

CHARACTERISTIC CLASSES FOR FLAT $\text{DIFF}(M)$ -FOLIATIONS

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ABSTRACT. In this work we relate the known results about the homotopy type of classifying spaces for smooth foliations, with the homology and cohomology of the discrete group of diffeomorphisms of a smooth compact connected oriented manifold. The Mather-Thurston Theorem forms a bridge between the results on the homotopy types of classifying spaces and the various homology and cohomology groups that are studied. We introduce the algebraic K-theory of a manifold M that is derived from the discrete group of diffeomorphisms of M , and observe that the calculations of homotopy groups in this work are about these K-theory groups. We include a variety of remarks and open problems related to the study of the diffeomorphism groups and their homological invariants using the Mather-Thurston Theorem.

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1. INTRODUCTION

Let M be a smooth connected orientable manifold of dimension q . Let $\text{Diff}_c(M)$ denote the subgroup of compactly supported diffeomorphisms of the topological group $\text{Diff}(M)$, and let $\text{Diff}_c^+(M)$ and $\text{Diff}^+(M)$ be the respective subgroups of orientation-preserving diffeomorphisms.

For a topological group G , we let G^δ denote the group with the discrete topology. The cohomology groups $H^*(BG^\delta; \mathcal{A})$ are the universal characteristic classes for flat bundles associated with G^δ . For the identity map $\iota: G^\delta \rightarrow G$, there is a natural induced map $\iota^*: H^*(BG; \mathcal{A}) \rightarrow H^*(BG^\delta; \mathcal{A})$. When G is a connected Lie group, Milnor calculated this map for many cases in [99]. For example, when the coefficient ring \mathcal{A} is finite and G is solvable, the map is an isomorphism. In contrast, when \mathcal{A} has characteristic 0 and G is a compact or complex semisimple Lie group, Milnor showed that the map is trivial. One can ask for analogous results in the case of the groups $\text{Diff}_c(M)$ and $\text{Diff}_c^+(M)$.

The classifying space $B\text{Diff}(M)$ classifies M -bundles over a base space X , and the classifying space $B\text{Diff}(M)^\delta$ classifies M -bundles with discrete structure group, that is, with a foliation on the total space over X that is transverse to the fibers. The cohomology groups $H^*(B\text{Diff}(M)^\delta; \mathcal{A})$, for a coefficient group \mathcal{A} , give invariants for foliated M -bundles, which is one motivation for the study of the topological properties of the spaces $B\text{Diff}(M)^\delta$.

The inclusion map $\iota_\delta: \text{Diff}_c^+(M)^\delta \rightarrow \text{Diff}_c^+(M)$ induces a map of the classifying spaces, which has a homotopy theoretic fiber $\overline{B\text{Diff}_c(M)}$, and there is a fibration

$$(1) \quad \overline{B\text{Diff}_c(M)} \xrightarrow{\iota} B\text{Diff}_c^+(M)^\delta \xrightarrow{\iota_\delta} B\text{Diff}_c^+(M) .$$

The space $\overline{B\text{Diff}_c(M)}$ classifies foliations on a product space $X \times M$ which are transverse to the factor M , and are a product outside of a compact subset of $X \times M$. The cohomology of the fiber space $\overline{B\text{Diff}_c(M)}$ is viewed as comparing the cohomology of the group $\text{Diff}_c^+(M)$ equipped with two topologies, one continuous and the other discrete (see [111, 128].) The following is the most basic problem, which remains for the most part unsolved:

PROBLEM 1.1. *Calculate the groups $H_*(\overline{B\text{Diff}_c(M)}; \mathcal{A})$ and $H_*(B\text{Diff}_c^+(M)^\delta; \mathcal{A})$, and the map between them, for M a connected oriented manifold, and \mathcal{A} an abelian group.*

The direct approach to the study of $H_*(B\text{Diff}_c^+(M)^\delta; \mathbb{Q})$ is to construct a group action of a finitely generated group Γ on M , that is, a homomorphism $\Phi: \Gamma \rightarrow \text{Diff}_c^+(M)$, and show that the induced map $\Phi_*: H_*(B\Gamma; \mathbb{Q}) \rightarrow H_*(B\text{Diff}_c^+(M)^\delta; \mathbb{Q})$ has non-trivial classes in its image. The celebrated work of Thurston in [135] did this for the case $M = \mathbb{S}^1$. The various constructions for higher dimensional manifolds described in Section 6 also use this approach for $M = \mathbb{S}^q$ a higher dimensional sphere, and for a few other cases where M is a homogeneous space. There are very few other examples where this approach has been carried through. Nariman considered in [115] the case where $M = G$ is a connected Lie group, and related $H_2(B\text{Diff}_c^+(G)^\delta; \mathbb{Q})$ with the algebraic K-theory group $K_2(\mathbb{C})^+$.

Another approach to Problem 1.1 is to study the spectral sequence associated to the fibration (1), which relates the cohomology groups $H^*(B\text{Diff}_c^+(M); \mathcal{A})$ and $H^*(\overline{B\text{Diff}_c(M)}; \mathcal{A})$ with the groups $H^*(B\text{Diff}_c^+(M)^\delta; \mathcal{A})$. Nariman studied this spectral sequence in [115], with a focus on the induced map on cohomology $\iota_\delta^*: H^*(B\text{Diff}_c^+(M); \mathbb{Q}) \rightarrow H^*(B\text{Diff}_c^+(M)^\delta; \mathbb{Q})$.

The approach in this work uses the Mather-Thurston Theorem 3.2 to reformulate the problem in terms of the homotopy groups of a space associated to $\text{Diff}_c^+(M)^\delta$. The works of Mather [94, 95] and Thurston [136] show that there is a map $\tau: \overline{B\text{Diff}_c(M)} \rightarrow \mathcal{S}(M)$ which induces isomorphisms in homology, where $\mathcal{S}(M)$ is the space of compactly supported sections of a bundle with fiber the classifying space $B\overline{\Gamma}_q$ of framed smooth codimension- q foliations. The following notion is key to our study of Problem 1.1:

DEFINITION 1.2. *The algebraic K-theory groups of M are defined as*

$$(2) \quad \mathcal{K}_*(M) = \pi_* \left(\left\{ \overline{B\text{Diff}_c(M)} \right\}^+ \right) \cong \pi_*(\mathcal{S}(M)) .$$

Here, $\{B\text{Diff}_c^+(M)_\delta\}^+$ denotes the space obtained by applying the Quillen +-construction. These are called algebraic K-theory groups, in analogy with the definition of the algebraic K-theory groups of a ring [143]. We denote these groups by $\mathcal{K}_*(M)$ to distinguish them from the usual K-theory of M denoted by $K^*(M)$, and the K-homology denoted by $K_*(M)$.

For $M = \mathbb{R}^q$, one has $\mathcal{S}(\mathbb{R}^q) \cong \Omega^q B\bar{\Gamma}_q$ is a q -fold *based loop* space, and $\mathcal{K}_n(\mathbb{R}^q) \cong \pi_{q+n}(B\bar{\Gamma}_q)$. These groups are absolutely enormous, as seen for example in Theorem 7.4. The Mather-Thurston Theorem 3.2 says there is an isomorphism $H_*(B\overline{\text{Diff}}_c(\mathbb{R}^q); \mathbb{Z}) \cong H_*(\mathcal{S}(\mathbb{R}^q); \mathbb{Z})$, and as $\mathcal{S}(\mathbb{R}^q)$ is an H-space, its rational homology is a free graded algebra over the generating set $\mathcal{K}_*(\mathbb{R}^q) \otimes \mathbb{Q}$.

For $M = \mathbb{S}^q$, one has $\mathcal{S}(\mathbb{S}^q) \cong \Lambda^q B\bar{\Gamma}_q$ is a q -fold *free loop* space, so $\mathcal{K}_n(\mathbb{S}^q) \cong \pi_n(B\bar{\Gamma}_q) \oplus \pi_{q+n}(B\bar{\Gamma}_q)$. There is an inclusion of the based loops into the free loops, $i: \Omega^q B\bar{\Gamma}_q \rightarrow \Lambda^q B\bar{\Gamma}_q$ which induces an inclusion map $i_*: \mathcal{K}(\mathbb{R}^q) \rightarrow \mathcal{K}(\mathbb{S}^q)$. The calculation of the homology groups $H_*(\mathcal{S}(\mathbb{S}^q); \mathcal{A})$ is no longer straightforward, even with rational coefficients $\mathcal{A} = \mathbb{Q}$, as it requires unknown properties of the homotopy theory of $B\bar{\Gamma}_q$ to calculate the differentials in the minimal model for the space. Haefliger describes the construction of this model in [51].

For M a connected orientable manifold in general, the approach to studying $\mathcal{K}(M)$ and $H_*(\mathcal{S}(M); \mathcal{A})$ using minimal models encounters even more difficulties.

The strategy of our approach to Problem 1.1 is to construct elements of $\mathcal{K}_*(M)$ which pair non-trivially with cohomology classes in $H^*(\mathcal{S}(M); \mathbb{R})$ that are derived from the secondary characteristic classes in $H^*(B\bar{\Gamma}_q; \mathbb{R})$. This is used to show that their images under the Hurewicz homomorphism $h: \mathcal{K}_*(M) \rightarrow H_*(\mathcal{S}(M); \mathbb{Q}) \cong H_*(B\overline{\text{Diff}}_c(M); \mathbb{Q})$ are non-trivial. Additional arguments in special cases are used to show that some of these classes then map non-trivially into $H_*(B\overline{\text{Diff}}_c(M)^\delta; \mathbb{Q})$.

This method yields uncountably many independent cycles in $H_*(B\overline{\text{Diff}}_c(M); \mathbb{Q})$, each of which is represented by a homomorphism $\rho: \Gamma \rightarrow \text{Diff}_c(M)$ of a finitely generated group Γ , as discussed in Section 8. The constructions reveal no information on the group Γ or how it acts on M . In fact, Freedman estimates in [32] that using the approach to the proof of the Mather-Thurston Theorem via foliated surgery in Meigniez [97], the number of generators for Γ can be ‘‘exponentially large’’.

The results obtained by the approach using the secondary invariants of foliations depend upon the dimension q and the topology of M , and are given in detail in later sections of this work. We formulate here some general results, as an overview of the results obtained.

THEOREM 1.3. *Let M be a smooth connected orientable manifold of dimension q . Then there exists a sequence of integers $v_q \geq 1$ which tend to infinity as q tends to infinity, and there exists an uncountably generated subgroup $\mathbb{V}_q \subset \mathcal{K}_{q+1}(M)$, and for $1 \leq i \leq v_q$, linear functionals $\phi_i: \mathcal{K}_{q+1}(M) \rightarrow \mathbb{R}$, such that the vector-valued homomorphism*

$$(3) \quad \vec{\phi} = (\phi_1, \dots, \phi_{v_q}): \mathcal{K}_{q+1}(M) \otimes \mathbb{Q} \longrightarrow \mathbb{R}^{v_q}$$

is a surjection when restricted to the rational subspace $\mathbb{V}_q \otimes \mathbb{Q}$.

COROLLARY 1.4. *Let M be a smooth connected orientable manifold of dimension q . Then there exists a sequence of integers $v_q \geq 1$ which tend to infinity as q tends to infinity, and there exists an uncountably generated subgroup $\mathbb{H}_q \subset H_{q+1}(B\overline{\text{Diff}}_c^+(M)^\delta; \mathbb{Q})$, and for $1 \leq i \leq v_q$, cohomology classes $\alpha_i \in H^{q+1}(B\overline{\text{Diff}}_c^+(M)^\delta; \mathbb{R})$, such that the vector-valued homomorphism*

$$(4) \quad \vec{\alpha} = (\alpha_1, \dots, \alpha_{v_q}): H_{q+1}(B\overline{\text{Diff}}_c^+(M)^\delta; \mathbb{Q}) \longrightarrow \mathbb{R}^{v_q}$$

is a surjection when restricted to the rational subspace $\mathbb{H}_q \otimes \mathbb{Q}$.

When the manifold M has trivial tangent bundle, there are homotopy operations on $\mathcal{K}_*(M)$ which generate many more non-trivial classes. These are the brace products described in Proposition A.4, and the results now apply to the space $B\overline{\text{Diff}}_c(M)$. In Section 12, integers $\widehat{v}_{q,k}$ and κ_q are defined based on the number of independent secondary classes which are variable. We always have $\widehat{v}_{q,k} = v_q$ when $k = 2q + 1$, and for $q \geq 3$ we have $\widehat{v}_{q,k} = v_q$ for $k = 2q + 4$, and both indices tend to infinity as q tends to infinity. Other values of k are also tending to infinity, as the dimension q increases. Note that in the following theorem, we now assume that M is compact.

THEOREM 1.5. *Let M be a smooth compact connected manifold of dimension q with trivial tangent bundle. Then there exists a sequence of integers $\widehat{v}_{q,k}$, for $k \geq 2q+1$, and there exists an uncountably generated subgroup $\mathbb{V}_{q,k} \subset \mathcal{K}_k(M)$, and for $1 \leq i \leq \widehat{v}_{q,k}$, linear functionals $\widehat{\phi}_i: \mathcal{K}_k(M) \rightarrow \mathbb{R}$, such that the vector-valued homomorphism*

$$(5) \quad \vec{\phi} = (\widehat{\phi}_1, \dots, \widehat{\phi}_{\widehat{v}_{q,k}}): \mathcal{K}_k(M) \otimes \mathbb{Q} \longrightarrow \mathbb{R}^{\widehat{v}_{q,k}}$$

is a surjection when restricted to the rational subspace $\mathbb{V}_{q,k} \otimes \mathbb{Q}$.

Then again using the Mather-Thurston Theorem, we obtain:

COROLLARY 1.6. *Let M be a smooth compact connected manifold of dimension q with trivial tangent bundle. Then there exists a sequence of integers $\widehat{v}_{q,k}$, for $k \geq 2q+1$, and there exists an uncountably generated subgroup $\mathbb{H}_{q,k} \subset H_k(\overline{BDiff}(M); \mathbb{Q})$, and for $1 \leq i \leq \widehat{v}_{q,k}$, cohomology classes $\widehat{\alpha}_i \in H^k(\overline{BDiff}(M); \mathbb{R})$, such that the vector-valued homomorphism*

$$(6) \quad \vec{\alpha} = (\widehat{\alpha}_1, \dots, \widehat{\alpha}_{\widehat{v}_{q,k}}): H_k(\overline{BDiff}(M); \mathbb{Q}) \longrightarrow \mathbb{R}^{\widehat{v}_{q,k}}$$

is a surjection when restricted to the rational subspace $\mathbb{H}_{q,k} \otimes \mathbb{Q}$.

There are many further constructions of non-trivial families of classes in $H_*(\overline{BDiff}_c(M); \mathbb{Q})$ beyond those above, obtained using the techniques developed in Theorems 4.4 and 4.7. A selection of these constructions is given in later sections.

We give a brief overview of the contents of the work. Section 2 recalls the definition of the spaces $B\overline{\Gamma}_q$ and $B\Gamma_q$ which classify smooth foliations of codimension q .

Section 3 introduces the space $\mathcal{S}(M)$ and discusses some basic properties about its homotopy type. Then in Section 4 we give some general constructions of non-trivial classes in $\mathcal{K}_*(M)$, assuming the existence of spherical cohomology classes in $H^*(B\overline{\Gamma}_q; \mathbb{R})$.

The construction of the secondary characteristic classes in $H^*(B\Gamma_q^+; \mathbb{R})$ and $H^*(B\overline{\Gamma}_q; \mathbb{R})$ is reviewed in Section 5, and in Section 6 we recall the examples from the literature showing that many of these classes are non-trivial. Then in Section 7, we apply the results of Sections 4, 5 and 6 to show the existence of non-trivial spherical cohomology classes in $H^*(B\overline{\Gamma}_q; \mathbb{R})$. The current understanding of the topological types of $B\overline{\Gamma}_q$ and $B\Gamma_q$ as described in Section 7 remains essentially unchanged since the 1980's, as described in [68], and also the survey [74].

Constructions of flat $\text{Diff}_c^+(M)$ -bundles using the results of Section 7 and the ideas from the proof of the Mather-Thurston Theorem are given in Section 8. In particular, we consider which of these homology classes in $H_*(\overline{BDiff}_c(M); \mathbb{Q})$ map non-trivially into $H_*(\text{Diff}_c^+(M)^\delta; \mathbb{Q})$.

In Section 9, we consider what is known for Problem 1.1 when $q = 1$, so either $M = \mathbb{R}$ or $M = \mathbb{S}^1$. Morita has obtained the most comprehensive results in this case in the publications [104, 106].

In Section 10, we consider the case when $q = 2$, and the special cases when $M = \mathbb{R}^2, \mathbb{S}^2, \mathbb{T}^2$ and for closed surfaces Σ_g with genus $g \geq 2$. Our analysis uses the results of Rasmussen in [122]. The case when $M = \Sigma_g$ is a compact surface of genus $g \geq 2$ has been studied in depth in the works of Bowden [18] and Nariman [113, 114], and we recall some of their results as well.

In Section 11, we consider the case for orientable 3-manifolds, where the use of the James brace product described in Appendix A.3 now enters into the constructions. In Section 12, we describe the results for higher dimensional manifolds.

For dimensions for $q = 2$ and 3, the structure of the groups $\text{Diff}_c^+(M)$ is mostly well-understood, but for higher dimensions these topological groups become increasingly mysterious. There have also been many recent developments in the study of the homology and cohomology of the group of diffeomorphisms for manifolds with dimensions $q \geq 5$, as discussed by Kupers [86]. Understanding the relations between these classes and the groups $H^*(\text{Diff}_c^+(M)^\delta; \mathbb{Q})$ is a fundamental question, which is mostly unsolved except in special cases. Problems 12.5 and 12.6 at the end of this section were one of the original motivations for this work.

In the author's thesis [65], a variety of basic techniques from homotopy theory of spaces and fibrations were used to extract homotopy results from the non-vanishing of the secondary invariants. These were recalled and extended in the work [68], and the applications to the study the homology of diffeomorphism groups was developed in the unpublished manuscript [69]. The formulation of the K-theory of a smooth manifold $\mathcal{K}_*(M)$ was given in the author's 1987 lectures in Berlin [71] as applications of these calculations of $\pi_*(B\bar{\Gamma}_q)$. Throughout this text, we liberally cite references to works on foliation theory and related topics, so that the bibliography is extensive as a result.

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2. FOLIATION CLASSIFYING SPACES

We give an overview of the definitions and basic properties of the classifying spaces for foliations introduced by Haefliger in [49, 50]. All foliations and maps are assumed to be smooth.

Let Γ_q denote the pseudogroup generated by local C^∞ -diffeomorphisms of \mathbb{R}^q . A codimension q -foliation \mathcal{F} on M , along with a good open covering of a paracompact manifold M , defines a 1-cocycle with values in Γ_q . Haefliger observed in [49] that there is a classifying space $B\Gamma_q$ for these Γ_q -cocycles, so that the foliation \mathcal{F} induces a continuous map $\chi_{\mathcal{F}}: M \rightarrow B\Gamma_q$ whose homotopy class is independent of the choice of the good covering. One says that $\chi_{\mathcal{F}}$ is the classifying map for \mathcal{F} . The Jacobian map of a local diffeomorphism induces a universal map $\nu: B\Gamma_q \rightarrow BO(q)$, which classifies the normal bundle on the universal foliated bundle over $B\Gamma_q$.

There are alternate models for the classifying space $B\Gamma_q$ which are all weak homotopy equivalent, by Greenberg [46], Jekel [77], McDuff [96], Moerdijk [101], and Segal [125]. Any of these models suffice for "classifying" foliations of codimension- q on paracompact manifolds.

The work by Bott, Shulman and Stasheff [14] (see also the text by Dupont [28]) constructed a semi-simplicial model for $B\Gamma_q$ and a semi-simplicial de Rham complex of forms whose cohomology groups are isomorphic to $H^*(B\Gamma_q, \mathbb{R})$. The existence of a de Rham complex for $B\Gamma_q$ allows one to use de Rham homotopy methods in the study of the topology of the space, as for example in [15].

There are various alternate subclasses of groupoids that arise for foliations with some additional structure, as discussed by Haefliger in [50]. We consider here the pseudogroup Γ_q^+ of orientable local C^∞ -diffeomorphisms of \mathbb{R}^q and its classifying space $B\Gamma_q^+$. There is a natural map $\nu: B\Gamma_q^+ \rightarrow BSO(q)$ classifying the normal bundle on $B\Gamma_q^+$. Also considered in the literature are groupoids of germs of maps preserving a volume form on \mathbb{R}^q , or preserving a symplectic form, or a Riemannian metric, to cite some of the cases which have been studied.

There is a canonical basepoint $w \in B\Gamma_q^+$ given by the image of the point foliation of \mathbb{R}^q . Let $B\bar{\Gamma}_q$ denote the homotopy fiber of $\nu: B\Gamma_q^+ \rightarrow BSO(q)$ over the basepoint $w_0 = \nu(w) \in BSO(q)$. Then $B\bar{\Gamma}_q$ is realized as the pull-back in the diagram (7):

$$(7) \quad \begin{array}{ccccc} \mathrm{SO}(q) & \xrightarrow{\iota} & B\bar{\Gamma}_q & \xrightarrow{\iota} & B\Gamma_q^+ \\ \downarrow \simeq & & \downarrow \bar{\nu} & & \downarrow \nu \\ \mathrm{SO}(q) \simeq \Omega(\mathrm{BSO}(q)) & \xrightarrow{\iota} & \mathcal{P}\mathrm{BSO}(q) & \xrightarrow{\mathrm{eval}} & \mathrm{BSO}(q) \end{array}$$

The map $\bar{\nu}$ induces a trivialization of the universal normal bundle over $B\bar{\Gamma}_q$ so that $B\bar{\Gamma}_q$ can also be interpreted as the classifying space of foliations with trivialized normal bundles. Haefliger observed in [49, Theorem 3] that this implies the space $B\bar{\Gamma}_q$ is $(q-1)$ -connected, as the Gromov-Phillips immersion theory [48, 118, 119] implies there is a unique foliation by points on \mathbb{R}^q . More is true, as Haefliger showed that $\pi_q(B\bar{\Gamma}_q)$ is trivial, that there is a unique homotopy class of Γ_q -structures on a foliated microbundle over \mathbb{S}^q .

PROBLEM 2.1. *Show that the space $B\bar{\Gamma}_q$ is $2q$ -connected, or determine the least value of $k \leq 2q$ for which $\pi_k(B\bar{\Gamma}_q)$ is non-trivial.*

The best partial result to date on this problem is Theorem 3.3 below.

3. MATHER-THURSTON THEORY

Let M be a connected smooth orientable manifold of dimension q . We recall the Mather-Thurston Theory, and then introduce the algebraic K-theory groups associated to this theory.

