TYPE INVARIANTS FOR SOLENOIDAL MANIFOLDS

STEVEN HURDER AND OLGA LUKINA

Abstract. A solenoidal manifold is the inverse limit space of a tower of proper coverings of a compact base manifold. In this work, we introduce new invariants for solenoidal manifolds, their type and their typeset. These are collections of equivalence classes of asymptotic Steinitz orders associated to the monodromy Cantor action associated to the fibration of the solenoidal manifold over its base. We show the type of a solenoidal manifold is an invariant of its homeomorphism class. We introduce the notion of commensurable typesets, and show that homeomorphic solenoidal manifolds have commensurable typesets. When the base manifold in question is an n-torus, then there is a finite rank subgroup of $\mathbb{Q}^n$ associated to the solenoidal manifold, and the type and typesets for subgroups of $\mathbb{Q}^n$ are isomorphism invariants that have been well-studied in the literature. Examples are given to illustrate the properties of the type and typeset invariants.

1. Introduction

A 1-dimensional solenoid is defined as the inverse limit of a sequence of finite covering maps of the circle. The dyadic (2-fold coverings) solenoid was introduced by Vietoris [56] in 1927, and again later independently by van Danzig [52]. Bing observed in [9] that a 1-dimensional solenoid is determined up to homeomorphism by the supernatural number associated to the sequence of coverings defining it. McCord showed in [35, Section 2] the converse, that if two 1-dimensional solenoids are homeomorphic spaces, then the associated supernatural (or Steinitz) numbers are "equivalent", or more precisely they determine the same *type invariant*.

A *solenoidal manifold* of dimension n is defined as the inverse limit of a sequence of finite covering maps of a compact manifold of dimension n. These natural generalizations of the Vietoris solenoids were first studied by McCord [35] and Schori [46], who called them *weak solenoids*. The classification of solenoidal manifolds, up to homeomorphism, is studied by the authors in their works [12, 26, 27, 28, 29]. A solenoidal manifold can be considered as a generalized manifold, which is the point of view of the works of Sullivan [47, 48, 49] and Verjovsky [53, 54], where they investigate their differential geometric properties.

The type of a Steinitz number was introduced by Baer in [6, Section 2], where the terminology *genus* was used. Baer used the genus to analyze the classification problem for rank n subgroups $\mathbb{Q}^n$. The classification problem for Vietoris solenoids corresponds exactly with the classification problem for dense subgroups of $\mathbb{Q}$. Giordano, Putnam and Skau in [24] related the classification problem for solenoidal manifolds over $\mathbb{T}^n$ with the classification problem for dense subgroups of $\mathbb{Q}^n$. It is thus natural to ask whether there are well-defined type invariants for more general solenoidal manifolds, and to what extent they determine their homeomorphism types.

In this work, we introduce the *type* and *typeset* for solenoidal manifolds. We show that the type is a homeomorphism invariant. We introduce the notion of *commensurable typesets* for Cantor actions and show that the commensurable class of the typeset is a homeomorphism invariant. In Section 6 we give examples of Cantor actions of finitely generated groups which illustrate basic cases of the results of this work. In Section 7, we consider the typeset invariants for Cantor actions induced on the boundaries of *spherically homogeneous* trees.

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1.1. **Solenoidal manifolds.** We recall the definition and some properties of solenoidal manifolds. A *presentation* is a sequence of *proper finite covering* maps $\mathcal{P} = \{\, q_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq 1\}$, where each M_ℓ is a compact connected manifold without boundary of dimension n. (This terminology was introduced in [20].) The inverse limit

$$\mathcal{S}_\mathcal{P} \equiv \lim_{\longleftarrow} \, \{q_\ell \colon M_\ell \to M_{\ell-1}\} \ \subset \prod_{\ell \geq 0} \ M_\ell \tag{1}$$

is the *weak solenoid*, or *solenoidal manifold*, associated to $\mathcal{P}$. The set $\mathcal{S}_\mathcal{P}$ is given the relative topology, induced from the product topology, so that $\mathcal{S}_\mathcal{P}$ is compact and connected. By the definition of the inverse limit, for a sequence $\{x_\ell \in M_\ell \mid \ell \geq 0\}$, we have

$$x = (x_0, x_1, \ldots) \in \mathcal{S}_\mathcal{P} \iff q_\ell(x_\ell) = x_{\ell-1} \text{ for all } \ell \geq 1 \ . \tag{2}$$

For each $\ell \geq 0$, there is a fibration $\widehat{q}_\ell \colon \mathcal{S}_\mathcal{P} \to M_\ell$, given by projection onto the ℓ-th factor in (1), so $\widehat{q}_\ell(x) = x_\ell$. For each $\ell > 0$ there is a covering map denoted by $\overline{q}_\ell = q_\ell \circ q_{\ell-1} \circ \cdots \circ q_1 \colon M_\ell \to M_0$. Note that $\widehat{q}_0 = \overline{q}_\ell \circ \widehat{q}_\ell$. The initial factor M_0 is called the *base manifold* of the presentation $\mathcal{P}$.

McCord showed in [35] that a solenoidal manifold $\mathcal{S}_\mathcal{P}$ is a foliated space with foliation $\mathcal{F}_\mathcal{P}$, in the sense of [36], where the leaves of $\mathcal{F}_\mathcal{P}$ are coverings of the base manifold M_0 via the projection map $\widehat{q}_0 \colon \mathcal{S}_\mathcal{P} \to M_0$ restricted to the path-connected components of $\mathcal{S}_\mathcal{P}$. Solenoidal manifolds are *matchbox manifolds* of dimension n in the terminology of [12]. The authors' works [12, 13, 26] and the survey by Verjovsky [54] discuss many examples of solenoidal manifolds.

Given a presentation $\mathcal{P}$, define the truncated presentation $\mathcal{P}_m = \{\, q_\ell \colon M_\ell \to M_{\ell-1} \mid \ell > m\}$, then it is a formality that the solenoidal manifolds $\mathcal{S}_\mathcal{P}$ and $\mathcal{S}_{\mathcal{P}_m}$ are homeomorphic. Thus, homeomorphism invariants for solenoidal manifolds have an "asymptotic" character in terms of their presentations.

1.2. **Types of solenoidal manifolds.** We next define the notion of *type*. More detailed discussions can be found Section 3 below, and in the works by Arnold [3, Section 1]), by Wilson [55, Chapter 2] and by Ribes and Zalesskii [43, Chapter 2.3]. Let $\vec{m} = \{m_i \mid 1 \leq i < \infty\}$ be an infinite collection of positive integers. The *supernatural number*, or *Steinitz number*, defined by $\vec{m}$ is the infinite product

$$\xi(\vec{m}) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\} \ , \tag{3}$$

where lcm denotes the least common multiple of the collection of integers. A Steinitz number ξ can be written uniquely as the formal product over the set of primes Π,

$$\xi = \prod_{p \in \Pi} \ p^{\chi_\xi(p)} \ , \tag{4}$$

where the *characteristic function* $\chi_\xi \colon \Pi \to \{0, 1, \ldots, \infty\}$ counts the multiplicity with which a prime p appears in the infinite product ξ.

DEFINITION 1.1. *Two Steinitz numbers ξ and ξ' are said to be* asymptotically equivalent *if there exists finite integers $m, m' \geq 1$ such that $m \cdot \xi = m' \cdot \xi'$, and we then write $\xi \overset{a}{\sim} \xi'$. A* type *is an asymptotic equivalence class of Steinitz numbers. Let $\tau[\xi]$ denote the type of a Steinitz number ξ.*

The notion of the *prime spectrum* associated to a Steinitz number is useful in classification problems, the study of the dynamical properties of solenoidal manifolds [30], and the calculation of the mapping class group of a solenoidal manifold [31].

DEFINITION 1.2. *Let $\Pi = \{2, 3, 5, \ldots\}$ denote the set of primes. Given $\xi = \prod_{p \in \Pi} p^{\chi(p)}$, define:*

$$\begin{aligned}
\pi(\xi) &= \{p \in \Pi \mid \chi(p) > 0\} \ , \text{ the prime spectrum of } \xi \ ; \\
\pi_f(\xi) &= \{p \in \Pi \mid 0 < \chi(p) < \infty\} \ , \text{ the finite prime spectrum of } \xi \ ; \\
\pi_\infty(\xi) &= \{p \in \Pi \mid \chi(p) = \infty\} \ , \text{ the infinite prime spectrum of } \xi \ .
\end{aligned}$$

Note that if $\xi \overset{a}{\sim} \xi'$, then $\pi_\infty(\xi) = \pi_\infty(\xi')$. The property that $\pi_f(\xi)$ is an *infinite* set is also preserved by asymptotic equivalence of Steinitz numbers, so is an invariant of type.

Let $\mathcal{P}$ be a presentation defining the solenoidal manifold $\mathcal{S}_{\mathcal{P}}$. For each $\ell > 0$, let $m_\ell > 1$ denote the degree of the covering map q_ℓ in (1).

DEFINITION 1.3. *The* Steinitz order *of a presentation $\mathcal{P}$ is the Steinitz number*

$$\xi(\mathcal{P}) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\}. \tag{5}$$

The type *of $\mathcal{P}$ is the type class associated to $\xi(\mathcal{P})$, denoted by $\tau[\mathcal{P}]$.*

As noted above, the type is a complete invariant for 1-dimensional solenoids. Given a sequence of covering maps of the circle, their inverse limit $\mathcal{S}(\vec{m})$ is a Vietoris solenoid,

$$\mathcal{S}(\vec{m}) \stackrel{\text{def}}{=} \varprojlim \, \{ \, q_\ell : \mathbb{S}^1 \to \mathbb{S}^1 \mid \ell \geq 1\} \; . \tag{6}$$

Bing observed in [9] that for 1-dimensional solenoids $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{m}')$, if $\xi(\vec{m}) \overset{a}{\sim} \xi(\vec{m}')$ then the solenoids are homeomorphic. McCord showed in [35, Section 2] the converse, that if $\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{m'})$ are homeomorphic spaces, then $\xi(\vec{m}) \overset{a}{\sim} \xi(\vec{m}')$. Aarts and Fokkink gave an alternative proof in [1].

THEOREM 1.4. [1, 9, 35] *$\mathcal{S}(\vec{m})$ and $\mathcal{S}(\vec{m}')$ are homeomorphic if and only if $\xi(\vec{m}) \overset{a}{\sim} \xi(\vec{m}')$.*

In this work, we generalize McCord's result to all solenoidal manifolds.

THEOREM 1.5. *The type $\tau[\mathcal{P}]$ depends only on the homeomorphism class of $\mathcal{S}_{\mathcal{P}}$ as a continuum.*

COROLLARY 1.6. *The* infinite prime spectrum $\pi_\infty(\xi(\mathcal{P}))$ *of a weak solenoid $\mathcal{S}_{\mathcal{P}}$ depends only on the homeomorphism class of $\mathcal{S}_{\mathcal{P}}$, as does also the property that $\pi_f(\xi(\mathcal{P}))$ is infinite.*

1.3. **Typesets for solenoidal manifolds.** The classification problem of solenoidal manifolds over tori is equivalent to the classification problem for finite rank, dense subgroups $\mathcal{A} \subset \mathbb{Q}^n$ for $n \geq 2$, which leads to the introduction of the *typeset* invariants for such subgroups, as discussed in Remark 3.9. We extend the notion of typeset to solenoidal manifolds.

Let M_0 be the base manifold of a presentation $\mathcal{P}$ and let $\Gamma = \pi_1(M_0, x_0)$ be its fundamental group, for some choice of a basepoint $x_0 \in M_0$. For each non-trivial $\gamma \in \Gamma$ we associate a type $\tau[\gamma]$, whose definition requires some technical preparations, and is deferred to Definition 3.7.

There is an intuitive description of the type $\tau[\gamma]$ when γ is represented by a simple closed curve $c_\gamma : \mathbb{S}^1 \to M_0$. Recall the projection map $\widehat{q}_0 : \mathcal{S}_{\mathcal{P}} \to M_0$ which restricts to a covering map on each leaf of $\mathcal{F}_{\mathcal{P}}$. When the preimage $\mathcal{S}_\gamma = \widehat{q}_0^{\,-1}(c_\gamma(\mathbb{S}^1))$ is connected, it is a 1-dimensional solenoid, then $\tau[\gamma]$ is the type associated to the Vietoris solenoid $\mathcal{S}_\gamma$.

DEFINITION 1.7. *The* typeset *of $\mathcal{P}$ is the countable collection of types $\mathcal{T}[\mathcal{P}] = \{\tau[\gamma] \mid \gamma \in \Gamma\}$.*

The notion of the *commensurable class* of a typeset is discussed in Section 3.5, with the proper definition given in Definition 3.12, after some technical preparations. Intuitively, this notion of equivalence is modeled on the notion of the abstract commensurable equivalence between groups, as presented for example in the works [2, 8]. Then our second main result is the following:

THEOREM 1.8. *Suppose that the solenoidal manifolds $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{P}'}$ are homeomorphic, then their typesets $\tau[\mathcal{P}]$ and $\tau[\mathcal{P}']$ are commensurable.*

The proofs of Theorems 1.5 and 1.8 proceed by associating to a presentation $\mathcal{P}$ for the solenoidal manifold $\mathcal{S}_{\mathcal{P}}$ its monodromy action $\Phi : \Gamma \times X_\infty \to X_\infty$, which is a minimal, equicontinuous action of Γ on a Cantor space X_∞. This is discussed in Section 4.1. Theorem 4.1 implies that if two solenoidal manifolds are homeomorphic, then their monodromy Cantor actions are return equivalent (see Definition 2.6).

Given a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ where Γ is finitely presented, and so is the fundamental group of a compact closed connected manifold M, the suspension construction (see [11, 12, 13]) yields a solenoidal manifold $\mathcal{S}_\Phi$ whose monodromy action is $(\mathfrak{X}, \Gamma, \Phi)$ with base M. In particular, the invariants for Cantor actions of finitely presented groups translate into properties of solenoidal spaces. That is, the study of type invariants for solenoidal manifolds is a special case of the study of type invariants for Cantor actions. The following is the main result of this article.

THEOREM 1.9. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action. Then:*

1. *There is a well defined Steinitz order associated to the action, $\xi(\mathfrak{X}, \Gamma, \Phi)$.*
2. *There is a well-defined set of Steinitz orders $\{\xi(\gamma) \mid \gamma \in \Gamma\}$.*
3. *If the Cantor action $(\mathfrak{X}', \Gamma', \Phi')$ is return equivalent to $(\mathfrak{X}, \Gamma, \Phi)$, then*
   - (a) *their types are equal, $\tau[\mathfrak{X}, \Gamma, \Phi] = \tau[\mathfrak{X}', \Gamma', \Phi']$;*
   - (b) *their typesets $\mathcal{T}[\mathfrak{X}, \Gamma, \Phi]$ and $\mathcal{T}[\mathfrak{X}', \Gamma', \Phi']$ are commensurable.*

Various classes of Cantor actions admit stronger results, which follow from the method of proof of Theorem 1.9. The next result that follows directly from the proof of Theorem 1.9.

