ESSENTIAL HOLONOMY OF CANTOR ACTIONS

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Abstract. A group action has essential holonomy if the set of points with non-trivial holonomy has positive measure. If such an action is topologically free, then having essential holonomy is equivalent to the action not being essentially free, that is, the set of points with non-trivial stabilizer has positive measure. In the paper, we investigate the relation between the property of having essential holonomy and structure of the acting group for minimal equicontinuous actions on Cantor sets. We show that if such a group action is locally quasi-analytic and has essential holonomy, then every commutator subgroup in the group lower central series has elements with positive measure set of points with non-trivial holonomy. We deduce that all minimal equicontinuous Cantor actions by nilpotent groups have no essential holonomy. We also introduce a local version of the Farber criterion, which allows to determine when a locally quasi-analytic action has no essential holonomy.

1. Introduction

A measure-preserving action of a finitely generated group Γ on a metric space $\mathfrak{X}$ is *essentially free* if the set of points of $\mathfrak{X}$ with non-trivial stabilizers has measure zero. In this paper, we are interested in the case when $\mathfrak{X}$ is totally disconnected, more precisely, $\mathfrak{X}$ is a Cantor set, and the action of Γ is minimal. We call such actions *Cantor actions.* We always assume that a Cantor action is effective, and often that it is equicontinuous, see Section 2.1 for definitions. Equicontinuous Cantor actions arise, for instance, as monodromy actions on the fibres of towers of covers of smooth manifolds, and the property that such an action is essentially free is useful for the study of the associated von Neumann algebras. A Cantor action can only be essentially free if the space $\mathfrak{X}$ has points with trivial stabilizers. Then the minimality and continuity of such an action implies that the set of points with trivial stabilizers is G_δ-dense (residual) in $\mathfrak{X}$, and so the action is *topologically free.*

Bergeron and Gaboriau [6] showed that a non-amenable group which is a free product of two residually finite groups admits an equicontinuous Cantor action which is topologically free and not essentially free. Abért and Elek [1] proved a similar result for finitely generated non-abelian free groups Γ. More recently, Joseph [14] proved that any non-amenable surface group admits a continuum of pairwise non-conjugate and non-measurably isomorphic equicontinuous Cantor actions which are topologically free and not essentially free.

A Cantor action which is not topologically free cannot be essentially free, as such actions admit open sets of fixed-points for non-trivial group elements. Examples of actions which are not topologically free include the actions of branch and weakly branch groups on rooted trees [12, 25], many actions of nilpotent groups [20], and actions of topological full groups [8]. The work by Gröger and the second author [13] proposed an approach to generalizing the notion of essentially free actions to actions where each point has a non-trivial stabilizer by looking at the behavior in small neighborhoods of fixed points. Here the notion of *trivial or non-trivial holonomy* is a key concept. In this more general

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setting, the analog of the set of points with trivial (resp. non-trivial) stabilizers is the set of points with trivial (resp. non-trivial) holonomy, and the analog of an essentially free action is an action which has *no essential holonomy*, see Definition 1.3. Gröger and Lukina constructed in [13] further examples of equicontinuous Cantor actions which are not essentially free or have essential holonomy, and introduced a criterion for when such an action has no essential holonomy. The following is an important goal of the study of the topological dynamics of Cantor actions:

PROBLEM 1.1. *Find conditions on a finitely-generated group* Γ *which implies that every minimal equicontinuous Cantor action of* Γ *has trivial essential holonomy.*

For example, if Γ is a finitely-generated amenable group, is the conclusion of Problem 1.1 true?

It is elementary to show that an effective action of an abelian group Γ is always free, and so essentially free. Joseph [14, Corollary 2.4] proved that topologically free Cantor actions of groups with the Noetherian property (i.e. those which only have a countable number of subgroups) are always essentially free. Kambites, Silva and Steinberg [21] showed that the action of a group generated by finite automata on a rooted tree is topologically free if and only if it is essentially free. This result is a special case of a later result by Gröger and the second author [13], who proved that the action of a group generated by finite automata on a rooted tree has no essential holonomy (note that [13] does not require the Cantor action to be topologically free). Vorobets [26] showed that the action of the Grigorchuk group has only a countable set of points with non-trivial holonomy, which implies that it has no essential holonomy; this is also a special case of the result in [13].

We approach Problem 1.1 from a new angle in this work. Theorem 1.6 below relates the non-trivial essential holonomy property with the lower central series of the acting group Γ. Namely, we show that if an equicontinuous Cantor action of Γ is locally quasi-analytic (see Definition 1.4) and has essential holonomy, then every commutator subgroup in the lower central series of Γ has elements with positive measure sets of points with non-trivial holonomy. In particular, by Theorem 1.5 every Cantor action by a finitely-generated nilpotent group is quasi-analytic, so combined with Theorem 1.6, we obtain Corollary 1.7, which generalizes the result of Joseph [14, Corollary 2.4] for topologically free actions of nilpotent groups, to all nilpotent Cantor actions.

We also introduce the *local Farber criterion*, which allows to determine if an equicontinuous locally quasi-analytic action has no essential holonomy. (The Farber criterion available in the literature applies to equicontinuous essentially free actions.)

We now recall the basic definitions, then state our main results rigorously.

We say that $(\mathfrak{X}, \Gamma, \Phi)$ is a *Cantor action* if Γ is a countable group, $\mathfrak{X}$ is a Cantor space, and $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is a minimal action by homeomorphisms. Cantor actions arise in a variety of contexts, ranging from the study of minimal sets for smooth group actions, to actions of profinite groups that appear in arithmetic dynamics [23], and as sources of examples for the construction of classes of C^*-algebras [15]. We assume that $(\mathfrak{X}, \Gamma, \Phi)$ is measure-preserving and ergodic, with ergodic invariant measure μ.

Below we consider topological as well as measure-theoretical properties of group actions. When we define or discuss a property which does not require a measure, we denote the action by $(\mathfrak{X}, \Gamma, \Phi)$. When we speak about measure-theoretical properties of the same action, we denote it by $(\mathfrak{X}, \Gamma, \Phi, \mu)$.