The space $\overline{B\text{Diff}}_c(M)$ classifies foliations on a product space $X \times M$, for X compact, which are transverse to the second factor M , and are horizontal outside of a compact subset of $X \times M$. Thus, there is a natural classifying map

$$(8) \quad \tilde{\nu}_M: \overline{B\text{Diff}}_c(M) \times M \rightarrow B\Gamma_q^+$$

Let $\nu_M: M \rightarrow BSO(q)$ denote the classifying map of the tangent bundle to M . Then we have a commutative diagram:

$$\begin{array}{ccc} & & B\Gamma_q^+ \\ & \nearrow \tilde{\nu}_M & \downarrow \nu \\ \overline{B\text{Diff}}_c(M) \times M & \xrightarrow{\nu_M} & BSO(q) \end{array}$$

Let $\sigma_M: M \rightarrow B\Gamma_q^+$ be the classifying map of the point foliation on M . Then $\mathcal{S}(M)$ denotes the space of compactly supported liftings of the map $\nu_M: M \rightarrow BSO(q)$ to $B\Gamma_q^+$. That is, $g: X \rightarrow \mathcal{S}(M)$ is a map $g: X \times M \rightarrow B\Gamma_q^+$ such that $\nu \circ g = \nu_M$, and $g = \sigma_M: X \times M \rightarrow B\Gamma_q^+$ outside of a compact subset of $X \times M$. (See [95, Section 6].) Then by the works of Mather and Thurston [94, 95, 136]:

THEOREM 3.1. *The adjunct map $\tau: \overline{B\text{Diff}}_c(M) \rightarrow \mathcal{S}(M)$ induces an isomorphism in homology.*

For the special case $M = \mathbb{R}^q$, the mapping $\tilde{\nu}_M$ can be considered as being defined on the q -fold suspension $\Sigma^q \overline{B\text{Diff}}_c(\mathbb{R}^q)$, and so its adjoint induces a map from $\overline{B\text{Diff}}_c(\mathbb{R}^q)$ to the iterated loop space on $B\bar{\Gamma}_q$. As a special case, there is the remarkable consequence of Theorem 3.1:

THEOREM 3.2. *The adjunct map $\tau: \overline{B\text{Diff}}_c(\mathbb{R}^q) \rightarrow \Omega^q B\bar{\Gamma}_q$ is a homology isomorphism.*

There are several proofs of Theorem 3.1 in the literature, beyond the initial works of Mather and Thurston [94, 95, 136]. McDuff gave a proof using categorical methods in [96], following ideas of Segal in [125]. Mitsumatsu and Vogt gave a geometric proof in [100] for foliations whose leaves have dimension 2. Meigniez gave a proof in [97] using bordism techniques, for $q \geq 2$.

The Mather-Thurston Theorem 3.2 is key to the proof of the following fundamental result, due to Mather [93, 94] for $q = 1$, and to Thurston [136] for $q > 1$.

THEOREM 3.3 (Mather [94, 95], Thurston [136]). *The space $B\bar{\Gamma}_q$ is $(q+1)$ -connected. That is, for $q \geq 1$, $\pi_k(B\bar{\Gamma}_q) = 0$ for all $0 \leq k \leq q+1$.*

COROLLARY 3.4. *The space $\mathcal{S}(M)$ is simply connected.*

Let X be a connected topological space, and assume that $\pi_1(X)$ is a perfect group. Let X^+ denote the space obtained by applying the Quillen $+$ -construction to X ; see [45] or [143, Chapter IV]. Then X^+ is a connected space with a natural map $Q: X \rightarrow X^+$ which induces isomorphisms on all homology groups. The space X^+ with these properties is unique up to weak homotopy equivalence.

Given a smooth connected manifold M of dimension q , the adjunct map $\tau: \overline{B\text{Diff}}_c(M) \rightarrow \mathcal{S}(M)$ is a homology isomorphism, and $\mathcal{S}(M)$ is simply connected. Thus, there is a weak homotopy equivalence between $\mathcal{S}(M)$ and $\{\overline{B\text{Diff}}_c(M)\}^+$, and so these spaces have isomorphic homotopy groups. We thus have the equality stated in Definition 1.2,

$$(9) \quad \mathcal{K}_\ell(M) = \pi_\ell \left(\left\{ \overline{B\text{Diff}}_c(M) \right\}^+ \right) \cong \pi_\ell(\mathcal{S}(M)), \text{ for all } \ell \geq 1.$$

We thus can formulate the problem addressed in this work:

PROBLEM 3.5. *For a smooth connected manifold M of dimension q , construct classes which map to non-zero classes in the image of the Hurewicz map*

$$(10) \quad h_* : \pi_\ell(\mathcal{S}(M)) \longrightarrow H_\ell(\mathcal{S}(M); \mathbb{Q}) .$$

We will first introduce techniques for constructing non-trivial classes in $\mathcal{K}_*(M)$, and then study which of these classes pair non-trivially with cohomology classes in $H^*(\mathcal{S}(M); \mathbb{Q})$. The second step is critical, as the results of Section 7 imply that the groups $\pi_\ell(\mathcal{S}(M))$ are uncountable for an infinite range of $\ell \rightarrow \infty$, but only some of these classes are known to yield non-trivial homology classes. Thus, the refined strategy for constructing non-trivial classes in $H_*(B\text{Diff}(M); \mathbb{Q})$ is to first construct elements of $\mathcal{K}_*(M)$ using techniques from rational homotopy theory and properties of the homotopy type of $B\overline{\Gamma}_q$, then showing the secondary classes in $H^*(B\overline{\Gamma}_q; \mathbb{R})$ detect these classes. Theorem 3.1 implies these calculations yield non-trivial homology classes defined by flat $\text{Diff}_c^+(M)$ -bundles, and as discussed in Section 8, one can show that some of these classes are nontrivial in the image of the map $\iota_* : H_*(B\overline{\text{Diff}}(M); \mathbb{Q}) \rightarrow H_*(B\text{Diff}_c^+(M)^\delta; \mathbb{Q})$.

4. CALCULATIONS OF THE K-THEORY GROUPS

We first make some general remarks about the calculation of the groups $\mathcal{K}_*(M)$, then give three methods for constructing classes homotopy classes in $\pi_*(\mathcal{S}(M))$ which are detected by cohomology classes in $H^*(\mathcal{S}(M); \mathbb{R})$. A major difficulty with the study of $\mathcal{S}(M)$ is that there is very incomplete knowledge of the topological type of the space $B\overline{\Gamma}_q$, so one typically works with an approximation to the fiber group $B\overline{\Gamma}_q$ which suffices to construct non-trivial classes in $H_*(\mathcal{S}(M); \mathbb{Q})$.

If the classifying map $\nu_M : M \rightarrow BSO(q)$ is not trivial, then the calculation of the cohomology of the mapping space $\mathcal{S}(M)$ can be quite complicated. The works by Haefliger [51, 53], by Shibata [126, 127] and by Tsujisita [141, 142] use rational homotopy techniques to study analogs of $\mathcal{S}(M)$.

If the tangent bundle to M is trivial, so the map classifying ν_M can be assumed to be the constant map to the basepoint $w_0 \in BSO(q)$, then $\mathcal{S}(M) \cong \text{Maps}_c(M, B\overline{\Gamma}_q)$ which greatly simplifies the study of $\mathcal{S}(M)$. The study of a space of maps is a classical topic in topology, as discussed by Cohen and Taylor in [24, 25] and by Bendersky and Gitler in [5]; see also Félix and Tanré [31] which compares these two approaches.

The case when M is an open manifold, so not compact, then there is a weak homotopy equivalence $\mathcal{S}(M) \cong \mathcal{S}(\widehat{M}, x_\infty)$ where \widehat{M} denotes the 1-point compactification of M , and x_∞ is the point at infinity, which is taken as the basepoint. Then $\mathcal{S}_\infty(\widehat{M}, x_\infty)$ denotes the sections which map the basepoint x_∞ to the basepoint $\sigma_M \in B\Gamma_q^+$. That is, for a compact space X , the Γ_q^+ structure induced on $X \times M$ by a map $f : X \rightarrow \mathcal{S}(\widehat{M}, x_\infty)$ is that of the horizontal product foliation outside of a compact subset of $X \times M$. In the case when $M = \mathbb{R}^q$, which was discussed in the introduction, we obtain $\mathcal{S}(\mathbb{R}^q) \cong \text{Maps}(\mathbb{S}^q, B\overline{\Gamma}_q, x_\infty) \cong \Omega^q B\overline{\Gamma}_q$. For other manifolds, such as when M is the interior of a compact manifold with boundary, then other interesting calculations arise.

We now give some methods for analyzing the space $\mathcal{S}(M)$. Given a non-trivial class ϕ in $H_\pi^*(B\Gamma_q^+; \mathbb{R})$ or $H_\pi^*(B\overline{\Gamma}_q; \mathbb{R})$, we construct classes in $\mathcal{K}_{*-k}(M)$ which are detected by cohomology classes in $H^{*-k}(\mathcal{S}(M); \mathbb{R})$, where the degree is decreased by an integer k which depends the construction.

Recall the following standard notion.

DEFINITION 4.1. *Let X be a topological space. A class $\phi \in H^n(X; \mathbb{R})$ is said to be spherical if there exists a map $f : \mathbb{S}^n \rightarrow X$ such that $f^*(\phi) \in H^n(\mathbb{S}^n; \mathbb{R})$ is non-trivial. Let $H_\pi^*(X; \mathbb{R})$ denote the subspace of spherical cohomology classes.*

Here is the first method of constructing classes.

THEOREM 4.2. *Let M be a connected oriented manifold of dimension q . Let $\phi \in H_\pi^n(B\Gamma_q^+; \mathbb{R})$ be a spherical class, for $n > 2q$. Then for $\tilde{\alpha} = \int_M \tilde{\nu}^*(\phi) \in H^{n-q}(\mathcal{S}(M); \mathbb{R})$ there exists $[f_0] \in \mathcal{K}_{n-q}(M)$ such that the evaluation $\langle \tilde{\alpha}, [f_0] \rangle \neq 0$.*

Proof. Let \mathcal{F} denote the product foliation on $N = \mathbb{S}^{n-q} \times M$ with leaves $\mathbb{S}^{n-q} \times \{y\}$ for $y \in M$. Let $\tilde{\nu}_{\mathcal{F}}: N \rightarrow B\Gamma_q^+$ classify the foliation \mathcal{F} , which is a lifting of the map $\nu_{\mathcal{F}}: N \rightarrow BSO(q)$ classifying the normal bundle of \mathcal{F} . We construct a compactly supported perturbation of $\tilde{\nu}$.

Choose basepoints $x_0 \in M$ and $\theta_{n-q} \in \mathbb{S}^{n-q}$, and set $N = \mathbb{S}^{n-q} \times M$ with basepoint $(\theta_{n-q}, x_0) \in N$. Set $w_0 = \nu_{\mathcal{F}}(\theta_{n-q}, m_0) \in BSO(q)$ and set the basepoint $w = \tilde{\nu}_M(\theta_{n-q}, m_0) \in B\Gamma_q^+$. Let $\iota_w: B\bar{\Gamma}_q \rightarrow B\Gamma_q^+$ denote the inclusion of the fiber over the basepoint w_0 . Then $w \in B\bar{\Gamma}_q \subset B\Gamma_q^+$.

Let $f': \mathbb{S}^n \rightarrow B\Gamma_q^+$ be a map such that $\langle \phi, \hat{f}' \rangle \neq 0$. The group $\pi_n(BSO(q)) \otimes \mathbb{Q} = 0$ as $n > 2q$, so there exists a multiple of f' in the image of the map $\pi_n(B\bar{\Gamma}_q) \rightarrow \pi_n(B\Gamma_q^+)$. Thus we may assume that $f' = \iota_w \circ f$ for $f: \mathbb{S}^n \rightarrow B\bar{\Gamma}_q$ and that f is a pointed map, with $f(\theta_{n-q}, m_0) = w$.

Choose a contractible disk neighborhood $\mathbb{D}^n \cong U \subset N$ with the basepoint $(\theta_{n-q}, m_0) \in \partial U$, where ∂U is homeomorphic to \mathbb{S}^{n-1} . Let N/U denote the quotient space of N by the set U , then N and N/U are homeomorphic. Note that the restriction of $\tilde{\nu}_{\mathcal{F}}|_U: U \rightarrow B\Gamma_q^+$ is homotopic to the constant map to the basepoint $w \in B\Gamma_q^+$. Also, the quotient $U/\partial U \cong \mathbb{S}^n$ so collapsing the boundary ∂U yields a map $b: N \rightarrow \mathbb{S}^n \vee N$. In colloquial terms, this process ‘‘bubbles off’’ a copy of \mathbb{S}^n from N .

Define $\hat{f} = (\hat{f}_1 \vee \hat{f}_2) \circ b: N \rightarrow B\Gamma_q^+$ as the join of two maps, where $\hat{f}_1 = \iota_w \circ f: \mathbb{S}^n \rightarrow B\bar{\Gamma}_q \rightarrow B\Gamma_q^+$ and $\hat{f}_2 = \tilde{\nu}_{\mathcal{F}} \circ \pi: N \rightarrow N/U \cong N \rightarrow B\Gamma_q^+$. Calculate the pairing of the cohomology class ϕ with the homology class of \hat{f} :

$$(11) \quad \langle \phi, [\hat{f}] \rangle = \langle \phi, [\hat{f}_1] \rangle + \langle \phi, [\hat{f}_2] \rangle = \langle \phi, [\hat{f}_1] \rangle = \langle \iota_w^*(\phi), [f] \rangle \neq 0,$$

where we use that $\hat{f}_2: N \rightarrow B\Gamma_q^+$ is homotopic to a map into the q -skeleton, so the evaluation of the class ϕ vanishes as its degree $n > q$. Thus the pull-back class $\hat{f}^*(\phi) \in H^n(N; \mathbb{R})$ is non-vanishing. The map $\hat{f}: N = \mathbb{S}^{n-q} \times M \rightarrow B\Gamma_q^+$ is a lift of the map $\nu_M: N \rightarrow BSO(q)$, so induces maps $f_0: \mathbb{S}^{n-q} \rightarrow \mathcal{S}(M)$ and $f_1: \mathbb{S}^{n-q} \times M \rightarrow \mathcal{S}(M) \times M$. We then have the commutative diagram in cohomology, where the maps \int_M are integration over the fiber (see Appendix A.2):

$$\begin{array}{ccccc} H^n(\mathbb{S}^{n-q} \times M; \mathbb{R}) & \xleftarrow{f_1^*} & H^n(\mathcal{S}(M) \times M; \mathbb{R}) & \xleftarrow{\tilde{\nu}^*} & H^n(B\Gamma_q^+; \mathbb{R}) \\ & & \downarrow \int_M & & \\ H^{n-q}(\mathbb{S}^{n-q}; \mathbb{R}) & \xleftarrow{f_0^*} & H^{n-q}(\mathcal{S}(M); \mathbb{R}) & & \end{array}$$

Then $\hat{f} = \tilde{\nu} \circ f_1$ so $\hat{f}^*(\phi) \neq 0$ implies that $f_0^* \int_M \tilde{\nu}^*(\phi) = \int_M f_1^* \circ \tilde{\nu}^*(\phi) \neq 0$. Set $\tilde{\alpha} = \int_M \tilde{\nu}^*(\phi) \in H^{n-q}(\mathcal{S}(M); \mathbb{R})$ which then pairs nontrivially with the class $[f_0] \in \pi_{n-q}(\mathcal{S}(M))$. \square

REMARK 4.3. The construction in the above proof is equivalent to a well-known construction of Thurston which first appeared in [136]. Given a basepoint preserving map $f: \mathbb{S}^n \rightarrow B\bar{\Gamma}_q$, it determines a delooped map $f_0: \mathbb{S}^{n-q} \rightarrow \Omega^q B\bar{\Gamma}_q$ and a corresponding map $f_1: \mathbb{S}^{n-q} \times \mathbb{R}^q \rightarrow B\bar{\Gamma}_q$ which maps the complement of a compact subset to the basepoint of $B\bar{\Gamma}_q$. Choose a basepoint $x_0 \in M$, and choose disk neighborhood $x_0 \in U \subset M$ and a diffeomorphism $\psi: \mathbb{R}^q \cong \mathbb{D}^q \rightarrow U \subset M$. Then ψ induces a map $f_\psi: \mathbb{S}^{n-q} \times \mathbb{R}^q \rightarrow \mathcal{S}(M) \times M$ which is just the map f_1 in the proof.

For the next results, we assume that the tangent bundle to M is framed, so the classifying map $\nu_M: M \rightarrow BSO(q)$ is the constant map, and the framing defines a lifting $\tilde{\nu}_M: M \rightarrow B\bar{\Gamma}_q$ to the fiber. Then $\mathcal{S}(M) = \text{Maps}_c(M, B\bar{\Gamma}_q)$ is a space of maps with basepoint $\tilde{\nu}_M$, and there are additional tricks for constructing non-trivial homotopy classes in $\mathcal{K}_*(M)$. We first give a construction which requires the additional assumption that M is compact.

THEOREM 4.4. *Let M be a compact connected oriented manifold of dimension q , and assume the tangent bundle TM is trivial. Let $\phi \in H_\pi^n(B\bar{\Gamma}_q; \mathbb{R})$ be a spherical class, for $n \geq q + 2$. For $\tilde{\beta} = \sigma^* \circ \tilde{\nu}^*(\phi) \in H^n(\mathcal{S}(M); \mathbb{R})$ there exists $[g_0] \in \mathcal{K}_n(M)$ such that the evaluation $\langle \tilde{\beta}, [g_0] \rangle \neq 0$.*

Proof. Let $\phi \in H^n(B\bar{\Gamma}_q; \mathbb{R})$ be a spherical class, which pairs non-trivially with the homology class defined by a map $f: \mathbb{S}^n \rightarrow B\bar{\Gamma}_q$. Assume that f maps $\theta_n \in \mathbb{S}^n$ to $w \in B\bar{\Gamma}_q$. Let $\hat{f}: N = \mathbb{S}^n \times M \rightarrow B\bar{\Gamma}_q$ be the induced map on the product, which is a compact perturbation of $\tilde{\nu}_M$ as M is compact.

Then $\hat{f}^*(\phi) \in H^n(\mathbb{S}^n \times M; \mathbb{R})$ is non-vanishing, and supported on the first factor of the decomposition $H^*(\mathbb{S}^n \times M; \mathbb{R}) \cong H^*(\mathbb{S}^n; \mathbb{R}) \otimes H^*(M; \mathbb{R})$. The map \hat{f} is a lift of the constant map $\nu_{\mathcal{F}}: N \rightarrow BSO(q)$, and as M is compact, \hat{f} induces maps $f_0: \mathbb{S}^{n-q} \rightarrow \mathcal{S}(M)$ and $f_1: \mathbb{S}^{n-q} \times M \rightarrow \mathcal{S}(M)$.

Let $\sigma_1: \mathbb{S}^n \rightarrow \mathbb{S}^n \times \{x_0\} \subset \mathbb{S}^n \times M$ be the inclusion of the first factor. Then we have a commutative diagram in cohomology:

$$\begin{array}{ccccc} H^n(\mathbb{S}^n \times M; \mathbb{R}) & \xleftarrow{f_1^*} & H^n(\mathcal{S}(M) \times M; \mathbb{R}) & \xleftarrow{\tilde{\nu}^*} & H^n(B\bar{\Gamma}_q; \mathbb{R}) \\ \downarrow \sigma_1^* & & \downarrow \sigma_1^* & & \\ H^n(\mathbb{S}^n; \mathbb{R}) & \xleftarrow{f_0^*} & H^n(\mathcal{S}(M); \mathbb{R}) & & \end{array}$$

Then $\hat{f} = \tilde{\nu} \circ f_1$ so $\hat{f}^*(\phi) \neq 0$ implies $f_0^* \circ \sigma_1^* \circ \tilde{\nu}^*(\phi) = \sigma_1^* \circ f_1^* \circ \tilde{\nu}^*(\phi) \neq 0$. Set $\tilde{\beta} = \sigma_1^* \circ \tilde{\nu}^*(\phi) \in H^{n-q}(\mathcal{S}(M); \mathbb{R})$ which then pairs nontrivially with the class $[f_0] \in \pi_n(\mathcal{S}(M))$. \square

REMARK 4.5. Note that the construction of the classes $[f_0]$ and $\tilde{\beta}$ depends on the existence of the classifying map $\hat{f}: N = \mathbb{S}^n \times M \rightarrow B\Gamma_q^+$ that has image in the fiber $B\bar{\Gamma}_q$. When the tangent bundle of M is not trivial, the map \hat{f} has to cover the classifying map $\nu_{\mathcal{F}}: N \rightarrow BSO(q)$ which is not trivial. Thus, the fiber over the image of $\nu_{\mathcal{F}}$ need not be constant, and it is no longer obvious how to construct the lifted map \hat{f} . In fact, the twisting of the fiber over $\nu_{\mathcal{F}}$ may result in a twisting of the chains in $B\Gamma_q^+$ which may yield a non-trivial differential in the spectral sequence for the fibration $B\bar{\Gamma}_q \rightarrow B\Gamma_q^+ \rightarrow BSO(q)$ killing an extension of the class $\phi \in H^n(B\bar{\Gamma}_q; \mathbb{R})$ to $H^n(B\Gamma_q^+; \mathbb{R})$.

When X is not compact, recall that \hat{X} denotes the 1-point compactification of X , and otherwise $\hat{X} = X$. For the next constructions, we use the following notion.

DEFINITION 4.6. Let X be a connected topological space. A homology class $C \in H_k(\hat{X}; \mathbb{Z})$ is said to be co-spherical if there exists a map $\xi: \hat{X} \rightarrow \mathbb{S}^k$ such that $\xi_*(C) \in H_k(\mathbb{S}^k; \mathbb{Z})$ is non-trivial. Let $H_*^\pi(\hat{X}; \mathbb{Z})$ denote the subgroup of co-spherical classes.

The fundamental class of M is co-spherical, as seen by the mapping $\xi: M \rightarrow \mathbb{S}^q$ of M to its quotient by its $(q-1)$ -skeleton, and if M is non-compact, composed with the collapse of a neighborhood of infinity to the basepoint, which is then homotopy equivalent to \mathbb{S}^q . This map was the key to the construction in the proof of Theorem 4.2.

The next construction uses a combination of the ideas behind the proofs of Theorems 4.2 and 4.4. We assume there is given a co-spherical class $[C] \in H_k^\pi(\hat{M}; \mathbb{Z})$ for some $0 < k < q$, and use this to make a compact perturbation of the base map $\tilde{\nu}_M$. However, the perturbation is no longer assumed to be on a contractible disc in $N = \mathbb{S}^k \times M$, so an additional assumption is needed on the classifying map for the tangent bundle over the perturbation. The simplest assumption is that M has trivialized tangent bundle, but it suffices to assume that the tangent bundle is trivial over the compact support of the perturbation. For example, for M compact, the homology $H_1(M; \mathbb{Q})$ is generated by co-spherical classes, as $H^1(M; \mathbb{Z}) \cong [M, K(\mathbb{Z}, 1)] \cong [M, \mathbb{S}^1]$, and a tubular neighborhood of each generator $\mathbb{S}^1 \subset M$ will have trivial tangent bundle. We formulate the next result with the assumption that TM is trivial, and leave extensions of the method to more general cases as exercises. Theorem 10.11 illustrates the application of these remarks.