COROLLARY 1.10. *Suppose that Γ and Γ' are abelian. If minimal equicontinuous Cantor actions $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are return equivalent, then $\mathcal{T}[\mathfrak{X}, \Gamma, \Phi] = \mathcal{T}[\mathfrak{X}', \Gamma', \Phi']$.*

Example 6.3 shows that the typesets need not be equal for return equivalent Cantor actions, if one of the groups is virtually abelian, and the other is abelian. Section 6 also gives a selection of abelian Cantor actions to illustrate the definitions of types and typesets.

A finitely-generated group Γ is said to be *renormalizable* if there exists a proper self-embedding $\varphi\colon \Gamma \to \Gamma$ whose image has finite index [32]. Another name for this property is that Γ is *finitely non-co-Hopfian*. The embedding φ defines a group chain in Γ which gives rise to a minimal equicontinuous Cantor action, denoted by $(X_\varphi, \Gamma, \Phi_\varphi)$. The properties of the Cantor actions obtained this way are studied in the work [32]; see also Sabitova [45].

THEOREM 1.11. *Let $\varphi\colon \Gamma \to \Gamma$ be a renormalization with associated Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$. Then the action has a well-defined type $\tau[X_\varphi, \Gamma, \Phi_\varphi]$ and typeset $\mathcal{T}[X_\varphi, \Gamma, \Phi_\varphi]$. Let $\varphi'\colon \Gamma' \to \Gamma'$ be a renormalization of the group Γ', and assume the actions $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X_{\varphi'}, \Gamma', \Phi_{\varphi'})$ are return equivalent, then their types and typesets are equal.*

Many finitely-generated nilpotent groups admit renormalizations, as well as some other classes of groups, as discussed in [32]. Examples 6.4 and 6.5 give constructions using renormalizable groups which illustrate the conclusions of Theorem 1.11.

The proofs of Corollary 1.10 and Theorem 1.11 are given in Section 5.3.

1.4. **Cardinality of typesets.** Section 7.1 introduces *d-regular Cantor actions*, which are Cantor actions that have faithful representations as actions on d-regular trees.

THEOREM 1.12. *A minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ which is* d-regular *has finite typeset $\mathcal{T}[X_\infty, \Gamma, \Phi]$. More precisely, suppose $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to an action on a d-ary rooted tree, for some $d \geq 2$. Let P_d be the set of distinct prime divisors of the integers $\{2, \ldots, d\}$, and let $N_d = |P_d|$. Then*

$$|\mathcal{T}[X_\infty, \Gamma, \Phi]| \leq \sum_{k=0}^{N_d} \binom{N_d}{k} = \sum_{k=0}^{N_d} \frac{N_d!}{(N_d - k)!k!} \ . \tag{7}$$

Moreover, each equivalence class $\tau \in \mathcal{T}[X_\infty, \Gamma, \Phi]$ is represented by a Steinitz number ξ with empty finite prime spectrum $\pi_f(\xi)$, and so $\pi(\xi) = \pi_\infty(\xi)$.

Example 7.8 gives examples which realize the typesets arising in Theorem 1.12.

PROBLEM 1.13 (Realization)**.** *Given a finitely-generated torsion free group Γ, what (if any) restrictions are there on the types and typesets which can be realized by minimal equicontinuous actions of Γ?*

The solution to Problem 1.13 is known for the case when $\Gamma = \mathbb{Z}^n$ (see []), and is likely solvable when Γ is a nilpotent group. The solution for a general finitely generated group Γ is unknown.

PROBLEM 1.14 (Classification)**.** *For a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ with type invariants $\tau[\mathfrak{X}, \Gamma, \Phi]$ and $\mathcal{T}[\mathfrak{X}, \Gamma, \Phi]$, classify the Cantor actions with the same type and typesets.*

2. Cantor actions

We recall some of the basic properties of Cantor actions, as required for the proofs of the results in Section 1. More complete discussions of the properties of equicontinuous Cantor actions are given in the text by Auslander [5], the papers by Cortez and Petite [15], Cortez and Medynets [16], and the authors' works, in particular [19, 20] and [28, Section 3].

2.1. **Basic concepts.** Let $(\mathfrak{X}, \Gamma, \Phi)$ denote an action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$. We write $g \cdot x$ for $\Phi(g)(x)$ when appropriate. The orbit of $x \in \mathfrak{X}$ is the subset $\mathcal{O}(x) = \{g \cdot x \mid g \in \Gamma\}$. The action is *minimal* if for all $x \in \mathfrak{X}$, its orbit $\mathcal{O}(x)$ is dense in $\mathfrak{X}$.

An action $(\mathfrak{X}, \Gamma, \Phi)$ is *equicontinuous* with respect to a metric $d_{\mathfrak{X}}$ on $\mathfrak{X}$, if for all $\varepsilon > 0$ there exists $\delta > 0$, such that for all $x, y \in \mathfrak{X}$ and $g \in \Gamma$ we have that $d_{\mathfrak{X}}(x, y) < \delta$ implies $d_{\mathfrak{X}}(g \cdot x, g \cdot y) < \varepsilon$. The property of being equicontinuous is independent of the choice of the metric on $\mathfrak{X}$ which is compatible with the topology of $\mathfrak{X}$.

Now assume that $\mathfrak{X}$ is a Cantor space. Let $\mathrm{CO}(\mathfrak{X})$ denote the collection of all clopen (closed and open) subsets of $\mathfrak{X}$, which forms a basis for the topology of $\mathfrak{X}$. For $\phi \in \mathrm{Homeo}(\mathfrak{X})$ and $U \in \mathrm{CO}(\mathfrak{X})$, the image $\phi(U) \in \mathrm{CO}(\mathfrak{X})$. The following result is folklore, and a proof is given in [27, Proposition 3.1].

PROPOSITION 2.1. *For $\mathfrak{X}$ a Cantor space, a minimal action $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is equicontinuous if and only if the Γ-orbit of every $U \in \mathrm{CO}(\mathfrak{X})$ is finite for the induced action $\Phi_* \colon \Gamma \times \mathrm{CO}(\mathfrak{X}) \to \mathrm{CO}(\mathfrak{X})$.*

All Cantor actions in this work are assumed to be minimal and equicontinuous

We say that $U \subset \mathfrak{X}$ is *adapted* to the action $(\mathfrak{X}, \Gamma, \Phi)$ if U is a *non-empty clopen* subset, and for any $g \in \Gamma$, if $\Phi(g)(U) \cap U \neq \emptyset$ implies that $\Phi(g)(U) = U$. The proof of [27, Proposition 3.1] shows that given $x \in \mathfrak{X}$ and clopen set $x \in W$, there is an adapted clopen set U with $x \in U \subset W$.

For an adapted set U, the set of "return times" to U,

$$\Gamma_U = \{g \in \Gamma \mid g \cdot U \cap U \neq \emptyset\} \tag{8}$$

is a subgroup of Γ, called the *stabilizer* of U. Then for $g, g' \in \Gamma$ with $g \cdot U \cap g' \cdot U \neq \emptyset$ we have $g^{-1}\, g' \cdot U = U$, hence $g^{-1}\, g' \in \Gamma_U$. Thus, the translates $\{g \cdot U \mid g \in \Gamma\}$ form a finite clopen partition of $\mathfrak{X}$, and are in 1-1 correspondence with the quotient space $X_U = \Gamma/\Gamma_U$. Then Γ acts by permutations of the finite set X_U and so the stabilizer group $\Gamma_U \subset G$ has finite index. Note that this implies that if $V \subset U$ is a proper inclusion of adapted sets, then the inclusion $\Gamma_V \subset \Gamma_U$ is also proper.

DEFINITION 2.2. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a Cantor action. A properly descending chain of clopen sets $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell > 0\}$ is said to be an* adapted neighborhood basis *at $x \in \mathfrak{X}$ for the action Φ, if $x \in U_{\ell+1} \subset U_\ell$ is a proper inclusion for all $\ell > 0$, with $\cap_{\ell > 0} U_\ell = \{x\}$, and each U_ℓ is adapted to the action Φ.*

Given $x \in \mathfrak{X}$ and $\varepsilon > 0$, Proposition 2.1 implies there exists an adapted clopen set $U \in \mathrm{CO}(\mathfrak{X})$ with $x \in U$ and $\mathrm{diam}(U) < \varepsilon$. Thus, one can choose a descending chain $\mathcal{U}$ of adapted sets in $\mathrm{CO}(\mathfrak{X})$ whose intersection is x, from which the following result follows:

PROPOSITION 2.3. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a Cantor action. Given $x \in \mathfrak{X}$, there exists an adapted neighborhood basis $\mathcal{U}$ at x for the action Φ.*

Combining the above remarks, we have:

COROLLARY 2.4. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a Cantor action, and $\mathcal{U}$ be an adapted neighborhood basis. Set $\Gamma_\ell = \Gamma_{U_\ell}$, with $\Gamma_0 = \Gamma$, then there is a nested chain of finite index subgroups, $\mathcal{G}_{\mathcal{U}} = \{\Gamma_0 \supset \Gamma_1 \supset \cdots\}$.*

2.2. **Equivalence of Cantor actions.** We next recall the notions of equivalence of Cantor actions which we use in this work. The first and strongest is that of isomorphism of Cantor actions, which is a generalization of the usual notion of conjugacy of topological actions. The definition below agrees with the usage in the papers [16, 27, 34].

DEFINITION 2.5. *Cantor actions* $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ *and* $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ *are said to be* isomorphic *if there is a homeomorphism* $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ *and group isomorphism* $\Theta \colon \Gamma_1 \to \Gamma_2$ *so that*

$$\Phi_1(g) = h^{-1} \circ \Phi_2(\Theta(g)) \circ h \in \mathrm{Homeo}(\mathfrak{X}_1) \textit{ for all } g \in \Gamma_1 \ . \tag{9}$$

The notion of *return equivalence* for Cantor actions is weaker than the notion of isomorphism, and is natural when considering the Cantor systems defined by the holonomy actions for solenoidal manifolds, as considered in the works [26, 27, 28].

For a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ and an adapted set $U \subset \mathfrak{X}$, by an abuse of notation, we use Φ_U to denote both the restricted action $\Phi_U \colon \Gamma_U \times U \to U$, and the induced quotient action $\Phi_U \colon \mathcal{H}_U \times U \to U$, where $\mathcal{H}_U = \Phi(\Gamma_U) \subset \mathrm{Homeo}(U)$. Then $(U, \mathcal{H}_U, \Phi_U)$ is called the *restricted holonomy action* for Φ, in analogy with the case where U is a transversal to a solenoidal manifold, and $\mathcal{H}_U$ is the holonomy group for this transversal. A technical issue that often arises though, is that while $\Gamma_U \subset \Gamma$ has finite index, the action map $\Phi_U \colon \Gamma_U \to \mathcal{H}_U$ need not be injective, and can in fact can have a large kernel as is the case for example for the actions of weakly branch groups (see Example 7.10).

DEFINITION 2.6. *Cantor actions* $(\mathfrak{X}, \Gamma, \Phi)$ *and* $(\mathfrak{X}', \Gamma', \Phi')$ *are* return equivalent *if there exists an adapted set* $U \subset \mathfrak{X}$ *for the action* Φ*, and an adapted set* $U' \subset \mathfrak{X}'$ *for the action* Φ'*, such that the restricted actions* $(U, \mathcal{H}_U, \Phi_U)$ *and* $(U', \mathcal{H}'_{U'}, \Phi'_{U'})$ *are isomorphic.*

Note that if we take $U = \mathfrak{X}$ and $U' = \mathfrak{X}'$ in Definition 2.6, then return equivalence may still be weaker than isomorphism in Definition 2.5, unless the actions Φ and Φ' are topologically free [27, 34, 41].

2.3. **Algebraic Cantor actions.** Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a descending chain of finite index subgroups. Let $X_\ell = \Gamma/\Gamma_\ell$ and note that Γ acts transitively on the left on the finite set X_ℓ. The inclusion $\Gamma_{\ell+1} \subset \Gamma_\ell$ induces a natural Γ-invariant quotient map $p_{\ell+1} \colon X_{\ell+1} \to X_\ell$. Introduce the inverse limit

$$\begin{aligned} X_\infty \quad &\equiv \quad \varprojlim \, \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \\ &= \quad \{(x_0, x_1, \ldots) \in X_\infty \mid p_{\ell+1}(x_{\ell+1}) = x_\ell \text{ for all } \ell \geq 0 \ \} \ \subset \prod_{\ell \geq 0} \ X_\ell \ . \end{aligned} \tag{10}$$

Then X_∞ is a Cantor space with the Tychonoff topology, where the actions of Γ on the factors X_ℓ induce a minimal equicontinuous action denoted by $\Phi \colon \Gamma \times X_\infty \to X_\infty$. There is a natural basepoint $x_\infty \in X_\infty$ given by the cosets of the identity element $e \in \Gamma$, so $x_\infty = (e\Gamma_\ell)$. An adapted neighborhood basis of x_∞ is given by the clopen sets

$$U_\ell = \{x = (x_i) \in X_\infty \mid x_i = e\Gamma_i \in X_i \ , \ 0 \leq i \leq \ell \ \} \subset X_\infty \ . \tag{11}$$

Then there is the tautological identity $\Gamma_\ell = \Gamma_{U_\ell}$.

Suppose that we are given a Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, and an adapted neighborhood basis $\mathcal{U}$. Define subgroups $\Gamma_\ell = \Gamma_{U_\ell}$, with $\Gamma_0 = \Gamma$, which form the group chain $\mathcal{G}_{\mathcal{U}} = \{\Gamma_0 \supset \Gamma_1 \supset \cdots\}$. Then we have the folklore result:

THEOREM 2.7. *Let* $(\mathfrak{X}, \Gamma, \Phi)$ *be a minimal equicontinuous Cantor action, and* $\mathcal{U}$ *an adapted neighborhood basis. Then the action* $(\mathfrak{X}, \Gamma, \Phi)$ *is isomorphic to the Cantor action* (X_∞, Γ, Φ) *constructed from the group chain* $\mathcal{G}_{\mathcal{U}}$.

3. Type and typeset

The notion of type was introduced in 1937 by Baer in [6, Section 2] as part of the study of the classification problem for finite rank dense subgroups of $\mathbb{Q}^n$. The work of Butler [10] introduced a restricted class of subgroups in $\mathbb{Q}^n$ now called *Butler groups*. The classification theory for Butler groups was further developed by Richman [44] and Mutzbauer [37], and the works of Arnold (see [3, Section 1]), and Arnold and Vinsonhaler [4]. For a comprehensive treatment of these ideas, see the monograph by Fuchs [23]. Thomas applied the type invariants in his analysis of the classification complexity of these groups in the work [51, Section 3]. The applications of type invariants to profinite groups are discussed in the works by Ribes [42, Chapter 1, Section 4], Wilson [55, Chapter 2] and Ribes and Zalesskii [43, Chapter 2.3]. In this section, we recall basic notions and properties of Steinitz numbers and their types, and their definitions for solenoidal manifolds.