Recall the following notion from the theory of pseudogroup actions:

DEFINITION 1.2. *Let* $(\mathfrak{X}, \Gamma, \Phi)$ *be a Cantor action. Say that* $x \in \mathfrak{X}$ *is a* point of non-trivial holonomy for $g \in \Gamma$ *if* $\Phi(g)(x) = x$*, and for each open set* $U \subset \mathfrak{X}$ *with* $x \in U$*, there exists* $y \in U$ *such that* $\Phi(g)(y) \neq y$*.*

We say that x is a *a point of trivial holonomy for* g if Definition 1.2 is not satisfied for $g \in \Gamma$ and $x \in \mathfrak{X}$, that is, x has an open neighborhood $U_{x,g}$ where every point is fixed by the action of g. We call $x \in \mathfrak{X}$ a *point of trivial holonomy*, if x is a point of trivial holonomy for all $g \in \Gamma$. We call $x \in \mathfrak{X}$ is a *point of non-trivial holonomy* if x is a point of non-trivial holonomy for at least one $g \in \Gamma$.

Let $\mathfrak{X}_g$ denote the set of fixed-points for $\Phi(g)\colon \mathfrak{X} \to \mathfrak{X}$, and let $\mathfrak{X}_g^{hol} \subset \mathfrak{X}_g$ denote the subset of points of non-trivial holonomy. Then let

$$\mathfrak{X}_\Gamma = \bigcup_{e \neq g \in \Gamma} \mathfrak{X}_g, \qquad \mathfrak{X}_\Gamma^{hol} = \bigcup_{g \in \Gamma} \mathfrak{X}_g^{hol} \tag{1}$$

denote the set of all points fixed by some non-identity element $g \in \Gamma$, and the set of all points of non-trivial holonomy for some $g \in \Gamma$, respectively. Note that if $x \in \mathfrak{X}_g$ is an interior point, then $x \notin \mathfrak{X}_g^{hol}$, so that $\mathfrak{X}_g^{hol}$ consists of the subset of frontier points in $\mathfrak{X}_g$. It follows that the frontier of $\mathfrak{X}_\Gamma$ is contained in $\mathfrak{X}_\Gamma^{hol}$ but this inclusion is not necessarily equality. Also note that the set $\mathfrak{X}_\Gamma^{hol}$ is invariant under the action of Γ, so if the measure μ is ergodic then it has either μ-measure 0 or 1.

If Γ is abelian, and $x \in \mathfrak{X}$ is a fixed-point for $\Phi(g)$, then minimality of the action Φ implies that $\Phi(g)$ acts as the identity on $\mathfrak{X}$ and thus $\mathfrak{X}_g^{hol}$ is empty, as is $\mathfrak{X}_\Gamma^{hol}$.

Our interest is in Cantor actions for which $\mathfrak{X}_\Gamma^{hol}$ is "dynamically large", i.e. of μ-measure 1. This concept is captured by the following notion which we introduce in this paper, and which was considered without a giving it a specific name in [13].

DEFINITION 1.3. *A group action* $(\mathfrak{X}, \Gamma, \Phi, \mu)$ has essential holonomy *if the set* $\mathfrak{X}_\Gamma^{hol}$ *of points with non-trivial holonomy has positive* μ*-measure, and otherwise it* has no essential holonomy*.*

From now on we assume that $(\mathfrak{X}, \Gamma, \Phi)$ is an equicontinuous action, and so $(\mathfrak{X}, \Gamma, \Phi)$ has a unique ergodic invariant measure μ, see Section 2 for definitions and details.

The decomposition (1) of $\mathfrak{X}_\Gamma^{hol}$ as a countable union of fixed-point sets of elements implies that an action has essential holonomy if and only if, for at least one $g \in \Gamma$, the set $\mathfrak{X}_g^{hol}$ has positive μ-measure. The assumption that μ is a probability measure, and that the action is minimal, implies that μ is a continuous measure; that is, the measure of any point $x \in \mathfrak{X}$ is zero. Thus the intersection of $\mathfrak{X}_g^{hol}$ with the support of μ must be an uncountable set. Also note, it is easy to show that the property that an action has essential holonomy is an invariant of continuous orbit equivalence, see [8] for definitions.

As an aside, consider the case of a C^1-diffeomorphism $f\colon M \to M$ of a manifold M which preserves a smooth probability measure μ. Then it is elementary to show that the subset of the fixed points of M_f^{hol} with non-trivial linear holonomy has Lebesgue measure 0. That is, the set of points for which the derivative map $D_x f$ is not the identity always has Lebesgue measure zero. This is a special case of [16, Proposition 7.1], which showed that the set of points in a C^1-pseudogroup with non-trivial linear holonomy has Lebesgue measure zero. In analogy to the case of C^1-actions, the property that a Cantor action has essential holonomy is a measure of the "local non-regularity" of the action.

Another property which characterizes the topological complexity of actions on Cantor sets, and which has been studied in [17, 18, 19, 20], is the property that $(\mathfrak{X}, \Gamma, \Phi)$ is *locally quasi-analytic*. This notion generalized to actions on totally disconnected spaces the notion of a *quasi-analytic action*, initially studied by Haefliger for actions of pseudogroups of real-analytic diffeomorphisms. The generalization was introduced by Álvarez López and Candel in [3], and the interpretation of this property we use in our works is as follows.

DEFINITION 1.4. [3, Definition 9.4] *A topological action* $(\mathfrak{X}, \Gamma, \Phi)$ *on a metric Cantor space* $\mathfrak{X}$*, is* locally quasi-analytic *if there exists* $\varepsilon > 0$ *such that for any non-empty open set* $U \subset \mathfrak{X}$ *with* diam$(U) < \varepsilon$, *and for any non-empty open subset* $V \subset U$, *and elements* $g_1, g_2 \in \Gamma$

$$\text{if the restrictions } \Phi(g_1)|V = \Phi(g_2)|V, \quad \text{then } \Phi(g_1)|U = \Phi(g_2)|U. \tag{2}$$

The action is said to be quasi-analytic *if* (2) *holds for* $U = \mathfrak{X}$.

In other words, $(\mathfrak{X}, \Gamma, \Phi)$ is locally quasi-analytic if for every $g \in \Gamma$ the homeomorphism $\Phi(g)$ has unique extensions on the sets of diameter $\varepsilon > 0$ in $\mathfrak{X}$, with ε uniform over $\mathfrak{X}$. We note that an effective action $(\mathfrak{X}, \Gamma, \Phi)$ is topologically free if and only if it is quasi-analytic, see [18, Proposition 2.2]. By Theorem 1.5, a minimal equicontinuous action $(\mathfrak{X}, \Gamma, \Phi)$ of a group Γ with Noetherian property is always locally quasi-analytic. Examples of locally quasi-analytic actions which are not topologically free are easy to construct, see for instance [18, Example A.4]. On the other hand, we have the following result.