THEOREM 4.7. Let M be a connected oriented manifold of dimension q , and assume the tangent bundle TM is trivial. Let $\phi \in H_\pi^n(B\bar{\Gamma}_q; \mathbb{R})$ be a spherical class, for $n \geq q+2$. Given a co-spherical class $C \in H_k^\pi(\hat{M}; \mathbb{Z})$, there is a class $\tilde{\gamma}_C = \int_C \tilde{\nu}^*(\phi) \in H^{n-k}(\mathcal{S}(M); \mathbb{R})$ and $[f_C] \in \mathcal{K}_{n-k}(M)$ such that the evaluation $\langle \tilde{\gamma}_C, [f_C] \rangle \neq 0$.

Proof. Let $f: \mathbb{S}^n \rightarrow B\bar{\Gamma}_q$ whose homology class pairs non-trivially with $\phi \in H^n(B\bar{\Gamma}_q; \mathbb{R})$.

Let $\xi: \widehat{M} \rightarrow \mathbb{S}^k$ be such that $\xi_*(C) \in H^k(\mathbb{S}^k; \mathbb{Z})$ is non-trivial. Note that ξ maps the complement of some compact set $K \subset M$ to the basepoint $\theta_k \in \mathbb{S}^k$. The classifying map $\nu_M: M \rightarrow BSO(q)$ is homotopic to a constant, and the fiber $B\bar{\Gamma}_q$ is $(q+1)$ -connected, so we can assume that ν_M is constant on some open neighborhood of K .

Let $N = \mathbb{S}^{n-k} \times M$. Let $\wedge: \mathbb{S}^{n-k} \times \mathbb{S}^k \rightarrow \mathbb{S}^n$ denote the smash product, then let

$$(12) \quad \widehat{f}_C: N = \mathbb{S}^{n-k} \times M \xrightarrow{id \times \xi} \mathbb{S}^{n-k} \times \mathbb{S}^k \xrightarrow{\wedge} \mathbb{S}^n \xrightarrow{f} B\bar{\Gamma}_q.$$

Then $\widehat{f}_C^*(\phi) \in H_c^n(\mathbb{S}^{n-k} \times M; \mathbb{R}) \cong H^{n-k}(\mathbb{S}^{n-k} \times M, x_0; \mathbb{R})$ is non-vanishing in the cohomology with compact supports, and pairs non-trivially with the cycle defined by $(id \times \xi): \mathbb{S}^{n-k} \times C \rightarrow \mathbb{S}^{n-k} \times \widehat{M}$. Let $f_C: \mathbb{S}^{n-k} \rightarrow \mathcal{S}(M)$ denote the adjoint to \widehat{f}_C .

Let $\int_C: H^n(\mathbb{S}^{n-k} \times M; \mathbb{R}) \rightarrow H^{n-k}(\mathbb{S}^{n-k}; \mathbb{R})$ denote the integration over the cycle C , which is just the slant product with the homology class C followed by projection to $H^{n-k}(\mathbb{S}^{n-k}; \mathbb{R})$. Then as before, we have the commutative diagram in cohomology:

$$\begin{array}{ccccc} H_c^n(\mathbb{S}^{n-k} \times M; \mathbb{R}) & \xleftarrow{(f_C \times id)^*} & H_c^n(\mathcal{S}(M) \times M; \mathbb{R}) & \xleftarrow{\tilde{\nu}^*} & H^n(B\bar{\Gamma}_q; \mathbb{R}) \\ \downarrow \int_C & & \downarrow \int_C & & \\ H^{n-k}(\mathbb{S}^{n-k}; \mathbb{R}) & \xleftarrow{f_C^*} & H^{n-k}(\mathcal{S}(M); \mathbb{R}) & & \end{array}$$

As $\widehat{f}_C^* = (f_C \times id)^* \circ \tilde{\nu}^*$, we then have $f_C^* \circ \int_C \tilde{\nu}^*(\phi) = \int_C \widehat{f}_C^* \phi \neq 0$.

Set $\tilde{\gamma}_C = \int_C \tilde{\nu}^*(\phi) \in H^{n-k}(\mathcal{S}(M); \mathbb{R})$ which then pairs nontrivially with $[f_C] \in \pi_{n-k}(\mathcal{S}(M))$. \square

5. SECONDARY CHARACTERISTIC CLASSES

The techniques in the previous section for constructing classes in $\mathcal{K}_*(M)$ assume there is a spherical class $\phi \in H_\pi^n(B\Gamma_q^+; \mathbb{R})$. The secondary classes for smooth foliations provides the canonical (and essentially only) approach to showing the existence of spherical classes for $B\Gamma_q^+$ for degrees $n > 2q$. We give a brief introduction to their construction. Lawson in his survey [88] gives a more extensive discussion of the secondary invariants and their properties.

The normal bundle Q to a smooth-foliation \mathcal{F} , when restricted to a leaf L_x of \mathcal{F} , has a natural flat connection ∇^{L_x} defined by the leafwise parallel transport on Q restricted to L_x . An *adapted connection* $\nabla^{\mathcal{F}}$ on $Q \rightarrow M$ is any connection whose restrictions to leaves equals this natural flat connection. An adapted connection need not be flat over M . The connection data provided by $\nabla^{\mathcal{F}}$ can be thought of as a ‘‘linearization’’ of the normal structure to \mathcal{F} along the leaves. Thus, $\nabla^{\mathcal{F}}$ captures aspects of the data provided by the Haefliger groupoid $\Gamma_{\mathcal{F}}^r$ of \mathcal{F} – it is a ‘‘partial linearization’’ of the highly nonlinear data which defines the homotopy type of $B\Gamma_{\mathcal{F}}$. In this section, we discuss the applications of this partial linearization to the study of the space $B\Gamma_q$.

The seminal observation was made by Bott around 1970. The cohomology ring $H^*(BO(q); \mathbb{R}) \cong \mathbb{R}[p_1, \dots, p_k]$ where $2k \leq q$, and p_j has graded degree $4j$.

THEOREM 5.1 (Bott Vanishing [9]). *Let \mathcal{F} be a codimension- q , C^2 -foliation. Let $\nu_Q: M \rightarrow BO(q)$ be the classifying map for the normal bundle Q . Then $\nu_Q^*: H^\ell(BO(q); \mathbb{R}) \rightarrow H^\ell(M; \mathbb{R})$ is the trivial map for $\ell > 2q$.*

Morita observed in [103, Section 3] that there exists a codimension-2 smooth foliation on a closed 4-manifold M for which $p_1(Q) = \nu_Q^*(p_1) \in H^4(M; \mathbb{R})$ is non-zero. By taking products of this example, one observes that the classes $\nu^*(p_1^k) \in H^{4k}(B\Gamma_q; \mathbb{R})$ are non-vanishing for $4k \leq 2q$. Moskowitiz gave in [110] an alternate construction of a compact 4-manifold with a trivial plane field that has $p_1(Q) \neq 0$ for its normal bundle. The existence of the codimension-2 foliation in the examples of both Morita

and Moskowicz uses [137, Theorem 1] of Thurston to construct the foliation on the manifold. A simplified version of Thurston's proof for the case of 2-dimensional foliations is given by Mitsumatsu and Vogt in their work [100, Theorem 1.1], which is more "elementary" in its methods. In any case, it seems there is no explicit construction of a C^2 -foliation for which $\nu_Q^*: H^\ell(BO(q); \mathbb{R}) \rightarrow H^\ell(M; \mathbb{R})$ is non-trivial in the range $q < \ell \leq 2q$.

It is a remarkable observation that Theorem 5.1 is false for integral coefficients:

THEOREM 5.2 (Bott-Heitsch [12]). *The universal normal bundle map,*

$$(13) \quad \nu^*: H^\ell(BO(q); \mathbb{Z}) \rightarrow H^\ell(B\Gamma_q; \mathbb{Z})$$

is injective for all $\ell \geq 0$.

Theorem 5.2 implies that for $q \geq 2$ and $r \geq 2$, the space $B\Gamma_q$ does not have the homotopy type of a finite type CW complex. More is true, that for all $\ell > q/2$, a CW model for $B\Gamma_q$ must have infinitely many cells in dimension $4\ell - 1$.

We next recall the construction of the secondary characteristic classes for C^2 -foliations. The key observation is that the Bott Vanishing Theorem holds at the level of the differential forms representing the characteristic classes of the normal bundle, and not just for their cohomology classes.

Denote by $I(\mathfrak{gl}(q, \mathbb{R}))$ the graded ring of adjoint-invariant polynomials on the Lie algebra $\mathfrak{gl}(q, \mathbb{R})$ of the real general linear group $GL(q, \mathbb{R})$. As a ring, $I(\mathfrak{gl}(q, \mathbb{R})) \cong \mathbb{R}[c_1, c_2, \dots, c_q]$ is a polynomial algebra on q generators, where the i^{th} -Chern polynomial c_i has graded degree $2i$. Let $I(\mathfrak{gl}(q, \mathbb{R}))^{(q+1)}$ denote the ideal of polynomials of degree greater than q , and introduce the quotient ring,

$$I_q \equiv I(\mathfrak{gl}(q, \mathbb{R}))_q = I(\mathfrak{gl}(q, \mathbb{R})) / I(\mathfrak{gl}(q, \mathbb{R}))^{(q+1)} \cong \mathbb{R}[c_1, c_2, \dots, c_q]_{2q}$$

which is isomorphic to a polynomial ring truncated in graded degrees larger than $2q$. Associate to each generator c_i the closed $2i$ -form $c_i(\Omega(\nabla^{\mathcal{F}})) \in \Omega_{deR}^{2i}(M)$, so that one obtains the Chern homomorphism $\Delta^{\mathcal{F}}: \mathbb{R}[c_1, c_2, \dots, c_q] \rightarrow \Omega_{deR}^{ev}(M)$.

THEOREM 5.3 (Strong Bott Vanishing [10]). *Let \mathcal{F} be a codimension- q , C^2 -foliation, and $\nabla^{\mathcal{F}}$ an adapted connection on the normal bundle Q . Then for any polynomial $c_J \in \mathbb{R}[c_1, c_2, \dots, c_q]$ of graded degree $\deg(c_J) > 2q$, the Chern form $c_J(\Omega(\nabla^{\mathcal{F}}))$ is identically zero. Thus, there is an induced map of differential graded algebras:*

$$(14) \quad \Delta^{\mathcal{F}}: \mathbb{R}[c_1, c_2, \dots, c_q]_{2q} \longrightarrow \Omega_{deR}^{ev}(M) .$$

Associated to the DGA map $\Delta^{\mathcal{F}}$ is a map of minimal models, whose homotopy class is itself an invariant under concordance for foliations, and determines additional invariants for foliations [65, 66].

Now assume that the normal bundle to the foliation \mathcal{F} on M is trivial. The choice of a framing of Q , denoted by s , induces an isomorphism $Q \cong M \times \mathbb{R}^q$. Let ∇^s denote the connection for the trivial bundle, then the curvature forms of ∇^s vanish identically. It follows that the transgression form $y_i = Tc_i(\nabla^{\mathcal{F}}, \nabla^s) \in \Omega_{deR}^{2i-1}(M)$ of c_i satisfies the equation $dy_i = c_i(\Omega(\nabla^{\mathcal{F}}))$. Consider the DGA complex

$$(15) \quad W_q = \Lambda(y_1, y_2, \dots, y_q) \otimes \mathbb{R}[c_1, c_2, \dots, c_q]_{2q} ,$$

where the differential is defined by $d(h_i \otimes 1) = 1 \otimes c_i$.

The data $(\mathcal{F}, s, \nabla^{\mathcal{F}})$ determine a map of differential algebras $\Delta^{\mathcal{F}, s}: W_q \rightarrow \Omega^*(M)$. The induced map in cohomology, $\Delta^{\mathcal{F}, s}: H^*(W_q) \rightarrow H^*(M)$, depends only on the homotopy class of the framing s and the framed concordance class of \mathcal{F} , and moreover, this construction is functorial.

THEOREM 5.4. *There is a well-defined universal characteristic map*

$$(16) \quad \Delta: H^*(W_q) \rightarrow H^*(B\bar{\Gamma}_q; \mathbb{R})$$

Given a codimension- q foliation \mathcal{F} with framing s , the classifying map $\Delta^{\mathcal{F}, s}: H^(W_q) \rightarrow H^*(M; \mathbb{R})$ factors through the universal map:*

$$\begin{array}{ccc}
& & H^*(B\bar{\Gamma}_q; \mathbb{R}) \\
& \nearrow \Delta & \downarrow h_{\mathcal{F}}^* \\
H^*(W_q) & \xrightarrow{\Delta^{\mathcal{F}, s}} & H^*(M; \mathbb{R})
\end{array}$$

The *Vey basis* for $H^*(W_q)$ was given in [44]:

PROPOSITION 5.5. *The following set of monomials in W_q form a basis for $H^*(W_q)$:*

$$\begin{aligned}
(17) \quad y_I c_J \quad \text{such that} \quad & I = (i_1, \dots, i_s) \text{ with } 1 \leq i_1 < \dots < i_s \leq q \\
& J = (j_1, \dots, j_\ell) \text{ with } j_1 \leq \dots \leq j_\ell, \quad j_1 + \dots + j_\ell \leq q \\
& i_1 + j_1 + \dots + j_\ell \geq q + 1, \quad i_1 \leq j_1.
\end{aligned}$$

Now consider a smooth foliation \mathcal{F} of a manifold M , with no assumption that its normal bundle is trivial. In place of the flat connection ∇^s associated to a framing of Q , let ∇^g be the connection on Q associated to a Riemannian metric on Q . Then the closed forms $c_{2i}(\Omega(\nabla^g)) \in \Omega_{deR}^{4i}(M)$ need not vanish, and in fact their cohomology classes define the Pontrjagin forms for the normal bundle. On the other hand, the forms $c_{2i+1}(\Omega(\nabla^g)) \in \Omega_{deR}^{4i+2}(M)$ vanish by the skew symmetry of the curvature matrix $\Omega(\nabla^g)$. Thus, we can repeat the above construction of a characteristic map for the subcomplex W_q of W_q defined by

$$(18) \quad WO_q = \Lambda(y_1, y_3, \dots, y_{q'}) \otimes \mathbb{R}[c_1, c_2, \dots, c_q]_{2q},$$

where $q' \leq q$ is the largest odd integer with $q' \leq q$.

The data (\mathcal{F}, ∇^g) determine a map of differential algebras $\Delta^{\mathcal{F}}: WO_q \rightarrow \Omega^*(M)$. The induced map in cohomology, $\Delta^{\mathcal{F}}: H^*(WO_q) \rightarrow H^*(M)$, depends only on the framed concordance class of \mathcal{F} , but not on the choice of the Riemannian metric on Q . It follows that there is a well-defined universal characteristic map $\Delta: H^*(WO_q) \rightarrow H^*(B\bar{\Gamma}_q; \mathbb{R})$.

The *Vey basis* for $H^*(WO_q)$ was given in [44]:

PROPOSITION 5.6. *The following set of monomials in WO_q form a basis for $H^*(WO_q)$:*

$$\begin{aligned}
(19) \quad y_I c_J \quad \text{such that} \quad & I = (i_1, \dots, i_s) \text{ with } 1 \leq i_1 < \dots < i_s \leq q, \text{ each } i_k \text{ is odd} \\
& J = (j_1, \dots, j_\ell) \text{ with } j_1 \leq \dots \leq j_\ell, \quad j_1 + \dots + j_\ell \leq q \\
& i_1 + j_1 + \dots + j_\ell \geq q + 1, \quad i_1 \leq \text{any odd } j_k.
\end{aligned}$$

Note that $H^{2q+1}(WO_q) \cong H^{2q+1}(W_q)$ when q is even.

The *secondary characteristic classes* of foliations are defined as those in the image of the maps Δ for degree at least $2q + 1$, in either the framed or unframed cases. There are special subclasses of the secondary invariants corresponding to special properties, which we mention:

DEFINITION 5.7. *Let $h_I \wedge c_J$ be an element of the Vey basis for $H^*(WO_q)$ or $H^*(W_q)$.*

- $h_I \wedge c_J$ is said to be a generalized Godbillon-Vey class.
- $h_I \wedge c_J$ is said to be residual if $j_1 + \dots + j_\ell = q$.
- $h_I \wedge c_J$ is said to be rigid if $i_1 + j_1 + \dots + j_\ell \geq q + 2$.

The study of the image of the universal maps Δ have been the primary source of information about the (non-trivial) homotopy type of $B\bar{\Gamma}_q$ and $B\bar{\Gamma}_q$. The outstanding problem remains:

PROBLEM 5.8. *Show that the map $\Delta: H^*(W_q) \rightarrow H^*(B\bar{\Gamma}_q; \mathbb{R})$ is injective for $q \geq 1$.*

6. FOLIATED MANIFOLDS WITH NON-TRIVIAL SECONDARY CLASSES

We survey the literature on examples of foliated manifolds which are used to show the non-triviality of some of the universal classes in $H^*(B\bar{\Gamma}_q; \mathbb{R})$ and $H^*(B\bar{\Gamma}_q; \mathbb{R})$.

First, we recall the case for $q = 1$ where there $H^3(WO_1) \cong \mathbb{R}$ is generated by the class y_1c_1 , and for all other degrees, the cohomology vanishes. Let \mathcal{F} be an oriented codimension-one foliation of a manifold M , then $\Delta^{\mathcal{F}}(h_1) = \frac{1}{2\pi}\eta \in \Omega_{deR}^1(M)$ is the Reeb class, and so $\Delta^{\mathcal{F}}(h_1c_1) = \frac{1}{4\pi^2}\eta \wedge d\eta \in \Omega_{deR}^3(M)$ represents the Godbillon-Vey class [43].

THEOREM 6.1 (Godbillon-Vey [43]). *Let \mathcal{F} be a codimension-one, C^2 foliation on M with trivial normal bundle. Then the 3-form $\Delta^{\mathcal{F}}(h_1c_1)$ is closed, and its cohomology class $GV(\mathcal{F}) \in H^3(M; \mathbb{R})$ is independent of all choices. Moreover, $GV(\mathcal{F})$ depends only on the concordance class of \mathcal{F} .*

The paper [43] also included an example by Roussarie to show this class was non-zero. Soon afterwards, Thurston gave a construction in [135] of families of examples of foliations on the 3-sphere \mathbb{S}^3 , such that the $GV(\mathcal{F}) \in H^3(\mathbb{S}^3; \mathbb{Z}) \cong \mathbb{R}$ assumes a continuous range of real values. As a consequence:

THEOREM 6.2 (Thurston [135]). *The Godbillon-Vey class is a surjection $GV: \pi_3(B\Gamma_1) \rightarrow \mathbb{R}$.*

The appendix by Brooks to [15] gives a detailed explanation of the examples Thurston constructed. The Seminaire Bourbaki article by Ghys [42] is a basic reference for the properties of the Godbillon-Vey class; the author's survey [73] is a more recent update.

For a foliation \mathcal{F} of codimension $q > 1$ with normal bundle framing s , the secondary classes of \mathcal{F} are spanned by the images $\Delta^{\mathcal{F},s}(y_{Ic_J})$, where y_{Ic_J} satisfies (17). Without the assumption that the normal bundle is framed, we restrict to monomials y_{Ic_J} as in (19).

The original example of Roussarie [43], and its extensions to codimension $q > 1$, start with a semi-simple Lie group G . Choose closed subgroups $K \subset H \subset G$, with K compact. Then G/K is foliated by the left cosets of H/K . Choose a cocompact, torsion-free lattice $\Gamma \subset G$, then the foliation of G/K descends to a foliation \mathcal{F} on the compact manifold $M = \Gamma \backslash G/K$ which is a locally homogeneous space. The calculation of the secondary invariants for such foliations then follows from explicit calculations in Lie algebra cohomology, using Cartan's approach to the cohomology of homogeneous spaces. Examples of this type are studied in [2, 34, 80, 81, 82, 83, 117, 120, 121, 132, 133, 134, 145]. For example, Baker shows in [2]:

THEOREM 6.3. *Let $q = 2m > 4$. Then the set of residual classes*

$$\{h_1h_2h_{i_1} \cdots h_{i_k}c_1^q; h_1h_2h_{i_1} \cdots h_{i_k}c_1^{q-2}c_2 \mid 2 < i_1 < \cdots < i_k \leq m\}$$

in $H^(W_q)$ map under Δ to linearly independent classes in $H^*(B\bar{\Gamma}_q; \mathbb{R})$.*

The non-vanishing results of Kamber and Tondeur follow a similar format, but are more extensive, as given in Theorem 7.95 of [81] for example. The conclusion of all these approaches is to show that the universal map Δ is injective on various subspaces of $H^*(W_q)$ and $H^*(WO_q)$.

For the case of foliations with codimension $q = 2$, Rasmussen [122] modified the construction by Thurston of codimension-one foliations with varying Godbillon-Vey class. Thurston's construction used the weak-stable foliation of the geodesic flow on a compact Riemann surface (with boundary) of constant negative curvature. Rasmussen extended these ideas to the case of compact hyperbolic 3-manifolds. He showed there exists families $\{\mathcal{F}_\lambda \mid \lambda \in \mathbb{R}\}$ of smooth foliations in codimension-2 on oriented compact 5-manifolds M_1 and M_2 for which the secondary classes $\{h_1c_1^2, h_1c_2\}$ vary continuously and independently when evaluated on the fundamental classes of M_1 and M_2 .

For the case of foliations with codimension $q \geq 3$, Heitsch developed a residue theory for the residual secondary classes of smooth foliations [61, 62], and used this to construct extensive families of foliations whose secondary classes varied continuously [62, Theorems 6.1, 6.2, 6.3].

Together, these examples yield that for a fixed collection of classes in the image of Δ , there are continuous families of cycles $h_{\mathcal{F}_\lambda}: M \rightarrow B\Gamma_q^+$ or $h_{\mathcal{F}_\lambda, s}: M \rightarrow B\bar{\Gamma}_q$ such that the evaluation of these fixed secondary classes on the cycles defined by the \mathcal{F}_λ vary continuously. Thus, the groups $H_*(B\Gamma_q^+; \mathbb{Z})$ and $H_*(B\bar{\Gamma}_q; \mathbb{Z})$ must be a truly enormous.

There are two very remarkable results of a different nature, one for foliations of codimension 1 by Tsuboi, and the other for foliations of codimension 2 by Boullay, on the existence of divisible subgroups in the homology of $B\Gamma_q^+$.

THEOREM 6.4. [139, Theorem 1.2] $H_3(B\Gamma_1^+; \mathbb{Z})$ contains a subgroup isomorphic to \mathbb{R} .