3.1. **Types and typesets.** Recall that a Steinitz number ξ can be written uniquely as the formal product over the set of primes,

$$\xi = \prod_{p \in \Pi} p^{\chi_\xi(p)} \ , \tag{12}$$

where the *characteristic function* $\chi_\xi \colon \Pi \to \{0, 1, \ldots, \infty\}$ counts the multiplicity with which a prime p appears in the infinite product ξ.

Recall from Definition 1.1 that two Steinitz numbers ξ and ξ' are said to be *asymptotically equivalent* if there exists finite integers $m, m' \geq 1$ such that $m \cdot \xi = m' \cdot \xi'$, and we then write $\xi \overset{a}{\sim} \xi'$. Recall that the type $\tau[\xi]$ associated to a Steinitz number ξ is the asymptotic equivalence class of ξ.

LEMMA 3.1. *ξ and ξ' satisfy $\xi \overset{a}{\sim} \xi'$ if and only if their characteristic functions χ_1, χ_2 satisfy*

- *$\chi_1(p) = \chi_2(p)$ for all but finitely many primes $p \in \Pi$,*
- *$\chi_1(p) = \infty$ if and only iff $\chi_1(p) = \infty$ for all primes $p \in \Pi$.*

Given two types τ and τ', we write $\tau \leq \tau'$ if there exists representatives $\xi \in \tau$ and $\xi' \in \tau'$ such that their characteristic functions satisfy $\chi_\xi(p) \leq \chi_{\xi'}(p)$ for all primes $p \in \Pi$. Then two Steinitz numbers ξ and ξ' are asymptotically equivalent if and only if $\chi_\xi \leq \chi_{\xi'}$ and $\chi_{\xi'} \leq \chi_\xi$.

DEFINITION 3.2. *A* typeset $\mathcal{T}$ *is a collection of types.*

There are three operations on types τ and τ': sum, join and intersection. Let χ (respectively χ') be the characteristic function for a representative $\xi \in \tau$ (respectively $\xi' \in \tau'$), then:

$$\begin{array}{llllll} \text{Sum}: & \tau + \tau' & \text{type defined by} & \chi^{+}(p) & = & \chi(p) + \chi'(p) \\ \text{Join}: & \tau \vee \tau' & \text{type defined by} & \chi^{\vee}(p) & = & \max\{\chi(p), \chi'(p)\} \\ \text{Intersection}: & \tau \wedge \tau' & \text{type defined by} & \chi^{\wedge}(p) & = & \min\{\chi(p), \chi'(p)\} \end{array} .$$

Note that $\tau \wedge \tau' \leq \tau \vee \tau' \leq \tau + \tau'$. A typeset $\mathcal{T}$ need not be closed under the operations of sum, join or intersection. However, a typeset $\mathcal{T}$ always admits a partial ordering.

3.2. **Type for profinite groups.** The *Steinitz order* $\Pi[\mathfrak{G}]$ of a profinite group $\mathfrak{G}$ is defined by the supernatural number associated to a presentation of $\mathfrak{G}$ as an inverse limit of finite groups (see [55, Chapter 2] or [43, Chapter 2.3]).

For a profinite group $\mathfrak{G}$, an open subgroup $\mathfrak{U} \subset \mathfrak{G}$ has finite index [43, Lemma 2.1.2]. Let $\mathfrak{D} \subset \mathfrak{G}$ be a closed subgroup, and $\mathfrak{N} \subset \mathfrak{G}$ is an open normal subgroup, then $\mathfrak{N} \cdot \mathfrak{D}$ is an open subgroup of $\mathfrak{G}$ and $\mathfrak{N} \cap \mathfrak{D}$ is an open normal subgroup of $\mathfrak{D}$.

DEFINITION 3.3. *Let $\mathfrak{D} \subset \mathfrak{G}$ be a closed subgroup of the profinite group $\mathfrak{G}$. Define the Steinitz orders for the pair as follows:*

(1) $\xi(\mathfrak{G}) = \mathrm{lcm}\{\# \ \mathfrak{G}/\mathfrak{N} \mid \mathfrak{N} \subset \mathfrak{G}$ *open normal subgroup*$\}$*;*
(2) $\xi(\mathfrak{D}) = \mathrm{lcm}\{\# \ \mathfrak{D}/(\mathfrak{N} \cap \mathfrak{D}) \mid \mathfrak{N} \subset \mathfrak{G}$ *open normal subgroup*$\}$*,*
(3) $\xi(\mathfrak{G} : \mathfrak{D}) = \mathrm{lcm}\{\# \ \mathfrak{G}/(\mathfrak{N} \cdot \mathfrak{D}) \mid \mathfrak{N} \subset \mathfrak{G}$ *open normal subgroup*$\}$*.*

The Steinitz orders satisfy the Lagrange identity, where the multiplication is taken in the sense of supernatural numbers (see [43, 55]), and we have

$$\xi(\mathfrak{G}) = \xi(\mathfrak{G} : \mathfrak{D}) \cdot \xi(\mathfrak{D}) \ . \tag{13}$$

In particular, we always have $\tau[\mathfrak{D}] \leq \tau[\mathfrak{G}]$.

3.3. **Type for Cantor actions.** Let (X_∞, Γ, Φ) be a Cantor action, defined by a group chain $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \cdots\}$. Recall that the *normal core* of Γ_ℓ is the largest normal subgroup $C_\ell \subset \Gamma_\ell$. The profinite group $\widehat{\Gamma}(\mathcal{G})$ associated to the action (X_∞, Γ, Φ) defined by $\mathcal{G}$ is given by:

DEFINITION 3.4. *Let $\mathcal{G}$ be a group chain with associated normal core chain $\{C_\ell \mid \ell \geq 0\}$. Set*

$$\widehat{\Gamma}(\mathcal{G}) = \varprojlim \{\Gamma/C_{\ell+1} \to \Gamma/C_\ell \mid \ell \geq 0\} \ . \tag{14}$$

The profinite group $\widehat{\Gamma}(\mathcal{G})$ is a quotient of the full profinite completion of Γ, but it is typically not equal. The properties of the group $\widehat{\Gamma}(\mathcal{G})$ associated to a Cantor action are studied extensively in the authors' works [19, 20, 26, 28]. The Steinitz order of $\widehat{\Gamma}(\mathcal{G})$ is well-defined, and given by

$$\xi(\widehat{\Gamma}(\mathcal{G})) = \mathrm{lcm}\{\#(\Gamma/C_\ell) \mid \ell > 0\} \ . \tag{15}$$

The type $\tau[\widehat{\Gamma}(\mathcal{G})] = \tau[\xi(\widehat{\Gamma}(\mathcal{G}))]$ need not be an invariant of return equivalence of the associated Cantor action (X_∞, Γ, Φ), as explained below.

DEFINITION 3.5. *Let (X_∞, Γ, Φ) be a minimal equicontinuous Cantor action. The type $\tau[X_\infty, \Gamma, \Phi]$ of the action is the equivalence class of the Steinitz order*

$$\xi(X_\infty, \Gamma, \Phi) = \mathrm{lcm}\{\#X_\ell = \#(\Gamma/\Gamma_\ell) \mid \ell > 0\} \ . \tag{16}$$

Recall that there is a transitive action $\widehat{\Phi} \colon \widehat{\Gamma}(\mathcal{G}) \times X_\infty \to X_\infty$ induced by the minimal action of Γ. The action $(X_\infty, \widehat{\Gamma}(\mathcal{G}), \widehat{\Phi})$ is free precisely when the isotropy subgroup $\mathcal{D}(\mathcal{G}) \subset \widehat{\Gamma}(\mathcal{G})$ of the action at the basepoint $x_\infty \in X_\infty$ is trivial. In this case, we have a homeomorphism $X_\infty \cong \widehat{\Gamma}(\mathcal{G})$ that commutes with the action, and so X_∞ inherits the structure of a Cantor group from the action.

However, when the action $(X_\infty, \widehat{\Gamma}(\mathcal{G}), \widehat{\Phi})$ is not free, then $\mathfrak{D}(\mathcal{G})$ is not trivial, and we have:

PROPOSITION 3.6. *Let (X_∞, Γ, Φ) be a Cantor action. Then $\xi(X_\infty, \Gamma, \Phi) = \xi(\widehat{\Gamma}(\mathcal{G}) : \mathfrak{D}(\mathcal{G}))$.*

We omit the proof of this, as it is a direct consequence of the definitions, and the result is not needed for the proofs of our main theorems. However, Proposition 3.6 provides some insights on the properties of the type $\tau[X_\infty, \Gamma, \Phi]$ of an action with respect to return equivalence.

The Lagrange Theorem for profinite groups (13) implies that $\tau[\widehat{\Gamma}(\mathcal{G})] = \tau[X_\infty, \Gamma, \Phi] \ \tau[\mathcal{D}(\mathcal{G})]$. The type $\tau[\mathcal{D}(\mathcal{G})]$ need not be invariant under restriction to adapted subsets, and so $\tau[\mathcal{D}(\mathcal{G})]$ need not be invariant under the relation of return equivalence, and thus the same is true for $\tau[\widehat{\Gamma}(\mathcal{G})]$. On the other hand, the proof of Theorem 1.9 shows that the relative type $\tau[\widehat{\Gamma}(\mathcal{G}) : \mathcal{D}(\mathcal{G})] = \tau[\mathcal{G}]$ is invariant under restriction to adapted subsets.

3.4. **Typesets for Cantor actions.** Next, we introduce the definition of the type of an element $\gamma \in \Gamma$, and its properties under restriction, which leads to the the notion of commensurable typesets.

Let (X_∞, Γ, Φ) be a Cantor action defined by a group chain $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$. Recall that $C_\ell \subset \Gamma_\ell$ is the normal core.

Let $\gamma \in \Gamma$, and let $\langle\gamma\rangle \subset \Gamma$ denote the subgroup it generates. For $\ell > 0$, the intersection $\langle\gamma\rangle_\ell = \langle\gamma\rangle \cap C_\ell$ is a subgroup of finite index in $\langle\gamma\rangle \cong \mathbb{Z}$. We thus obtain a group chain in $\langle\gamma\rangle$, denoted

$$\mathcal{C}_\gamma = \{\langle\gamma\rangle \supset \langle\gamma\rangle_1 \supset \langle\gamma\rangle_2 \supset \cdots\} \ . \tag{17}$$

DEFINITION 3.7. *Let (X_∞, Γ, Φ) be the Cantor action defined by a group chain $\mathcal{G}$. For $\gamma \in \Gamma$, the type $\tau[\gamma]$ of γ is the asymptotic equivalence class of the Steinitz order*

$$\xi(\gamma) = \mathrm{lcm}\{\#(\langle\gamma\rangle/\langle\gamma\rangle_\ell) \mid \ell > 0\} \ . \tag{18}$$

The typeset *of the action is the collection*

$$\mathcal{T}[X_\infty, \Gamma, \Phi] = \{\tau[\gamma] \mid \gamma \in \Gamma\} \ . \tag{19}$$

Note that we allow γ to be the identity in this definition, which has type $\tau[e] = \{0\}$, and it may happen as well that there exists $\gamma \in \Gamma$ such that $\gamma \in C_\ell$ for all ℓ sufficiently large, and then we also have that $\tau[\gamma] = \{0\}$. For example, if γ has finite order, or the action of $\Phi(\gamma)$ on X_∞ is periodic with bounded period, then $\tau[\gamma] = \{0\}$.

Observe that if the group chain $\mathcal{C}_\gamma$ in (17) does not stabilize, that is, $\langle\gamma\rangle_{\ell+1} \subset \langle\gamma\rangle_\ell$ is a proper inclusion for infinitely many values of ℓ, then $\mathcal{C}_\gamma$ defines an inverse limit Cantor space X_γ as in (10), with a minimal equicontinuous action by $\mathbb{Z} \cong \langle\gamma\rangle$. Then X_γ is isomorphic to the closure $\overline{\langle\gamma\rangle}$ of the subgroup $\langle\gamma\rangle \subset \widehat{\Gamma}(\mathcal{G})$, which is a profinite abelian group. The type $\tau[\gamma]$ equals the type of the 1-dimensional action of $\mathbb{Z}$ on the profinite torus $\overline{\langle\gamma\rangle}$.

For all $\ell > 0$, we have that $\langle\gamma\rangle_\ell \subset C_\ell$, so the order of the subgroup $\langle\gamma\rangle/\langle\gamma\rangle_\ell$ divides the order of Γ/C_ℓ by Lagrange's Theorem. Thus we have:

PROPOSITION 3.8. *For a group chain $\mathcal{G}$, $\tau[\mathcal{G}] \leq \tau[\widehat{\Gamma}(\mathcal{G})]$, and for each $\gamma \in \Gamma$, $\tau[\gamma] \leq \tau[\widehat{\Gamma}(\mathcal{G})]$.*

The definition of typesets for Cantor actions is not an intuitively obvious notion, except for the special case of abelian Cantor actions. We give an extended remark concerning this case.

REMARK 3.9. Let $\Gamma = \mathbb{Z}^n$ and $\mathcal{G} = \{\Gamma_\ell \mid \ell > 0\}$ be a group chain in $\mathbb{Z}^n$. The chain $\mathcal{G}$ defines a presentation $\mathcal{P}$ of covers of the torus $\mathbb{T}^n$, and so a solenoidal manifold $\mathcal{S}_\mathcal{P}$ with base manifold $\mathbb{T}^n$. Then by Theorem 1.3 in [13], the classification of $\mathcal{S}_\mathcal{P}$ up to homeomorphism is equivalent to the classification up to return equivalence of the action $(X_\infty, \Phi, \mathbb{Z}^n)$ defined by the group chain $\mathcal{G}$.

The group chain $\mathcal{G}$ in $\mathbb{Z}^n$ is a descending chain of rank n subgroups $\Gamma_\ell \subset \mathbb{Z}^n$, so for each $\ell > 0$ there exists an $n \times n$ integer matrix so that multiplication yields an isomorphism $A_\ell \colon \mathbb{Z}^n \to \Gamma_\ell$. Then the type of the Cantor action is

$$\xi(X_\infty, \mathbb{Z}, \Phi) = \mathrm{lcm}\{\det(A_1 A_2 \cdots A_\ell) \mid \ell > 0\} \ .$$

Thus, the well-defined property of $\xi(X_\infty, \mathbb{Z}, \Phi)$ is just the independence of the asymptotic determinant of the composite of the maps A_ℓ on the choice of basis. Moreover, all the subgroups Γ_ℓ in a group chain in $\mathbb{Z}^n$ are normal, so the notion of restricted type is the same as the type. However, it is not easy to give a geometric interpretation of the type of an individual element $\gamma \in \mathbb{Z}^n$.