THEOREM 1.5. [18, Theorem 1.6] *Let* Γ *be a Noetherian group. Then a minimal equicontinuous Cantor action* $(\mathfrak{X}, \Gamma, \Phi)$ *is locally quasi-analytic.*

A finitely-generated nilpotent group is Noetherian, so as a corollary we obtain that all Cantor actions by finitely-generated nilpotent groups are quasi-analytic.

Recall the construction of the lower central series for Γ. Set $\gamma_1(\Gamma) = \Gamma$, and for $i \geq 1$, let $\gamma_{i+1}(\Gamma) = [\Gamma, \gamma_i(\Gamma)]$ be the commutator subgroup, which is a normal subgroup of Γ. Then for $a \in \gamma_i(\Gamma)$ and $b \in \gamma_j(\Gamma)$ the commutator $[a, b] \in \gamma_{i+j}(\Gamma)$. Moreover, these subgroups form a descending chain

$$\Gamma = \gamma_1(\Gamma) \supset \gamma_2(\Gamma) \supset \cdots \supset \gamma_n(\Gamma) \supset \cdots . \tag{3}$$

Our main result relates the dynamical properties of a Cantor action of a finitely-generated group Γ with the lower central series for Γ.

THEOREM 1.6. *Let* Γ *be a countable group, and let* $(\mathfrak{X}, \Gamma, \Phi, \mu)$ *be a minimal equicontinuous Cantor action with unique ergodic invariant probability measure* μ. *If the action has essential holonomy, then for every* $n \geq 1$ *there exists* $\phi_n \in \gamma_n(\Gamma)$ *such that the action of* ϕ_n *has a positive measure set of points with non-trivial holonomy.*

For a nilpotent group, the lower central series terminates, and thus we obtain the following consequence of Theorem 1.6 which includes the result of [14, Corollary 2.4]. The proof in [14] uses the properties of invariant random subgroups in the space of all subgroups of Γ.

COROLLARY 1.7. *Let* $(\mathfrak{X}, \Gamma, \Phi, \mu)$ *be a minimal equicontinuous Cantor action with unique ergodic invariant measure* μ. *If* Γ *is a finitely-generated nilpotent group, then* $(\mathfrak{X}, \Gamma, \Phi, \mu)$ *has no essential holonomy.*

Finally, we discuss the question how one can determine that an action $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is essentially free, or more generally, has no essential holonomy? For topologically free actions this can be done using the *Farber criterion* [11, 6, 2]. The Farber criterion is defined using decreasing sequences of finite index subgroups of Γ associated to the action, see Section 2.2 for details. In Section 3 we discuss various formulations of the Farber criterion in the literature, and its relation to the property that a minimal equicontinuous action $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is essentially free. In a few words, if a minimal equicontinuous action $(\mathfrak{X}, \Gamma, \Phi, \mu)$ satisfies the Farber criterion, then it is topologically free and essentially free.

As we explain in Section 3, if a minimal equicontinuous action $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is locally quasi-analytic but not topologically free, then the Farber criterion is not satisfied even if the set of points with non-trivial holonomy has zero measure. We now introduce a *local Farber criterion* which is applicable to locally quasi-analytic actions, and allows to determine if an action has no essential holonomy.

Given a minimal equicontinuous action $(\mathfrak{X}, \Gamma, \Phi)$ and a point $x \in \mathfrak{X}$, there is a descending sequence $\mathcal{G} : \Gamma = \Gamma_1 \supset \Gamma_2 \supset \Gamma_3 \supset \cdots$ of finite index subgroups which are stabilizer groups of a decreasing sequence of clopen (closed and open) neighborhoods of x, see Section 2.2 for details. Such a chain $\mathcal{G}$ depends on x and on the choice of a sequence of clopen neighborhoods of x, but the fact that the action satisfies the Farber criterion does not depend on these choices.

If $\mathcal{G}$ can be chosen so that every subgroup in $\mathcal{G}$ is normal in Γ then the action $(\mathfrak{X}, \Gamma, \Phi)$ is free. Therefore, we can assume that every subgroup in $\mathcal{G}$ is not normal in $\mathcal{G}$. Recall that given a finite index subgroup $H \subset \Gamma$, the intersection $\bigcap_{g \in \Gamma} gHg^{-1}$ is the maximal normal subgroup of H in Γ, and has finite index in Γ.

DEFINITION 1.8. *(Local Farber criterion) Let Γ be an infinite discrete group, and let $\mathcal{G} : \Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be a sequence of subgroups of finite index. Then $\mathcal{G}$* satisfies the local Farber criterion, *if there exists $k \geq 0$ and a group chain $H_{k,0} \supset H_{k,1} \supset H_{k,2} \supset \cdots$, where*

$$H_{k,i-k} = \Gamma_i/\mathcal{C}, \text{ for } i \geq k, \text{ and } \mathcal{C} = \bigcap_{i \geq k} \left(\bigcap_{g \in \Gamma_k} g\Gamma_i g^{-1} \right), \tag{4}$$

such that for all $g \neq e \in H_{k,0}$

$$\lim_{j \to \infty} \frac{\#\{hH_{k,i-k} \mid g \cdot hH_{k,i-k} = hH_{k,i-k}\}}{|H_{k,0} : H_{k,i-k}|} = 0, \tag{5}$$

where $hH_{k,i-k}$ denotes the coset of h in $H_{k,0}$, and $|H_{k,0} : H_{k,i-k}|$ is the index of $H_{k,i-k}$ in $H_{k,0}$.

The Farber criterion is obtained from the local Farber criterion in Definition 1.8 by setting $k = 0$. Under this assumption $\mathcal{C}$ in (4) is the trivial group, and (5) must be satisfied for $H_{0,0} = \Gamma = \Gamma_0$ and $H_{0,i} = \Gamma_i$, for $i \geq 1$. We have the following result.

THEOREM 1.9. *Let $(\mathfrak{X}, \Gamma, \Phi, \mu)$ be a minimal equicontinuous Cantor action, and let $\mathcal{G}$ be a sequence of finite index subgroups associated to the action. If $\mathcal{G}$ satisfies the local Farber criterion in Definition 1.8, then:*

(1) $(\mathfrak{X}, \Gamma, \Phi)$ is locally quasi-analytic.
(2) $(\mathfrak{X}, \Gamma, \Phi, \mu)$ has no essential holonomy.

