MAPPING CLASS GROUPS OF SOLENOIDAL MANIFOLDS

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Abstract. The goal of this work is to calculate the mapping class group of a solenoidal manifold. We introduce the dynamical commensurator group associated to an odometer action, and show its role in the calculation of the mapping class group. Our results generalize the previous results of Odden for the mapping class group of the universal solenoid over a Riemann surface, and of Bering and Studenmund for the mapping class group of the universal solenoid over a compact aspherical manifold. We include a selection of examples.

1. Introduction

Let $\mathfrak{M}$ be a compact connected metric space, and $\mathrm{Homeo}(\mathfrak{M})$ denote its group of homeomorphisms, equipped with the compact open topology. Let $\mathrm{Homeo}_0(\mathfrak{M})$ denote the connected component of the identity, which is a closed normal subgroup. The *mapping class group* of $\mathfrak{M}$ is the quotient

$$\mathrm{Mod}(\mathfrak{M}) = \mathrm{Homeo}(\mathfrak{M})/\mathrm{Homeo}_0(\mathfrak{M}) \ . \tag{1}$$

The algebraic group $\mathrm{Mod}(\mathfrak{M})$ can be calculated explicitly in many cases. For example, when $\mathfrak{M} = \Sigma_g$ is a closed surface of genus $g \geq 0$, then the group $\mathrm{Mod}(\Sigma_g)$ has been extensively studied [26]. The goal of this work is to calculate $\mathrm{Mod}(\mathfrak{M})$ for another class of spaces, the *solenoidal manifolds*.

Let M be a compact connected manifold without boundary, and set $\Gamma = \pi_1(M,x)$ for choice of $x \in M$. A *group chain* in Γ is a strictly descending chain of subgroups of finite index, $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$. A group chain $\mathcal{G}$ and basepoint $x \in M$ determine a tower of finite index coverings of $M_0 = M$ by closed manifolds:

$$\mathcal{P}_{\mathcal{G}} \equiv \left\{ M_0 \xleftarrow{p_1} M_1 \xleftarrow{p_2} M_2 \xleftarrow{p_3} \cdots \right\} \tag{2}$$

where M_0 is called the base of the tower. The inverse limit space

$$\mathfrak{M}_{\mathcal{P}} = \varprojlim \, \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \subset \prod_{\ell \geq 0} M_\ell \tag{3}$$

is called a *weak solenoid* by McCord [44], or a *solenoidal manifold* by Sullivan [58, 60].

The tower of coverings $\mathcal{P}_{\mathcal{G}}$ is said to be a *presentation* for $\mathfrak{M}_{\mathcal{P}}$. For each $\ell \geq 0$, the coordinate projection $\widehat{q}_\ell \colon \mathfrak{M}_{\mathcal{P}} \to M_\ell$ is a fibration, where the fiber $\mathfrak{X}_\ell = \widehat{q}_\ell^{-1}(x_\ell)$ a Cantor space.

PROBLEM 1.1. *Let $\mathfrak{M}_{\mathcal{P}}$ be a solenoidal manifold. Give an algebraic model for $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ which is computable in terms of the topology of M and the algebraic properties of the group chain $\mathcal{G}$.*

The approach to this problem, following the works of Odden [50] and Bering and Studenmund [7], is to first calculate the *pointed mapping class group* of $\mathfrak{M}_{\mathcal{P}}$. Fix a basepoint $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$, and let $\mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ denote the subgroup of $\mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}})$ which fixes $\widehat{x}$. Set

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) = \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})/\mathrm{Homeo}_0(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \ . \tag{4}$$

Then $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ is an algebraic extension of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$, as shown in Section 8, so if suffices to develop techniques for the study of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$. The key idea is to relate $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ with the abstract commensurator group of $\Gamma = \pi(M,x)$, as in [50, 7].

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A *commensurator* for a countable group Γ is a pair of finite-index subgroups $H, K \subset \Gamma$ and an isomorphism $\phi\colon H \to K$. Two commensurators $\phi_1\colon H_1 \to K_1$ and $\phi_2\colon H_2 \to K_2$ are equivalent, $\phi_1 \sim \phi_2$, if there exists a finite index subgroup $H_3 \subset H_1 \cap H_2$ such that $\phi_1|H_3 = \phi_2|H_3$.