There is a duality between group chains in $\mathbb{Z}^n$ and direct limit subgroups of $\mathbb{Q}^n$. Define the dual lattice chain $\mathcal{G}^* = \{\Gamma_\ell^* \mid \ell \geq 0\}$ where $\Gamma_\ell^* = \mathrm{Hom}(\Gamma^\ell, \mathbb{Z})$. This is an ascending chain and admits an embedding into the vector space $\mathbb{Q}^n$ with $\Gamma^* \cong \mathbb{Z}^n \subset \mathbb{Q}^n$. Let $\Lambda(\mathcal{G}) \subset \mathbb{Q}^n$ denote the direct limit of this chain of subgroups. Moreover, in this case X_∞ is now a profinite abelian group, as each subgroup $\Gamma_\ell \subset \mathbb{Z}^n$ is obviously normal. Giordano, Putnam and Skau gave in [24, Theorem 1.5] a series of equivalent forms of the classification problem for effective, minimal equicontinuous $\mathbb{Z}^n$-Cantor actions, one of which is the classification of rank n subgroups of $\mathbb{Q}^n$ up to commensurability.

Thus, the classification problem for solenoidal manifolds over $\mathbb{T}^n$ is equivalent to the classification of rank n dense subgroups on $\mathbb{Q}^n$, which was the original motivation for Baer [6] for the introduction of the type invariants. It is natural to impose further restrictions on the properties of the subgroup $\Lambda(\mathcal{G}) \subset \mathbb{Q}^n$, such as the class of Butler groups introduced in [10], which have been extensively studied and classified [4]; see also the compendium by Fuchs [23]. Moreover, for general groups, the typeset has a poset structure, and Thomas discusses the role of the poset structure on a typeset in [51, Section 4] for the classification of subgroups of $\mathbb{Q}^n$. He observes that by the work of Nazarova [38], the classification of representations of finite posets is itself an intractable problem.

The study of abelian Cantor actions suggests new lines of investigation of general Cantor actions.

PROBLEM 3.10. Are there special classes of finitely-generated groups, beyond the abelian case, for which there are analogs of the Butler groups that lead to effective classification results? Are there assumptions on the poset structure of the typeset which lead to new classification results?

3.5. **Commensurable typesets.** We next examine the behavior of the typeset of a Cantor action (X_∞, Γ, Φ) under return equivalence. This leads to the notion of commensurable typesets.

Let $U \subset X_\infty$ be an adapted set for the action. The translates $\{\delta \cdot U \mid \delta \in \Gamma\}$ define a finite partition of X_∞. Assume $\Phi(\gamma)$ leaves each set in the partition invariant. Then one can consider the Steinitz order of the restricted action $\Phi(\gamma)$ on each set in the partition $\delta \cdot U$, and it may happen that these Steinitz orders depend on the choice of δ. This is the case, for example, with the actions of branch groups on the boundary of the tree on which they are defined; see Example 7.10. On the other hand, the collection of all types for elements acting on U includes the types of the group elements $\delta\gamma\delta^{-1}$ for all δ, which belong to the restricted typeset for the action. We show that the typeset is invariant on restriction to adapted sets, modulo an equivalence relation of commensurable typesets.

Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be the subgroup chain associated to the Cantor action, and let $H \subset \Gamma$ be a subgroup of finite index such that $\Gamma_\ell \subset H$ for some $\ell > 0$. By omitting some initial terms, we can assume that $\Gamma_1 \subset H$. Then for each $\ell > 0$, define

$$C_\ell^H = \bigcap_{\delta \in H} \delta \Gamma_\ell \delta^{-1} \ . \tag{20}$$

Then $C_\ell^H \subset \Gamma_\ell$ is a subgroup of finite index, as Γ_ℓ has finite index in Γ. In particular, $C_\ell = C_\ell^\Gamma$ is the normal core of Γ_ℓ, and there is an inclusion $C_\ell \subset C_\ell^H$ for any choice of H. Also note that $C_{\ell+1}^H \subset C_\ell^H$ for all $\ell \geq 0$.

Given $\gamma \in \Gamma$ define $C_{\gamma,\ell}^H = \langle\gamma\rangle \cap C_\ell^H$, and let $\mathcal{C}_\gamma^H = \{\langle\gamma\rangle \cap H = C_{\gamma,0}^H \supset C_{\gamma,1}^H \supset C_{\gamma,2}^H \supset \cdots\}$ denote the resulting subgroup chain in $\langle\gamma\rangle$.

DEFINITION 3.11. *Given a group chain $\mathcal{G}$ and $H \subset \Gamma$ a subgroup with $\Gamma_1 \subset H$, the H-*restricted order *of $\gamma \in \Gamma$ is the Steinitz order with respect to the chain $\mathcal{C}_\gamma^H$,*

$$\xi^H(\gamma) = \mathrm{lcm}\{\#(\langle\gamma\rangle / C_{\gamma,\ell}^H) \mid \ell > 0\} \ . \tag{21}$$

*The $H-$*restricted typeset *for the chain $\mathcal{G}$ is the collection*

$$\mathcal{T}_H[\mathcal{G}] = \{\tau[\xi^H(\gamma)] \mid \gamma \in \Gamma\} \ . \tag{22}$$

Note that when $H = \Gamma$ we recover the typeset $\mathcal{T}[\mathcal{G}]$ in Definition 3.7. Also, the restriction $\Gamma_1 \subset H$ is not significant, as will be seen in the proofs below, as one can always conjugate the chain $\mathcal{G}$ by an element of Γ so that this condition is satisfied.

We can now formulate the notion of commensurable typesets for group chains $\mathcal{G}$ and $\mathcal{G}'$.

DEFINITION 3.12. *Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ and $\mathcal{G}' = \{\Gamma' = \Gamma'_0 \supset \Gamma'_1 \supset \Gamma'_2 \supset \cdots\}$ be group chains. We say that their typesets $\mathcal{T}[\mathcal{G}]$ and $\mathcal{T}[\mathcal{G}']$ are* commensurable *if there exists finite index subgroups $\Gamma_1 \subset H \subset \Gamma$ and $\Gamma'_1 \subset H' \subset \Gamma'$ such that $\mathcal{T}_H[\mathcal{G}] = \mathcal{T}_{H'}[\mathcal{G}']$.*

4. Invariance of types

In this section, we give the proof of Theorem 1.5, and begin the proof of Theorem 1.9. We first reduce the study of equivalence of solenoidal manifolds to the study of Cantor actions defined by their monodromy actions. We then investigate the invariance of the types for Cantor actions.

4.1. **Monodromy actions.** Let $\mathcal{P} = \{\, q_\ell \colon M_\ell \to M_{\ell-1} \mid \ell \geq 1\}$ be a presentation, and recall from Section 1.1 that we have

$$\mathcal{S}_\mathcal{P} \equiv \lim_{\longleftarrow} \, \{q_\ell \colon M_\ell \to M_{\ell-1}\} \ \subset \prod_{\ell \geq 0} \ M_\ell \ . \tag{23}$$

For each $\ell \geq 0$, there is a fibration $\widehat{q}_\ell \colon \mathcal{S}_\mathcal{P} \to M_\ell$, given by projection onto the ℓ-th factor in (23). Also, there is a covering map denoted by $\overline{q}_\ell = q_\ell \circ q_{\ell-1} \circ \cdots \circ q_1 \colon M_\ell \to M_0$, such that $\widehat{q}_0 = \overline{q}_\ell \circ \widehat{q}_\ell$. Choose a basepoint $x_0 \in M_0$ and basepoint $x \in \mathfrak{X} = \widehat{q}_0^{\,-1}(x_0)$, the fiber over x_0. Then for each $\ell > 0$, this defines the basepoint $x_\ell = \widehat{q}_\ell(x) \in M_\ell$. Let $\Gamma_\ell = (\overline{q}_\ell)_\#(\Gamma_0)$ denote the image Γ_ℓ of the fundamental group $\pi_1(M_\ell, x_\ell)$ in $\Gamma = \pi_1(M_0, x_0)$.

The fiber $\overline{q}_\ell^{-1}(x_0) \subset M_\ell$ is identified with the quotient set $X_\ell = \Gamma/\Gamma_\ell$ which is a left Γ-space. In this way, the subspace $\mathfrak{X} = \widehat{q}_0^{-1}(x_0)$ of the inverse limit in (23) is identified with the Γ-space X_∞ in (10).

The covering degree m_ℓ of $q_\ell \colon M_\ell \to M_{\ell-1}$ equals the index of the subgroup $[\Gamma_{\ell-1} : \Gamma_\ell]$, and so the covering degree of $\overline{q}_\ell \colon M_\ell \to M_0$ is given by

$$\deg(\overline{q}_\ell) = m_\ell \cdot m_{\ell-1} \cdots m_1 = [\Gamma : \Gamma_\ell]. \tag{24}$$

By Definition 1.3 and (16) we have

$$\xi(\mathcal{P}) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\} = \mathrm{lcm}\{[\Gamma : \Gamma_\ell] \mid \ell > 0\} = \xi(X_\infty, \Gamma, \Phi) \ . \tag{25}$$

Given a second presentation $\mathcal{P}'$ with solenoidal manifold $\mathcal{S}_{\mathcal{P}'}$, we similarly obtain subgroups Γ'_ℓ and covering indices m'_ℓ, with

$$\xi(\mathcal{P}') = \mathrm{lcm}\{m'_1 m'_2 \cdots m'_\ell \mid \ell > 0\} = \mathrm{lcm}\{[\Gamma' : \Gamma'_\ell] \mid \ell > 0\} = \xi(X'_\infty, \Gamma', \Phi') \ . \tag{26}$$

Let $h \colon \mathcal{S}_{\mathcal{P}} \to \mathcal{S}_{\mathcal{P}'}$ be a homeomorphism. The homeomorphism h cannot be assumed to be fiber-preserving, that is to satisfy $h(\mathfrak{X}) = \mathfrak{X}'$, and it need not even be continuously deformable into a fiber-preserving map [13]. However, we can assume that on a sufficiently small transversal, the map h sends a clopen subset of a fiber to a clopen subset of a fiber. This suffices to imply that the homeomorphism induces a Morita equivalence between the pseudogroups defined by the foliations $\mathcal{F}_{\mathcal{P}}$ and $\mathcal{F}_{\mathcal{P}'}$, as discussed in [12, 13, 26]. Then we have:

THEOREM 4.1. *Let $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}_{\mathcal{P}'}$ be homeomorphic solenoidal manifolds. Then the corresponding monodromy Cantor actions $\Phi \colon \Gamma \times X_\infty \to X_\infty$ and $\Phi' \colon \Gamma' \times X'_\infty \to X'_\infty$ are return equivalent.*

To prove Theorem 1.5, we must show that if two solenoidal manifolds $\mathcal{S}_{\mathcal{P}}$ and $\mathcal{S}'_{\mathcal{P}'}$ are homeomorphic, then the Steinitz orders $\xi(\mathcal{P})$ and $\xi(\mathcal{P}')$ of the presentations $\mathcal{P}$ and $\mathcal{P}'$ belong to the same asymptotic equivalence class and so have equal types, $\tau[\mathcal{P}] = \tau[\mathcal{P}']$, defined by Definition 1.3. By Theorem 4.1 this reduces to showing that the Steinitz orders $\xi(X_\infty, \Gamma, \Phi)$ and $\xi(X'_\infty, \Gamma', \Phi')$, associated to return equivalent actions (X_∞, Γ, Φ) and $(X'_\infty, \Gamma', \Phi')$ are asymptotically equivalent.

To prepare for the proof of part (1) of Theorem 1.9, that is, that the Steinitz order $\xi(\mathfrak{X}, \Gamma, \Phi)$ is well-defined, recall one other fact about representing a Cantor action by an algebraic model. Given a minimal equicontinuous Cantor action $(\mathfrak{X}, \Gamma, \Phi)$, the choice of an adapted neighborhood basis $\mathcal{U}$ determines a subgroup chain $\mathcal{G}_{\mathcal{U}}$ as in Corollary 2.4, and so a Cantor action (X_∞, Γ, Φ) as in Section 2.3. The choice of another adapted basis $\mathcal{U}'$ yields an associated group chain $\mathcal{G}_{\mathcal{U}'}$ and associated Cantor action $(X'_\infty, \Gamma, \Phi')$. The following result follows from Theorem 1.4 in [19].

THEOREM 4.2. *Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous Cantor action, and assume that $\mathcal{G}_{\mathcal{U}}$ and $\mathcal{G}_{\mathcal{U}'}$ are adapted neighborhood bases for the action. Then the corresponding algebraic models of the action, (X_∞, Γ, Φ) and $(X'_\infty, \Gamma, \Phi')$, are isomorphic in the sense of Definition 2.5 with $\Theta \colon \Gamma \to \Gamma$ the identity map.*

Thus the proof of part (1) of Theorem 1.9 proceeds similarly to the proof of Theorem 1.5, except the assumption that Cantor actions (X_∞, Γ, Φ) and $(X'_\infty, \Gamma', \Phi')$ are return equivalent is replaced by the assumption that the actions are isomorphic. The proof of part (3a) of Theorem 1.9 is analogous to that of Theorem 1.5, after the application of Theorem 4.1.

4.2. **Invariance of type.** The proofs of Theorem 1.5, and of parts (1) and (3a) of Theorem 1.9 start similarly, as below.

Suppose that (X_∞, Γ, Φ) and $(X'_\infty, \Gamma', \Phi')$ are return equivalent. Then there exists adapted sets $U \subset X_\infty$ and $U' \subset X'_\infty$ and a homeomorphism $h \colon U \to U'$ so that the induced homomorphism $h_* \colon \mathrm{Homeo}(U) \to \mathrm{Homeo}(U')$ restricts to an isomorphism between the image $\mathcal{H}_U = \Phi(\Gamma_U)$ with the image $\mathcal{H}'_{U'} = \Phi'(\Gamma'_{U'})$.

Recall a standard construction of a diagram of maps between adapted sets [22], [19, Theorem 3.3]:

$$\begin{array}{ccccccccccccc} X_\infty & \supset & U & \supset & \gamma_1 \cdot U_{\ell_1} & \supset & h^{-1}(\gamma_1' \cdot U'_{\ell_1'}) & \supset & \gamma_2 \cdot U_{\ell_2} & \supset & h^{-1}(\gamma_2' \cdot U'_{\ell_2'}) & \supset & \cdots \\ & & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & & \downarrow h & & \\ X'_\infty & \supset & U' & \supset & h(\gamma_1 \cdot U_{\ell_1}) & \supset & \gamma_1' \cdot U'_{\ell_1'} & \supset & h(\gamma_2 \cdot U_{\ell_2}) & \supset & \gamma_2' \cdot U'_{\ell_2'} & \supset & \cdots \end{array} \tag{27}$$

The sets U_ℓ are defined in (11), and adapted to the action of Γ, and similarly for the sets $U'_{\ell'}$ which are adapted to the action of Γ'. The subscripts and intermediate adapted sets are defined iteratively in the following. We adopt the "$\cdot$" notation for the actions, as it is clear from the context which action is being applied. Recall that the action of Γ_U on U is minimal, as is the action of $\Gamma'_{U'}$ on U'. Denote $e_\infty = (e\Gamma_\ell) \in X_\infty$, and $e'_\infty = (e\Gamma'_{\ell'}) \in X'_\infty$, where Γ_ℓ is the isotropy subgroup of U_ℓ and Γ'_ℓ is the isotropy subgroup of $U'_{\ell'}$, for each $\ell \geq 0$ and each $\ell' \geq 0$.