The rest of the paper is organized as follows. In Section 2 we recall some basic properties and describe some basic constructions for minimal equicontinuous actions on Cantor sets. The proofs of Theorem 1.6 and Corollary 1.7 are given in Section 4. The discussion of the local Farber criterion and the proof of Theorem 1.9 is given in Section 3.

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2. Cantor actions

We recall some basic properties of Cantor actions. More details can be found in [4, 7, 8, 18, 19, 22].

2.1. **Basic notions.** Let $(\mathfrak{X}, \Gamma, \Phi)$ denote a topological action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$. We write $g \cdot x$ for $\Phi(g)(x)$ when appropriate. The orbit of $x \in \mathfrak{X}$ is the subset $\mathcal{O}(x) = \{g \cdot x \mid g \in \Gamma\}$. The action is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x)$ is dense in $\mathfrak{X}$. The action is said to be *effective* if the homomorphism $\Phi \colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ is injective, and one also says that the action is faithful. We assume throughout the paper that $(\mathfrak{X}, \Gamma, \Phi)$ is effective and minimal.

An action $(\mathfrak{X}, \Gamma, \Phi)$ is *equicontinuous* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ and $g \in \Gamma$ we have that $d_{\mathfrak{X}}(x, y) < \delta$ implies $d_{\mathfrak{X}}(g \cdot x, g \cdot y) < \varepsilon$. The property of being equicontinuous is independent of the choice of the metric on $\mathfrak{X}$ which is compatible with the topology of $\mathfrak{X}$.

We say that $U \subset \mathfrak{X}$ is *adapted* to the action $(\mathfrak{X}, \Gamma, \Phi)$ if U is a *non-empty clopen* subset, and for any $g \in \Gamma$, if $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$. The proof of [18, Proposition 3.1] shows that given $x \in \mathfrak{X}$ and clopen set $x \in W$, there is an adapted clopen set U with $x \in U \subset W$.

For an adapted set U, the set of "return times" to U,

$$\Gamma_U = \{g \in \Gamma \mid g \cdot U \cap U \neq \emptyset\} \tag{6}$$

is a subgroup of Γ, called the *stabilizer* of U. Then for $g, g' \in \Gamma$ with $g \cdot U \cap g' \cdot U \neq \emptyset$ we have $g^{-1}\, g' \cdot U = U$, hence $g^{-1}\, g' \in \Gamma_U$. Thus, the translates $\{g \cdot U \mid g \in \Gamma\}$ form a finite clopen partition of $\mathfrak{X}$, and are in 1-1 correspondence with the quotient space $X_U = \Gamma/\Gamma_U$. Then Γ acts by permutations of the finite set X_U and so the stabilizer group $\Gamma_U \subset \Gamma$ has finite index. Note that this implies that if $V \subset U$ is a proper inclusion of adapted sets, then the inclusion $\Gamma_V \subset \Gamma_U$ is also proper.

We recall a basic property of equicontinuous Cantor actions (see [18, Section 3].)

PROPOSITION 2.1. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action, and let $d_{\mathfrak{X}}$ be an invariant metric on $\mathfrak{X}$ compatible with the topology on $\mathfrak{X}$. Given $x \in \mathfrak{X}$ and $\varepsilon > 0$, there exists an adapted clopen set $U \subset \mathfrak{X}$ with $x \in U$ and* $\mathrm{diam}(U) < \varepsilon$.

2.2. **Group chains.** Given a basepoint x, by iterating the process in Proposition 2.1 one can always construct the following:

DEFINITION 2.2. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a Cantor action. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ is said to be an* adapted neighborhood basis *at $x \in \mathfrak{X}$ for the action Φ if $x \in U_{\ell+1} \subset U_\ell$ for all $\ell \geq 1$ with $\cap\, U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ.*

Let $\Gamma_\ell = \Gamma_{U_\ell}$ denote the stabilizer group of U_ℓ given by (6). Then we obtain a descending chain of finite index subgroups

$$\mathcal{G}^x_{\mathcal{U}} : \Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots \ . \tag{7}$$

Note that each Γ_ℓ has finite index in Γ, and is not assumed to be a normal subgroup. Also note that while the intersection of the chain $\mathcal{U}$ is a single point $\{x\}$, the intersection of the stabilizer groups in $\mathcal{G}^x_{\mathcal{U}}$ need not be the trivial group.

Next, set $X_\ell = \Gamma/\Gamma_\ell$ and note that Γ acts transitively on the left on X_ℓ. The inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a natural Γ-invariant quotient map $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. Introduce the inverse limit

$$X \equiv \varprojlim \, \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell > 0\} = \{(x_\ell) = (x_0, x_1, \ldots) \mid p_{\ell+1}(x_{\ell+1}) = x_\ell\} \tag{8}$$

which is a Cantor space with the Tychonoff topology. Thus elements of X are infinite sequences with entries in X_ℓ, $\ell \geq 0$. The actions of Γ on the factors X_ℓ induce a minimal equicontinuous action, denoted by $\Phi_x \colon \Gamma \times X \to X$, which reads

$$(g, (x_\ell)) \mapsto g \cdot (x_\ell) = (g \cdot x_\ell) = (g \cdot x_0, g \cdot x_1, \ldots). \tag{9}$$

For each $\ell \geq 0$, we have the "partition coding map" $\Theta_\ell \colon \mathfrak{X} \to X_\ell$ which is Γ-equivariant. The maps $\{\Theta_\ell\}$ are compatible with the map on quotients in (8), and so define a limit map $\Theta_x \colon \mathfrak{X} \to X$. The fact that the diameters of the clopen sets $\{U_\ell\}$ tend to zero, implies that Θ_x is a homeomorphism. This is proved in detail in [9, Appendix A]. Moreover, $\Theta_x(x) = e_\infty = (e\Gamma_\ell) \in X$, the basepoint of the inverse limit (8), where $e\Gamma_\ell = \Gamma_\ell$ is the coset of the identity $e \in \Gamma$. Let X have a metric such that Γ acts on X by isometries, for instance, let

$$d_X\left((x_\ell), (y_\ell)\right) = 2^{-m}, \quad \text{where } m = \max\{\ell \mid x_\ell = y_\ell,\ \ell \geq 0\}. \tag{10}$$

Then let $d_{\mathfrak{X}}$ be the metric on $\mathfrak{X}$ induced from d_X by the homeomorphism $\widehat{\Theta}_x$. The minimal equicontinuous action (X, Γ, Φ_x) is called the *odometer model* centered at x for the action $(\mathfrak{X}, \Gamma, \Phi)$.