THEOREM 6.5. [17, Corollary 4] $H_5(B\Gamma_2^+; \mathbb{Z})$ contains a subgroup isomorphic to \mathbb{R}^2 .

The work of Boullay uses the results on divisibility in algebraic K-groups to construct his examples. Nariman discusses this connection in [115].

As mentioned above, Morita constructed in [103] a compact 4-manifold M with a codimension 2 foliation that has non-trivial first Pontrjagin class $p_1 \in H^4(M; \mathbb{R})$. By taking products of the examples constructed by Tsuboi or Boullay, depending on whether the codimension q is odd or even, with an appropriate number of copies of this manifold, one obtains cycles in arbitrary codimension q that produce divisible subgroups in the homology of degree $2q + 1$. Using the same calculations as in [103], one obtains:

COROLLARY 6.6. For all $q \geq 1$, $H_{2q+1}(B\Gamma_q^+; \mathbb{Z})$ contains a subgroup isomorphic to \mathbb{R} .

Actually, much more can be shown using homotopy techniques, as discussed in [75].

The notion of *discontinuous invariants* for foliations was developed by Morita in his works [104, 106] (and see also [108, Chapter 3]) give an approach for the study of the variable foliation invariants, and is also analogous to the divisibility property for the integer homology groups. The universal Godbillon-Vey class $GV \in H^3(B\Gamma_1^+; \mathbb{R})$ defines a map $\overline{GV}: B\Gamma_1^+ \rightarrow K(\mathbb{R}^\delta, 3)$ where \mathbb{R}^δ denotes the group of reals with the discrete topology, and $K(\mathbb{R}^\delta, 3)$ is the Eilenberg-MacLane space in dimension 3 for \mathbb{R}^δ . While the homotopy groups of $K(\mathbb{R}^\delta, 3)$ are prescribed, the cohomology of this space is absolutely enormous. The image of $\overline{GV}^*: H^*(K(\mathbb{R}^\delta, 3); \mathbb{Z}) \rightarrow H^*(B\Gamma_1^+; \mathbb{Z})$ are the *discontinuous invariants* for codimension one foliations.

PROBLEM 6.7 (Morita [106]). Determine the image of \overline{GV}^* in $H^*(B\Gamma_1^+; \mathbb{Z})$.

There are an analogous constructions of discontinuous invariants in each codimension $q > 1$ as well, using the variable secondary classes to define maps to products of Eilenberg-MacLane spaces. The non-triviality of products of the discontinuous invariants is completely unknown for smooth foliations, though the work of Tsuboi in [140] shows there are non-trivial products for foliations which are transversally Lipschitz.

What is striking, looking back at the roughly 15 years between 1971 and 1986 during which the study of secondary invariants for foliations was most actively researched, is the many approaches and expositions of the subject, especially for example [2, 6, 7, 11, 13, 15, 10, 33, 34, 40, 41, 79, 79, 81, 82, 88, 108, 120, 121]. The notation in these papers is not consistent.

It is also striking how limited are the types of examples presented in these works. All of the “explicit constructions” of foliations in the literature with non-trivial secondary characteristic classes are either locally homogeneous, or deformations of locally homogeneous actions. Essentially, they are all generalizations and/or modifications of the original constructions of Roussarie [43] and Thurston [135] for the Godbillon-Vey class in codimension-one. Nothing is known about *explicit* constructions of examples with non-vanishing secondary classes beyond these “almost homogeneous” examples.

7. SPHERICAL SECONDARY CLASSES

In this section, we give a review the known results about the spherical secondary classes for the spaces $B\Gamma_q^+$ and $B\overline{\Gamma}_q$. A key tool is the Rational Hurewicz Theorem in A.1. We also use the *brace product* in Section A.3 to extend the non-triviality results for $B\overline{\Gamma}_q$ from degree $2q + 1$ to higher degrees. First, we give the prototypical construction of spherical cohomology classes.

THEOREM 7.1. Let $\phi \in H^{2q+1}(B\overline{\Gamma}_q; \mathbb{R})$ be a cohomology class such that there exists a continuous family of framed of codimension- q foliations $\{(\mathcal{F}_\lambda, s_\lambda) \mid \lambda \in \Lambda\}$ on a manifold Y , such that the pullback $\chi_{\mathcal{F}_\lambda}^*(\phi) \in H^{2q+1}(Y; \mathbb{R})$ is non-trivial, and varies continuously with λ if Λ is not an isolated set. Then for each $\lambda \in \Lambda$, there exists $[f_\lambda] \in \pi_{2q+1}(B\overline{\Gamma}_q)$ so that the pairing $\langle \phi, [f_\lambda] \rangle$ is non-vanishing and varies continuously with λ . That is, ϕ induces a map $\phi: \pi_{2q+1}(B\overline{\Gamma}_q) \rightarrow \mathbb{R}$ which is surjective.

Proof. First note that as the evaluation map $\phi: H_{2q+1}(B\bar{\Gamma}_q; \mathbb{Z}) \rightarrow \mathbb{R}$ is a homomorphism, if it is onto an open set, then it is onto all of \mathbb{R} .

Now let $\{F_\lambda \mid \lambda \in \Lambda\}$ be a continuous family of codimension- q foliations on Y . Suppose that the normal bundles to the foliations admit trivialisations s_λ that depend continuously on λ . Then there exists a continuous family of classifying maps for these foliations, denoted by $\chi_{\mathcal{F}_\lambda}: Y \rightarrow B\bar{\Gamma}_q$, where we suppress the dependence on the framing s_λ . Let $\phi \in H^{2q+1}(B\bar{\Gamma}_q; \mathbb{R})$ be a cohomology class such that the pull-back $\chi_{\mathcal{F}_\lambda}^*(\phi) \in H^\ell(Y; \mathbb{R})$ is non-trivial for $\lambda \in \Lambda$.

Since Y has the homotopy type of a CW complex, there is a CW complex $Y_{(q+1)}$ with all cells in dimension at least $q+2$, and a map $\xi: Y \rightarrow Y_{(q+1)}$ such that the map $H_{2q+1}(Y; \mathbb{Z}) \rightarrow H_{2q+1}(Y_{(q+1)}; \mathbb{Z})$ is injective. The space $Y_{(q+1)}$ is obtained by collapsing the cells of dimension less than $q+2$ in a CW decomposition of Y . As a consequence of Theorem 3.3 and basic obstruction theory, the classifying maps $\chi_{\mathcal{F}_\lambda}$ descend to a family of maps $\bar{\chi}_{\mathcal{F}_\lambda}: Y_{(q+1)} \rightarrow B\bar{\Gamma}_q$ such that $\bar{\chi}_{\mathcal{F}_\lambda}^*(\phi) \in H^{2q+1}(Y_{(q+1)}; \mathbb{R})$ is non-trivial for $\lambda \in \Lambda$.

By the Rational Hurewicz Theorem A.2 as applied to the space $Y_{(q+1)}$ for $r = q + 1$, the map

$$h: \pi_{2q+1}(Y_{(q+1)}) \otimes \mathbb{Q} \longrightarrow H_{2q+1}(Y_{(q+1)}; \mathbb{Q}) \cong H_{2q+1}(Y; \mathbb{Q})$$

is an isomorphism. The class $\bar{\chi}_{\mathcal{F}_\lambda}^*(\phi) \in H^{2q+1}(Y_{(q+1)}; \mathbb{R})$ is non-vanishing, so there exists a cycle $[Z] \in H_{2q+1}(Y_{(q+1)}; \mathbb{Z})$ such that the pairing $\langle \bar{\chi}_{\mathcal{F}_\lambda}^*(\phi), [Z] \rangle \neq 0$. Moreover, if Λ does not consist of isolated points, the evaluation pairing varies continuously with the parameter $\lambda \in \Lambda$, so $\langle \bar{\chi}_{\mathcal{F}_{\lambda'}}^*(\phi), [Z] \rangle \neq 0$ for $\lambda' \in \Lambda$ sufficiently close to λ . Choose a map $f_{[Z]}: \mathbb{S}^{2q+1} \rightarrow Y_{(q+1)}$ such that the image $h([f_{[Z]}]) \in H_{2q+1}(Y_{(q+1)}; \mathbb{Z})$ is a non-zero rational multiple of $[Z]$. Then the composition

$$(20) \quad \bar{\chi}_{\mathcal{F}_\lambda} \circ f_{[Z]}: \mathbb{S}^{2q+1} \rightarrow B\bar{\Gamma}_q$$

defines a family of spherical $(2q+1)$ -cycles in $B\bar{\Gamma}_q$ which pair non-trivially with the class ϕ . \square

Next, consider the homotopy properties for the space $B\bar{\Gamma}_q$ derived from the fibration sequences (7). The first observation is that $B\bar{\Gamma}_q$ is $(q+1)$ -connected implies that the map $\nu: B\Gamma_q^+ \rightarrow BSO(q)$ admits a section over the $(q+2)$ -skeleton $B^{(q+2)} \subset BSO(q)$ for a CW decomposition of $BSO(q)$. For $q = 4\ell - 1$ or $q = 4\ell - 2$, there is a class $\xi_\ell: \mathbb{S}^{4\ell} \rightarrow BSO(q)$ such that $\xi_\ell^*(p_\ell) \in H^\ell(\mathbb{S}^{4\ell}, \mathbb{Z})$ is non-vanishing, and ξ_ℓ lifts to a map $\hat{\xi}_\ell \rightarrow B\Gamma_q^+$ such that $\hat{\xi}_\ell^* \circ \nu^*(p_\ell) \neq 0$, hence $\nu^*(p_\ell) \neq 0$.

By the Bott Vanishing Theorem, $\nu^*(p_\ell^2) = 0$, so the map $\hat{\xi}_\ell \vee \hat{\xi}_\ell: \mathbb{S}^{4\ell} \vee \mathbb{S}^{4\ell} \rightarrow B\Gamma_q^+$ cannot be extended to the product $\mathbb{S}^{4\ell} \times \mathbb{S}^{4\ell}$. The obstruction to this extension is the Whitehead product $[\hat{\xi}_\ell, \hat{\xi}_\ell] \in \pi_{8\ell-1}(B\Gamma_q^+)$ which must be non-vanishing in rational homotopy theory. Schweitzer and Whitman observed that this homotopy class is non-zero in the note [124], using a theory of residues for the Pontrjagin classes.

Haefliger observed in the note [52] the following more general formulation. Suppose there is given a collection of forms $\{\phi_i \mid i \in \mathcal{I}\} \subset \Omega_{deR}^*(B\Gamma_q^+)$ in the de Rham complex of $B\Gamma_q^+$ which satisfy $\phi_i \wedge \phi_j = 0$ for $i \neq j$. Moreover, assume the linear functionals they define $[\phi_i]: \pi_*(B\Gamma_q^+) \rightarrow \mathbb{R}$ are rationally independent. Choose a collection of maps $\{f_i: \mathbb{S}^{n_i} \rightarrow B\Gamma_q^+ \mid i \in \mathcal{I}, n_i = \deg(\phi_i)\}$ for which their classes are independent when restricted to the homology classes they determine. Then for the induced map on their wedge product,

$$(21) \quad \hat{f}: X = \bigvee_{i \in \mathcal{I}} \mathbb{S}^{n_i} \rightarrow B\Gamma_q^+,$$

the induced map in rational homotopy theory, $\hat{f}_\#: \pi_*(X) \otimes \mathbb{Q} \rightarrow \pi_*(B\Gamma_q^+) \otimes \mathbb{Q}$, is injective. The rational homotopy groups $\pi_*(X) \otimes \mathbb{Q}$ have the structure of a free graded Lie algebra, with multiplication given by the Whitehead product, and this space is infinite dimensional if the index set \mathcal{I} is not a singleton. Note that the rational Hurewicz homomorphism $h_*: \pi_*(B\Gamma_q^+) \rightarrow H_*(B\Gamma_q^+; \mathbb{Q})$ maps all of these Whitehead product classes to zero.

Both of the observations, by Schweitzer and Whitman and by Haefliger, are subsumed by the author's thesis work [65, 66] using Sullivan's approach to rational homotopy theory using differential forms to

construct higher order invariants, the dual homotopy classes. There are many expositions of rational homotopy theory, but the original papers [27, 47, 129, 131] explain what is used below. We briefly recall some aspects of this approach.

Recall that a model for a differential graded algebra \mathcal{A} (over \mathbb{Q} or \mathbb{R}) is a DGA map $\psi: \mathfrak{M} \rightarrow \mathcal{A}$ where \mathfrak{M} is a free DGA and the map ψ induces an isomorphism on cohomology groups. Then \mathfrak{M} is a minimal model for \mathcal{A} if it is the minimal such DGA. Minimal models exist under some basic restrictions. Let $\mathfrak{M}(\mathcal{A}) \rightarrow \mathcal{A}$ denote a minimal model for \mathcal{A} .

For a semi-simplicial space X , let $\Omega_{deR}^*(X)$ denote the DGA of semi-simplicial forms on X , and let $\mathfrak{M}(X)$ denote a minimal model for $\Omega_{deR}^*(X)$. Let $\pi^*(\mathfrak{M}(X))$ denote the space of indecomposable elements in $\mathfrak{M}(X)$. A basic result of minimal model theory is that there is a natural isomorphism $\pi^*(\mathfrak{M}(X)) \cong \text{Hom}(\pi_*(X), \mathbb{R})$ so that the space $\pi^*(\mathfrak{M}(X))$ consists of “dual homotopy” classes.

The map (14) defined by the Strong Bott Vanishing Theorem 5.3 is natural, so induces a DGA map

$$(22) \quad \Delta: I_q \longrightarrow \Omega_{deR}^*(B\Gamma_q^+),$$

where recall that we defined $I_q \equiv \mathbb{R}[c_1, c_2, \dots, c_q]_{2q}$ to be the truncated polynomial algebra. This determines a map on minimal models $\mathfrak{M}(\Delta): \mathfrak{M}(I_q) \rightarrow \mathfrak{M}(B\Gamma_q^+)$ which induces a map on dual homotopy, denoted by $\Delta^\#: \pi^*(I_q) \rightarrow \text{Hom}(\pi_*(B\Gamma_q^+), \mathbb{R})$. The classes in the image of $\Delta^\#$ are called the *dual homotopy invariants* of foliations in [66]. For example, corresponding to the indecomposable element of $\mathfrak{M}(I_q)$ defined by the Pontrjagin class $p_\ell = c_{2\ell} \in I_q$, there is an indecomposable element $y_{8\ell-1} \in \mathfrak{M}(I_q)$, and the image $\Delta^\#(y_{8\ell-1}) \in \text{Hom}(\pi_{8\ell-1}(B\Gamma_q^+), \mathbb{R})$ detects the Whitehead product constructed by Schweitzer and Whitman. Moreover, it was shown that the class $\Delta^\#(y_{8\ell-1})$ is equal to the evaluation of the secondary class $\Delta(y_{2\ell}c_{2\ell}) \in H^{8\ell-1}(B\bar{\Gamma}_q; \mathbb{R})$. Similar results hold for other elements of $\pi^*(I_q)$ under the image of $\Delta^\#$, so the study of the dual homotopy invariants for foliations gives a useful method of constructing non-trivial spherical cohomology classes in $H^*(B\bar{\Gamma}_q; \mathbb{R})$. Many more such examples are given in the works [65, 66, 67, 68]. It should be noted that the complete structure of the model $\mathfrak{M}(I_q)$ is unknown, as its calculation becomes exponentially more complicated in higher degrees as the codimension q increases. It is known that $\pi^*(I_q)$ contains graded Lie algebras of increasingly large size, which are the analogues of the examples studied by Haefliger in [52].

Next, we construct additional spherical classes in $H^*(B\bar{\Gamma}_q; \mathbb{R})$ using the *brace product* in Section A.3. Recall that $\iota: B^{(q+2)} \subset BSO(q)$ denotes the $(q+2)$ -skeleton for a CW decomposition of $BSO(q)$, and let $\sigma_q: B^{(q+2)} \rightarrow B\Gamma_q^+$ be a section to the fibration $\nu: B\Gamma_q^+ \rightarrow BSO(q)$. Introduce the pull-back E_q of this fibration over the inclusion $B_{(q+2)} \subset BSO(q)$, so we obtain a fibration

$$(23) \quad B\bar{\Gamma}_q \xrightarrow{\iota} E_q \xrightarrow{\nu} B^{(q+2)}$$

which admits a section denoted by $\sigma: B^{(q+2)} \rightarrow E_q$. Given classes $\alpha \in \pi_i(B^{(q+2)})$ and $\beta \in \pi_j(B\bar{\Gamma}_q)$ the brace product $\{\alpha, \beta\} \in \pi_{i+j-1}(B\bar{\Gamma}_q)$. We apply this construction to spherical secondary classes, to construct additional spherical secondary classes of higher degree.

Let $0 < 4\ell \leq q+1$ then there exists a map $\tau_\ell: \mathbb{S}^{4\ell} \rightarrow B^{(q+2)}$ such that the Pontrjagin class $p_\ell \in H^{4\ell}(BSO(q); \mathbb{Z})$ pairs non-trivially with the image of the composition $\iota \circ \tau_\ell \rightarrow BSO(q)$. Then the composition $\sigma \circ \tau_\ell: \mathbb{S}^{4\ell} \rightarrow E_q$ is a lift of this map, and the brace product defines a mapping $\{\tau_\ell, \cdot\}: \pi_j(B\bar{\Gamma}_q) \rightarrow \pi_{4\ell+j-1}(B\bar{\Gamma}_q)$. We use this mapping to show:

THEOREM 7.2. *Assume $q \geq 3$. Let $y_{i_1}c_J \in WO_q$ be an element of the Vey basis of degree $2q+1$, and suppose there exists a map $f: \mathbb{S}^{2q+1} \rightarrow B\bar{\Gamma}_q$ such that $\Delta(y_{i_1}c_J)$ pairs non-trivially on the homology class $[f] \in H_{2q+1}(B\Gamma_q^+; \mathbb{R})$. That is, $\Delta(y_{i_1}c_J) \in H_\pi^{2q+1}(B\Gamma_q^+; \mathbb{R})$. Assume that $I' = (i'_1, \dots, i'_r)$ satisfies*

$$(24) \quad i_1 < i'_1 < i'_2 < \dots < i'_r, \quad 2i'_r \leq q+1, \quad \text{and } i'_\ell \text{ is even for } 1 \leq \ell \leq r.$$

Then the secondary class $\Delta(y_I y_{I'} c_J) \in H_\pi^{2q+k}(B\bar{\Gamma}_q; \mathbb{R})$. That is, it is also non-trivial and spherical.

Proof. We construct a mapping $g: \mathbb{S}^k \rightarrow B\bar{\Gamma}_q$ such that the pairing $\langle \Delta(y_{i_1} y_{I'} c_J), [g] \rangle \neq 0$. We have $2i'_1 \leq q+1$, so there exists a lifting $\widehat{\tau}_{2i'_1}: \mathbb{S}^{2i'_1} \rightarrow B\Gamma_q^+$ that pairs non-trivially with $\nu^*(c_{2i'_1}) \in H^{2i'_1}(B\Gamma_q^+; \mathbb{R})$. For the map $f: \mathbb{S}^{2q+1} \rightarrow B\Gamma_q^+$, form the brace product $\{[\widehat{\tau}_{2i'_1}], [f]\}_\sigma \in \pi_{2q+2i'_1}(B\bar{\Gamma}_q)$.

The class $[\widehat{\tau}_{2i'_1}]$ pairs non-trivially with $\nu^*(c_{2i'_1})$ and $[f]$ pairs non-trivially with $\Delta(y_{i_1} c_J)$, so if the brace product vanishes in $\pi_{2q+2i'_1}(B\overline{\Gamma}_q) \otimes \mathbb{Q}$, then the product $\Delta(y_{i_1} c_{2i'_1} c_J) \in H^{2q+2i'_1+1}(B\overline{\Gamma}_q^+; \mathbb{R})$ must be non-vanishing. But the product $c_{2i'_1} c_J = 0$ in WO_q so this is a contradiction.

Now proceed recursively on the index set $I' = (i'_1, \dots, i'_r)$. Set $f_0 = f: \mathbb{S}^{2q+1} \rightarrow B\overline{\Gamma}_q$ and $f_1: \mathbb{S}^{2q+2i'_1} \rightarrow B\overline{\Gamma}_q$ which represents the brace product $\{[\widehat{\tau}_{2i'_1}], [f]\}_\sigma$. Now assume for $\ell < r$ that maps $f_\ell: \mathbb{S}^{2q+k} \rightarrow B\overline{\Gamma}_q$ have been constructed which pair non-trivially with the class $\Delta(y_{i_1} y_{i'_1} \cdots y_{i'_\ell} c_J) \in H^{2q+k}(B\overline{\Gamma}_q; \mathbb{R})$. Let $f_{\ell+1}: \mathbb{S}^{2q+k+2i'_{\ell+1}-1} \rightarrow B\overline{\Gamma}_q$ represent the brace product $\{[\widehat{\tau}_{2i'_{\ell+1}}], [f_\ell]\}_\sigma$. We show this class pairs non-trivially with the cohomology class

$$(25) \quad \Delta(y_{i_1} y_{i'_1} \cdots y_{i'_{\ell+1}} c_J) \in H^{2q+k+2i'_{\ell+1}-1}(B\overline{\Gamma}_q; \mathbb{R}).$$

Use the map $\widehat{\tau}_{2i'_{\ell+1}}$ to form a pull-back fibration $B\overline{\Gamma}_q \rightarrow E_{\ell+1} \rightarrow \mathbb{S}^{2i'_{\ell+1}}$.

The space $E_{\ell+1}$ is $(2i'_{\ell+1} - 1)$ -connected, so the Pontrjagin classes $\Delta(c_{i'_j})$ vanish when restricted to $E_{\ell+1}$, for $1 \leq j \leq \ell$. Thus the class $\Delta(y_{i_1} y_{i'_1} \cdots y_{i'_\ell} c_J)$ is well-defined on $E_{\ell+1}$. It follows that the brace product $\{[\widehat{\tau}_{2i'_{\ell+1}}], [f_\ell]\}_\sigma$ is non-vanishing in $\pi_{2q+k+2i'_{\ell+1}-1}(B\overline{\Gamma}_q)$ by the same argument as before, and pairs non-trivially with the secondary class in (25).

We thus obtain the map $g = f_r: \mathbb{S}^{2q+k} \rightarrow B\overline{\Gamma}_q$ which pairs non-trivially with $\Delta(y_I y_I c_J)$. \square

REMARK 7.3. There is a more formal proof of Theorem 7.2 using the calculation of the minimal model for the space E_q in (23). The steps are the same, using what amounts to an iterated Hirsch construction (that is, a Postnikov tower) as is typical for building minimal models. The above description has a more geometric flavor.