We define the maps and indices in Diagram (27) recursively.

The set U is clopen, so there exists $\gamma_1 \in \Gamma$ such that $\gamma_1 \cdot e_\infty \in U$.
Choose $\ell_1 > 0$ such that $\gamma_1 \cdot U_{\ell_1} \subset U$.

The image $h(\gamma_1 \cdot U_{\ell_1}) \subset U'$ is clopen, so choose $\gamma_1' \in \Gamma'$ such that $\gamma_1' \cdot e'_\infty \in h(\gamma_1 \cdot U_{\ell_1})$.
Choose $\ell_1' > 0$ such that $\gamma_1' \cdot U'_{\ell_1'} \subset h(\gamma_1 \cdot U_{\ell_1})$.

The image $h^{-1}(\gamma_1' \cdot U'_{\ell_1'}) \subset \gamma_1 \cdot U_{\ell_1}$ is clopen, so choose $\gamma_2 \in \Gamma$ such that $\gamma_2 \cdot e_\infty \in h^{-1}(\gamma_1' \cdot U'_{\ell_1'}) \subset \gamma_1 \cdot U_{\ell_1}$.
Choose $\ell_2 > \ell_1$ such that $\gamma_2 \cdot U_{\ell_2} \subset h^{-1}(\gamma_1' \cdot U'_{\ell_1'})$.

The image $h(\gamma_2 \cdot U_{\ell_2}) \subset \gamma_1' \cdot U'_{\ell_1'}$ is clopen, , so choose $\gamma_2' \in \Gamma'$ such that $\gamma_2' \cdot e'_\infty \in h(\gamma_2 \cdot U_{\ell_2})$.
Choose $\ell_2' > \ell_1'$ such that $\gamma_2' \cdot U'_{\ell_2'} \subset h(\gamma_2 \cdot U_{\ell_2})$.

Continue this procedure recursively to obtain Diagram (27) where we have:

- increasing sequences $0 < \ell_1 < \ell_2 < \ell_3 < \cdots$ and $0 < \ell_1' < \ell_2' < \ell_3' < \cdots$,
- a sequence $\{\gamma_1, \gamma_2, \gamma_3, \ldots\} \subset \Gamma$,
- a sequence $\{\gamma_1', \gamma_2', \gamma_3', \ldots\} \subset \Gamma'$.

Observe that by choice we have $\gamma_{i+1} \cdot U_{\ell_{i+1}} \subset \gamma_i \cdot U_{\ell_i}$ and thus $\gamma_{\ell_i}^{-1}\gamma_{\ell_{i+1}} \in \Gamma_{\ell_i}$ for all $i \geq 1$. This recursion relation implies that the sequence $\{\gamma_i \mid i \geq 0\}$ converges in the profinite topology on Γ induced by the subgroup chain $\{\Gamma_\ell \mid \ell \geq 0\}$, and similarly for $\{\gamma_i' \mid i \geq 0\}$ in the profinite topology on Γ'. If we denote the respective limits by $\overline{\{\gamma_i\}} \in \overline{\Gamma}$ and $\overline{\{\gamma_i'\}} \in \overline{\Gamma}'$, then we have

$$\overline{\{\gamma_i\}} \cdot e_\infty = h^{-1}(\overline{\{\gamma_i'\}} \cdot e'_\infty), \text{ and } \overline{\{\gamma_i'\}} \cdot e'_\infty = h(\overline{\{\gamma_i\}} \cdot e_\infty).$$

All of the sets appearing in Diagram (27) are adapted for their respective actions, so the set inclusions induce corresponding subgroup chains of their isotropy groups, and these chains are interlaced:

$$\begin{array}{ccccccccccccc} \Gamma & \supset & H = \Gamma_U & \supset & \gamma_1 \Gamma_{\ell_1} \gamma_1^{-1} & \supset & \Gamma_{h^{-1}(\gamma_1' \cdot U'_{\ell_1'})} & \supset & \gamma_2 \Gamma_{\ell_2} \gamma_2^{-1} & \supset & \Gamma_{h^{-1}(\gamma_2' \cdot U'_{\ell_2'})} & \supset & \cdots \\ & & \downarrow h_* & & \downarrow h_* & & \downarrow h_* & & \downarrow h_* & & \downarrow h_* & & \\ \Gamma' & \supset & H' = \Gamma'_{U'} & \supset & \Gamma'_{h(\gamma_1 \cdot U_{\ell_1})} & \supset & \gamma_1' \Gamma'_{\ell_1'} (\gamma_1')^{-1} & \supset & \Gamma'_{h(\gamma_2 \cdot U_{\ell_2})} & \supset & \gamma_2' \Gamma'_{\ell_2'} (\gamma_2')^{-1} & \supset & \cdots \end{array} \tag{28}$$

Conjugation does not change the index of a subgroup, so $[\Gamma : \Gamma_{\ell_j}] = [\Gamma : \gamma_j \Gamma_{\ell_j} \gamma_j^{-1}]$ for all $j \geq 1$, and likewise we have $[\Gamma' : \Gamma'_{\ell_j'}] = [\Gamma' : \gamma_j' \Gamma'_{\ell_j'} (\gamma_j')^{-1}]$. It then follows that

$$[\Gamma : \Gamma_{\ell_j}] = [\Gamma : \Gamma_{\ell_1}][\Gamma_{\ell_1} : \Gamma_{\ell_j}] = [\Gamma : \Gamma_{\ell_1}][\gamma_1 \Gamma_{\ell_1} \gamma_1^{-1} : \gamma_j \Gamma_{\ell_j} \gamma_j^{-1}] \ , \tag{29}$$

$$[\Gamma' : \Gamma'_{\ell_j'}] = [\Gamma' : \Gamma'_{\ell_1}][\Gamma'_{\ell_1'} : \Gamma'_{\ell_j'}] = [\Gamma' : \Gamma'_{\ell_1}][\gamma_1' \Gamma'_{\ell_1'} (\gamma_1')^{-1} : \gamma_j' \Gamma'_{\ell_j'} (\gamma_j')^{-1}] \ . \tag{30}$$

We now show the key fact for the proof of Theorem 1.5.

LEMMA 4.3. *For all $j > i > 0$,*

$$[\Gamma_{\ell_i} : \Gamma_{\ell_j}] = [\Gamma'_{h(\gamma_i \cdot U_{\ell_i})} : \Gamma'_{h(\gamma_j \cdot U_{\ell_j})}]. \tag{31}$$

Proof. The isotropy group Γ_{ℓ_i} acts minimally on the clopen set U_{ℓ_i}, and its action translates the clopen subset $U_{\ell_j} \subset U_{\ell_i}$ to give a partition of this set. The index $[\Gamma_{\ell_i} : \Gamma_{\ell_j}]$ equals the number of clopen subsets in this partition. It follows that the action of $\gamma_i \Gamma_{\ell_i} \gamma_i^{-1}$ on $\gamma_i \cdot U_{\ell_i}$ partitions this set into $[\Gamma_{\ell_i} : \Gamma_{\ell_j}]$ translates of the clopen subset $\gamma_j \cdot U_{\ell_j}$.

The homeomorphism $h\colon U \to U'$ restricts to a homeomorphism $h\colon \gamma_i \cdot U_{\ell_i} \to h(\gamma_i \cdot U_{\ell_i})$ which induces a conjugacy of the action of $\Gamma_{\gamma_i \cdot U_{\ell_i}}$ with the action of $\Gamma'_{h(\gamma_i \cdot U_{\ell_i})}$ on $h(\gamma_i \cdot U_{\ell_i}) \subset U'$. Thus, the translates of the clopen subset $h(\gamma_j \cdot U_{\ell_j})$ partition $h(\gamma_i \cdot U_{\ell_i})$ into $[\Gamma_{\ell_i} : \Gamma_{\ell_j}]$ clopen subsets, and so the identity (31) follows. □

Proof of Theorem 1.5 and part (3a) of Theorem 1.9. Recall that

$$\xi(\mathcal{P}) = \mathrm{lcm}\{[\Gamma : \Gamma_\ell] \mid \ell > 0\}] = \mathrm{lcm}\{[\Gamma : \Gamma_{\ell_j}] \mid j > 0\}\ , \tag{32}$$

$$\xi(\mathcal{P}') = \mathrm{lcm}\{[\Gamma' : \Gamma'_\ell] \mid \ell > 0\}] = \mathrm{lcm}\{[\Gamma' : \Gamma'_{\ell'_j}] \mid j > 0\}\ . \tag{33}$$

Then by (31), for $j > 1$ we have

$$[\Gamma : \Gamma_{\ell_j}] = [\Gamma : \Gamma_{\ell_1}]\ [\Gamma_{\ell_1} : \Gamma_{\ell_j}] = [\Gamma : \Gamma_{\ell_1}]\ [\Gamma'_{h(\gamma_1 \cdot U_{\ell_1})} : \Gamma'_{h(\gamma_j \cdot U_{\ell_j})}]\ . \tag{34}$$

Then calculate, recalling the inclusions $U' \supset h(\gamma_1 \cdot U_{\ell_1}) \supset \gamma'_1 \cdot U'_{\ell_1} \supset h(\gamma_j \cdot U_{\ell_j}) \supset \gamma'_j \cdot U'_{\ell'_j}$ from (27):

$$\begin{aligned}
[\Gamma' : \Gamma'_{\ell'_j}] &= [\Gamma' : \gamma'_1 \Gamma'_{\ell'_1} (\gamma'_1)^{-1}] \\
&= [\Gamma' : \Gamma'_{h(\gamma_1 \cdot U_{\ell_1})}]\ [\Gamma'_{h(\gamma_1 \cdot U_{\ell_1})} : \Gamma'_{\gamma'_1 \cdot U'_{\ell'_1}}]\ [\Gamma'_{\gamma'_1 \cdot U'_{\ell'_1}} : \Gamma'_{h(\gamma_j \cdot U_{\ell_j})}]\ [\Gamma'_{h(\gamma_j \cdot U_{\ell_j})} : \Gamma'_{\gamma'_j \cdot U'_{\ell'_j}}] \\
&= [\Gamma' : \Gamma'_{h(\gamma_1 \cdot U_{\ell_1})}]\ [\Gamma'_{h(\gamma_1 \cdot U_{\ell_1})} : \Gamma'_{h(\gamma_j \cdot U_{\ell_j})}]\ [\Gamma'_{h(\gamma_j \cdot U_{\ell_j})} : \Gamma'_{\gamma'_j \cdot U'_{\ell'_j}}].
\end{aligned} \tag{35}$$

Then by (34), the last line of (35) yields

$$[\Gamma : \Gamma_{\ell_1}]\ [\Gamma' : \Gamma'_{\ell'_j}] = [\Gamma' : \Gamma'_{h(\gamma_1 \cdot U_{\ell_1})}]\ [\Gamma : \Gamma_{\ell_j}]\ [\Gamma'_{h(\gamma_j \cdot U_{\ell_j})} : \Gamma'_{\gamma'_j \cdot U'_{\ell'_j}}]. \tag{36}$$

Thus, the index $[\Gamma : \Gamma_{\ell_j}]$ divides $[\Gamma : \Gamma_{\ell_1}]\ [\Gamma' : \Gamma'_{\ell'_j}]$ for all $j > 1$. In particular, $[\Gamma : \Gamma_{\ell_j}]$ divides $[\Gamma : \Gamma_{\ell_1}]\ \xi(\mathcal{P}')$ for all $j > 1$, and so $\xi(\mathcal{P})$ divides $[\Gamma : \Gamma_{\ell_1}]\ \xi(\mathcal{P}')$, hence $\tau[\mathcal{P}] \leq \tau[\mathcal{P}']$.

Next, repeat these calculations for $j > 1$, starting with

$$[\Gamma : \Gamma_{\ell_{j+1}}] = [\Gamma : \Gamma_{h^{-1}(\gamma'_1 \cdot U_{\ell'_1})}]\ [\Gamma_{h^{-1}(\gamma'_1 \cdot U_{\ell'_1})} : \Gamma_{h^{-1}(\gamma'_j \cdot U'_{\ell'_j})}]\ [\Gamma_{h^{-1}(\gamma'_j \cdot U'_{\ell'_j})} : \Gamma_{\gamma_{j+1} \cdot U_{\ell_{j+1}}}]\ . \tag{37}$$

Lemma 4.3 can be applied to the inverse map $h^{-1}\colon U' \to U$ as well, to obtain that for all $j > i > 0$,

$$[\Gamma'_{\ell'_i} : \Gamma'_{\ell'_j}] = [\Gamma_{h^{-1}(\gamma'_i \cdot U_{\ell'_i})} : \Gamma_{h^{-1}(\gamma'_j \cdot U'_{\ell'_j})}] \tag{38}$$

and so

$$[\Gamma : \Gamma_{\ell_{j+1}}] = [\Gamma : \Gamma_{h^{-1}(\gamma'_1 \cdot U_{\ell'_1})}]\ [\Gamma'_{\ell'_1} : \Gamma'_{\ell'_j}]\ [\Gamma_{h^{-1}(\gamma'_j \cdot U'_{\ell'_j})} : \Gamma_{\gamma_{j+1} \cdot U_{\ell_{j+1}}}]\ . \tag{39}$$

Thus, the index $[\Gamma'_{\ell'_1} : \Gamma'_{\ell'_j}]$ divides $[\Gamma : \Gamma_{\ell_{j+1}}]$ for all $j > 1$ and so $[\Gamma'_{\ell'_1} : \Gamma'_{\ell'_j}]$ divides $\xi(\mathcal{P})$. It follows that $\xi(\mathcal{P}')$ divides $\xi(\mathcal{P})$, hence $\tau[\mathcal{P}'] \leq \tau[\mathcal{P}]$. Thus, $\tau[\mathcal{P}] = \tau[\mathcal{P}']$. This completes the proof of Theorem 1.5 and part (3a) of Theorem 1.9. □

Proof of part (1) of Theorem 1.9. Note that $\Gamma = \Gamma'$, but the group chains may be different, giving rise to the inverse limit spaces X_∞ and X'_∞. In the above calculations, take $U = X_\infty$ and $U' = X'_\infty$, $\gamma_1 = e \in \Gamma$ with $\ell_1 = 0$, and $\gamma'_1 = e' \in \Gamma'$ with $\ell'_1 = 0$. Then the terms $[\Gamma : \Gamma_{\ell_1}] = 1$ and $[\Gamma' : \Gamma'_{\ell'_1}] = 1$. Then by (36) we have $[\Gamma : \Gamma_{\ell_j}]$ divides $[\Gamma' : \Gamma'_{\ell'_j}]$, and by (39) we have $[\Gamma' : \Gamma'_{\ell'_j}]$ divides $[\Gamma : \Gamma_{\ell_{j+1}}]$.

It follows that $\xi(\mathcal{G}) = \xi(\mathcal{G}')$. That is, the Steinitz order is independent of the choice of the group chain defining the action. This completes the proof of part (1) of Theorem 1.9. $\square$

The proof of part (2) and (3b) of Theorem 1.9 will follow from the calculations in the next section, for $U = X_\infty$ and $U' = X'_\infty$ as in the proof of (1) above.

