on U . Let $\bar{\Phi}_U: H_U \times U \rightarrow U$ denote the induced action of $H_U = \Phi(G_U) \subset \text{Homeo}(U)$.

Similarly, for a minimal equicontinuous action $(\mathfrak{X}', G', \Phi')$ with adapted set $V \subset \mathfrak{X}'$, let $(V, H'_V, \bar{\Psi}_V)$ denote the induced action by $\Phi'_V: G'_V \times V \rightarrow V$, where $H'_V = \Phi'_V(G'_V) \subset \text{Homeo}(V)$.

DEFINITION 5.2. *Two minimal equicontinuous actions (\mathfrak{X}, G, Φ) and $(\mathfrak{X}', G', \Phi')$ are return equivalent if there exists*

- (1) *an adapted set $U \subset \mathfrak{X}$ for the action Φ ,*
- (2) *an adapted set $V \subset \mathfrak{X}'$ for the action Φ' ,*
- (3) *an isomorphism $\theta_U: H_U \rightarrow H'_V$,*

and a homeomorphism $h_U: U \rightarrow V$ which induces a θ_U -conjugacy between the action of H_U on U and the action of H'_V on V .

6. PROOF OF THEOREM 1.1

We are given that G and G' are finitely generated groups, and (\mathfrak{X}, G, Φ) is a minimal, equicontinuous action. Let $(\mathfrak{X}', G', \Phi')$ be an action which is continuously orbit equivalent to (\mathfrak{X}, G, Φ) by a map $h: \mathfrak{X} \rightarrow \mathfrak{X}'$. Then $(\mathfrak{X}', G', \Phi')$ is minimal by the existence of the orbit preserving homeomorphism h .

Proposition 3.2 shows that $(\mathfrak{X}', G', \Phi')$ is an equicontinuous action.

Theorem 4.5 implies that $(\mathfrak{X}', G', \Phi')$ is also a stable action.

Finally, we recall the following result from [17] which is a generalization of the Theorem 3.3 in [10], which applies to free actions.

THEOREM 6.1. [17, Theorem 1.5] *Let G and G' be finitely generated groups, and suppose that (\mathfrak{X}, G, Φ) and $(\mathfrak{X}', G', \Phi')$ are stable, minimal and equicontinuous actions on Cantor spaces. If the actions are continuously orbit equivalent, then the actions are return equivalent.*

The assumptions of Theorem 1.1 imply that the conditions of Theorem 6.1 are satisfied, and so the conditions of Definition 5.2 are satisfied. Let $U \subset \mathfrak{X}$ be adapted for the action Φ , let $V \subset \mathfrak{X}'$ be adapted for the action Φ' , and let $\theta_U: H_U \rightarrow H'_V$ be an isomorphism.

The assumption that G is nilpotent implies that the subgroup G_U is nilpotent, and hence also its image H_U . As θ_U is an isomorphism, H'_V is also nilpotent.

Let $K'_V \subset G'_V \subset G'$ be the kernel of Φ'_V . Then K'_V is normal in G'_V , but need not be normal in G' .

The quotient $G'_V/K'_V \cong H'_V$ so is nilpotent. We claim that the kernel K'_V is also nilpotent.

Let $X'_V = G'/G'_V$ be the finite set of cosets of G'_V , with a transitive left G' action. The action Φ' induces a map $\pi'_V: G' \rightarrow \text{Perm}(X'_V)$ to the group of permutations of the set, and G'_V is the stabilizer of the identity coset eG'_V . Let $C'_V = \ker(\pi'_V) \subset G'$ be the kernel of this representation, so C'_V is a normal subgroup of G' with finite index.

Choose representatives $\{h_i \in G' \mid 1 \leq i \leq \nu\}$ of the cosets of G'/G'_V then $\mathfrak{X}' = V \cup h_1 \cdot V \cup \dots \cup h_\nu \cdot V$.

Note that the action of $h \in C'_V$ on \mathfrak{X} leaves each clopen set $h_i \cdot V$ invariant, and the action is conjugate to the action of $h_i^{-1} h h_i$ on V . Thus, the kernel of the action $\Phi: C'_V \times (h_i \cdot V) \rightarrow (h_i \cdot V)$ is the subgroup $h_i (K'_V \cap C'_V) h_i^{-1} = h_i K'_V h_i^{-1} \cap C'_V$. Thus, the kernel of the action $\Phi'|_{C'_V}: C'_V \times \mathfrak{X} \rightarrow \mathfrak{X}$ is the intersection

$$(8) \quad \ker(\Phi'|_{C'_V}) = \bigcap_{1 \leq i \leq \nu} \{h_i K'_V h_i^{-1} \cap C'_V\} = \bigcap_{1 \leq i \leq \nu} \{h_i K'_V h_i^{-1}\} \cap C'_V = \bigcap_{1 \leq i \leq \nu} \{h_i K'_V h_i^{-1}\}.$$

Now assume that the action Φ' is effective, then $\ker(\Phi'|_{C'_V})$ is the trivial group, and hence the intersection in (8) is the trivial group.