We conclude by recalling a theorem from [68], which summarizes some of the known results on the non-triviality of the higher degree homotopy groups $\pi_k(B\overline{\Gamma}_q)$ that follows using the dual homotopy invariants. The conclusion is that the homotopy groups of $B\overline{\Gamma}_q$ are non-trivial in an infinite range and uncountably generated, as the degree ℓ tends to infinity. The integers $\{v_{q,\ell}\}$ in the statement below are defined in Section 2.8 of [68], and are defined as the ranks of various free graded Lie algebras in the minimal model for the truncated polynomial algebra $\mathbb{R}[c_1, c_2, \dots, c_q]_{2q}$.

THEOREM 7.4. [68, Theorem 1] *For $q > 1$ and $\ell > 2q$, there is an epimorphism of abelian groups*

$$(26) \quad h^*: \pi_\ell(B\overline{\Gamma}_{2q+1}) \rightarrow \mathbb{R}^{v_{q,\ell}},$$

where the sequences of non-negative integers $\{v_{q,\ell}\}$ satisfy:

- (1) For $q = 2$, $\lim_{k \rightarrow \infty} v_{2,4k+1} = \infty$, with $v_{2,4k+1} > 0$ for all $k > 0$;
- (2) For $q = 3$, $\lim_{k \rightarrow \infty} v_{3,3k+1} = \infty$, with $v_{3,3k+1} > 0$ for all $k > 0$;
- (3) For $q > 3$, $\lim_{k \rightarrow \infty} v_{q,k} = \infty$.

This has as an immediate consequence.

COROLLARY 7.5. *For each $q \geq 1$, the homology groups $H_*(\overline{BDiff}_c(\mathbb{R}^q); \mathbb{Q})$ contains a free exterior algebra whose basis elements in degree ℓ correspond to a Hamel basis for $\mathbb{R}^{v_{q,\ell}}$.*

8. FLAT DIFF(M)-BUNDLES WITH NON-TRIVIAL CHARACTERISTIC CLASSES

In this section, we describe the explicit construction of non-trivial classes in $H_*(\overline{BDiff}_c(M); \mathbb{Q})$ starting with the cycles in $H_*(\mathcal{S}(M); \mathbb{Q})$ constructed in the previous sections. We then consider the problem of determining the image of these classes in $H_*(BDiff_c^+(M)^\delta; \mathbb{Q})$.

PROPOSITION 8.1. *Let M be a compact, connected oriented manifold of dimension q . Given $\phi \in H_\pi^n(B\overline{\Gamma}_q^+; \mathbb{R})$ and a map $f: \mathbb{S}^n \rightarrow B\overline{\Gamma}_q$ whose homology class pairs non-trivially with ϕ , and also assume there is given a co-spherical class $C \in H_k^\pi(\widehat{M}; \mathbb{Z})$. Then there exists $\psi_C \in$*

$H^{n-k}(\overline{B\text{Diff}}_c(M); \mathbb{R})$, a compact oriented manifold Z_C and map $g_C: Z_C \rightarrow \overline{B\text{Diff}}_c(M)$, and rational number $r \neq 0$ such that

$$(27) \quad \langle \psi_C, [g_C(Z_C)] \rangle = r \cdot \langle \phi, [f(\mathbb{S}^n)] \rangle,$$

Proof. Recall that $\tilde{\nu}_M: M\text{Diff}_c^+(M)^\delta \rightarrow B\Gamma_q^+$ classifies the foliation on the total space over $B\text{Diff}_c^+(M)^\delta$, and by restriction also gives a map $\tilde{\nu}_M: \overline{B\text{Diff}}_c(M) \times M \rightarrow B\Gamma_q^+$.

The proof of Theorem 4.7 constructs a class $\tilde{\gamma}_C \in H_\pi^{n-k}(\mathcal{S}(M); \mathbb{R})$ and defines as in (12) a map $\hat{f}_C: N = \mathbb{S}^{n-k} \times M \rightarrow B\overline{\Gamma}_q$ such that $\hat{f}_C^*(\phi) \in H^n(\mathbb{S}^{n-k} \times M; \mathbb{R})$ is non-vanishing.

Let $f_C: \mathbb{S}^{n-k} \rightarrow \mathcal{S}(M)$ be the adjoint to \hat{f}_C , then $\langle \tilde{\gamma}_C, [f_C] \rangle \neq 0$.

The Mather-Thurston Theorem 3.1 says that for the adjoint map $\tau: \overline{B\text{Diff}}_c(M) \rightarrow \mathcal{S}(M)$, the induced map $\tau_*: H_*(\overline{B\text{Diff}}_c(M); \mathbb{Z}) \rightarrow H_*(\mathcal{S}(M); \mathbb{Z})$ is an isomorphism. We make this isomorphism explicit for the spherical class determined by $f_C: \mathbb{S}^{n-k} \rightarrow \mathcal{S}(M)$.

The map $\hat{f}_C: N = \mathbb{S}^{n-k} \times M \rightarrow B\overline{\Gamma}_q$ defines a Haefliger structure on $\mathbb{S}^{n-k} \times M$ whose normal bundle is the lift of the tangent bundle to M . By the Mather-Thurston Theorem 3.1, there is

- a cycle $g_C: Z_C \rightarrow \overline{B\text{Diff}}_c(M)$;
- an oriented simplicial complex W_C of dimension $(n - k + 1)$;
- a map $b_C: W_C \rightarrow \mathcal{S}(M)$ with boundary components $\partial W_C = \partial W_C^+ \cup \partial W_C^-$, such that $b_C(W_C^-) = f(\mathbb{S}^{n-k})$ and $b_C(W_C^+) = g_C(Z_C)$

That is, $b_C(W_C)$ is a boundary between the cycles $\tau \circ g_C(Z_C)$ and $f_C(\mathbb{S}^{n-k})$ in $\mathcal{S}(M)$.

REMARK 8.2. Since $H_*(B\overline{\Gamma}_q; \mathbb{Q})$ and the oriented bordism group $\Omega_*^{SO}(B\overline{\Gamma}_q) \otimes \mathbb{Q}$ are isomorphic, we can assume that both Z_C and W_C are oriented compact manifolds if desired, replacing f_C with a positive multiple of itself, if necessary. For $q \geq 2$, the proof of Mather-Thurston by Meigniez [97] then constructs the simplicial complex W_C via a sequence of foliated surgeries on $\mathbb{S}^{n-k} \times M$. The cycle $g_C: Z_C \rightarrow \overline{B\text{Diff}}_c(M)$ then arises as the boundary of the space W_C so constructed. The other proofs of the Mather-Thurston Theorem in [94, 95, 96, 136] prove that τ_* is an isomorphism by more abstract methods, so reveal even less information about the space Z_C and the action of its fundamental group on M .

Let $\hat{b}_C: W_C \times M \rightarrow B\overline{\Gamma}_q$ denote the adjoint of b_C , and set $\hat{g}_C = \tilde{\nu}_M \circ (g_C \times id): Z_C \times M \rightarrow B\overline{\Gamma}_q$. Then $\hat{b}_C(W_C \times M)$ is a foliated concordance $\tilde{\nu}_{W_C}: W_C \times M \rightarrow B\overline{\Gamma}_q$ between $\hat{f}_C(\mathbb{S}^k \times M)$ and $\hat{g}_C(Z_C \times M)$ in $B\overline{\Gamma}_q$. That is, we have the commutative diagram:

$$\begin{array}{ccccc}
 & & \overline{B\text{Diff}}_c(M) \times M & \xrightarrow{\tilde{\nu}_M} & B\Gamma_q^+ \\
 & & \uparrow g_C \times id & \nearrow \hat{b}_C & \uparrow \hat{f}_C \\
 & & Z_C \times M & \xrightarrow{\tilde{\nu}_{W_C}} & \mathbb{S}^{n-k} \times M \\
 & & \downarrow \pi \quad \uparrow \sigma & & \downarrow \pi \\
 & & Z_C & & \mathbb{S}^{n-k} \\
 & \swarrow & \leftarrow g_C & & \\
 & & \overline{B\text{Diff}}_c(M) & &
 \end{array}$$

If the tangent bundle TM is trivial, then we can substitute $B\overline{\Gamma}_q$ for $B\Gamma_q^+$ in the above diagram.

Set $\hat{\phi}_C = \hat{f}_C^*(\phi) \in H^n(\mathbb{S}^{n-k} \times M; \mathbb{R})$ and $\hat{\psi}_C = (g_C^* \circ \tilde{\nu}_M^*)(\phi) \in H^n(Z_C \times M; \mathbb{R})$. By the Kunneth formula, we have

$$(28) \quad H^*(\mathbb{S}^{n-k} \times M; \mathbb{R}) \cong H^*(\mathbb{S}^{n-k}; \mathbb{R}) \otimes H^*(M; \mathbb{R}), \quad H^*(Z_C \times M; \mathbb{R}) \cong H^*(Z_C; \mathbb{R}) \otimes H^*(M; \mathbb{R}).$$

The slant product with the class $C \in H_k(M; \mathbb{Z})$ defines $\setminus C: H^*(M; \mathbb{R}) \rightarrow H^{*-k}(M; \mathbb{R})$, and we compose this with projection Π_1 to the first factor in (28) to define:

$$(29) \quad \phi_C = \Pi_1(\hat{\phi}_C \setminus C) \quad , \quad \psi_C = \Pi_1(\hat{\psi}_C \setminus C).$$

Then as the coboundary \widehat{b}_C is induced on the product on factor of M , we have:

$$(30) \quad \langle \psi_C, [Z_C] \rangle = \langle \widehat{\psi}_C, [Z_C] \otimes C \rangle = \langle \widehat{\phi}_C, [\mathbb{S}^{n-k}] \otimes C \rangle = \langle \phi_C, [\mathbb{S}^{n-k}] \rangle = \langle \phi, [f(\mathbb{S}^n)] \rangle$$

This completes the proof of the claim in (27). \square

We give a simple extension of Proposition 8.1, which is used in the following sections to construct uncountable families of linearly independent homology classes in $H_*(\overline{BDiff}(M); \mathbb{Q})$.

PROPOSITION 8.3. *Let M be a compact, connected oriented manifold of dimension q , and assume there is given a co-spherical class $C \in H_k^\pi(\widehat{M}; \mathbb{Z})$. Let $\{\phi_1, \dots, \phi_m\} \subset H_n^\pi(B\Gamma_q^+; \mathbb{R})$ for $n > 2q$, and let $\{f_i: \mathbb{S}^n \rightarrow B\Gamma_q^+ \mid i \in \mathcal{I}\}$ be a collection of maps such that the homology classes they define, $\{[f_i] \mid i \in \mathcal{I}\} \subset H_n(B\Gamma_q^+; \mathbb{Q})$, are rationally linearly independent for the vector mapping $\vec{\phi} = (\phi_1, \dots, \phi_m): H_n(B\Gamma_q^+; \mathbb{Q}) \rightarrow \mathbb{R}^m$. Then there exists $\{\psi_1, \dots, \psi_m\} \in H^{n-k}(\overline{BDiff}_c(M); \mathbb{R})$ and for each $i \in \mathcal{I}$, there exists a compact oriented $(n-k)$ -manifold N_i and map $g_i: N_i \rightarrow \overline{BDiff}_c(M)$ such that the classes $\{[g_i(N_i)] \mid i \in \mathcal{I}\} \subset H_{n-k}(\overline{BDiff}_c(M); \mathbb{Q})$ are rationally linearly independent in the image of the vector mapping $\vec{\psi}: H_{n-k}(\overline{BDiff}_c(M); \mathbb{Q}) \rightarrow \mathbb{R}^m$.*

Proof. For $\tilde{\nu}_M: \overline{BDiff}_c(M) \times M \rightarrow B\Gamma_q^+$ and $1 \leq \ell \leq m$, set $\widehat{\psi}_\ell = \tilde{\nu}_M^*(\psi_\ell) \in H^n(\overline{BDiff}_c(M) \times M; \mathbb{R})$ and $\psi_\ell = \widehat{\psi}_\ell \setminus C \in H^{n-k}(\overline{BDiff}_c(M); \mathbb{R})$.

Observe that $\pi_n(BSO(q)) \otimes \mathbb{Q}$ is trivial for $n > 2q$. Thus for each $i \in \mathcal{I}$, there exists an integer $m_i > 0$ such that the class $m_i \cdot [f_i] \in \pi_n(B\Gamma_q^+)$ is in the image of $\pi_n(B\overline{\Gamma}_q) \rightarrow \pi_n(B\Gamma_q^+)$. So without loss of generality we can assume that each map f_i factors through $B\overline{\Gamma}_q$.

For each $i \in \mathcal{I}$, apply the method of proof of Proposition 8.1 to the map f_i and cycle C , to construct a manifold N_i of dimension $n-k$, and map $g_i: N_i \rightarrow \overline{BDiff}_c(M)$.

Suppose that the class $A = \sum_{i \in \mathcal{I}} r_i \cdot [g_i(N_i)] \in H_{n-k}(\overline{BDiff}_c(M); \mathbb{Q})$ is zero, where $r_i \in \mathbb{Q}$ and only finitely many are non-zero. Then the class $\tilde{\nu}_M(A \cup C) \in H_n(B\Gamma_q^+; \mathbb{Q})$ is also zero. Then by (27),

$$\sum_{i \in \mathcal{I}} r'_i \cdot f_i(\mathbb{S}^n) = \tilde{\nu}_M(A \cup C) \in H_n(B\Gamma_q^+; \mathbb{Q}),$$

where r'_i is a non-zero rational multiple of r_i hence is again rational. But this contradicts our assumption that these classes are rationally linearly independent. \square

Next, consider the problem whether the classes constructed in Proposition 8.1 are non-trivial in the image of the map $(\iota_0)_*: H_*(\overline{BDiff}_c(M); \mathbb{Q}) \rightarrow H_*(BDiff_c^+(M)^\delta; \mathbb{Q})$. We have the commutative diagram:

$$\begin{array}{ccccc} \overline{BDiff}_c(M) \times M & \xrightarrow{\iota_0 \times id} & M\text{Diff}_c^+(M)^\delta & \xrightarrow{\tilde{\nu}_M} & B\Gamma_q^+ \\ \pi \downarrow & & \downarrow \pi & & \\ \overline{BDiff}_c(M) & \xrightarrow{\iota_0} & BDiff_c^+(M)^\delta & \xrightarrow{\iota_\delta} & BDiff_c^+(M) \end{array}$$

Given a cycle $g_C: Z_C \rightarrow \overline{BDiff}_c(M)$ we obtain a cycle $\iota_0 \circ g_C: Z_C \rightarrow BDiff_c^+(M)^\delta$, and so the question is whether the homology class it defines in $H_{n-k}(BDiff_c^+(M)^\delta; \mathbb{Q})$ is non-trivial.

For the case $C = M$, then the class $[g_M(Z_M)] \in H_{n-q}(BDiff_c^+(M)^\delta; \mathbb{Q})$ pairs non-trivially with the cohomology class $\psi_M = \int_M (\iota_0 \times id)^* \circ \tilde{\nu}_M^*(\phi) \in H^{n-q}(\overline{BDiff}_c(M); \mathbb{R})$. The integration over the fiber map is well-defined for the space $M\text{Diff}_c^+(M)^\delta$ as well, so there is a class $\tilde{\psi}_M = \int_M \tilde{\nu}_M^*(\phi) \in H^{n-q}(BDiff_c(M)^\delta; \mathbb{R})$ such that $\iota_0^*(\tilde{\psi}_M) = \psi_M$. Thus, the image $\iota_0(Z_M) \in H_{n-q}(BDiff_c(M)^\delta; \mathbb{Q})$ is non-trivial.

For the homology classes in $H_{n-k}(\overline{BDiff}_c(M); \mathbb{Q})$ constructed using the methods of Theorems 4.4 and 4.7, for $0 \leq k < q$, the relation between the groups $H_n(M\text{Diff}_c^+(M)^\delta; \mathbb{Q})$ and $H_{n-k}(BDiff_c^+(M)^\delta; \mathbb{Q})$

is more complicated, and depends on the homology spectral sequence associated to the fibration (1). Even for the case when $M = \mathbb{S}^1$, the above argument fails for the case $k = 0$, as there is no class in degree 3 in $H_*(B\text{Diff}^+(\mathbb{S}^1)^\delta; \mathbb{Q})$; see Theorem 9.7 below.

Next, recall from Remark 4.3 the alternative interpretation of the construction in Theorem 4.2, which expands on a construction of Thurston in [136].

Choose $x \in M$ and let $V \subset M$ be an open neighborhood of x . Choose distinct points $\{x_1, \dots, x_m\}$ in V , and for each $1 \leq \ell \leq m$ choose an open disk neighborhood $x_\ell \in U_\ell \subset V$, such that their closures are disjoint, $\overline{U}_i \cap \overline{U}_j = \emptyset$ for $i \neq j$. Moreover, assume that each closure \overline{U}_ℓ is diffeomorphic to the closed unit disk $\mathbb{D}^q \subset \mathbb{R}^q$, and choose a diffeomorphism $h_\ell: \mathbb{D}^q \rightarrow \overline{U}_\ell$. Then the restriction $h_\ell: \mathbb{B}^q \rightarrow U_\ell$ is a diffeomorphism of the open unit ball to U_ℓ .

Fix a diffeomorphism $\lambda: \mathbb{R}^q \rightarrow \mathbb{B}^q$, then define the inclusion mapping $\rho_\ell: \overline{\text{Diff}_c(\mathbb{R}^q)} \rightarrow \overline{\text{Diff}(M)}$, where a map $g \in \overline{\text{Diff}_c(\mathbb{R}^q)}$ is conjugated by the map $h_\ell \circ \lambda$ to a diffeomorphism of U_ℓ with compact support in U_ℓ , then extended as the identity on the rest of M . Note that the diffeomorphisms in the images of ρ_i and ρ_j commute if $i \neq j$, and thus we obtain a homomorphism

$$(31) \quad \Upsilon_m = \rho_1 \times \cdots \times \rho_m: \overline{\text{Diff}_c(\mathbb{R}^q)} \times \cdots \times \overline{\text{Diff}_c(\mathbb{R}^q)} \rightarrow \overline{\text{Diff}(M)}.$$

Then the classifying map for R_m yields a map on homology

$$(32) \quad (B\Upsilon_m)_*: H_*(\overline{\text{Diff}_c(\mathbb{R}^q)}; \mathbb{Q}) \otimes \cdots \otimes H_*(\overline{\text{Diff}_c(\mathbb{R}^q)}; \mathbb{Q}) \rightarrow H_*(\overline{\text{Diff}(M)}; \mathbb{Q}).$$

By Theorem 3.2, the adjoint induces an isomorphism $\tau_*: H_*(B\overline{\text{Diff}_c(\mathbb{R}^q)}; \mathbb{Q}) \rightarrow H_*(\Omega^q B\overline{\Gamma}_q; \mathbb{Q})$, where Ω^q denotes the q -fold based loops.

Recall that if X is a simply connected topological space, then the loop space ΩX is an H-space, and using the Pontrjagin product, its homology $H_*(\Omega X; \mathbb{Q})$ is a free polynomial algebra over $\pi_*(\Omega X) \otimes \mathbb{Q}$. Thus, $H_*(B\overline{\text{Diff}_c(\mathbb{R}^q)}; \mathbb{Q})$ is a free polynomial algebra over $\pi_{*+q}(B\overline{\Gamma}_q) \otimes \mathbb{Q}$. As we showed in Section 7, and Theorem 7.4 in particular, the spaces $\pi_*(B\overline{\Gamma}_q)$ are uncountably large, and so the same follows even more so for the groups $H_*(B\overline{\text{Diff}_c(\mathbb{R}^q)}; \mathbb{Q})$.

Note that the representations ρ_ℓ and $\rho_{\ell'}$ are conjugate in $\overline{\text{Diff}(M)}$, so the product (32) also induces a multiplication for the homology classes in the image of each map ρ_ℓ . Thus, as observed by Morita in [104, Section 4], the Pontrjagin product on $H_*(B\overline{\text{Diff}_c(\mathbb{R}^q)}; \mathbb{Q})$ coincides with the ‘‘spacial product’’ induced from the representations ρ_ℓ .

The proof of Proposition 8.1 and the arguments above shows that each induced map

$$(B\rho_\ell)_*: H_{q+1}B\overline{\text{Diff}_c(\mathbb{R}^q)}; \mathbb{Q}) \rightarrow H_{q+1}(B\text{Diff}^+(M)^\delta; \mathbb{Q})$$

is injective. For the case when $M = \mathbb{S}^1$, Morita shows in [104] that the image of the product map (32) is injective in $H_*(B\text{Diff}^+(\mathbb{S}^1)^\delta; \mathbb{Q})$ for all $m > 1$ as well. For the case when $M = \Sigma_g$ is a closed surface of sufficiently high genus $g \gg 2$, Nariman shows in [113, Theorem 0.17] that again, the image of the product map is injective for all $m > 1$. The image of the product map (32) is unknown for $m > 1$ in all other cases.

9. CHARACTERISTIC CLASSES FOR DIMENSION 1

The results for the groups $H_*(B\overline{\text{Diff}}(\mathbb{R}); \mathbb{Z})$, $H_*(B\overline{\text{Diff}}(\mathbb{S}^1); \mathbb{Z})$ and $H_*(B\text{Diff}^+(\mathbb{S}^1)^\delta; \mathbb{Z})$ are all the result of the construction of examples by Thurston, which show that the Godbillon-Very class is spherical. Morita gave a thorough analysis of these groups in his works [104, 106, 108]. In this section we recall these results, and the relation to the techniques developed in previous sections.

Recall that $B\overline{\Gamma}_1 = B\Gamma_1^+$ so there is a map $\hat{\nu}: B\overline{\text{Diff}_c(\mathbb{R})} \times \mathbb{R} \rightarrow B\Gamma_1^+$ where $\hat{\nu}$ maps infinity to the basepoint given by the product foliation. Mather proved in [92, 93] that the adjunct map $\tau: B\overline{\text{Diff}_c(\mathbb{R})} \rightarrow \Omega B\Gamma_1^+$ is a homology isomorphism, which is the first case of Theorem 3.2. As $\overline{\text{Diff}_c(\mathbb{R})}$ is a simple group [136], the group $H_1(\Omega B\Gamma_1^+; \mathbb{Z})$ vanishes, and it follows that $\pi_1(\Omega B\Gamma_1^+)$ vanishes, so that $\pi_3(B\Gamma_1^+)$ is the first non-vanishing homotopy group of $B\Gamma_1^+$, hence there is an isomorphism $H_3(B\Gamma_1^+; \mathbb{Z}) \cong \pi_3(B\Gamma_1^+)$ by the Hurewicz Theorem.