5. Invariance of typesets

In this section, we complete the proof of Theorem 1.9. The proofs of Corollary 1.10 and Theorem 1.11 are given in Section 5.3.

5.1. **Standard form.** We first reduce to a standard form, which is then analyzed. Let $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ be a minimal equicontinuous Cantor actions, and assume the actions are return equivalent. Then there exists adapted sets $U \subset \mathfrak{X}$ and $U' \subset \mathfrak{X}'$ and a homeomorphism $h \colon U \to U'$ that induces an isomorphism between the actions $(U, \mathcal{H}_U, \Phi_U)$ and $(U', \mathcal{H}'_{U'}, \Phi'_{U'})$. Recall that $\mathcal{H}_U = \Phi(\Gamma_U) \subset \mathrm{Homeo}(U)$ and $\mathcal{H}'_{U'} = \Phi'(\Gamma'_{U'}) \subset \mathrm{Homeo}(U')$. In particular, h induces an isomorphism between the groups $\mathcal{H}_U$ and $\mathcal{H}'_{U'}$.

Next, choose basepoints $x \in U$ and $x' \in U'$. Choose an adapted neighborhood basis $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell > 0\}$ at x for the action Φ as in Definition 2.2, and an adapted neighborhood basis $\mathcal{U}' = \{U'_\ell \subset \mathfrak{X}' \mid \ell > 0\}$ at x' for the action Φ'. We can assume that $U_1 \subset U$ and $U'_1 \subset U'$.

Form the group chain $\mathcal{G} = \{\Gamma_0 \supset \Gamma_1 \supset \cdots\}$, where $\Gamma_\ell = \Gamma_{U_\ell}$ with $\Gamma_0 = \Gamma$, and the group chain $\mathcal{G}' = \{\Gamma'_0 \supset \Gamma'_1 \supset \cdots\}$, where $\Gamma'_\ell = \Gamma'_{U'_\ell}$ with $\Gamma'_0 = \Gamma'$.

Set $H = \Gamma_U$ and $H' = \Gamma'_{U'}$. We will show that the H-restricted typeset $\mathcal{T}_H[X_\infty, \Gamma, \Phi]$ and the H'-restricted typeset $\mathcal{T}_{H'}[X'_\infty, \Gamma', \Phi']$ are equal.

Given $\gamma \in \Gamma$, for any positive integer $m > 0$ the group $\langle \gamma^m \rangle$ is a subgroup of finite index in $\langle \gamma \rangle$. Thus $\xi^H(\gamma) = m\xi^H(\gamma^m)$. Thus, $\tau[\xi^H(\gamma)] = \tau[\xi^H(\gamma^m)]$. Since H has finite index in Γ, for any $\gamma \in \Gamma$ there exists $m > 0$ such that $\gamma^m \in H$. Thus we have $\mathcal{T}_H[\mathcal{G}] = \mathcal{T}_H[\{H \cap \Gamma_\ell \mid \ell \geq 0\}]$, so it suffices to consider $\gamma \in H$, and likewise for $\gamma' \in H'$.

5.2. **Invariance of typesets.** We next show that typesets for (X_∞, Γ, Φ) and $(X'_\infty, \Gamma', \Phi')$ are invariant under isomorphism, and under return equivalence up to the commensurable relation in Definition 3.12, which yields the proofs of parts (2) and (3b) of Theorem 1.9. The proofs of both parts start similarly.

Assume that (X_∞, Γ, Φ) and $(X'_\infty, \Gamma', \Phi')$ are return equivalent, so there exists adapted sets $U \subset X_\infty$ and $U' \subset X'_\infty$ and a homeomorphism $h \colon U \to U'$ conjugating the action Φ of H on U, with the action Φ' of H' on U'. Let sequences $\{\gamma_i \mid i \geq 1\}$, $\{\gamma'_i \mid i \geq 1\}$, $\{\ell_i \mid i \geq 1\}$ and $\{\ell'_i \mid i \geq 1\}$ be chosen as in Section 4.2, resulting in the diagram (27) of adapted sets, and the diagram (28) of group inclusions. Note that as we assume $e_\infty \in U$ we can choose γ_1 to be the identity, and likewise as $e'_\infty \in U'$ we choose γ'_1 to be the identity. By choice we have $\gamma_{i+1} \cdot U_{\ell_{i+1}} \subset \gamma_i \cdot U_{\ell_i}$ and thus $\gamma_{\ell_i}^{-1}\gamma_{\ell_{i+1}} \in \Gamma_{\ell_i}$ for all $i \geq 1$. In particular, this implies that $\gamma_i \in H$ for all $i \geq 1$. The analogous conclusion holds, that $\gamma'_i \in H'$ for all $i \geq 1$.

For notational convenience, set $H_i = \Gamma_{\gamma_i \cdot U_{\ell_i}} = \gamma_i \Gamma_{\ell_i} \gamma_i^{-1}$ and $H'_i = \Gamma'_{h(\gamma_i \cdot U_{\ell_i})}$ for $i > 0$. Then $H_j \subset H_i \subset H = \Gamma_U$ and $H'_j \subset H'_i \subset H' = \Gamma'_{U'}$ for $j > i > 0$.

Set $m = \#(\Gamma/H)!$ which is the order of the group of permutations on the set $\{X_\infty : U\} = \Gamma/H$. Then for $\gamma \in \Gamma$ the action of γ^m on the set $\{X_\infty : U\}$ is the identity. This implies that $\gamma^m \in C_H^\Gamma$ the core of H. We have $\xi^H(\gamma^m) = m\xi^H(\gamma)$ and thus $\tau[\xi^H(\gamma^m)] = \tau[\xi^H(\gamma)]$. So without loss of generality it suffices to consider $\gamma \in C_H^\Gamma \subset H$.

The homeomorphism $h \colon U \to U'$ induces an isomorphism $h_* \colon \mathcal{H}_U \to \mathcal{H}'_{U'}$. Thus there exists $\gamma' \in H'$ whose action $\Phi'_{U'}(\gamma')$ on U' equals the image $\phi'_\gamma = h_*(\Phi_U(\gamma))$. We show that the H'-restricted type of γ' equals the H-restricted type of γ.

Let $V \subset W \subset X_\infty$ be adapted subsets. The restricted action of Γ_W on W is minimal, hence the translates of V define a clopen partition of W. Let $\{W : V\}$ denote the set of elements in this partition, and let $|W : V| = \#\{W : V\}$ denote the cardinality of the set of translates.

The action Φ induces a map $\Phi_V^W : \Gamma_W \to \mathrm{Aut}(\{W : V\})$ into the permutations of the set $\{W : V\}$. The kernel of the map Φ_V^W is denoted by $C_V^W \subset \Gamma_V$ and equals the normal core of Γ_V as a subgroup of Γ_W. Thus the index $[\Gamma_W : \Gamma_V] = |W : V|$, and $[\Gamma_W : C_V^W] = \#\mathrm{Image}(\Phi_V^W(\Gamma_W))$.

Now apply this observation to the action of $H = \Gamma_U$ on U. For each $j \geq 1$, the action Φ induces a map $\widehat{\Phi}^U_{U_{\ell_j}} : H \to \mathrm{Aut}(\{U : \gamma_j \cdot U_{\ell_j}\})$ which permutes the elements of this partition. The kernel of the action map $\widehat{\Phi}^U_{U_{\ell_j}}$ is the subgroup

$$C_j^H = \bigcap_{\delta \in H} \delta \Gamma_{\gamma_j \cdot U_{\ell_j}} \delta^{-1} = \bigcap_{\delta \in H} (\delta\gamma_j)\Gamma_{\ell_j}(\delta\gamma_j)^{-1} = \bigcap_{\delta \in H} \delta \Gamma_{\ell_j} \delta^{-1} \ . \tag{40}$$

Set $C^H_{\gamma,j} = \langle\gamma\rangle \cap C_j^H$, then the subgroup $\langle\gamma\rangle / C^H_{\gamma,j} \subset H/C_j^H$ is mapped injectively into $\mathrm{Aut}(\{U : U_{\ell_j}\})$. Thus, $\#(\langle\gamma\rangle/C^H_{\gamma,j}) = \#\mathrm{Image}(\Phi^U_{U_{\ell_j}}(\langle\gamma\rangle))$, and so we have

$$\xi^H(\gamma) = \mathrm{lcm}\{\#(\langle\gamma\rangle/C^H_{\gamma,j}) \mid j > 0\} = \mathrm{lcm}\{\#\mathrm{Image}(\widehat{\Phi}^U_{U_{\ell_j}}(\langle\gamma\rangle)) \mid j > 0\} \ . \tag{41}$$

We next use the conjugation by h to relate the order of a subgroup of $\mathrm{Aut}(\{U : V\})$ with the order of a subgroup of $\mathrm{Aut}(\{U' : V'\})$ for appropriate choices of adapted subsets V and V'. This is analogous to the idea behind the proof of Lemma 4.3.

From (27), for $j \geq 1$ we have

$$U' \supset h(\gamma_1 \cdot U_{\ell_1}) \supset \gamma_1' \cdot U'_{\ell_1'} \supset \cdots \supset h(\gamma_j \cdot U_{\ell_j}) \supset \gamma_j' \cdot U'_{\ell_j'} \supset \cdots \ . \tag{42}$$

Using that $\gamma_j' \cdot U'_{\ell_j'} \subset h(\gamma_j \cdot U_{\ell_j})$ we have

$$\gamma_j' \Gamma'_{\ell_j} (\gamma_j')^{-1} = \Gamma'_{\gamma_j' \cdot U'_{\ell_j'}} \subset \Gamma'_{h(U_{\ell_j})} = H_j' \subset H' \ , \tag{43}$$

and so $C_j^{H'} = C^{H'}_{\gamma_j'\Gamma'_{\ell_j}(\gamma_j')^{-1}} \subset C^{H'}_{H_j'}$. Then

$$\langle\gamma'\rangle \supset \left(\langle\gamma'\rangle \cap C^{H'}_{H_j'}\right) \supset \left(\langle\gamma'\rangle \cap C_j^{H'}\right) = C^{H'}_{\gamma',j}. \tag{44}$$

The action of $\Phi'_{U'}(\gamma')$ on U' translates the clopen set $h(\gamma_j \cdot U_{\ell_j})$ within the clopen set U', thus

$$\#(\langle\gamma\rangle/C^H_{\gamma,j}) = \#\mathrm{Image}(\widehat{\Phi}^U_{U_{\ell_j}}(\langle\gamma\rangle)) = \#\mathrm{Image}(\widehat{\Phi}^{U'}_{h(U_{\ell_j})}(\langle\gamma'\rangle)) = \#(\langle\gamma'\rangle/(\langle\gamma'\rangle \cap C^{H'}_{H_j'})) \ . \tag{45}$$

Using that $\gamma_j' \cdot U'_{\ell_j'} \subset h(\gamma_j \cdot U_{\ell_j})$ we have

$$\gamma_j' \Gamma'_{\ell_j} (\gamma_j')^{-1} = \Gamma'_{\gamma_j' \cdot U'_{\ell_j'}} \subset \Gamma'_{h(U_{\ell_j})} = H_j' \subset H' \ , \tag{46}$$

and so $C_j^{H'} = C^{H'}_{\gamma_j'\Gamma'_{\ell_j}(\gamma_j')^{-1}} \subset C^{H'}_{H_j'}$. We then have, using (44),

$$\begin{aligned} \#(\langle\gamma'\rangle/C^{H'}_{\gamma',j}) &= \#(\langle\gamma'\rangle/(\langle\gamma'\rangle \cap C^{H'}_{H_j'})) \cdot \#((\langle\gamma'\rangle \cap C^{H'}_{H_j'})/C^{H'}_{\gamma',j}) \\ &= \#(\langle\gamma\rangle/C^H_{\gamma,j}) \cdot \#((\langle\gamma'\rangle \cap C^{H'}_{H_j'})/(\langle\gamma'\rangle \cap C_j^{H'})) \ . \end{aligned} \tag{47}$$

Thus we have that $\#(\langle\gamma\rangle/C^H_{\gamma,\ell_j})$ divides $\#(\langle\gamma'\rangle/C^{H'}_{\gamma',j})$ for all $j \geq 1$. It follows that $\tau_H[\gamma] = \tau[\xi^H(\gamma)] \leq \tau[\xi^{H'}(\gamma')] = \tau_{H'}[\gamma']$. Then by reversing these calculations, starting with γ' chosen as above, we obtain $\tau_{H'}[\gamma'] \leq \tau_H[\gamma]$ and hence $\tau_H[\gamma] = \tau_{H'}[\gamma']$.

We have thus shown that for each $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma'$ with the same restricted type. Reversing this process, we deduce that $\mathcal{T}_H[X_\infty, \Gamma, \Phi] = \mathcal{T}_{H'}[X'_\infty, \Gamma', \Phi']$.

Proof of part (2) of Theorem 1.9. We must show that for $\gamma \in \Gamma$, there is a well-defined Steinitz order $\xi(\gamma)$ which is independent of the choice of an adapted basis $\mathcal{U}$ for the action. Given two adapted bases $\mathcal{U}$ and $\mathcal{U}'$ for the action $(\mathfrak{X}, \Gamma, \Phi)$, then by Theorem 2.7 the algebraic models they define, (X_∞, Γ, Φ) and $(X'_\infty, \Gamma, \Phi')$, are isomorphic. Thus, in the above we can take $U = X_\infty$ and $U' = X'_\infty$ and so $H = \Gamma_U = \Gamma$ and $H' = \Gamma'_{U'} = \Gamma'$. Then observe that for $\ell > 0$ the normal subgroups $C^H_\ell = C_\ell$ and $C^{H'}_\ell = C'_\ell$. It follows that for $\gamma \in \Gamma$ and γ' chosen as in the proof of part (3b) we have $\xi^H(\gamma) = \xi(\gamma)$ and $\xi^{H'}(\gamma') = \xi(\gamma')$, and the proof shows that $\xi(\gamma) = \xi(\gamma')$. It follows that the set of orders $\{\xi(\gamma) \mid \gamma \in \Gamma\}$ is independent of the choice of an algebraic model for the action, and so is an invariant of the isomorphism class of the action $(\mathfrak{X}, \Gamma, \Phi)$. □

Proof of part (3b) of Theorem 1.9. We conclude that the sets $\mathcal{T}_H[X_\infty, \Gamma, \Phi]$ and $\mathcal{T}_{H'}[X'_\infty, \Gamma', \Phi']$ are independent of the choice of adapted bases, and so have the equality $\mathcal{T}_H[X_\infty, \Gamma, \Phi] = \mathcal{T}_{H'}[X'_\infty, \Gamma', \Phi']$. This completes the proof of part (3b) of Theorem 1.9. □

5.3. **Invariance for special cases.** We next deduce Corollary 1.10 and Theorem 1.11.