Define a representation $\hat{\rho}$ of C'_V into a product of ν copies of H'_V by setting for $h \in C'_V$,

$$(9) \quad \hat{\rho}: C'_V \rightarrow H'_V \times \dots \times H'_V \quad , \quad \hat{\rho}(h) = \Phi'_V(h_1^{-1} h h_1) \times \dots \times \Phi'_V(h_\nu^{-1} h h_\nu)$$

Then the kernel of the map $\hat{\rho}$ in (9) is the intersection of the kernels in (8) which is trivial. Thus, $\hat{\rho}$ is an injection of C'_V into a product of nilpotent groups, which is again nilpotent, and so C'_V is a nilpotent group. Hence, G' is virtually nilpotent, as was to be shown.

REMARK 6.2. In the statement of Theorem 1.1, if the actions are both assumed to be topologically free, then essentially the same proof as given by Cortez and Medynets of [10, Theorem 3.3] for free actions, shows that the nilpotent subgroups can be chosen to have the same index; that is $[G : H] = [G' : H']$ and moreover, $H \cong H'$. However, when the actions are just stable, then the return equivalence relationship does not seem sufficient to make these conclusions. The problem arises in the proof of Theorem 1.1, where it is shown that the normal subgroup $C'_V \subset G'$ is nilpotent, and there is no obvious estimate on the index $[G'_V : C'_V]$. Furthermore, it is not obvious how to estimate the degree of nilpotency of C'_V .

REMARK 6.3. The conclusions of Theorem 1.1 can be applied to the geometry of matchbox manifolds, following the ideas of the authors' work [13]. Let \mathfrak{M} be a matchbox manifold (as discussed in [8, 13]) whose leaves have polynomial growth. Then the monodromy action for \mathfrak{M} has polynomial growth, and hence is given by the action of a group H with polynomial growth on a transversal. If this Cantor action is continuously orbit equivalent to the monodromy action of an equicontinuous matchbox manifold \mathfrak{M}' , then the methods of [13] show that the leaves of the foliation of \mathfrak{M}' also have polynomial growth.

7. EXAMPLES

EXAMPLE 7.1. First, here is a simple example. Let G and G' be finite groups of the same order, $N = |G| = |G'| \geq 4$, and suppose that G and G' are not isomorphic. Let $X = \{1, 2, \dots, N\}$ be a finite set, and choose an isomorphism $\varphi: G \rightarrow X$ and define the action (X, G, Φ) by using φ to conjugate the left action of G on itself to an action on X . Likewise, choose an isomorphism $\varphi': G' \rightarrow X$ and define the action (X, G', Φ') by using φ' to conjugate the left action of G' on itself to an action on X . For both of these actions, there is only one orbit of the action. The identity map $X \rightarrow X$ is a continuous orbit equivalence, but the actions Φ and Φ' cannot be conjugate as there is no group isomorphism $\theta: G \rightarrow G'$.

Modify this simple example by taking products to obtain Cantor actions which show that the conclusion of Theorem 1.1 cannot be improved to obtain a conjugacy of the actions. Let (\mathfrak{Y}, H, Ψ) be any minimal equicontinuous Cantor action of a free abelian group H . Set $\mathfrak{X} = X \times \mathfrak{Y}$ which is a Cantor space. Then the product actions $(\mathfrak{X}, G \times H, \Phi \times \Psi)$ and $(\mathfrak{X}, G' \times H, \Phi' \times \Psi)$ are locally

quasi-analytic and continuously orbit equivalent. If G is an abelian group, then the product $G \times H$ is abelian, so the actions satisfy the hypotheses of Theorem 1.1, but cannot be conjugate.

There are many variations possible for Example 7.1, where for example the product group $G \times H$ is replaced by a cross-product of a nilpotent normal subgroup by a finite quotient group, so that two actions are continuous orbit equivalent but are not θ -conjugate actions. Examples 7.5 and 8.8 in [12] and Example 7.1 in [13] are of this type.

EXAMPLE 7.2. An important and less trivial class of examples are obtained from the odometer construction using group chains in a finitely-generated nilpotent group G . Lightwood, Şahin and Ugarcovici [20] discuss the construction of *normal* group chains in the Heisenberg group, yielding actions of the Heisenberg group G on a pro-finite completion $\mathfrak{X} \cong \widehat{G}$. These actions are always free. On the other hand, Dyer gives a criterion in her thesis [11, Example A.5] for the construction of *non-normal* group chains in the Heisenberg group G , using a normality criteria from [20], so that the resulting action G is stable but not free. This example is also described in [12, Example 8.5].

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STEVEN HURDER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT CHICAGO, 322 SEO (M/C 249), 851 S. MORGAN STREET, CHICAGO, IL 60607-7045

Email address: hurder@uic.edu

OLGA LUKINA, FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, 1090 VIENNA, AUSTRIA

Email address: olga.lukina@univie.ac.at