DEFINITION 1.2. *The collection of all commensurators for* Γ*, modulo the equivalence* $\sim$*, is called the* abstract commensurator group $\mathrm{Comm}(\Gamma)$.

Let $\Gamma' \subset \Gamma$ be a finite index subgroup. Given a commensurator $\phi\colon H \to K$ in $\mathrm{Comm}(\Gamma)$, observe that $\phi\colon H \cap \Gamma' \to K \cap \Gamma'$ is a commensurator in $\mathrm{Comm}(\Gamma')$. The converse clearly holds, hence $\mathrm{Comm}(\Gamma') \cong \mathrm{Comm}(\Gamma)$. Thus, $\mathrm{Comm}(\Gamma)$ can be intuitively viewed as the group of "germs" of isomorphisms between finite-index subgroups of Γ. See [55, 56] for surveys by Studenmund of properties and calculations of abstract commensurator groups.

The fundamental idea behind the calculations of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ is a result from pro-homotopy theory. Mioduszewski showed in [45] that maps between inverse limits of polyhedra have a special form, which up to homotopy are a "zig-zag correspondence" between the towers of maps. Rogers and Tollefson gave a sharper version of this result in [52] for inverse limits of covering spaces, and in particular for a homeomorphism $h \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$. This result implies that h induces a pointed map between covering spaces of the base M, which naturally leads to the role of commensurators in the study of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$.

THEOREM 1.3. *Let* $\mathfrak{M}_{\mathcal{P}}$ *be a solenoidal manifold with base* M *defined by a group chain* $\mathcal{G}$ *in* $\Gamma = \pi_1(M, x)$*. Then there is a well-defined characteristic map*

$$\chi\colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \to \mathrm{Comm}(\Gamma) \ . \tag{5}$$

The construction of the map χ is given in Section 3. In general, the map χ need not be surjectiive, as given a commensurator $\phi \in \mathrm{Comm}(\Gamma)$ it is in the image of χ only if it preserves the group chain defining the coverings in $\mathcal{P}$, up to zig-zag equivalence. The map χ also need not be injective, as there may be non-trivial $h \in \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ which act as the identity on a finite index subgroup of Γ. The strategy for calculating $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ is thus to first calculate $\mathrm{Comm}(\Gamma)$, which has been done for many classes of groups [56].

PROBLEM 1.4. *Determine the image of the charatceristic map* χ*. Give conditions on the manifold* M *and group chain* $\mathcal{G}$ *which imply that the map* χ *injective.*

There is one case where this program has been carried through, which is for the *universal solenoids*. Let M be a closed connected manifold with fundamental group $\Gamma = \pi_1(M, x)$. The collection of all finite-index subgroups of Γ forms a partially ordered set $\mathcal{G}_u(\Gamma)$, with maps defined by inclusions. Then there is a "universal" group chain $\mathcal{G}_\nu(\Gamma) \subset \mathcal{G}_u(\Gamma)$ which is cofinal, where each Γ_ℓ in $\mathcal{G}_\nu(\Gamma)$ is a characteristic normal subgroup. The *universal solenoid* over M is defined to be $\widehat{M} = \mathfrak{M}_{\mathcal{P}}$ where $\mathcal{P}$ is the tower of coverings of M defined by $\mathcal{G}_\nu(\Gamma)$. The fiber of the projection $q_0\colon \widehat{M} \to M$ is the profinite completion $\widehat{\Gamma}$ of Γ, and there is a natural *right* action of $\widehat{\Gamma}$ by deck transformations on $\widehat{M}$.

For a closed surface Σ_g of genus $g \geq 2$, the space $\widehat{\Sigma_g}$ is called the *universal hyperbolic solenoid.* This space was introduced in the works of Biswas, Nag and Sullivan [9, 10, 11, 57] in their studies of the universal Teichmüller space of $\widehat{\Sigma_g}$. Odden proved in his thesis the seminal result:

THEOREM 1.5 (Odden, [50])**.** *The map* $\chi\colon \mathrm{Mod}(\widehat{\Sigma_g}, \widehat{x}) \to \mathrm{Comm}(\Gamma)$ *is an isomorphism, and induces an isomorphism* $\mathrm{Mod}(\widehat{\Sigma_g}) \cong \mathrm{Comm}(\Gamma) \times \widehat{\Gamma}$.