Proof of Corollary 1.10. Suppose that $(\mathfrak{X}, \Gamma, \Phi)$ and $(\mathfrak{X}', \Gamma', \Phi')$ are minimal equicontinuous Cantor actions, where both Γ and Γ' are abelian, and the actions are return equivalent. Then every group Γ_ℓ in a subgroup chain $\mathcal{G}_{\mathcal{U}}$ in Γ is normal. Then for any subgroup $H \subset \Gamma$, the normal cores satisfy $C^H_\ell = C_\ell = \Gamma_\ell$. Thus, in (41), the restricted type $\xi^H(\gamma) = \xi(\gamma)$. Similarly, we have $\xi^{H'}(\gamma') = \xi(\gamma')$ for $\gamma' \in \Gamma'$. Thus, the above proof shows in this case that the typesets $\mathcal{T}[\mathfrak{X}, \Gamma, \Phi]$ and $\mathcal{T}[\mathfrak{X}', \Gamma', \Phi']$ are equal, which is the claim of Corollary 1.10. □

It is surprising perhaps, that the conclusion of Corollary 1.10 need not hold if one of the groups is virtually abelian, but not abelian, as illustrated in Example 6.3. The idea of Example 6.3 is that we add to an abelian group a single element that normalizes it, does not commute with it, andt destroys the equality $C^H_\ell = C_\ell$. Then the types of elements are no longer equal to their H-restricted types.

Recall that Theorem 1.11 concerns Cantor actions defined by a self-embedding $\varphi\colon \Gamma \to \Gamma$.

Proof of Theorem 1.11. Recall from [32] that the embedding φ defines a group chain $\mathcal{G}_\varphi$ by setting $\Gamma_0 = \Gamma$ and then inductively defining $\Gamma_{\ell+1} = \varphi(\Gamma_\ell) \subset \Gamma_\ell$ for $\ell \geq 0$. Then with $H = \Gamma_k$ for $k > 0$ and $\ell > k$,

$$C^H_\ell = \bigcap_{\delta \in H} \delta^{-1}\Gamma_\ell\delta = \bigcap_{\delta \in \varphi^k(\Gamma)} \delta^{-1}\varphi^\ell(\Gamma)\delta = \varphi^k\left(\bigcap_{\delta \in \Gamma} \delta^{-1}\varphi^{\ell-k}(\Gamma)\delta\right) = \varphi^k(C_{\ell-k}) \ , \tag{48}$$

and so $\Gamma_\ell / C^H_\ell = \varphi^k(\Gamma_{\ell-k}/C_{\ell-k})$. Then for $\gamma \in H = \Gamma_k$ set $\overline{\gamma} = \varphi^{-k}(\gamma)$, then have

$$\xi^H(\gamma) = \mathrm{lcm}\{\#(\langle\gamma\rangle / C^H_{\gamma,\ell}) \mid \ell > k\} = \mathrm{lcm}\{\#(\langle\overline{\gamma}\rangle / C_{\overline{\gamma},\ell-k}) \mid \ell - k > 0\} = \xi(\overline{\gamma}) \ . \tag{49}$$

Then as $\Gamma = \varphi^{-k}(H)$, we have $\mathcal{T}_H[X_\varphi, \Gamma, \Phi_\varphi] = \mathcal{T}[X_\varphi, \Gamma, \Phi_\varphi]$.

Let $(X_\varphi, \Gamma, \Phi_\varphi)$ and $(X_{\varphi'}, \Gamma', \Phi_{\varphi'})$ be Cantor actions associated to renormalizations $\varphi\colon \Gamma \to \Gamma$ and $\varphi'\colon \Gamma' \to \Gamma'$. Assume the Cantor actions are return equivalent by a homeomorphism $h\colon U \to U'$. Then choose $\delta \in \Gamma$ with $\delta \cdot e_\infty \in U$ and $\delta' \in \Gamma'$ with $\delta' \cdot e'_\infty \in U'$. Then $\varphi^\delta = \Phi(\delta) \circ \varphi \circ \Phi(\delta^{-1})$ is a renormalization of Γ with group chain $\Gamma^\delta_\ell = \delta\Gamma_\ell\delta^{-1}$. The group Γ^δ_ℓ stabilizes the translate $\delta \cdot U_\ell$. As $\delta \cdot e_\infty \in U$, there exists $k > 0$ such that $\delta \cdot U_k \subset U$. Repeat this argument for the renormalization φ' to obtain a $\delta' \in \Gamma'$ and $k' > 0$ such that $\delta' \cdot U'_{k'} \subset U'$.

Then proceed as in the proof in Section 4.2 with $H = \delta\Gamma_k\delta^{-1}$ and $H' = \delta'\Gamma'_{k'}(\delta')^{-1}$ to obtain that

$$\mathcal{T}[X_\varphi, \Gamma, \Phi_\varphi] = \mathcal{T}_H[X_\varphi, \Gamma, \Phi_\varphi] = \mathcal{T}_{H'}[X'_{\varphi'}, \Gamma', \Phi_{\varphi'}] = \mathcal{T}[X'_{\varphi'}, \Gamma', \Phi_{\varphi'}],$$

as claimed in Theorem 1.11. In particular, this implies that the typeset $\mathcal{T}[X_\varphi, \Gamma, \Phi_\varphi]$ is an invariant of the isomorphism class of the renormalizable Cantor action $(X_\varphi, \Gamma, \Phi_\varphi)$. □

6. Basic Examples

6.1. **Virtually abelian actions.** A group Γ is *virtually abelian* if it admits a finite-index subgroup $A \subset \Gamma$ which is abelian. It is straightforward to construct Cantor actions of virtually abelian groups, and yet they illustrate several basic properties of the type and typeset invariants.

EXAMPLE 6.1. Consider the case $n = 1$. Choose two disjoint sets of distinct primes,

$$\pi_f = \{q_1, q_2, \ldots\} \quad , \quad \pi_\infty = \{p_1, p_2, \ldots\}$$

where π_f and π_∞ can be chosen to be finite or infinite sets, and either π_f is infinite, or π_∞ is non-empty. Choose multiplicities $n(q_i) \geq 1$ for the primes in π_f. For each $\ell > 0$, define a subgroup of $\Gamma = \mathbb{Z}$ by

$$\Gamma_\ell = \{q_1^{n(q_1)} q_2^{n(q_2)} \cdots q_\ell^{n(q_\ell)} \cdot p_1^\ell p_2^\ell \cdots p_\ell^\ell \cdot n \mid n \in \mathbb{Z}\} \ . \tag{50}$$

If π_∞ is a finite set, then we use the convention that $p_\ell = 1$ in (50) when p_ℓ is not defined by the listing of π_∞. The completion $\widehat{\Gamma}$ of $\mathbb{Z}$ with respect to this group chain admits a product decomposition into its Sylow p-subgroups

$$\widehat{\Gamma} \ \cong \ \prod_{i=1}^{\infty} \mathbb{Z}/q_i^{n(q_i)}\mathbb{Z} \ \cdot \ \prod_{p \in \pi_\infty} \widehat{\mathbb{Z}}_{(p)} \ , \tag{51}$$

where $\widehat{\mathbb{Z}}_{(p)}$ denotes the p-adic completion of $\mathbb{Z}$. Thus $\pi(\xi(\widehat{\Gamma})) = \pi_f \cup \pi_\infty$. As $\mathbb{Z}$ is abelian, $X_\infty = \widehat{\Gamma}$. By Theorem 1.4, the type of this action classifies it up to return equivalence.

EXAMPLE 6.2. The diagonal actions of $\mathbb{Z}^n$ for $n \geq 2$ are direct extensions of Example 6.1. Make n choices of prime spectra as in Example 6.1, then take the product action on the individual factors. The type of the action no longer determines the isomorphism class of the Cantor actions obtained. The typeset is an invariant under return equivalence by Corollary 1.10, but only in special cases does the typeset determine the isomorphism class of the action. The interested reader can consult the works [3, 4, 10, 23, 37].

EXAMPLE 6.3. We next give an example that illustrates that the commensurable relation on typesets is optimal. We construct the simplest example which shows this, and it is clear that many more similar constructions are possible.

Let $\Gamma = \mathbb{Z}^2 \rtimes \mathbb{Z}_2$ be the semi-direct product of $\mathbb{Z}^2$ with the order 2 group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$, where the generator $\sigma \in \mathbb{Z}_2$ acts on $\mathbb{Z}^2$ by permuting the summands. The group $\Gamma' = \mathbb{Z}^2$ is abelian.

Chose distinct primes $p, q > 1$. Define the subgroup chain in Γ and Γ' as follows

$$\Gamma_\ell = \Gamma'_\ell = \{(p^\ell k, q^\ell m, id) \mid (h, m) \in \mathbb{Z}^2\} \quad , \quad \Gamma'_\ell = \{(p^\ell k, q^\ell m) \mid (k, m) \in \mathbb{Z}^2\} \ . \tag{52}$$

Note that the resulting actions (X_∞, Γ, Φ) and $(X'_\infty, \Gamma', \Phi')$ are return equivalent.

Observe that $C'_\ell = \Gamma'_\ell$, while Γ_ℓ is not normal in Γ, and we have

$$C_\ell = \{((pq)^\ell k, (pq)^\ell m, id) \mid (h, m) \in \mathbb{Z}^2\} \subset \Gamma_\ell \ . \tag{53}$$

The actions of Γ and Γ' have the same type, with characteristic functions $\chi(p) = \chi(q) = \infty$, and all other values are zero. However, we have the typesets

$$\mathcal{T}[X_\infty, \Gamma, \Phi] = \{[(pq)^\infty]\} \quad \text{and} \quad \mathcal{T}[X'_\infty, \Gamma', \Phi'] = \{[p^\infty], [q^\infty], [(pq)^\infty]\} \ , \tag{54}$$

so the typeset is not invariant under return equivalence.

This example easily generalizes, where we take $\Gamma' = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ to be the direct sum of n copies of $\mathbb{Z}$, and replace $\mathbb{Z}_2$ with any non-trivial subgroup $\Delta \subset \mathrm{Perm}(n)$ of the permutation group on n elements. Let $\Gamma = \Gamma' \rtimes \Delta$ be the semi-direct product of Γ' with Δ. Choose the subgroup chain $\{\Gamma'_{\ell'}\}$ in Γ' as in Example 6.2 and use the chain $\{\Gamma_\ell = \Gamma'_\ell \times id\}$ in Γ. Then the resulting actions are return equivalent, and one can obtain a wide variety of finite typesets $\mathcal{T}[X'_\infty, \Gamma', \Phi']$ for the abelian action. If the action of Δ is transitive, then we have that $\mathcal{T}[X_\infty, \Gamma, \Phi]$ consists of a single element. When Δ does not act transitively, there is even more variation on the typesets of the two actions.

6.2. **Nilpotent Cantor actions.** The next examples use the actions associated to a renormalization of a finitely generated group Γ. Many finitely generated nilpotent groups admit a renormalization [14, 17, 18, 21, 33, 40], which yield many examples of Cantor actions with well-defined typesets by Theorem 1.11.

The integer Heisenberg group is simplest non-abelian nilpotent group, and is represented as the upper triangular matrices in $\mathrm{GL}(n, \mathbb{Z})$. That is,

$$\Gamma = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\} . \tag{55}$$

We denote a 3×3 matrix in Γ by the coordinates as (a, b, c).

EXAMPLE 6.4. For a prime $p \geq 2$, define the self-embedding $\varphi_p \colon \Gamma \to \Gamma$ by $\varphi(a, b, c) = (pa, pb, p^2c)$. Then define a group chain in Γ by setting

$$\Gamma_\ell = \varphi_p^\ell(\Gamma) = \{(p^\ell a, p^\ell b, p^{2\ell} c) \mid a, b, c \in \mathbb{Z}\} \quad , \quad \bigcap_{\ell > 0} \Gamma_\ell = \{e\} \ .$$

For $\ell > 0$, the normal core for Γ_ℓ is given by $C_\ell = \mathrm{core}(\Gamma_\ell) = \{(p^{2\ell} a, p^{2\ell} b, p^{2\ell} c) \mid a, b, c \in \mathbb{Z}\}$, and so the quotient group $Q_\ell = \Gamma / C_\ell \cong \{(\overline{a}, \overline{b}, \overline{c}) \mid \overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}/p^{2\ell}\mathbb{Z}\}$. The profinite group $\widehat{\Gamma}_\infty$ is the inverse limit of the quotient groups Q_ℓ so we have $\widehat{\Gamma}_\infty = \{(\widehat{a}, \widehat{b}, \widehat{c}) \mid \widehat{a}, \widehat{b}, \widehat{c} \in \widehat{\mathbb{Z}}_{p^2}\}$. Thus, every non-trivial $\gamma \in \Gamma$ has type $\tau[\gamma] = \tau[p^\infty]$.

EXAMPLE 6.5. For distinct primes $p, q \geq 2$, define the self-embedding $\varphi_{p,q} \colon \Gamma \to \Gamma$ by $\varphi(a, b, c) = (pa, qb, pqc)$. Then define a group chain in Γ by setting

$$\Gamma_\ell = \varphi_{p,q}^\ell(\Gamma) = \{(p^\ell a, q^\ell b, (pq)^\ell c) \mid a, b, c \in \mathbb{Z}\} \quad , \quad \bigcap_{\ell > 0} \Gamma_\ell = \{e\} \ .$$

For $\ell > 0$, the normal core for Γ_ℓ is given by $C_\ell = \mathrm{core}(\Gamma_\ell) = \{((pq)^\ell a, (pq)^\ell b, (pq)^\ell c) \mid a, b, c \in \mathbb{Z}\}$, and so we obtain the quotient group $Q_\ell = \Gamma / C_\ell \cong \{(\overline{a}, \overline{b}, \overline{c}) \mid \overline{a}, \overline{b}, \overline{c} \in \mathbb{Z}/(pq)^\ell\mathbb{Z}\}$. The profinite group $\widehat{\Gamma}_\infty$ is the inverse limit of the quotient groups Q_ℓ so we have $\widehat{\Gamma}_\infty = \{(\widehat{a}, \widehat{b}, \widehat{c}) \mid \widehat{a}, \widehat{b}, \widehat{c} \in \widehat{\mathbb{Z}}_{pq}\}$. Thus, every non-trivial $\gamma \in \widehat{\Gamma}_\infty$ has type $\tau[\gamma] = \tau[(pq)^\infty]$.

Note that the typeset for the Cantor action defined by the $\varphi_{p,q}$-renormalization equals the typeset for the abelian action in Example 6.3, but the two actions are clearly not return equivalent.

A second source of examples for Cantor actions of nilpotent groups uses the decomposition of a profinite nilpotent group into its prime localizations, a technique that is especially adapted to realizing a given collection of primes as the spectrum of such an action. Moreover, these actions can be constructed to have special dynamical properties, as in [29]. The construction is necessarily more complex than for renormalizable actions, as there must be an infinite sequence of choices to make. These ideas were developed in [26, Section 9] for actions of $SL(n, \mathbb{Z})$, and the work [30] discusses this construction for nilpotent groups.