The group chain $\mathcal{G}^x_{\mathcal{U}}$ depends on x and $\mathcal{U}$, and one can introduce an equivalence relation which, for a given group Γ, identifies the class of group chains with topologically conjugate associated odometer models. We do not discuss this question now, since it is not needed for the further exposition, but refer the interested reader to [9].

2.3. **Unique ergodic invariant measure.** For a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, the closure $E = \overline{\Phi(\Gamma)} \subset Homeo(\Gamma)$ in the uniform topology is a profinite compact group, called the *Ellis*, or *enveloping group* [4, 10]. The group E acts on $\mathfrak{X}$, the isotropy group $E_x = \{g \in E \mid g \cdot x = x\}$ of this action at x is a closed subgroup of E, and we have $\mathfrak{X} = E/E_x$. The profinite group E has the Haar measure $\widehat{\mu}$, which is invariant with respect to the action of E on itself and is ergodic. The measure $\widehat{\mu}$ on E pushes down to a measure μ on $\mathfrak{X}$, and with this measure $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is uniquely ergodic.

Now choose an adapted neighborhood basis $\mathcal{U}$ and consider the corresponding group chain $\mathcal{G}^x_{\mathcal{U}}$ and the odometer model (8) - (9). The measure μ descends to a measure on X_ℓ, which by invariance must assign equal weight to every coset in X_ℓ, so

$$\mu(h\Gamma_k) = \frac{1}{|\Gamma : \Gamma_k|}, \text{ for all } h\Gamma_k \in \Gamma/\Gamma_k \text{ and all } k \geq 0, \tag{11}$$

where $|\Gamma : \Gamma_k|$ denotes the index of Γ_k in Γ.

2.4. **Lebesgue density theorem.** Let $(\mathfrak{X}, \Gamma, \Phi, \mu)$ be a minimal equicontinuous action with unique invariant probability measure μ and ultrametric $d_{\mathfrak{X}}$ induced from (10). Denote by

$$B(x, \epsilon) = \{y \in \mathfrak{X} \mid d_{\mathfrak{X}}(x, y) < \epsilon\}$$

an open ball around x of radius $\varepsilon > 0$. The proof of the Lebesgue Theorem below can be found, for instance, in [24, Proposition 2.10].

THEOREM 2.1. *Let $\mathfrak{X}$ be a Polish space, and suppose $\mathfrak{X}$ has an ultrametric $d_{\mathfrak{X}}$ compatible with its topology. Let μ be a probability measure on $\mathfrak{X}$, and let A be a Borel set of positive measure. Then the* Lebesgue density *of x in A, given by*

$$\lim_{\epsilon \to 0} \frac{\mu(A \cap B(x, \epsilon))}{\mu(B(x, \epsilon))} \tag{12}$$

exists and is equal to 1 *for μ-almost every $x \in A$.*

We give a consequence of the Lebesgue Density Theorem which is used in the proof of Theorem 1.6.

LEMMA 2.3. *Let $(\mathfrak{X}, \Gamma, \Phi, \mu)$ be a minimal equicontinuous Cantor action, with unique invariant probability measure μ. Assume there exists an element $g \in \Gamma$ for which $\mathfrak{X}^{hol}_g$ has positive μ-measure. Then there exists adapted set U with $x \in U$ and $\mu(U \cap \mathfrak{X}^{hol}_g) \geq 3/4 \cdot \mu(U)$.*

Proof. Since $\mathfrak{X}^{hol}_g$ has positive μ-measure, by the Lebesgue Density Theorem 2.1 there exists a point $x \in \mathfrak{X}^{hol}_g$ of full Lebesgue density. For this point, choose an adapted neighborhood basis $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell \geq 0\}$ with $\cap_{\ell > 0} U_\ell = \{x\}$. By the convergence of the limit in Theorem 2.1 there exists ℓ_0 so that $\mu(U_\ell \cap \mathfrak{X}^{hol}_g) \geq 3/4 \cdot \mu(U_\ell)$ for $\ell \geq \ell_0$. □

3. The Farber criterion

Recall that a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is topologically free if the set of points with trivial stabilizers is residual in $\mathfrak{X}$. For minimal actions by homeomorphisms, as in our setting, $(\mathfrak{X}, \Gamma, \Phi)$ is topologically free if and only if it has an orbit of points with trivial stabilizers. An effective action $(\mathfrak{X}, \Gamma, \Phi)$ is topologically free if and only if it is quasi-analytic, see [18, Proposition 2.2].

Farber introduced in [11] the criterion, which can be used to study the property of being essentially free for topologically free, equicontinuous Cantor actions. In this section, we generalize the Farber criterion so that it differentiates actions with no essential holonomy which are locally quasi-analytic (see Definition 1.4), but not topologically free.

3.1. **Farber criterion for topologically free actions.**

Denote by e the identity element of Γ.

DEFINITION 3.1. *Let Γ be an infinite discrete group, and let $\mathcal{G} : \Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be a sequence of subgroups of finite index. For $k \geq 1$, denote by n_k the number of subgroups of Γ conjugate to Γ_k. Given $g \in \Gamma$, denote by $n_k(g)$ the number of subgroups conjugate to Γ and containing g. Then $\mathcal{G}$* satisfies the Farber criterion*, or is a* Farber sequence*, if for all $g \in \Gamma$, $g \neq e$, we have*

$$\lim_{k\to\infty} \frac{n_k(g)}{n_k} = 0. \tag{13}$$

The Farber criterion is sometimes formulated in the literature in terms of fixed points of the action of Γ on the finite quotients Γ/Γ_k, see for instance [2]. The relation is as follows.

Let $\mathrm{Sub}(\Gamma)$ be the set of all closed subgroups of Γ. This set can be made into a compact totally disconnected topological space by giving it the Chaubaty-Fell topology, see for instance [5]. The group Γ acts on $\mathrm{Sub}(\Gamma)$ by conjugation.

Consider the mapping

$$s_k : \Gamma/\Gamma_k \to \mathrm{Sub}(\Gamma) : g\Gamma_k \mapsto g\Gamma_k g^{-1},$$

which assigns to each coset of Γ_k in Γ/Γ_k its stabilizer for the left action of Γ. Then the image of s_k consists of n_k subgroups, and s_k is a fibration with fibre of constant cardinality over image(s_k). The action of g on Γ/Γ_k fixes the coset $h\Gamma_k$ if and only if g is in the stabilizer $h\Gamma_k h^{-1}$ of $h\Gamma_k$. Thus we have for all $k \geq 1$

$$\frac{n_k(g)}{n_k} = \frac{\#\{h\Gamma_k \mid g \cdot h\Gamma_k = h\Gamma_k\}}{|\Gamma : \Gamma_k|},$$

where $|\Gamma : \Gamma_k|$ denotes the index of Γ_k in Γ, and the Farber criterion can be reformulated as follows.