Let M be a closed aspherical manifold, that is, the universal cover of M is contractible. Let $\widehat{M}$ denote the universal solenoid over M. The work of Belk and Forrest [6], and Bering and Studenmund [7] extended the ideas of Odden to the universal solenoid $\widehat{M}$. In place of the group $\mathrm{Mod}(\widehat{M}, \widehat{x})$ they consider the group $\mathcal{E}(\widehat{M}, \widehat{x})$ of pointed homotopy self-equivalences of $\widehat{M}$.

THEOREM 1.6 (Bering & Studenmund, [7])**.** *Assume that* M *be a closed aspherical manifold, and its fundamental group* Γ *is residually finite, then there is an isomorphism* $\chi\colon \mathcal{E}(\widehat{M}, \widehat{x}) \cong \mathrm{Comm}(\Gamma)$*, and consequently* $\mathcal{E}(\widehat{M}) \cong \mathrm{Comm}(\Gamma) \times \widehat{\Gamma}$.

For a general solenoidal manifold $\mathfrak{M}_{\mathcal{P}}$, the analysis of Problem 1.4 is more subtle. In particular, the solution depends on the dynamical properties of the topological Cantor action $\Phi\colon \Gamma \times X_{\mathcal{G}} \to X_{\mathcal{G}}$ defined by the monodromy action of the fibration $\widehat{q}_0\colon \mathfrak{M}_{\mathcal{P}} \to M_0$. This action is minimal and equicontinuous (see Section 4), so is an odometer in the sense of [19, 20].

Let $(X_{\mathcal{G}}, \Gamma, \Phi)$ be an odometer. A *dynamical commensurator* at $x \in X_{\mathcal{G}}$ is a homeomorphism $h_U\colon U \to V$, where $U, V \subset X_{\mathcal{G}}$ are adapted subsets with $x \in U \cap V$, such that $h_U(x) = x$, and h_U induces an isomorphism $\Theta_U\colon \mathcal{H}_U \to \mathcal{H}_V$ of their restricted holonomy actions. Commensurators $h_U\colon U \to V$ and $h_{U'}\colon U' \to V'$ are equivalent if there exists an adapted set U'' with $x \in U'' \subset U \cap U'$ such that $h_U|U'' = h'_{U'}|U''$. We write $(h_U, U, V) \overset{c}{\sim} (h'_{U'}, U', V')$.

DEFINITION 1.7. *Let $(X_{\mathcal{G}}, \Gamma, \Phi)$ be an odometer. For $x \in X_{\mathcal{G}}$, the* dynamical commensurator group *of $(X_{\mathcal{G}}, \Gamma, \Phi)$ at x is the set of germs of dynamical commensurators,*

$$\mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x) = \{h\colon U \to V \mid x \in U \cap V\}/ \overset{c}{\sim} \ . \tag{6}$$

A dynamical commensurator is the analog of “Morita equivalence” in the category of foliated spaces, and correspondingly, $\mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x)$ is the group of pointed Morita equivalences. The importance of this notion comes from the following result, which follows from the results in [17].

THEOREM 1.8. *Suppose that $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are solenoidal manifolds. If $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are homeomorphic, then their monodromy odometers $(X_{\mathcal{G}}, \Gamma, \Phi)$ and $(X'_{\mathcal{G}'}, \Gamma', \Phi')$ are return equivalent. Moreover, for $x \in X_{\mathcal{G}}$ corresponding to $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$, there is a well-defined characteristic map*

$$\sigma_{\mathcal{P},\widehat{x}}\colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \to \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x) \ . \tag{7}$$

Given a solenoidal manifold $\mathfrak{M}_{\mathcal{P}}$ with monodromy action $(X_{\mathcal{G}}, \Gamma, \Phi)$ and basepoint $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$, by Theorems 1.3 and 1.8, there are well-defined maps χ and $\sigma_{\mathcal{P},\widehat{x}}$ which form two edges in the diagram

$$\begin{array}{ccc} & & \mathrm{Comm}(\Gamma) \\ & \overset{\chi}{\nearrow} & \\ \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) & & \big\uparrow \chi_{\Phi}? \\ & \underset{\sigma_{\mathcal{P},\widehat{x}}}{\searrow} & \\ & & \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x) \end{array} \tag{8}$$

Given a group chain $\mathcal{G}$, the strategy for calculating $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ is to show the existence of a map $\chi_{\Phi}\colon \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x) \to \mathrm{Comm}(\Gamma)$ which makes the diagram (8) commute.