By Thurston [135] the Godbillon-Vey class is spherical, hence one has:

PROPOSITION 9.1. *There exists a collection of maps $\{f_i: \mathbb{S}^3 \rightarrow B\Gamma_1^+ \mid i \in \mathcal{I}\}$ where \mathcal{I} is an uncountable set, and their classes $\{[f_i] \mid i \in \mathcal{I}\} \subset \pi_3(B\Gamma_1^+)$ are linearly independent over \mathbb{Q} .*

Proof. The Godbillon-Vey class defines a surjection $GV: \pi_3(B\Gamma_1^+) \cong H^3(B\text{Diff}_c(\mathbb{R})^\delta; \mathbb{Z}) \rightarrow \mathbb{R}$. Choose a Hamel basis $\{a_i \mid i \in \mathcal{I}\} \subset \mathbb{R}$ of \mathbb{R} over \mathbb{Q} . Then for each $i \in \mathcal{I}$ choose a map $f_i: \mathbb{S}^3 \rightarrow B\Gamma_1^+$ such that $\langle GV, [f_i] \rangle = a_i$. \square

The next result follows using standard properties of homology and H-spaces.

THEOREM 9.2. *The cohomology ring $H^*(\Omega B\Gamma_1^+; \mathbb{Q})$ contains a free polynomial algebra with basis an uncountable collection $\{\phi_i \mid i \in \mathcal{I}\} \subset H^2(\Omega B\Gamma_1^+; \mathbb{Q})$.*

Proof. We have $\pi_2(\Omega B\Gamma_1^+) \cong H_2(\Omega B\Gamma_1^+; \mathbb{Z})$ so $H^2(\Omega B\Gamma_1^+; \mathbb{Q}) \cong \text{Hom}(\pi_2(\Omega B\Gamma_1^+), \mathbb{Q})$. For each $i \in \mathcal{I}$, let $\phi_i \in H^2(\Omega B\Gamma_1^+; \mathbb{Q})$ be such that $\phi_i([f_j]) = \delta_i^j$ where $f_j: \mathbb{S}^2 \rightarrow \Omega B\Gamma_1^+$ is the map chosen in Proposition 9.1 and δ_i^j is the indicatrix function.

Let \vec{n}_k denote an integer valued function on the k -tuples $I = (i_1, \dots, i_k)$ with $i_\ell \in \mathcal{I}$, which is zero except on a finite number of I . For each such I set $\phi_I = \phi_{i_1} \wedge \dots \wedge \phi_{i_k}$.

Let $\Phi(\vec{n}_k) = \sum_I n_I \phi_I \in H^{2k}(\Omega B\Gamma_1^+; \mathbb{Q})$ be the finite sum. Suppose that $\Phi(\vec{n}_k) = 0$, and let I be a k -tuple with $n_I \neq 0$. Let $f_I: \mathbb{S}^2 \times \dots \times \mathbb{S}^2 \rightarrow \Omega B\Gamma_1^+$ be the Pontrjagin product of the classes $\{f_{i_1}, \dots, f_{i_k}\}$, then $0 = \Phi([f_I]) = n_I a_{i_1} \dots a_{i_k}$ implies that $n_I = 0$, which is a contradiction. \square

COROLLARY 9.3. *The groups $H^{2k}(B\text{Diff}_c^+(\mathbb{R})^\delta; \mathbb{Q})$ are uncountable for all $k > 0$.*

Proof. By Mather's Theorem we have

$$H^{2k}(B\text{Diff}_c^+(\mathbb{R})^\delta; \mathbb{Q}) \cong \text{Hom}(H_{2k}(B\text{Diff}_c^+(\mathbb{R})^\delta; \mathbb{Z}); \mathbb{Q}) \cong \text{Hom}(\pi_{2k}(\Omega B\Gamma_1^+), \mathbb{Q}) \cong H^{2k}(\Omega B\Gamma_1^+; \mathbb{Q}).$$

For each non-zero polynomial $\Phi(\vec{n}_k)$ as above, let $\overline{\Phi(\vec{n}_k)} \in H^{2k}(B\text{Diff}_c^+(\mathbb{R})^\delta; \mathbb{Q})$ be the corresponding class. Then for each $k > 0$, the collection of classes $\{\overline{\Phi(\vec{n}_k)}\} \subset H^{2k}(B\text{Diff}_c^+(\mathbb{R})^\delta; \mathbb{Q})$ span a vector space with uncountable dimension. \square

Let X be a space with basepoint $x_0 \in X$, let ΩX be the space of loops based at x_0 . The reduced suspension $\Sigma \Omega X$ is the quotient space of $\Omega X \times [0, 1]$ modulo $(\Omega X \times \{0\}) \cup \Omega X \times \{1\} \cup \{\widetilde{x_0}\} \times [0, 1]$. The fundamental map $k: \Sigma \Omega X \rightarrow X$ is then given by $k(\omega, t) = \omega(t)$. The homology suspension map $E_*: H_*(\Omega X; \mathcal{A}) \rightarrow H_{*+1}(\Sigma \Omega X; \mathcal{A})$ is induced by the geometric suspension of singular simplices. The composition

$$(33) \quad \sigma = -k_* \circ E: H_n(\Omega X; \mathcal{A}) \rightarrow H_{n+1}(X; \mathcal{A})$$

is induced by the transgression map of the path space fibration $\mathcal{P}X \rightarrow X$. Barcus and Meyer show:

THEOREM 9.4. [4, Proposition 2.4] *Let X be r -connected, then σ is isomorphism for $0 < n < 2r$.*

Apply this result to the 2-connected space $X = B\Gamma_1^+$ to obtain:

COROLLARY 9.5. *The suspension $\sigma: H_2(\Omega B\Gamma_1^+; \mathbb{Z}) \rightarrow H_3(B\Gamma_1^+; \mathbb{Z})$ is an isomorphism.*

This is just a restatement of the isomorphisms used to construct the classes in Proposition 9.2.

PROBLEM 9.6. *Show that $\sigma: H_{2n}(\Omega B\Gamma_1^+; \mathbb{Z}) \rightarrow H_{2n+1}(B\Gamma_1^+; \mathbb{Z})$ is non-trivial for $n > 1$.*

A solution to Problem 9.6 is related to a solution of Problem 6.7. Morita observed that the non-triviality of the map $\overline{GV}^*: H^6(K(\mathbb{R}^\delta, 3); \mathbb{Z}) \rightarrow H^6(B\Gamma_1^+; \mathbb{Z})$ follows from the vanishing of the rational Whitehead products of the classes chosen in the proof of Proposition 9.1; see [106, Proposition 6.4]). By the Samelson result [123, Theorem] (or [106, Theorem 6.2]) the product vanishes if the images of the suspension map σ annihilates their Pontrjagin products.

Next, we have the commutative diagram:

$$\begin{array}{ccccc}
\mathbb{S}^1 & \longrightarrow & \mathbb{S}^1 & & \\
\downarrow & & \downarrow & & \\
\overline{BDiff}(\mathbb{S}^1) \times \mathbb{S}^1 & \xrightarrow{\iota_1} & MDiff^+(\mathbb{S}^1)^\delta & \xrightarrow{\tilde{\nu}} & B\Gamma_1^+ \\
\sigma \updownarrow \pi & & \downarrow \pi & & \\
\overline{BDiff}(\mathbb{S}^1) & \xrightarrow{\iota_0} & BDiff^+(\mathbb{S}^1)^\delta & \xrightarrow{\iota_\delta} & BDiff^+(\mathbb{S}^1) \cong BSO(2)
\end{array}$$

The composition $\tilde{\nu} \circ \iota_1$ induces the adjunction map

$$\tau: \overline{BDiff}(\mathbb{S}^1) \rightarrow \mathcal{S}(\mathbb{S}^1) \cong \text{Maps}(\mathbb{S}^1, B\Gamma_1^+) = \Lambda B\Gamma_1^+,$$

where $\Lambda B\Gamma_1^+$ denotes the free loop space, and the map τ induces isomorphisms in homology.

Recall there is a fibration $\Omega\Gamma_1^+ \rightarrow \Lambda B\Gamma_1^+ \rightarrow B\Gamma_1^+$, which admits a section $s: B\Gamma_1^+ \rightarrow \Lambda B\Gamma_1^+$ where $s(x)$ is the constant loop at x . It follows that $\pi_*(\Lambda B\Gamma_1^+) = \pi_*(\Omega B\Gamma_1^+) \oplus \pi_*(B\Gamma_1^+)$. Since $\pi_1(B\Gamma_1^+) = \pi_2(B\Gamma_1^+) = \{0\}$, by [130] the rational minimal model for $\Lambda B\Gamma_1^+$ contains as a DGA summand the free algebra $\Lambda[\pi_3(B\Gamma_1^+) \oplus \pi_2(\Omega B\Gamma_1^+)]$ with trivial differential.

Let $gv = \tilde{\nu}^*(GV) \in H^2(MDiff_c^+(\mathbb{S}^1)^\delta; \mathbb{R})$, then $\iota_1^*(gv)$ decomposes into classes α, β in the expression

$$(34) \quad \beta + \alpha \otimes [d\theta] \in H^3(\overline{BDiff}(\mathbb{S}^1) \times \mathbb{S}^1; \mathbb{R}) \cong H^3(\overline{BDiff}(\mathbb{S}^1); \mathbb{R}) \oplus H^2(\overline{BDiff}(\mathbb{S}^1); \mathbb{R}) \otimes H^1(\mathbb{S}^1; \mathbb{R}),$$

where $[d\theta] \in H^1(\mathbb{S}^1; \mathbb{R})$ is the generator, and α is obtained by taking the cap product with the fundamental class $[\mathbb{S}^1]$. That is, $\alpha = \int_{\mathbb{S}^1} \iota_1^*(GV) \in H^2(\overline{BDiff}(\mathbb{S}^1); \mathbb{R})$ is the class constructed in the proof of Theorem 4.2. Morita gave the following expression for α in [104, Corollary 3.2]

$$(35) \quad \alpha = \int_{\mathbb{S}^1} \iota_1^*(gv) = \iota_0^* \int_{\mathbb{S}^1} gv \in H^2(\overline{BDiff}(\mathbb{S}^1); \mathbb{R}).$$

Furthermore, $\beta = \sigma^* \circ \iota_1^*(gv) \in H^3(\overline{BDiff}(\mathbb{S}^1); \mathbb{R})$ is the class constructed in the proof of Theorem 4.4.

We recall two further results by Morita to complete the discussion of the dimension 1 case.

THEOREM 9.7. [104, Theorem 4.3] *For all $n \geq 1$, the classes $\{\alpha^n, \alpha^{n-1}\beta\} \in H^*(\overline{BDiff}(\mathbb{S}^1); \mathbb{R})$ are linearly independent, and give independently variable maps on the groups $H^*(\overline{BDiff}(\mathbb{S}^1); \mathbb{Z})$.*

COROLLARY 9.8. *For $n \geq 1$, there are surjective maps $\alpha^n: H_{2n}(\overline{BDiff}(\mathbb{S}^1); \mathbb{Z}) \rightarrow \mathbb{R}$ and $\alpha^{n-1}\beta: H_{2n+1}(\overline{BDiff}(\mathbb{S}^1); \mathbb{Z}) \rightarrow \mathbb{R}$.*

Finally, Morita showed that the image of the Euler class $\bar{\chi} \in H^2(BSO(2); \mathbb{Z})$ and its powers are mapped injectively, $\chi^n = \iota_\delta^*(\bar{\chi}^n) \in H^{2n}(BDiff^+(\mathbb{S}^1)^\delta; \mathbb{Z})$. Even more is true,

THEOREM 9.9. [104, Theorem 1.1] *For all $n \geq 1$, the classes $\{\alpha^n, \chi^n\} \in H^{2n}(BDiff^+(\mathbb{S}^1)^\delta; \mathbb{R})$ are linearly independent, and the evaluation $\alpha^n: H_{2n}(BDiff^+(\mathbb{S}^1)^\delta; \mathbb{Z}) \rightarrow \mathbb{R}$ is a surjection.*

The proof in [104, Section 7] uses that the space \mathbb{S}^1 admits finite coverings of arbitrary degree. Nariman uses an extension of this idea in his work [116].

10. CHARACTERISTIC CLASSES FOR DIMENSION 2

The analysis of the homology and cohomology groups of the spaces $\overline{BDiff}_c(M)$ and $BDiff_c^+(M)^\delta$ for an oriented surface M has several major differences from the 1-dimensional case, and the study of $\mathcal{S}(M)$ gets more interesting. We describe these results for $q = 2$ in some detail, to compare and contrast them from the results in Section 9.

The first difference is that the spaces of secondary invariants, $H^*(WO_2)$ and $H^*(W_2)$ as introduced in Section 5, now consists of more classes than just the Godbillon-Vey invariant:

$$\begin{aligned} H^{>4}(WO_2) &= \bigwedge (y_1 c_1^2, y_1 c_2), |y_1 c_1^2| = |y_1 c_2| = 5 \\ H^*(W_2) &= \bigwedge (y_1 c_1^2, y_1 c_2, y_2 c_2, y_1 y_2 c_2), |y_1 c_1^2| = |y_1 c_2| = 5, |y_2 c_2| = 7, |y_1 y_2 c_2| = 8. \end{aligned}$$

The author showed in [68, Theorem 2.5] that both $\{\Delta(y_1 c_1^2), \Delta(y_1 c_2)\} \subset H^5(B\Gamma_2^+; \mathbb{R})$ are spherical classes, and so give rise to generalized Godbillon-Vey classes $\{gv_1, gv_2\} \subset H^5(M\text{Diff}_c^+(M)^\delta; \mathbb{R})$ defined as in (39) below, and the classes $\{\alpha_1, \alpha_2\} \subset H^3(B\text{Diff}_c^+(M); \mathbb{R})$ defined by integration over the fiber in (40). In addition, there are also non-trivial classes in $H^4(\overline{B\text{Diff}}_c(M); \mathbb{R})$ defined using co-spherical classes in $H_1(M; \mathbb{Z})$.

The class $\Delta(y_2 c_2) \in H^7(B\overline{\Gamma}_2; \mathbb{R})$ is a ‘‘rigid’’ class, and it is unknown if it is non-zero, though Pittie showed in [121] that it must vanish on locally homogeneous foliations. The class $\Delta(y_1 y_2 c_1^2) \in H^8(B\overline{\Gamma}_2; \mathbb{R})$ is non-trivial, as shown by Baker [2] for example. However, it is unknown if this class is non-vanishing on the image of the Hurewicz map $\pi_8(B\overline{\Gamma}_2) \rightarrow H_8(B\overline{\Gamma}_2; \mathbb{Z})$.

The second difference is that for $M = \mathbb{S}^2$ and $M = \Sigma_g$ a surface of genus $g \geq 2$, the tangent bundle is not trivial, so the method of Theorem 4.4 does not apply, and the method of Theorem 4.7 must be adjusted for its application. On the other hand, for $M = \mathbb{T}^2$, all of the methods of Section 4 can be applied, yielding many additional non-trivial homology classes in $H_*(\overline{B\text{Diff}}(\mathbb{T}^2); \mathbb{Q})$.

We recall the known facts about the spherical cohomology groups $H_\pi^*(B\Gamma_2^+; \mathbb{R})$ and $H_\pi^*(B\overline{\Gamma}_2; \mathbb{R})$.

Let $\widehat{\nu}: \overline{B\text{Diff}}(\mathbb{R}^2) \times \mathbb{R}^2 \rightarrow B\overline{\Gamma}_2$ be the classifying map, where $\widehat{\nu}$ maps infinity to the basepoint w given by the point foliation. The adjunct map $\tau: B\text{Diff}(\mathbb{R}^2) \rightarrow \Omega^2 B\overline{\Gamma}_2$ is a homology isomorphism by Thurston. As $\overline{\text{Diff}}(\mathbb{R}^2)$ is a simple group [136], the group $H_1(B\overline{\Gamma}_2; \mathbb{Z})$ vanishes, and it follows that $\pi_1(\Omega^2 B\overline{\Gamma}_2)$ vanishes, so that $\pi_4(B\overline{\Gamma}_2)$ is the first possibly non-vanishing homotopy group of $B\overline{\Gamma}_2$.

By the Hurewicz Theorem, there is an isomorphism $\pi_4(B\overline{\Gamma}_2) \cong H_4(B\overline{\Gamma}_2; \mathbb{Z})$. In addition, the Rational Hurewicz Theorem A.2 yields:

PROPOSITION 10.1. *The Hurewicz map induces isomorphisms*

$$(36) \quad h: \pi_i(B\overline{\Gamma}_2) \otimes \mathbb{Q} \longrightarrow H_i(B\overline{\Gamma}_2; \mathbb{Q})$$

for $1 \leq i < 7$ and a surjection for $i = 7$.

Rasmussen constructed in [122] two different families of foliations, $\mathcal{F}_{1,\lambda}$ on Y_1 and $\mathcal{F}_{2,\lambda}$ on Y_2 , where Y_1 and Y_2 are oriented compact 5-manifolds without boundary such that evaluation of the secondary classes $\{y_1 c_1^2, y_1 c_2\}$ on these foliations vary independently, and thus evaluation of the universal secondary classes defines a map

$$(37) \quad \{\Delta(h_1 c_1^2), \Delta(h_1 c_2)\}: H_5(B\Gamma_2^+; \mathbb{Z}) \rightarrow \mathbb{R}^2.$$

The image contains an open subset of \mathbb{R}^2 , hence must be all of \mathbb{R}^2 . Note that there is no claim in [122] that the normal bundles to these foliations are trivial. Then by Theorem 7.1 we have:

THEOREM 10.2. [68, Theorem 2.5] *The classes $\{\Delta(h_1 c_1^2), \Delta(h_1 c_2)\}$ yield a surjection*

$$(38) \quad \{\Delta(h_1 c_1^2), \Delta(h_1 c_2)\}: \pi_5(B\overline{\Gamma}_2) \rightarrow \pi_5(B\Gamma_2^+) \rightarrow \mathbb{R}^2.$$

As in the proof of Proposition 9.1, we then have:

PROPOSITION 10.3. *Choose a Hamel basis $\{a_i \mid i \in \mathcal{I}\} \subset \mathbb{R}$ of \mathbb{R} over \mathbb{Q} . Then there exists a collection of maps $\{f_{i,j}: \mathbb{S}^5 \rightarrow B\overline{\Gamma}_2 \mid i, j \in \mathcal{I}\}$ such that*

$$\langle \Delta(h_1 c_1^2), [f_{i,j}] \rangle = a_i, \quad \langle \Delta(h_1 c_2), [f_{i,j}] \rangle = a_j.$$

In particular, the classes $\{[f_{i,j}] \mid i, j \in \mathcal{I}\} \subset \pi_5(B\overline{\Gamma}_2)$ are linearly independent over \mathbb{Q} .

As in the proof of Theorem 9.2, we obtain:

THEOREM 10.4. *The cohomology ring $H^*(\Omega^2 B\overline{\Gamma}_2; \mathbb{Q})$ contains a free exterior algebra with basis an uncountable collection $\{\phi_{i,j} \mid i, j \in \mathcal{I}\} \subset H^3(\Omega^2 B\overline{\Gamma}_2; \mathbb{Q})$.*

COROLLARY 10.5. *The groups $H^{3\ell}(\overline{BDiff}_c(\mathbb{R}^2); \mathbb{Q})$ are uncountable for all $\ell > 0$.*

We pose the analogous version of Problem 9.6 for the suspension map of codimension 2 foliations:

PROBLEM 10.6. *Show that $\sigma_2: H_{3\ell}(\Omega^2 B\overline{\Gamma}_2; \mathbb{Q}) \rightarrow H_{3\ell+2}(B\overline{\Gamma}_2; \mathbb{Q})$ is non-trivial for $\ell > 1$.*

Recall that $\tilde{\nu}_M: MDiff_c^+(M)^\delta \rightarrow B\overline{\Gamma}_2^+$ classifies the foliation on the total space over $BDiff_c^+(M)^\delta$. Define the generalized Godbillon-Vey classes

$$(39) \quad gv_1 = \tilde{\nu}_M^*(\Delta(h_1 c_1^2)) , \quad gv_2 = \tilde{\nu}_M^*(\Delta(h_1 c_2)) \in H^5(MDiff_c^+(M)^\delta; \mathbb{R}) .$$

Define classes $\{\alpha_1, \alpha_2\} \subset H^3(BDiff_c^+(M)^\delta; \mathbb{R})$ by integration over the fiber:

$$(40) \quad \alpha_1 = \int_M gv_1 , \quad \alpha_2 = \int_M gv_2 .$$

Next, let $\{f_{i,j}: \mathbb{S}^5 \rightarrow B\overline{\Gamma}_2 \mid i, j \in \mathcal{I}\}$ be the collection of maps constructed in Proposition 10.3.

By Proposition 8.3, for each $i, j \in \mathcal{I}$, there exists a closed compact oriented 3-manifold $N_{i,j}$ and map $g_{i,j}: N_{i,j} \rightarrow B\overline{Diff}_c(M)$ such that the classes $\{[g_{i,j}(N_{i,j})] \mid i, j \in \mathcal{I}\} \subset H_3(B\overline{Diff}_c(M); \mathbb{Q})$ are rationally linearly independent in the image of the map $\tilde{g}\tilde{\nu} = (gv_1, gv_2): H_3(B\overline{Diff}_c(M); \mathbb{Q}) \rightarrow \mathbb{R}^2$. By the comments at the end of Section 8, these classes inject into $H_3(BDiff_c^+(M)^\delta; \mathbb{Q})$, so we have:

THEOREM 10.7. *For a connected oriented surface M , the classes*

$$(41) \quad \{[g_{i,j}(N_{i,j})] \mid i, j \in \mathcal{I}\} \subset H_3(BDiff_c^+(M)^\delta; \mathbb{Q})$$

are linearly independent, and detected by the classes $\{gv_1, gv_2\} \subset H^3(BDiff_c^+(M)^\delta; \mathbb{R})$.

We next detail the constructions of classes for surfaces, on a case by case analysis by their genus.

For $M = \mathbb{S}^2$, the genus $g = 0$ case, the classifying map $\nu_{\mathbb{S}^2}: \mathbb{S}^2 \rightarrow BSO(2)$ is non-trivial, and $H_1(\mathbb{S}^2; \mathbb{Q})$ is trivial, so the only results are those in Theorem 10.7:

PROPOSITION 10.8. *$H_3(BDiff^+(\mathbb{S}^2)^\delta; \mathbb{Q})$ contains an uncountably generated rational subgroup which maps onto \mathbb{R}^2 by evaluation with the Godbillon-Vey classes $\{gv_1, gv_2\} \in H^3(BDiff^+(\mathbb{S}^2)^\delta; \mathbb{R})$.*

For $M = \mathbb{T}^2$, the genus $g = 1$ case, there are many more constructions of classes in $H_*(\overline{Diff}(\mathbb{T}^2); \mathbb{Q})$ but the classes in $H_*(BDiff^+(\mathbb{T}^2)^\delta; \mathbb{Q})$ are the same as for \mathbb{S}^2 . First, Theorems 4.4 and 10.2 yield:

PROPOSITION 10.9. *There is an injection*

$$(42) \quad \sigma^* \circ \tilde{\nu}^* : H_\pi^5(B\overline{\Gamma}_q; \mathbb{R}) \rightarrow H^5(B\overline{Diff}(\mathbb{T}^2); \mathbb{R}) .$$

In particular, for

$$(43) \quad \beta_1 = \sigma^* \circ \iota_1^* \circ \tilde{\nu}^*(\Delta(h_1 c_1^2)) , \quad \beta_2 = \sigma^* \circ \iota_1^* \circ \tilde{\nu}^*(\Delta(h_1 c_2))$$

the classes $\{\beta_1, \beta_2\} \subset H^5(B\overline{Diff}(\mathbb{T}^2); \mathbb{R})$ are linearly independent.