DEFINITION 3.2. *Let Γ be an infinite discrete group, and let $\mathcal{G} : \Gamma \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be a sequence of subgroups of finite index. Then $\mathcal{G}$* satisfies the Farber criterion*, or is a* Farber sequence*, if for all $g \in \Gamma$, $g \neq e$, we have*

$$\lim_{k\to\infty} \frac{\#\{h\Gamma_k \mid g \cdot h\Gamma_k = h\Gamma_k\}}{|\Gamma : \Gamma_k|} = 0. \tag{14}$$

We note that if Γ admits a Farber sequence, then it is residually finite. Indeed, as noted in [11], (14) implies that $\bigcap_{k\geq 1} C_k = \{e\}$, where $C_k = \bigcap_{g\in\Gamma} g\Gamma g^{-1}$. Here C_k is the maximal normal subgroup of Γ_k, and the action of every element of C_k fixes every coset in Γ/Γ_k. So if $g \in \bigcap_{k\geq 1} C_k$ and $g \neq e$, then the action of g fixes every coset in Γ/Γ_k, $k \geq 0$, and the limit in (14) is equal to 1.

Now let us consider the relation of the Farber criterion with the property that the action $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is essentially free. Suppose $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is topologically free. Choose $x \in \mathfrak{X}$ and an adapted neighborhood basis $\mathcal{U}$, and let $\mathcal{G}^x_{\mathcal{U}}$ be a group chain associated to the action as in Section 2.2. Denote $\mathcal{G} = \mathcal{G}^x_{\mathcal{U}}$ for simplicity, and consider the odometer model (8)-(9).

Then we have for every point $z \in \mathfrak{X}$ the corresponding sequence $(h_k\Gamma_k) = \Theta_x(z) \in X$ of cosets. The point z is fixed by the action of g if and only if $g \cdot h_k\Gamma_k = h_k\Gamma_k$ for all $k \geq 0$. The measure μ assigns equal weight to every coset in Γ/Γ_k, and so the ratio inside the limit in (14) approximates the measure of the set of fixed points of g in $\mathfrak{X}$. It follows that $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is essentially free, i.e. the set of points fixed by elements of Γ has measure zero with respect to μ if and only if the Farber criterion holds for the group chain $\mathcal{G}$.

We mentioned in Section 2.2 that the choice of a group chain $\mathcal{G}$ for the action $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is not unique. It is a technicality to show that the fact that the Farber criterion is satisfied for $(\mathfrak{X}, \Gamma, \Phi, \mu)$ does not depend on the choice of a group chain but only on $(\mathfrak{X}, \Gamma, \Phi, \mu)$; the argument boils down to considering conjugate group chains $\mathcal{G}' : \Gamma = \Gamma_0 \supset g_1\Gamma_1 g_1^{-1} \supset g_2\Gamma_2 g_2^{-1} \supset \cdots$ and noticing that $h\Gamma_k$ is fixed by the action of g if and only if the coset $g_k h \Gamma_k g_k^{-1}$ is fixed by the action of $g_k g g_k^{-1} \in g_k\Gamma_k g_k^{-1}$.

Now suppose the action $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is locally quasi-analytic but not topologically free. Then there exists $g \in \Gamma$ which fixes pointwise a clopen set $V \subset \mathfrak{X}$. It follows that the ratio inside the limit in (14) is bounded below by $\mu(V)$ and so the Farber criterion is not satisfied for this $g \in \Gamma$.

This argument gives the following corollary.

LEMMA 3.3. *Let* $(\mathfrak{X}, \Gamma, \Phi)$ *be a minimal equicontinuous action with an associated group chain* $\mathcal{G} : \Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots$. *If* $\mathcal{G}$ *satisfies the Farber criterion, then* $(\mathfrak{X}, \Gamma, \Phi)$ *is topologically free and essentially free.*

3.2. **Local Farber criterion.** We now introduce a *local* version of the Farber criterion which can be used to determine when a locally quasi-analytic action has no essential holonomy.

Let the action $(\mathfrak{X}, \Gamma, \Phi, \mu)$ be locally quasi-analytic for some $\varepsilon > 0$. Let $\mathcal{G}^x_{\mathcal{U}} : \Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots$ be a group chain as in (7). Choose $k \geq 0$ such that the adapted set U_k and its translates $g \cdot U_k$, $g \in \Gamma$, all have diameter less than $\varepsilon > 0$. Choose $g_1 = e, g_2, \ldots, g_{m_k}$ such that the translates $\mathfrak{X}_{k,\ell} = g_\ell \cdot U_k$ are all distinct and form a partition of $\mathfrak{X}$.

We consider the restricted actions of the groups $g_\ell \Gamma_k g_\ell^{-1}$ on the corresponding clopen sets $\mathfrak{X}_{k,\ell}$, $\ell = 1, 2, \ldots, m_k$. These actions are not effective since the action $(\mathfrak{X}, \Gamma, \Phi)$ is not quasi-analytic, and so for each $g_\ell \Gamma_k g_\ell^{-1}$ there are some elements which fix the set $\mathfrak{X}_{k,\ell}$ pointwise, for $\ell = 1, 2, \ldots, m_k$. Denote $\mathcal{H}_{k,\ell} = \Phi|_{\mathfrak{X}_{k,\ell}}(g_\ell \Gamma_k g_\ell^{-1})$, that is, $\mathcal{H}_{k,\ell} \cong g_\ell \Gamma_k g_\ell^{-1}/\ker(\Phi|_{\mathfrak{X}_{k,\ell}})$. Then by the choice of ε, the action of $\mathcal{H}_{k,\ell}$ on $\mathfrak{X}_{k,\ell}$ is topologically free.

For each action $(\mathfrak{X}_{k,\ell}, \mathcal{H}_{k,\ell}, \Phi|_{\mathfrak{X}_{k,\ell}})$, $\ell = 1, 2, \ldots, m_k$, we can build a group chain model by setting $H_{k,\ell,i-k} = g_\ell \Gamma_i g_\ell^{-1}/\ker(\Phi|_{\mathfrak{X}_{k,\ell}})$, for $i \geq k$, so $\mathcal{H}_{k,\ell} = H_{k,\ell,0}$. Then for $1 \leq \ell \leq m_k$ we have an isomorphism of Cantor spaces

$$\mathfrak{X}_{k,\ell} \cong \varprojlim \{\mathcal{H}_{k,\ell}/H_{k,\ell,i-k+1} \to \mathcal{H}_{k,\ell}/H_{k,\ell,i-k}, i \geq k\}, \tag{15}$$

equipped with the transitive action of $\mathcal{H}_{k,\ell}$.