7. Examples: d-regular actions

It is well-known that every minimal equicontinuous action $(\mathfrak{X}, \Gamma, \Phi)$ can be faithfully represented as an action of Γ on the boundary of a rooted tree. The study of actions on trees, especially the actions on d-ary (or d-regular) trees satisfying an additional condition of *self-similarity*, is an active topic in Geometric Group Theory, see [39, 25] for surveys. The tree models for Cantor actions are especially useful for constructing actions which are not topologically free, and thus for illustrating the commensurable relationship between types. In this section, we study the typesets for Cantor actions on boundaries of d-regular rooted trees.

7.1. **Actions on trees.** A *tree* T is an infinite graph with the set of vertices $V = \bigsqcup_{\ell \geq 0} V_\ell$ and the set of edges E. Each V_ℓ, $\ell \geq 0$ is a finite set, called the *set of vertices at level* ℓ. Edges in E join pairs of vertices in consecutive level sets $V_{\ell+1}$ and V_ℓ, $\ell \geq 0$, so that a vertex in $V_{\ell+1}$ is connected to a single vertex in V_ℓ by a single edge. A tree is *rooted* if $|V_0| = 1$.

DEFINITION 7.1. *A tree T is* spherically homogeneous *if there is a sequence $n = (n_1, n_2, \ldots)$, called the* spherical index *of T, such that for every $\ell \geq 1$ a vertex in $V_{\ell-1}$ is joined by edges to precisely n_ℓ vertices in V_ℓ. In addition, T is d-*ary*, or d-*regular*, if its spherical index $n = (n_1, n_2, \ldots)$ is* constant, *that is, $n_\ell = d$ for some positive integer d.*

We assume that $n_\ell \geq 2$ for $\ell \geq 1$. If $d = 2$, then a 2-ary tree T is also called a *binary* tree.

Let $(\mathfrak{X}, \Gamma, \Phi)$ be a minimal equicontinuous action, and let $\mathcal{U} = \{U_\ell \subset \mathfrak{X} \mid \ell > 0\}$ be a choice of an adapted neighborhood basis. By Corollary 2.4 there is a group chain $\mathcal{G}_\mathcal{U} = \{\Gamma_\ell = \Gamma_{U_\ell} \mid \ell \geq 0\}$ such that, associated to $\mathcal{G}_\mathcal{U}$ is a Cantor action (X_∞, Γ, Φ) which is isomorphic to $(\mathfrak{X}, \Gamma, \Phi)$. Here X_∞ is the inverse limit space of finite sets $X_\ell = \Gamma/\Gamma_\ell$, given by (10), see Section 2.3 for details.

We now build a tree model for the action $(\mathfrak{X}, \Gamma, \Phi)$, using the group chain $\mathcal{G}_\mathcal{U}$. For $\ell \geq 0$, let $V_\ell = X_\ell$, and join $v_\ell \in V_\ell$ and $v_{\ell+1} \in V_{\ell+1}$ by an edge if and only if $v_{\ell+1} \subset v_\ell$ as cosets. The obtained tree is spherically homogeneous, with spherical index entries $n_\ell = |\Gamma_{\ell-1} : \Gamma_\ell|$, for $\ell \geq 1$. The boundary ∂T of T is the collection of all infinite paths in T, that is,

$$\partial T = \{(v_\ell)_{\ell \geq 0} \subset \prod_{\ell \geq 0} V_\ell \mid v_{\ell+1} \text{ and } v_\ell \text{ are joined by an edge}\} \cong X_\infty,$$

and the induced action of Γ on ∂T, which we also denote by (X_∞, Γ, Φ).

DEFINITION 7.2. *A minimal equicontinuous action Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is d-*regular*, or just* regular, *if there exists $d \geq 2$ such that $(\mathfrak{X}, \Gamma, \Phi)$ is conjugate to an action of Γ on a rooted d-ary tree.*

REMARK 7.3. A Cantor action $(\mathfrak{X}, \Gamma, \Phi)$ is d-regular if the group chain $\mathcal{G}_\mathcal{U}$ above can be chosen so that each subgroup index $|\Gamma_\ell : \Gamma_{\ell-1}| = d$, for some $d \geq 2$ and all $\ell \geq 0$. Nilpotent actions given by a self-embedding $\varphi_p : \Gamma \to \Gamma$ in Example 6.4 are d-regular with $d = p^4$, and those in Example 6.5 are d-regular with $d = p^2 q^2$.

7.2. **Typesets of regular actions.** We show that the typeset of a d-regular action is always finite.

THEOREM 7.4. *A minimal equicontinuous Cantor actions $(\mathfrak{X}, \Gamma, \Phi)$ which is d-*regular *has finite typeset $\mathcal{T}[X_\infty, \Gamma, \Phi]$. More precisely, suppose $(\mathfrak{X}, \Gamma, \Phi)$ is isomorphic to an action on a d-ary rooted tree, for some $d \geq 2$. Let P_d be the set of distinct prime divisors of the elements in the collection* $\mathrm{lcm}\{2, \ldots, d\}$, *and let $N_d = |P_d|$. Then*

$$|\mathcal{T}[X_\infty, \Gamma, \Phi]| \leq \sum_{k=0}^{N_d} \binom{N_d}{k} = \sum_{k=0}^{N_d} \frac{N_d!}{(N_d - k)!k!} \ . \tag{56}$$

Moreover, each equivalence class $\tau \in \mathcal{T}[X_\infty, \Gamma, \Phi]$ is represented by a Steinitz number ξ with empty finite prime spectrum, $\pi_f(\xi) = \emptyset$, and so $\pi(\xi) = \pi_\infty(\xi)$.

Proof. Let N_d and P_d be as in the statement of the theorem, and let $L_d = \mathrm{lcm}\{P_d\}$.

LEMMA 7.5. *Let (X_∞, Γ, Φ) be an action on a rooted tree. Let $\gamma \in \Gamma$, let $\xi(\gamma)$ be the Steinitz order of γ defined in Definition 3.7, and let p be such that $\chi_\xi(p) \neq 0$. Then p divides L_d.*

Proof. If $\chi_\xi(p) \neq 0$, then there exists the smallest $\ell \geq 1$ such that p divides the order of the group $\langle\gamma\rangle/\langle\gamma\rangle_\ell$. The group $\langle\gamma\rangle/\langle\gamma\rangle_\ell = \langle\gamma\rangle/\langle\gamma\rangle \cap C_\ell$ is isomorphic to a subgroup of Γ/C_ℓ, where C_ℓ is the normal core of Γ_ℓ, and acts on the coset space X_ℓ by permutations. Let $\lambda_{\gamma,\ell}$ be the permutation of X_ℓ induced by γ. Then the order of $\langle\gamma\rangle/\langle\gamma\rangle_\ell$ is equal to the order of $\lambda_{\gamma,\ell}$, and so equal to the least common multiple of the length of the cycles in $\lambda_{\gamma,\ell}$.

Similarly, the order of $\langle\gamma\rangle/\langle\gamma\rangle_{\ell-1}$ is equal to the least common multiple of the length of the cycles in the permutation $\lambda_{\gamma,\ell-1}$ of $X_{\ell-1}$ induced by the action of γ. By the choice of ℓ the order of $\langle\gamma\rangle/\langle\gamma\rangle_{\ell-1}$ is not divisible by p. Therefore, for any cycle $c_{\ell-1}$ in $\lambda_{\gamma,\ell-1}$, the length $|c_{\ell-1}|$ is not divisible by p.

Consider the preimage $S_{c_{\ell-1}}$ of the set of elements in $c_{\ell-1}$ under the inclusion of cosets $X_\ell \to X_{\ell-1}$. Then $|S_{c_{\ell-1}}| = d|c_{\ell-1}|$, and γ permutes the elements in $S_{c_{\ell-1}}$. Let c_ℓ be a cycle in the permutation $\mu_{c_{\ell-1}}$ of $S_{c_{\ell-1}}$ induced by γ. Since the action of γ commutes with coset inclusions, $|c_\ell| = \alpha|c_{\ell-1}|$ for some $1 \leq \alpha \leq d$. Then p must divide such an α for one of the cycles in $\mu_{c_{\ell-1}}$, for some cycle $c_{\ell-1}$ in $\lambda_{\gamma,\ell-1}$. It follows that p divides L_d. □

Let $\xi(\gamma)$ be the Steinitz order of γ. Since every p for which $0 < \chi_\xi(p) < \infty$, divides L_d by Lemma 7.5, then the finite prime spectrum $\pi(\xi(\gamma))$ is finite, and the type $\tau(\gamma)$ has a representative $\widehat{\xi}$, such that if $\chi_{\widehat{\xi}}(p) \neq 0$ then $\chi_{\widehat{\xi}}(p) = \infty$. It follows that two types $\tau(\gamma)$ and $\tau(\gamma')$ with respective Steinitz orders ξ and ξ' are distinct if and only if there exists a prime p such that $\chi_\xi(p) = 0$ and $\chi_{\xi'}(p) \neq 0$. The bound in (56), which is the number of distinct collections of prime divisors of L_d, follows.

□

EXAMPLE 7.6. Let $d = 2$, then $L_2 = 2$ and $N_2 = 1$, and the upper bound on the cardinality of the typeset for 2-regular actions is 2, with possible types $\{[1], [2^\infty]\}$, where 1 denotes the type of the identity element.

Let $d = 3$, then $L_3 = 6$ and $N_3 = 2$. Then the upper bound on the cardinality of the typeset for 3-regular actions is 4, with possible types $\{[1], [2^\infty], [3^\infty], [(2 * 3)^\infty]\}$.

Let $d = 4$, then $P_4 = \{2, 3\}$, $L_4 = 6$ and $N_4 = 2$. Then the upper bound on the cardinality of the typeset for 4-regular actions is 4, with possible types $\{[1], [2^\infty], [3^\infty], [(2 * 3)^\infty]\}$.

EXAMPLE 7.7. Let p be an odd prime. The Gupta-Sidki p-group $GS(p)$ [39, Section 1.8.1] is a group acting on the rooted p-ary tree. The group is generated by a cyclic permutation of the set of p elements $\sigma = (0, 1, \ldots, p-1)$, and the recursively defined map

$$\tau = (\sigma, \sigma^{-1}, 1, \ldots, \tau).$$

Every element in the Gupta-Sidki group p-group has finite order, and so $\mathcal{T}[X_\infty, GS(p), \Phi] = \{[1]\}$, where [1] is the trivial type, i.e. the type of the identity element.

EXAMPLE 7.8. Let $d = 2$, and let (X_∞, Γ, Φ) be a 2-regular Cantor action. Then $|\mathcal{T}[X_\infty, \Gamma, \Phi]| \leq 2$, and either $\mathcal{T}[X_\infty, \Gamma, \Phi] = \{[1]\}$, or $\mathcal{T}[X_\infty, \Gamma, \Phi] = \{[1], [2^\infty]\}$.

An example of an action where the typeset $\mathcal{T}[X_\infty, \Gamma, \Phi] = \{[1]\}$, is the action of the Grigorchuk group, see for instance [25]. This group is an example of a Burnside group, which is an infinite group where every element has finite order, and so it has trivial type.

For the case $\mathcal{T}[X_\infty, \Gamma, \Phi] = \{[1], [2^\infty]\}$ we have two classes of examples. First, the examples of actions where $\tau(\gamma) = [2^\infty]$ for any non-trivial $\gamma \in \Gamma$, are the actions of iterated monodromy groups of quadratic polynomials for which the orbit of the critical point is periodic. Such groups are torsion-free, see [7], and so they do not have any finite order element except the identity in Γ. Such actions include, but are not limited to, the action of the odometer on the binary tree, and the Basilica group.

Second, the action may have non-trivial element which have trivial type, and also non-trivial elements with type $\{[2^\infty]\}$. These are given by the actions of the iterated monodromy groups of quadratic polynomials with strictly pre-periodic orbits of the critical point, see [7] for details.

7.3. **Typeset under the commensuration relation.** We now show that H-restricted types of a group element and of its conjugate need not coincide.

Given a tree T and a vertex $v \in V$, denote by T_v the subtree of T with root v.

DEFINITION 7.9. [25] *Let T be a spherically homogeneous tree, and let $\Gamma \subset \mathrm{Aut}(T)$. The action of Γ on ∂T is* weakly branch *if the following conditions hold:*

(1) The restriction of the action of Γ to each vertex level set V_n, $n \geq 1$, is transitive.
(2) For every $\ell \geq 1$ and every $v \in V_\ell$, there exists $g \in \Gamma$ such that $g \cdot v = v$, the restriction of the action of g on the subtree T_v is non-trivial, and the restriction of g to the complement $\partial T - \partial T_v$ is the identity map.

EXAMPLE 7.10. Let T be a spherically homogeneous tree and let $\Gamma \subset \mathrm{Aut}(T)$ be so that the action of Γ on ∂T is weakly branch. Choose a sequence (v_ℓ) of vertices in T, and let Γ_ℓ be the isotropy group (equivalently, the stabilizer) of $v_\ell \in V_\ell$. Then Γ_ℓ fixes v_ℓ while permuting other vertices in V_ℓ.

Choose $n \geq 1$. The normal core C_n of Γ_n consists of elements of Γ_n which fix every vertex in V_n. Let $g \in \Gamma_n$ be an element given by Definition 7.9 of a weakly branch group, namely, the restriction of g to T_{v_n} is non-trivial, and g is trivial on the complement of ∂T_{v_n} in ∂T. In particular, this means that $g \in C_n$. However, since g acts non-trivially on ∂T_{v_n}, then there exists $\ell > n$ such that $g \notin C_\ell$, and then the Steinitz order $\xi(g)$ is non-trivial.

Now let $\delta \in \Gamma$ be such that $\delta \cdot v_n \neq v_n$. Then the restriction of $\delta g \delta^{-1}$ to ∂T_{v_n} is the identity map. Since the Steinitz order is invariant under conjugation, $\xi(\delta g \delta^{-1})$ is non-trivial.

Now consider the Γ_n-restricted Steinitz orders of g and of $\delta g \delta^{-1}$. The subgroup $C_\ell^{\Gamma_n}$ is the normal core of Γ_ℓ in Γ_n, so it fixes every vertex in the set $V_\ell \cap T_{v_n}$, and it may permute the vertices of V_ℓ which are not in T_{v_n}. In particular, for all $\ell \geq n$ we have $\delta g \delta^{-1} \in C_\ell^{\Gamma_n}$, and the Γ_n-restricted Steinitz order $\xi^{\Gamma_n}(\delta g \delta^{-1})$ is trivial. At the same time, $\xi^{\Gamma_n}(g)$ is non-trivial, since the action of g permutes the vertices in $V_\ell \cap T_{v_n}$ for some $\ell \geq n$. Depending on the group Γ, $\xi^{\Gamma_n}(g)$ may have the trivial or non-trivial type; it is straightforward to construct examples of both situations.

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Steven Hurder, Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045

Email address: hurder@uic.edu

Olga Lukina, Mathematical Institute, Leiden University, P.O. Box 9512, 2300 RA Leiden, The Netherlands

Email address: o.lukina@math.leidenuniv.nl