Then by Theorem 10.2 and Proposition 10.3 we have:

THEOREM 10.10. *For each $i, j \in \mathcal{I}$, there exists a compact oriented 5-manifold $N'_{i,j}$ and map $\widehat{f'_{i,j}}: N'_{i,j} \rightarrow B\overline{Diff}(\mathbb{T}^2)$ such that the homology classes $\{[\widehat{f'_{i,j}}(N'_{i,j})] \mid i, j \in \mathcal{I}\} \subset H_5(B\overline{Diff}(\mathbb{T}^2); \mathbb{Q})$ are linearly independent, and detected by the cohomology classes $\{\beta_1, \beta_2\} \subset H^5(B\overline{Diff}(\mathbb{T}^2); \mathbb{R})$.*

Next, we use Theorem 4.7 and Proposition 10.3 to construct linearly independent classes in $H^4(B\overline{Diff}(\mathbb{T}^2); \mathbb{R})$. Chose a factorization $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, which defines the projections

$$p_1: \mathbb{T}^2 \rightarrow \mathbb{S}^1 \times \{\theta_1\} = \mathbb{S}^1 , \quad p_2: \mathbb{T}^2 \rightarrow \{\theta_1\} \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$$

and sections

$$\eta_1: \mathbb{S}^1 \cong \mathbb{S}^1 \times \{\theta_1\} \rightarrow \mathbb{T}^2 , \quad \eta_2: \mathbb{S}^1 \cong \{\theta_1\} \times \mathbb{S}^1 \rightarrow \mathbb{T}^2 .$$

The sections define co-spherical classes, again denoted by $\eta_1, \eta_2 \in H_1(\mathbb{T}^2; \mathbb{Z})$. Then by the proof of Theorem 4.7 define the classes

$$(44) \quad \gamma_1^1 = \int_{\eta_1} gv_1, \gamma_2^1 = \int_{\eta_1} gv_2, \gamma_1^2 = \int_{\eta_2} gv_1, \gamma_2^2 = \int_{\eta_2} gv_2.$$

Next, combine Proposition 8.3 using $C = \mathbb{S}^1 \times \{\theta_1\}$ or $C = \{\theta_1\} \times \mathbb{S}^1$, with Proposition 10.3 and Theorem 4.7 to conclude that the set $\{\gamma_1^1, \gamma_2^1, \gamma_1^2, \gamma_2^2\} \subset H^4(\overline{BDiff}(\mathbb{T}^2); \mathbb{R})$ is linearly independent, and there are four families of uncountably many linearly independent classes in $H_4(\overline{BDiff}(\mathbb{T}^2); \mathbb{Q})$ which pair non-trivially with the γ_j^i -classes.

Thus, for $M = \mathbb{T}^2$ we have defined the following linearly independent classes, for $1 \leq i, j \leq 2$, which pair non-trivially with corresponding uncountable generated subgroups of $H_*(\overline{BDiff}(\mathbb{T}^2); \mathbb{Q})$:

- $gv_i \in H^5(\overline{BDiff}(\mathbb{T}^2) \times \mathbb{T}^2; \mathbb{R})$.
- $\alpha_i = \int_{\mathbb{T}^2} gv_i \in H^3(\overline{BDiff}(\mathbb{T}^2); \mathbb{R})$.
- $\gamma_j^i = \int_{\eta_i} gv_j \in H^4(\overline{BDiff}(\mathbb{T}^2); \mathbb{R})$.
- $\beta_i = \sigma^* gv_i \in H^5(\overline{BDiff}(\mathbb{T}^2); \mathbb{R})$.

The basic fibration sequence (1) in this case becomes

$$\overline{BDiff}(\mathbb{T}^2) \xrightarrow{\iota} \overline{BDiff}^+(\mathbb{T}^2)^\delta \xrightarrow{\iota_\delta} \overline{BDiff}^+(\mathbb{T}^2) \cong BSL(2, \mathbb{Z}).$$

The fundamental group $SL(2, \mathbb{Z}) = \pi_1(\overline{BDiff}^+(\mathbb{T}^2))$ acts on the homology group $H_1(\overline{Diff}(\mathbb{T}^2); \mathbb{Q})$ and the only invariant subspace is the trivial one. Thus, the classes $\{\gamma_j^i\} \subset H^4(\overline{BDiff}(\mathbb{T}^2); \mathbb{R})$ defined above by slant product with the homology 1-cycles η_i are killed in the homology spectral sequence for the fibration (1). That is, their images in $H_1(\overline{BDiff}^+(\mathbb{T}^2)^\delta; \mathbb{Q})$ are trivial.

For $M = \Sigma_g$, the genus $g \geq 2$ case, there are many more constructions of classes in $H_*(\overline{Diff}(\Sigma_g); \mathbb{Q})$ and the complexity of the study of $\mathcal{S}(M)$ in higher dimensions $q \geq 2$ emerges.

The classes $\{\alpha_1, \alpha_2\} \subset H^3(\overline{BDiff}_c^+(\Sigma)^\delta; \mathbb{R})$ defined by (40) are non-trivial by Theorem 10.2, and by Theorem 10.7 they define a surjection of $H_3(\overline{BDiff}^+(\Sigma_g)^\delta; \mathbb{Q})$ onto \mathbb{R}^2 .

The classifying map $\nu_{\Sigma_g} : \Sigma_g \rightarrow BSO(2)$ is not trivial, as the surfaces have non-zero Euler class. However, by deleting a point, set $\Sigma_g^* = \Sigma_g \setminus \{m_0\}$, and now the tangent bundle of Σ_g^* is trivial. Also, the homology group $H_1(\Sigma_g^*; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ is generated by $2g$ simple closed curves $\{C_i \mid 1 \leq i \leq 2g\}$ based at a point $z_0 \neq m_0$ which give generators of $H_1(\Sigma_g^*; \mathbb{Z})$. We then use Theorem 4.7 to construct classes, for $1 \leq i \leq 2g$ and $j = 1, 2$,

$$(45) \quad \gamma_j^i = \int_{C_i} gv_j \in H^4(\overline{BDiff}_c(\Sigma_g^*); \mathbb{R})$$

Correspondingly, there are $4g$ families of linearly independent classes in $H_4(\overline{BDiff}_c(\Sigma_g^*); \mathbb{Q})$ such that the classes $\{\gamma_j^i \mid 1 \leq i \leq 2g, j = 1, 2\}$ evaluate on them to give a surjection onto \mathbb{R}^{4g} . Observe that there are inclusions

$$(46) \quad H^4(\overline{BDiff}_c(\Sigma_g^*); \mathbb{R}) \subset H^4(\overline{BDiff}(\Sigma_g); \mathbb{R}), \quad H_4(\overline{BDiff}_c(\Sigma_g^*); \mathbb{Q}) \subset H_4(\overline{BDiff}(\Sigma_g); \mathbb{Q}),$$

so we have:

THEOREM 10.11. *For $g \geq 2$, there exists linearly independent classes*

$$(47) \quad \{\gamma_j^i \mid 1 \leq i \leq 2g, j = 1, 2\} \subset H^4(\overline{BDiff}(\Sigma_g); \mathbb{R})$$

which evaluate on $H_4(\overline{BDiff}(\Sigma_g); \mathbb{Q})$ to yield a surjection onto \mathbb{R}^{4g} .

In addition, the curves $\{C_i \mid 1 \leq i \leq 2g\}$ can be chosen to be intersect only in the basepoint z_0 . It follows that the classes they define in $H_4(\overline{BDiff}(\Sigma_g); \mathbb{Q})$ have disjoint support, so we obtain a map analogous to the product map in (32). This yields non-trivial homology classes in $H_{4k}(\overline{BDiff}(\Sigma_g); \mathbb{Q})$ for k arbitrarily large.

The connected component $\text{Diff}_0(\Sigma_g)$ is contractible, and the quotient $\mathcal{MCG}_g = \text{Diff}^+(\Sigma_g)/\text{Diff}_0(\Sigma_g)$ is the *mapping class group* for genus g surfaces. The group $\text{Diff}_0(\Sigma_g)$ acts as the identity on $H_1(\Sigma_g; \mathbb{Z})$ so the action of $\text{Diff}^+(\Sigma_g)$ on $H_1(\Sigma_g; \mathbb{Q})$ factors through an action of \mathcal{MCG}_g . These actions are very well understood, and in particular there are no invariant subspaces for the action. Thus, the homology classes in Theorem 10.11 map to trivial classes in $H_1(\text{BDiff}^+(\Sigma_g)^\delta; \mathbb{Q})$.

Consider the map $\iota_\delta^*: H^*(\text{BDiff}^+(\Sigma_g); \mathbb{R}) \rightarrow H^*(\text{BDiff}^+(\Sigma_g)^\delta; \mathbb{R})$. We have $\text{BDiff}^+(\Sigma_g) \rightarrow \mathcal{BMCG}_g$ is a homotopy equivalence, and the cohomology groups $H^*(\mathcal{BMCG}_g; \mathbb{Q})$ have been extensively studied; see Margalit [91, Section 9] and Morita [109]. In a celebrated work [105, 107], Morita constructed many families of classes in $H^*(\text{Diff}^+(\Sigma_g); \mathbb{Q})$ where Σ_g is a compact surface of genus $g \geq 2$, and showed the map $\iota_\delta^*: H^*(\text{Diff}^+(\Sigma_g); \mathbb{Q}) \rightarrow H^*(\text{Diff}^+(\Sigma_g)^\delta; \mathbb{Q})$ is not injective.

We briefly recall the construction of the Morita-Mumford-Miller classes in $H^*(\text{BDiff}^+(\Sigma_g); \mathbb{Q})$. The tangent bundle to the fibers of the universal map $\Sigma_g \rightarrow \text{MDiff}^+(\Sigma_g) \rightarrow \text{BDiff}^+(\Sigma_g)$ are 2-dimensional, and let $e_g \in H^2(\text{MDiff}^+(\Sigma_g); \mathbb{Q})$ denote its Euler class. Then integrate the powers of e_g over the fibers to obtain the Morita-Mumford-Miller *tautological classes*:

$$(48) \quad \kappa_\ell = \int_{\Sigma_g} e_g^{\ell+1} \in H^{2\ell}(\text{BDiff}^+(\Sigma_g); \mathbb{Q}) \cong H^{2\ell}(\mathcal{BMCG}_g; \mathbb{Q}), \quad \ell \geq 1.$$

Miller [98] and Morita [107] proved the following seminal result:

THEOREM 10.12. [107, Theorem 6.1] *For $n \geq 1$ there exists a genus $g(n) \geq 2$, where $g(n) \leq 6n$, and canonical relations on the graded polynomial ring $\mathbb{Q}[\kappa_1, \dots, \kappa_{g-2}]$ such that the map*

$$(49) \quad \mathbb{Q}[\kappa_1, \dots, \kappa_{g-2}]/\text{relations} \rightarrow H^*(\text{BDiff}^+(\Sigma_g); \mathbb{Q})$$

is injective up to degree $2n$ for all $g \geq g(n)$.

A celebrated paper by Madsen and Weiss proved the remarkable result:

THEOREM 10.13. [90] *For $n \geq 1$ and g sufficiently large, there is an isomorphism*

$$(50) \quad \{\mathbb{Q}[\kappa_1, \dots, \kappa_{g-2}]/\text{relations}\}^{2n} \cong H^{2n}(\text{BDiff}^+(\Sigma_g); \mathbb{Q})$$

This result was given alternative, “more geometric” proofs by Eliashberg, Galatius and Mishachev [29] and Hatcher [59]. In contrast to the conclusion of Theorem 10.13, Morita showed in the celebrated work [107, 108] that the higher MMM-classes always vanish in $H^*(\text{BDiff}^+(\Sigma_g)^\delta; \mathbb{Q})$.

THEOREM 10.14. *The class $\kappa_\ell \in H^{6\ell}(\text{BDiff}^+(\Sigma_g)^\delta; \mathbb{Q})$ is trivial for $\ell \geq 3$.*

The proof that $\kappa_\ell = 0$ follows from the Bott Vanishing Theorem. Observe that $e_g^2 = p_i$ is the first Pontrjagin class of the tangent bundle to the fibers, and so $e_g^{\ell+1} \in H^{6\ell+2}(\text{BDiff}^+(\Sigma_g)^\delta; \mathbb{Q})$ vanishes for $\ell \geq 3$. It is unknown whether $\kappa_2 = 0$, as it is one of the intriguing open problems whether the cube of the Euler class $e_g^3 \in H^6(\text{B}\Gamma_2^+; \mathbb{Q})$ must vanish.

In contrast to this, Kotschick and Morita proved in [85] the following result, which was given an alternate proof by Nariman [113, Theorem 0.12]:

THEOREM 10.15. *There is an inclusion $\mathbb{Q}[\kappa_1] \subset H^{2\ell}(\text{BDiff}^+(\Sigma_g)^\delta; \mathbb{Q})$ stably. That is, given $\ell \geq 3$ there exists g_ℓ so that for $g \geq g_\ell$ the class $\kappa_1^\ell \neq 0$ in $H^{2\ell}(\text{BDiff}^+(\Sigma_g)^\delta; \mathbb{Q})$.*

As discussed by Margalit [91, Section 9], the cohomology above the range of Theorem 10.13 contains many classes not of the form of the MMM-classes, and in fact the constructions of these classes show they grow exponentially as g increases. The general problem is thus:

PROBLEM 10.16. *For fixed $g \geq 2$, determine which classes in $H^*(\mathcal{BMCG}_g; \mathbb{Q})$ have non-zero image under the map $\iota_\delta^*: H^*(\mathcal{BMCG}_g; \mathbb{Q}) \cong H^*(\text{BDiff}^+(\Sigma_g); \mathbb{Q}) \rightarrow H^*(\text{BDiff}^+(\Sigma_g)^\delta; \mathbb{Q})$.*

There are further results related to Problem 10.16 in the works by Kotschick and Morita in [85], Bowden in [18] and Nariman in [112, 113].

11. CHARACTERISTIC CLASSES FOR DIMENSION 3

Let M be a connected oriented 3-manifold. The tangent bundle to M is trivial, so the methods in Section 4 can be applied to construct families of classes in $H_*(B\overline{\text{Diff}}_c(M); \mathbb{Q})$. Moreover, for M of dimension $q \geq 3$, there are additional families of spherical classes in $H^*(B\Gamma_q^+; \mathbb{Q})$ beyond the Godbillon-Vey type classes in $H^{2q+1}(B\overline{\Gamma}_3; \mathbb{R})$. For the case when $q = 3$, these new spherical classes appear in degree $H^{10}(B\overline{\Gamma}_3; \mathbb{R})$ which give rise to even more families of homology classes in $H_*(B\overline{\text{Diff}}_c(M); \mathbb{Q})$. First, we recall the Vey basis [44]:

$$\begin{aligned} H^{*>6}(WO_3) &= \bigwedge (y_1 c_1^3, y_1 c_1 c_2, y_1 c_3, y_1 y_3 c_1^3, y_1 y_3 c_1 c_2, y_3 c_3) \\ H^*(W_3) &= \bigwedge (y_1 c_1^3, y_1 c_1 c_2, y_1 c_3, y_2 c_2, y_1 y_2 c_1^3, y_1 y_2 y_3 c_1 c_2, y_1 y_2 y_3 c_1^3, y_1 y_2 c_1 c_2, y_3 c_3) . \end{aligned}$$

Theorem 7.1 implies the classes $\{\Delta(y_1 c_1^3), \Delta(y_1 c_1 c_2), \Delta(y_1 c_3)\} \in H^7(B\Gamma_3^+; \mathbb{R})$ are spherical classes, as was previously observed in [65, 66]. The linear independence of the classes in the image of $\Delta: H^7(W_3) \rightarrow H^7(B\overline{\Gamma}_3; \mathbb{R})$ follows from the works cited for the proof of Theorem 6.3.

In addition, for dimension $q = 3$, Theorem 7.2 yields additional families of spherical classes as follows. The first Pontrjagin class $p_1 \in H^4(BSO(3); \mathbb{R})$ is spherical, detected by a map $\xi_4: \mathbb{S}^4 \rightarrow BSO(3)$. As $B\overline{\Gamma}_3$ is 4-connected, there is a lift $\widehat{\xi}_1: \mathbb{S}^4 \rightarrow B\Gamma_3^+$ with $\nu \circ \widehat{\xi}_1 = \xi_1$. The brace product with the class $[\widehat{\xi}_1] \in \pi_4(B\Gamma_3^+)$ as defined in (66) gives a mapping

$$(51) \quad \{[\widehat{\xi}_1], \cdot\}: \pi_j(B\overline{\Gamma}_3) \rightarrow \pi_{j+3}(B\overline{\Gamma}_3) .$$

The classes $\{\Delta(y_1 c_1^3), \Delta(y_1 c_1 c_2), \Delta(y_1 c_3)\} \subset H_7(B\overline{\Gamma}_3; \mathbb{R})$ are spherical, so using the mapping (51) the classes $\{\Delta(y_1 y_2 c_1^3), \Delta(y_1 y_2 c_1 c_2), \Delta(y_1 y_2 c_3)\} \subset H^{10}(B\overline{\Gamma}_3; \mathbb{R})$ are also spherical. Using the procedures of the previous sections, we then obtain:

THEOREM 11.1. *The classes $\{\Delta(y_1 c_1^3), \Delta(y_1 c_1 c_2), \Delta(y_1 c_3)\} \in H^7(B\Gamma_3^+; \mathbb{R})$ yield a surjection*

$$(52) \quad \{\Delta(y_1 c_1^3), \Delta(y_1 c_1 c_2), \Delta(y_1 c_3)\}: \pi_7(B\overline{\Gamma}_3) \rightarrow \pi_7(B\Gamma_3^+) \rightarrow \mathbb{R}^3 .$$

PROPOSITION 11.2. *Choose a Hamel basis $\{a_i \mid i \in \mathcal{I}\} \subset \mathbb{R}$ of \mathbb{R} over \mathbb{Q} . Then there exists a collection of maps $\{f_{i,j,k}: \mathbb{S}^7 \rightarrow B\overline{\Gamma}_3 \mid i, j, k \in \mathcal{I}\}$ such that*

$$\langle \Delta(y_1 c_1^3), [f_{i,j,k}] \rangle = a_i , \quad \langle \Delta(y_1 c_1 c_2), [f_{i,j,k}] \rangle = a_j , \quad \langle \Delta(y_1 c_3), [f_{i,j,k}] \rangle = a_k .$$

In particular, the classes $\{[f_{i,j,k}] \mid i, j, k \in \mathcal{I}\} \subset \pi_7(B\overline{\Gamma}_3)$ are linearly independent over \mathbb{Q} .

THEOREM 11.3. *The cohomology ring $H^*(\Omega^3 B\overline{\Gamma}_3; \mathbb{Q})$ contains a free exterior algebra with basis an uncountable collection $\{\phi_{i,j,k} \mid i, j, k \in \mathcal{I}\} \subset H^4(\Omega^3 B\overline{\Gamma}_3; \mathbb{Q})$.*

COROLLARY 11.4. *The groups $H^{4\ell}(\overline{B\text{Diff}}_c(\mathbb{R}^3); \mathbb{Q})$ are uncountable for all $\ell > 0$.*

PROBLEM 11.5. *Show that $\sigma_3: H_{4\ell}(\Omega^3 B\overline{\Gamma}_3; \mathbb{Q}) \rightarrow H_{4\ell+3}(B\overline{\Gamma}_3; \mathbb{Q})$ is non-trivial for $\ell > 1$.*

Recall that $\tilde{\nu}_M: M\overline{\text{Diff}}_c^+(M)^\delta \rightarrow B\overline{\Gamma}_3^+$ classifies the foliation on the total space over $B\overline{\text{Diff}}_c^+(M)^\delta$. Define the generalized Godbillon-Vey classes in $H^7(M\overline{\text{Diff}}_c^+(M)^\delta; \mathbb{R})$:

$$(53) \quad gv_1 = \tilde{\nu}_M^*(\Delta(h_1 c_1^3)) , \quad gv_2 = \tilde{\nu}_M^*(\Delta(h_1 c_1 c_2)) , \quad gv_3 = \tilde{\nu}_M^*(\Delta(h_1 c_3)) .$$

Define classes $\{\alpha_1, \alpha_2, \alpha_3\} \subset H^4(B\overline{\text{Diff}}_c^+(M)^\delta; \mathbb{R})$ by integration over the fiber:

$$(54) \quad \alpha_1 = \int_M gv_1 , \quad \alpha_2 = \int_M gv_2 , \quad \alpha_3 = \int_M gv_3 .$$

Next, $\{f_{i,j,k}: \mathbb{S}^7 \rightarrow B\overline{\Gamma}_3 \mid i, j, k \in \mathcal{I}\}$ be the collection of maps constructed in Proposition 11.2. By Proposition 8.3, for each $i, j, k \in \mathcal{I}$, there exists a compact oriented 4-manifold $N_{i,j,k}$ and map $g_{i,j,k}: N_{i,j,k} \rightarrow B\overline{\text{Diff}}_c(M)$ such that the classes $\{[g_{i,j,k}(N_{i,j,k})] \mid i, j, k \in \mathcal{I}\} \subset H_4(B\overline{\text{Diff}}_c(M); \mathbb{Q})$ are rationally linearly independent in the image of $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3): H_4(B\overline{\text{Diff}}_c(M); \mathbb{Q}) \rightarrow \mathbb{R}^3$. By the comments at the end of Section 8, the image of these classes inject into $H_4(B\overline{\text{Diff}}_c^+(M)^\delta; \mathbb{Q})$:

THEOREM 11.6. *For a connected oriented 3-manifold M , the cycles*

$$(55) \quad \{[g_{i,j,k}(N_{i,j,k})] \mid i, j, k \in \mathcal{I}\} \subset H_4(\overline{BDiff}_c^+(M)^\delta; \mathbb{Q})$$

are linearly independent, and evaluation of the classes $\{\alpha_1, \alpha_2, \alpha_3\} \subset H^4(\overline{BDiff}_c^+(M)^\delta; \mathbb{R})$ on these cycles yields a surjection onto \mathbb{R}^3 .