Note that the odometer models for the systems $(\mathfrak{X}_{k,1}, \mathcal{H}_{k,1}, \Phi|_{\mathfrak{X}_{k,1}})$ and $(\mathfrak{X}_{k,\ell}, \mathcal{H}_{k,\ell}, \Phi|_{\mathfrak{X}_{k,\ell}})$, $\ell = 2, 3, \ldots, m_k$ are conjugate (as dynamical systems, and this conjugacy is implemented by the algebraic conjugation by g_ℓ of subgroups in Γ). In particular, $g \in \mathcal{H}_{k,1}$ has a fixed point in $\mathcal{H}_{k,1}/H_{k,1,i-k}$ if and only if $g_\ell g g_\ell^{-1}$ has a fixed point in $\mathcal{H}_{k,\ell}/H_{k,\ell,i-k}$, and so the Farber criterion is satisfied for $(\mathfrak{X}_{k,1}, \mathcal{H}_{k,1}, \Phi|_{\mathfrak{X}_{k,1}})$ if and only if it is satisfied for $(\mathfrak{X}_{k,\ell}, \mathcal{H}_{k,\ell}, \Phi|_{\mathfrak{X}_{k,\ell}})$. We therefore can restrict to considering only one of these actions, say $(\mathfrak{X}_{k,1}, \mathcal{H}_{k,1}, \Phi|_{\mathfrak{X}_{k,1}})$. We do that below, thus eliminating the second index ℓ from the notation $H_{k,\ell,i-k}$ for subgroups in group chains.

To formulate the Farber criterion for locally quasi-analytic actions we only need to find a representation for $\ker(\Phi_{\mathfrak{X}_{k,1}})$ in terms of subgroups of Γ. For that, denote by

$$C_{k,i} = \bigcap_{g \in \Gamma_k} g\Gamma_i g^{-1}$$

the maximal normal subgroup of Γ_i in Γ_k, for $i \geq k$. The action of any $h \in C_{k,i}$ fixes every coset in the finite space $\mathcal{H}_{k,1}/H_{k,1,i-k}$, and elements in the intersection $\mathcal{C} = \bigcap_{i \geq k} C_{k,i}$ fix every point in $\mathfrak{X}_{k,1}$. Thus $\ker(\Phi|_{\mathfrak{X}_{k,1}}) \cong \mathcal{C}$. We now can formulate the criterion in Definition 1.8, which we repeat now in Definition 3.4 for the convenience of the reader.

DEFINITION 3.4. *(Local Farber criterion) Let Γ be an infinite discrete group, and let $\mathcal{G} : \Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots$ be a sequence of subgroups of finite index. Then $\mathcal{G}$* satisfies the local Farber criterion*, if there exists $k \geq 0$ and a group chain $H_{k,0} \supset H_{k,1} \supset H_{k,2} \supset \cdots$, where*

$$H_{k,i-k} = \Gamma_i/\mathcal{C}, \text{ for } i \geq 1, \text{ and } \mathcal{C} = \bigcap_{i \geq k} \left(\bigcap_{g \in \Gamma_k} g\Gamma_i g^{-1} \right), \tag{16}$$

such that for all $g \neq e \in H_{k,0}$

$$\lim_{j \to \infty} \frac{\#\{hH_{k,i-k} \mid g \cdot hH_{k,i-k} = hH_{k,i-k}\}}{|H_{k,0} : H_{k,i-k}|} = 0. \tag{17}$$

We note that $|H_{k,0} : H_{k,i-k}| = |\Gamma_k : \Gamma_i|$ for $i \geq k$. We can now prove Theorem 1.9, which says that if $(\mathfrak{X}, \Gamma, \Phi, \mu)$ satisfies the local Farber criterion, then it is locally quasi-analytic, and has no essential holonomy.

Proof. (of Theorem 1.9) Suppose $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is not locally quasi-analytic, let $x \in \mathfrak{X}$ and let $\mathcal{G} : \Gamma \supset \Gamma_1 \supset \cdots$ be an associated group chain. Then for any $k \geq 0$ in (16) the action of $H_{k,0}$ on the Cantor space defined by (15), is not topologically free, and so by Lemma 3.3 the Farber criterion cannot be satisfied for $H_{k,0}$. Thus if the local Farber criterion is satisfied, then $(\mathfrak{X}, \Gamma, \Phi, \mu)$ must be locally quasi-analytic. The second statement follows from the discussion preceding the introduction of the local Farber criterion in Definition 3.4. □

4. Proof of Theorem 1.6

Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action, with unique ergodic invariant probability measure μ. Recall the notation for the lower central series for a group Γ. Set $\gamma_1(\Gamma) = \Gamma$, and for $i \geq 1$, let $\gamma_{i+1}(\Gamma) = [\Gamma, \gamma_i(\Gamma)]$ be the commutator subgroup, which is a normal subgroup of Γ. Then for $a \in \gamma_i(\Gamma)$ and $b \in \gamma_j(\Gamma)$ the commutator $[a, b] \in \gamma_{i+j}(\Gamma)$. Moreover, these subgroups form a descending chain $\Gamma = \gamma_1(\Gamma) \supset \gamma_2(\Gamma) \supset \cdots \supset \gamma_n(\Gamma) \supset \cdots$. Note that each quotient group $\gamma_i(\Gamma)/\gamma_{i+1}(\Gamma)$ is abelian, but typically is not finite.

The group Γ is nilpotent if the lower central series terminates, i.e. there is a least $n_\Gamma \geq 0$ such that $\gamma_{n_\Gamma+1}(\Gamma) = \{e\}$, the trivial group. It follows that every element in $\gamma_{n_\Gamma}(\Gamma)$ commutes with any element of Γ; that is, $\gamma_{n_\Gamma}(\Gamma)$ is contained in the center of Γ.