Next, as M has trivial tangent bundle, Theorem 4.4 yields additional classes in $H^*(B\overline{\Gamma}_q^+; \mathbb{R})$. For the section $\sigma: \overline{BDiff}_c(M) \rightarrow \overline{BDiff}_c(M) \times M$, define classes in $H^7(\overline{BDiff}_c(M); \mathbb{R})$:

$$(56) \quad \beta_1 = \sigma^*(gv_1), \quad \beta_2 = \sigma^*(gv_2), \quad \beta_3 = \sigma^*(gv_3).$$

Then modify the maps $\{f_{i,j,k}: \mathbb{S}^7 \rightarrow B\overline{\Gamma}_3 \mid i, j, k \in \mathcal{I}\}$ constructed in Proposition 11.2, using the Mather-Thurston Theorem, to obtain:

THEOREM 11.7. *For $i, j, k \in \mathcal{I}$, there exists a compact oriented 7-manifold $N'_{i,j,k}$ and map $\widehat{f}'_{i,j,k}: N'_{i,j,k} \rightarrow \overline{BDiff}_c(M)$ such that the cycles $\{[\widehat{f}'_{i,j,k}(N'_{i,j,k})] \mid i, j, k \in \mathcal{I}\} \subset H_7(\overline{BDiff}_c(M); \mathbb{Q})$ are linearly independent, and evaluation of the classes $\{\beta_1, \beta_2, \beta_3\} \subset H^7(\overline{BDiff}_c(M); \mathbb{R})$ on these cycles yields a surjection onto \mathbb{R}^3 .*

Using the brace product and proceeding as before, we obtain:

THEOREM 11.8. $\{\Delta(y_1y_2c_1^3), \Delta(y_1y_2c_1c_2), \Delta(y_1y_2c_3)\} \in H^{10}(B\overline{\Gamma}_3; \mathbb{R})$ *yield a surjection*

$$(57) \quad \{\{\Delta(y_1y_2c_1^3), \Delta(y_1y_2c_1c_2), \Delta(y_1y_2c_3)\}: \pi_{10}(B\overline{\Gamma}_3) \rightarrow \pi_{10}(B\overline{\Gamma}_3^+) \rightarrow \mathbb{R}^3.$$

THEOREM 11.9. *For $i, j, k \in \mathcal{I}$, there exists a compact oriented 10-manifold $N''_{i,j,k}$ and map $\widehat{f}''_{i,j,k}: N''_{i,j,k} \rightarrow \overline{BDiff}_c(M)$ such that the cycles $\{[\widehat{f}''_{i,j,k}(N''_{i,j,k})] \mid i, j, k \in \mathcal{I}\} \subset H_{10}(\overline{BDiff}_c(M); \mathbb{Q})$ are linearly independent, and evaluation of the classes*

$$\{(\iota_0 \circ \sigma)^*(\Delta(y_1y_2c_1^3)), (\iota_0 \circ \sigma)^*(\Delta(y_1y_2c_1c_2)), (\iota_0 \circ \sigma)^*(\Delta(y_1y_2c_3))\} \subset H^{10}(\overline{BDiff}_c(M); \mathbb{R})$$

on these cycles yields a surjection onto \mathbb{R}^3 .

Using the methods of Theorem 4.7, when $H^1(M; \mathbb{Q})$ is non-trivial, additional families of classes can be constructed in $H_6(\overline{BDiff}_c(M); \mathbb{Q})$ and $H_9(\overline{BDiff}_c(M); \mathbb{Q})$, for each class $0 \neq Z \in H^1(M; \mathbb{Z})$ defined by a mapping $M \rightarrow \mathbb{S}^1$. Details are left to the reader.

Note that the cycles constructed in the proofs of Theorems 11.7 and 11.9, and using the methods of Theorem 4.7 are all supported in the interior of M . A 3-manifold admits a decomposition into its irreducible components

$$(58) \quad M = M_1 \#_{\mathbb{S}^2} M_2 \#_{\mathbb{S}^2} \cdots \#_{\mathbb{S}^2} M_k.$$

Then we can apply the above results to construct families of cycles for each irreducible component M_ℓ and they inject into $H_*(\overline{BDiff}_c(M); \mathbb{Q})$. One can thus apply these procedures in a ‘‘mix-and-match’’ construction.

The classes obtained using Theorem 11.6 are non-trivial in $H_4(\overline{BDiff}_c^+(M)^\delta; \mathbb{Q})$, but for the other classes constructed above in $H_*(\overline{BDiff}_c(M); \mathbb{Q})$, it seems to be completely unknown whether any of them survive for the map to $H_*(\overline{BDiff}_c^+(M)^\delta; \mathbb{Q})$.

Finally, assume that M is compact, and consider the basic fibration sequence (1) and its associated Leray-Serre spectral sequence, with E_2 -term

$$(59) \quad E_2^{r,s}(\overline{BDiff}_c^+(M)^\delta; \mathbb{Q}) \cong H^r(\overline{BDiff}_c^+(M); H^s(\overline{BDiff}_c(M)); \mathbb{Q}).$$

The topological type of $\overline{Diff}^+(M)$ has been determined for many compact 3-manifolds [60], so an analysis of (59) may be feasible for such manifolds. We consider some select cases.

Let $M = \mathbb{S}^3$, then the inclusion $\text{SO}(4) \subset \text{Diff}^+(\mathbb{S}^3)$ is a homotopy equivalence by Hatcher [58]. Thus $H^*(\overline{BDiff}_c^+(\mathbb{S}^3); \mathbb{Q}) \cong \mathbb{Q}[e_4, p_1]$ is a polynomial ring with generators the first Pontrjagin class p_1 and the Euler class e_4 , both of degree 4. Let $\tilde{p}_1 = \iota_\delta^*(p_1)$ and $\tilde{e}_4 = \iota_\delta^*(e_4)$ in $H^4(\overline{BDiff}_c^+(\mathbb{S}^3)^\delta; \mathbb{Q})$. Nariman has proved the following remarkable result:

THEOREM 11.10. [116, Theorem 1.2] *There is an inclusion $\mathbb{Q}[\tilde{e}_4, \tilde{p}_1] \subset H^4(B\text{Diff}^+(\mathbb{S}^3)^\delta; \mathbb{Q})$.*

Let $M = \mathbb{S}^1 \times \mathbb{S}^2$, Hatcher showed in [57] that the group $\text{Diff}(\mathbb{S}^1 \times \mathbb{S}^2)$ has the homotopy type of $O(2) \times O(3) \times \Omega O(3)$, though the H-space structure is not preserved.

PROBLEM 11.11. *Determine the image of $\iota_\delta^*: H^*(B\text{Diff}(\mathbb{S}^1 \times \mathbb{S}^2); \mathbb{Q}) \rightarrow H^*(B\text{Diff}(\mathbb{S}^1 \times \mathbb{S}^2)^\delta; \mathbb{Q})$.*

Let $M = \mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, Hamstrom showed in [55] that the homotopy groups $\pi_\ell(\text{Diff}(\mathbb{T}^3))$ are trivial for $\ell > 1$. On the other hand, there are inclusions $\mathbb{T}^3 \subset \text{Diff}_0(\mathbb{T}^3)$ and $\text{SL}(3, \mathbb{Z}) \rightarrow \text{Diff}^+(\mathbb{T}^3)$, and their images generate a subgroup isomorphic to the semi-direct product $G = \mathbb{T}^3 \rtimes \text{SL}(3, \mathbb{Z})$. The following problem seems approachable.

PROBLEM 11.12. *Determine the image of $\iota_\delta^*: H^*(B\text{Diff}(\mathbb{T}^3); \mathbb{Q}) \rightarrow H^*(B\text{Diff}(\mathbb{T}^3)^\delta; \mathbb{Q})$.*

Finally, let M_κ denote a compact oriented 3-manifold without boundary, and constant curvature $\kappa = \pm 1$. Then Bamler and Kleiner determine the homotopy type of $\text{Diff}^+(M_\kappa)$ in [3]; see also the monograph [63]. Again, it seems that nothing is known about the image of the map ι_δ^* in this case.

12. CHARACTERISTIC CLASSES FOR HIGHER DIMENSIONS

Let M be an oriented connected manifold of dimension $q > 3$. The constructions of classes in $H^*(B\text{Diff}_c^+(M); \mathbb{R})$ and $H_*(\overline{B\text{Diff}}_c(M); \mathbb{Q})$ proceeds using a combination of the methods described in Sections 6, 7 and 8, but with added complexity due to the increasing number of linearly independent spherical classes in $H^{2q+1}(B\Gamma_q^+; \mathbb{R})$ and $H^{2q+1}(B\overline{\Gamma}_q; \mathbb{R})$. In addition, there are multiple brace product operations on $\pi_*(B\overline{\Gamma}_q)$ that create additional spherical classes. We give a broad description of the constructions that are possible, but more precise statements require fixing the dimension and topological type of the manifold M .

For manifolds of dimension $q \geq 5$ there also arise a variety of non-trivial classes in $H^*(B\text{Diff}_c^+(M); \mathbb{Q})$ constructed using pseudo-isotopy theory. The relationship of these with $H^*(B\text{Diff}_c^+(M)^\delta; \mathbb{Q})$ is completely unknown. It would be especially interesting to understand the images under ι_δ^* of the exotic classes described in the works [36, 87, 138].

First, we describe the general form of the results obtained in previous sections. Let \mathcal{V}_q be the collection of elements $y_i c_J \in WO_q$ of the Vey basis for $H^{2q+1}(WO_q)$ which are independently variable as in Theorems 6.1 and 6.2, [62]. Let v_q denote the cardinality of the set \mathcal{V}_q .

Theorem 7.1 implies that $\Delta(\mathcal{V}_q)$ is a linearly independent set of spherical classes in $H^{2q+1}(B\Gamma_q^+; \mathbb{R})$, whose evaluation on $H_{2q+1}(B\Gamma_q^+; \mathbb{Q})$ define surjections onto \mathbb{R}^{v_q} .

Define $\alpha_{i,J} = \int_M \tilde{\nu}^*(\Delta(y_i c_J)) \in H^{q+1}(B\text{Diff}_c^+(M)^\delta; \mathbb{R})$ for $y_i c_J \in \mathcal{V}_q$.

By Proposition 8.3, there are v_q families of uncountably generated subgroups of $H_{q+1}(\overline{B\text{Diff}}_c(M); \mathbb{Q})$, and the evaluation of $\tilde{\alpha}$ on the images of these classes in $H_{q+1}(\overline{B\text{Diff}}_c(M); \mathbb{Q})$ yields a surjective map onto \mathbb{R}^{v_q} . In particular, these classes inject into $H_{q+1}(B\text{Diff}_c^+(M)^\delta; \mathbb{Q})$.

Next, consider the application of the brace product as in Theorem 7.2. Let κ_q be the greatest integer such that $4\kappa_q \leq q + 1$. The first solution $\kappa_3 = 1$ appears when $q \geq 3$, and a second solution $\kappa_7 = 2$ appears when $q \geq 7$, and so forth. Then for each $1 \leq \ell \leq \kappa_q$ there exists $\xi_\ell: \mathbb{S}^{4\ell} \rightarrow B\Gamma_q^+$ such that the pull-back $\xi_\ell^* \circ \nu^*(p_\ell) \neq 0$, where $p_\ell \in H^{4\ell}(BSO(q); \mathbb{Z})$ is the ℓ -th Pontrjagin class. Then the brace product defines a mapping $\{\xi_\ell, \cdot\}: \pi_j(B\overline{\Gamma}_q) \rightarrow \pi_{j+4\ell-1}(B\overline{\Gamma}_q)$.

Define the extended subset of the Vey basis for $H^*(WO_q)$:

$$(60) \quad \widehat{\mathcal{V}}_q = \{y_I y_{I'} c_J \mid y_I c_J \in \mathcal{V}_q, I' = (i'_1 < \dots < i'_r) \text{ each } i_\ell \text{ even with } 2i'_r \leq \kappa_q\}$$

Let $\widehat{\mathcal{V}}_{q,k} \subset \widehat{\mathcal{V}}_q$ denote the terms of degree k , and let $\widehat{v}_{q,k}$ denote the cardinality of the set $\widehat{\mathcal{V}}_{q,k}$. Then as in previous sections, we have the non-triviality results as follows.

THEOREM 12.1. *For $k \geq 2q + 1$, the classes $\Delta(\widehat{\mathcal{V}}_{q,k}) \in H^k(B\overline{\Gamma}_q; \mathbb{R})$ yield a surjection*

$$(61) \quad \Delta(\widehat{\mathcal{V}}_{q,k}): \pi_k(B\overline{\Gamma}_q) \rightarrow \mathbb{R}^{\widehat{\mathcal{V}}_{q,k}} .$$

PROPOSITION 12.2. *Choose a Hamel basis $\{a_i \mid i \in \mathcal{I}\} \subset \mathbb{R}$ of \mathbb{R} over \mathbb{Q} . Then for each $k \geq 2q + 1$ with $\widehat{\mathcal{V}}_{q,k} > 0$, there exists a collection of maps $\{f_{I,I',J,i}: \mathbb{S}^k \rightarrow B\overline{\Gamma}_q \mid y_i y_{I'} c_J \in \widehat{\mathcal{V}}_{q,k}, i \in \mathcal{I}\}$ such that $\langle \Delta(y_i y_{I'} c_J), [f_{I,I',J,i}] \rangle = a_i$, and evaluates to zero on these spherical cycles otherwise. In particular, the classes $\{[f_{I,I',J,i}]\} \subset \pi_k(B\overline{\Gamma}_q)$ are linearly independent over \mathbb{Q} .*

As before, this implies that the cohomology ring $H^*(\Omega^q B\overline{\Gamma}_q; \mathbb{Q})$ contains a free exterior algebra with basis indexed by the set $\widehat{\mathcal{V}}_{q,k} \times \mathcal{I}$, and correspondingly the groups $H^*(B\overline{\text{Diff}}_c(\mathbb{R}^q); \mathbb{Q})$ are uncountable.

In this manner, corresponding to the variable spherical classes in $H^{2q+1}(W_q)$ we obtain families of infinite dimensional subspaces of $\pi_k(B\overline{\Gamma}_q)$ for $k = 2q + 4, 2q + 8, 2q + 11, 2q + 12, \dots$, and thus subspaces of $H_{k-q}(B\overline{\text{Diff}}_c(M); \mathbb{Q})$ with an uncountable number of rationally independent generators. This just accounts for the classes obtained by integration over the fiber method in Theorem 4.2. There are additional homology classes corresponding to the methods in Theorems 4.4 and 4.7.

REMARK 12.3. The results on the homology groups $H_*(B\overline{\text{Diff}}_c(M); \mathbb{Q})$ and $H_*(B\overline{\text{Diff}}_c^+(M)^\delta; \mathbb{Q})$ show that they contain many non-trivial cycles, and each of these cycles is represented by a map $g: N \rightarrow B\overline{\text{Diff}}_c(M)$ which is detected by a secondary class in $H^*(B\overline{\Gamma}_q; \mathbb{R})$ or in $H^*(B\overline{\Gamma}_q^+; \mathbb{R})$. Each such cycle induces a representation $g_\#: \pi_1(N, x_0) \rightarrow \text{Diff}_c(M)$. It is unknown what is the ‘‘geometric meaning’’ of the non-triviality of the homology class of $g(N)$, or of its pairing with a secondary class used in its construction? The author’s works [70, 72] related the vanishing of the secondary invariants with ergodic properties of the action of $\pi_1(N, x_0)$ on M induced by $g_\#$. However, the construction of the cycles N using foliated surgery, as in Meigniez’s proof of the Mather-Thurston Theorem in [97], suggests that the groups $\pi_1(N, x_0)$ are absolutely enormous, and in particular not amenable, so the question remains essentially open, as to what the constructions in this paper mean in a geometric or dynamical context.

We conclude with a discussion of another direction of research, which is very much in the spirit of the recent works by Galatius, Kupers, Nariman and Randal-Williams. Let M be compact, and Consider the basic fibration sequence $B\overline{\text{Diff}}_c(M) \xrightarrow{i} B\overline{\text{Diff}}_c^+(M)^\delta \xrightarrow{\iota} B\overline{\text{Diff}}_c^+(M)$ and its associated Leray-Serre spectral sequence, with E_2 -term

$$(62) \quad E_2^{r,s}(B\overline{\text{Diff}}_c^+(M)^\delta; \mathbb{Q}) \cong H^r(B\overline{\text{Diff}}_c^+(M); H^s(B\overline{\text{Diff}}_c(M)); \mathbb{Q}) .$$

The first step in analyzing (62) is to understand the groups $H^*(B\overline{\text{Diff}}_c(M); \mathbb{Q})$, but this appears to be an almost completely unknown for dimensions $q \geq 4$. For example, according to Kupers [86], apparently nothing is known about $H^*(B\overline{\text{Diff}}(\mathbb{S}^4); \mathbb{Q})$. Even the simplest case when $M = \mathbb{D}^q$ for $q \geq 4$ is wide open. Kupers [86] writes:

Question 1.1.3. What is the homotopy type of $\text{Diff}_c(\mathbb{D}^q)$? For example, what are its path components? What are its homology groups? What are its homotopy groups? Is it homotopy equivalent to a Lie group? Are its homotopy groups finitely generated? Can we relate it to other objects in algebraic topology? Can we relate it to other objects in manifold theory?

One of the original motivations for the work in [69] was a result of Farrell and Hsiang [30, 64]:

THEOREM 12.4. *For $\ell \ll q$ and $\ell \equiv 3 \pmod{4}$ and q odd:*

$$(63) \quad \pi_\ell(\text{Diff}_0(\mathbb{T}^q)) \otimes \mathbb{Q} = \mathbb{Q} .$$

PROBLEM 12.5. *Show that the Farrell-Hsiang classes in Theorem 12.4 vanish in the image of $i_\delta^*: H^k(\text{Diff}_0(\mathbb{T}^q); \mathbb{Q}) \rightarrow H^k(\text{Diff}_0(\mathbb{T}^q)^\delta; \mathbb{Q})$.*

The vanishing of classes in the image of ι_δ^* implies the existence of non-trivial classes in some groups $H_\ell(\overline{B\text{Diff}}(\mathbb{T}^q); \mathbb{Q})$ for $1 < \ell \ll q$, and hence by the Mather-Thurston Theorem to the existence of non-trivial classes in $H_k(\overline{B\Gamma}_q; \mathbb{Q})$ in the “unknown range” $q + 1 < k \leq q$, so is of great interest.

The works of Galatius and Randal-Williams greatly extend this seminal result of Farrell and Hsiang in their works [35, 36, 38, 37, 39], and the techniques used there have intriguing aspects analogous to ideas in foliation theory. The (stabilized) homotopy groups of $\text{Diff}_0(M)$ can be calculated in terms of surgery groups and the Waldhausen K-theory of M , as described for example by Hatcher in [56, 60] and Loday [89]. The standard method of calculating the Waldhausen K-theory groups introduces the notion of *block automorphisms*, denoted $\tilde{\mathcal{A}}(M)$, that were introduced by Burghelea in [19]. A discussion of the relation between block automorphisms and the study of diffeomorphism groups is given in Section 2 of Hatcher [56]. (See also the works of Burghelea [20, 21, 22], Cohen [26] and Weiss and Williams [144].) At an intuitive level, the notion of a block automorphism is analogous to the fracturing technique in the proof of the Mather-Thurston Theorem, as described say in Mather’s paper [95]. It seems a very intriguing question whether this analogy has a formal mathematical explanation. This suggests a far-reaching and speculative research program:

PROBLEM 12.6. *Find geometric or algebraic correspondences between the Waldhausen K-theory groups of a manifold M and the groups $\mathcal{K}_*(M)$.*

APPENDIX A. BASIC HOMOTOPY TECHNIQUES

We recall some basic results from homotopy theory that are frequently used in our constructions.

A.1. Hurewicz Theorems. The Hurewicz Theorem relates the homotopy and homology groups.

THEOREM A.1. *Let X be an r -connected space for $r \geq 1$. That is, X is a simply connected space with $\pi_k(X) = 0$ for $0 \leq k \leq r$. Then the Hurewicz homomorphism $h_*: \pi_k(X) \rightarrow H_k(X; \mathbb{Z})$ is an isomorphism for $0 < k \leq r + 1$.*

The Rational Hurewicz Theorem relates $\pi_*(X) \otimes \mathbb{Q}$ with the groups $H_*(X; \mathbb{Q})$. Klaus and Kreck give an elementary proof of this result in [84], and provide additional interesting discussions on this fundamental theorem.

THEOREM A.2. *Let X be a simply connected space with $\pi_k(X) \otimes \mathbb{Q} = 0$ for $0 \leq k \leq r$. Then the Hurewicz map induces an isomorphism*

$$(64) \quad h: \pi_k(X) \otimes \mathbb{Q} \longrightarrow H_k(X; \mathbb{Q})$$

for $1 \leq k \leq 2r$, and a surjection for $k = 2r + 1$.

A.2. Integration over the fiber. The operation of “integration over the fiber” is often used in the context of sphere bundles, but it applies much more generally. It was first introduced in a paper by Chern [23], and stated more formally by Borel and Hirzebruch in [8, Section 8], where they explain the operation in terms of the differential on the E_2 -term of the spectral sequence associated to the fibration. This definition was related to the definition using differential forms by Auer [1]. Bott and Tu discuss integration over the fiber for vector bundles in [16, Section 6].

THEOREM A.3. *Let $p: E \rightarrow B$ be a fibration with fiber $X = p^{-1}(b)$, where B is connected and X is a compact oriented manifold of dimension q . Then for $n \geq q$, there exists a natural homomorphism*

$$(65) \quad \int_X : H^n(E; \mathbb{R}) \rightarrow H^{n-q}(B; \mathbb{R}),$$

called integration over the fiber. For $\alpha \in H^n(E; \mathbb{R})$ and $\omega \in H^\ell(B; \mathbb{R})$, $\omega \cdot \int_X(\alpha) = \int_X(p^*\omega \cdot \alpha)$.

A.3. Brace products. The *brace product* was introduced by James in [76].

PROPOSITION A.4. *Let $F \xrightarrow{\iota} E \xrightarrow{p} B$ be a fibration, and suppose there exists a section $\sigma: B \rightarrow E$. Then there is a well-defined product, called the brace product,*

$$(66) \quad \{\cdot, \cdot\}_\sigma: \pi_i(E) \times \pi_j(F) \rightarrow \pi_{i+j-1}(F) .$$

where the subscript σ indicates that the product depends upon the choice of the section σ .

Proof. Given classes $\alpha \in \pi_i(E)$ and $\beta \in \pi_j(F)$ the Whitehead product $[\sigma_\#(\alpha), \iota_\#(\beta)] \in \pi_{i+j-1}(E_q)$ is the image of a unique class denoted $\{\alpha, \beta\}_\sigma \in \pi_{i+j-1}(B\bar{\Gamma}_q)$. \square

The brace product vanishes when the space E is a product fibration, so the non-triviality of a pair $\{\alpha, \beta\}_\sigma$ is a measure of the “homotopy twisting” of the fibration. Kallel and Sjerve [78] relate the brace product to the structure of free loop space ΛE .

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