Given a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, denote by $\Phi_n : \gamma_n(\Gamma) \times \mathfrak{X} \to \mathfrak{X}$ the restriction of the action Φ to the subgroup $\gamma_n(\Gamma)$. We then make the following definition:

DEFINITION 4.1. *A minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi, \mu)$ is said to have* essential holonomy at depth n_t *if the restricted action $(\mathfrak{X}, \gamma_{n_t}(\Gamma), \Phi_{n_t}, \mu)$ has essential holonomy, but $(\mathfrak{X}, \gamma_n(\Gamma), \Phi_n, \mu)$ has no essential holonomy for $n > n_t$. The action has* essential holonomy at infinite depth *if for all $n \geq 1$, the restricted action $(\mathfrak{X}, \Gamma_{(n)}, \Phi_n, \mu)$ has essential holonomy.*

Here is our main technical result, which combined with Theorem 1.5 implies Theorem 1.6.

PROPOSITION 4.2. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action which is locally quasi-analytic. If the action is has essential holonomy, then it has essential holonomy of infinite depth.*

Proof. Suppose that $(\mathfrak{X}, \Gamma, \Phi)$ has essential holonomy at depth n_t. We show that this leads to a contradiction by localizing the action to a sufficiently small adapted set.

The set $\mathfrak{X}^{hol}_{\Gamma_{(n_t)}}$ has positive μ-measure, while the set $\mathfrak{X}^{hol}_{\gamma_{(n_t+1)}(\Gamma)}$ has μ-measure zero. Thus, there exists $g \in \gamma_{n_t}(\Gamma)$ such that $\mu(\mathfrak{X}^{hol}_g) > 0$. Moreover, for all $\tau \in \gamma_{(n_t+1)}(\Gamma)$ we have $\mu(\mathfrak{X}^{hol}_\tau) = 0$.

Let $x \in \mathfrak{X}^{hol}_g$ be a point of full Lebesgue density, so $g \cdot x = x$. Let $U \subset \mathfrak{X}$ be an adapted clopen set with $x \in U$, and so $g \in \Gamma_U$, and sufficiently small diameter such that the restricted action $\Phi_U \colon \Gamma_U \times U \to U$ is quasi-analytic, and we have $\mu(\mathfrak{X}^{hol}_g \cap U) \geq 3/4 \cdot \mu(U)$, see Lemma 2.3.

By the assumption that $x \in \mathfrak{X}^{hol}_g$, there exist $y \in U$ so that $z = g \cdot y \neq y$. Note that $z \in U$ and $z \neq x$ as $g \cdot x = x$. Let $V \subset U$ with $y \in V$ be sufficiently small clopen set such that $(g \cdot V) \cap V = \emptyset$. Then choose $k_y \in \Gamma_U$ such that $k_y \cdot x \in V$. Then replace y with $k_y \cdot x$ and we have $g \cdot y \neq y$ again.

Let $\tau = [g, k_y]$ be the commutator. Then $g \in \gamma_{n_t}(\Gamma)$ implies that $\tau \in \gamma_{(n_t+1)}(\Gamma) \cap \Gamma_U$. By definition we have $g \cdot k_y = \tau \cdot k_y \cdot g$. Then for $w \in \mathfrak{X}^{hol}_g \cap U$ calculate

$$g \cdot (k_y \cdot w) = \tau \cdot k_y \cdot g \cdot w = \tau \cdot (k_y \cdot w) \ . \tag{18}$$

We use the identity (18) to prove the key observation:

LEMMA 4.3. *The sets $k_y \cdot (\mathfrak{X}^{hol}_g \cap U)$ and $(\mathfrak{X}^{hol}_g \cap U)$ are μ-a.e. disjoint.*

Proof. We must show that $\mu\left(k_y \cdot (\mathfrak{X}^{hol}_g \cap U) \cap (\mathfrak{X}^{hol}_g \cap U)\right) = 0$.

Suppose that $w \in \mathfrak{X}^{hol}_g \cap U$ satisfies $k_y \cdot w \in \mathfrak{X}^{hol}_g \cap U$. Then $k_y \cdot w$ is a fixed-point for the action of g, and so by (18) we have $k_y \cdot w$ is a fixed point for the action of τ.

We have that $\tau \in \gamma_{(n_t+1)}(\Gamma)$, so for μ-almost all $w \in \mathfrak{X}^{hol}_g \cap U$, we have that $\Phi_U(\tau)$ has trivial germinal holonomy at $k_y \cdot w$, as the action of $\Phi(k_y)$ is a measure preserving homeomorphism. Thus, there exists a clopen set $W_w \subset U$ with $k_y \cdot w \in W_w$ such that $\Phi_U(\tau)$ acts as the identity on W_w. As we chose U so that the action Φ_U is quasi-analytic on U, this implies that $\Phi_U(\tau)$ acts as the identity on U. However, we also have $g \cdot y \neq y$, so using the identity (18) again, we have $\tau \cdot y = \tau \cdot (k_y \cdot x) \neq k_y \cdot x$, and thus $\Phi_U(\tau)$ does not act as the identity on U, which is a contradiction. It follows that for μ-almost all $w \in \mathfrak{X}^{hol}_g \cap U$ we have $k_y \cdot w \notin \mathfrak{X}^{hol}_g \cap U$. Hence $\mu\left(k_y \cdot (\mathfrak{X}^{hol}_g \cap U) \cap (\mathfrak{X}^{hol}_g \cap U)\right) = 0$. □

We now complete the proof of Proposition 4.2. As μ is invariant under the action of Φ_U we have $\mu\left(k_y \cdot (\mathfrak{X}^{hol}_g \cap U)\right) = \mu\left(\mathfrak{X}^{hol}_g \cap U\right) \geq 3/4 \cdot \mu(U)$. But then by Lemma 4.3 we obtain the contradiction

$$\mu(U) \geq \mu\left(k_y \cdot (\mathfrak{X}^{hol}_g \cap U)\right) + \mu\left(\mathfrak{X}^{hol}_g \cap U\right) \geq (3/4 + 3/4)\mu(U) \ . \tag{19}$$

Thus, the action $(\mathfrak{X}, \Gamma, \Phi)$ cannot have essential holonomy at finite depth. □

COROLLARY 4.4. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action, and suppose that Γ is a finitely-generated nilpotent group. Then the action has the zero measure set of points with non-trivial holonomy; that is, it does not have essential holonomy.*

Proof. By Theorem 1.5, the action $(\mathfrak{X}, \Gamma, \Phi)$ is locally quasi-analytic. As Γ is nilpotent, the commutator group $\gamma_{n_\Gamma}(\Gamma)$ is in the center of Γ, and hence its action on $\mathfrak{X}$ has measure 0 set of points with non-trivial holonomy; that is, it has no essential holonomy. □

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