DEFINITION 1.9. *An odometer action $(X_{\mathcal{G}}, \Gamma, \Phi)$ is regular if there exists $x \in X_{\mathcal{G}}$ and*

$$\chi_{\Phi}\colon \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x) \to \mathrm{Comm}(\Gamma) \tag{9}$$

such that the diagram (8) *commutes, and χ_{Φ} is an injection.*

The notion of a coherent odometer action is given in Definition 6.5, and is a dynamical property of the action $(X_{\mathcal{G}}, \Gamma, \Phi)$; see Sections 4 and 6. Then we have:

THEOREM 1.10. *If $(X_{\mathcal{G}}, \Gamma, \Phi)$ is a coherent, locally quasi-analytic and effective odometer, then the action is regular.*

COROLLARY 1.11. *If Γ is nilpotent, then an effective odometer action $(X_{\mathcal{G}}, \Gamma, \Phi)$ is regular.*

It then remains to show that the map $\sigma_{\mathcal{P},\widehat{x}}$ is onto. That is, given a class $\xi \in \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x)$, there exists a homeomorphism $h_{\xi} \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ which is mapped onto ξ. This requires two key assumptions, that the action is effective, or equivalently that there is a simply connected leaf in $\mathfrak{M}_{\mathcal{P}}$, and that the base manifold M_0 is strongly Borel, as discussed in Section 5.

Section 8 analyzes the group $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ as an extension of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$. In Section 4 we introduce the profinite groups $\mathfrak{D}_{\mathcal{G}} \subset \mathfrak{G}_{\mathcal{G}}$ associated to the odometer action $(X_{\mathcal{G}}, \Gamma, \Phi)$. Introduce the group $\mathfrak{N}_{\mathcal{G}} = \mathfrak{N}_{\mathfrak{G}_{\mathcal{G}}}(\mathfrak{D}_{\mathcal{G}})$ which is the normalizer of the subgroup $\mathfrak{D}_{\mathcal{G}}$ in $\mathfrak{G}_{\mathcal{G}}$. Let $\overline{g}$ denote the image of $g \in \Gamma$ in $\mathfrak{G}_{\mathcal{G}}$, define $\Gamma_{\mathfrak{N}} = \{g \in \Gamma \mid \overline{g} \in \mathfrak{N}_{\mathcal{G}}\}$, and introduce the coset space

$$\mathfrak{Z}_{\mathcal{G}} = \Gamma_{\mathfrak{N}} \backslash \mathfrak{G}_{\mathcal{G}} \ . \tag{10}$$

Then we show in Section 8:

THEOREM 1.12. *Let $\mathfrak{M}_{\mathcal{P}}$ be a solenoidal manifold. There exists a well-defined surjection*

$$\tau_{\mathcal{P}} \colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}) \to \mathfrak{Z}_{\mathcal{G}} \ . \tag{11}$$

Note that $\Gamma_{\mathfrak{N}}$ need not be normal in $\mathfrak{N}_{\mathcal{G}}$, so $\mathfrak{Z}_{\mathcal{G}}$ need not have a group structure.

Section 9 gives a collection of examples of calculations of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$. We recall some results of Kwapisz [37] in Section 9.1. For solenoids of higher dimension, we give a variety of examples that illustrate the above results, and extend the conclusions of Theorem 1.5 and 1.6.

We conclude with a summary of the results concerning the maps in Diagram 8, in terms of properties of the odometer $(X_{\mathcal{G}}, \Gamma, \Phi)$ and the group Γ.

THEOREM 1.13. *Assume that M is strongly Borel, and the action $(X_{\mathcal{G}}, \Gamma, \Phi)$ is effective. Then for the characteristic map $\chi \colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \to \mathrm{Comm}(\Gamma)$, we have:*

(1) *Suppose that $\mathcal{G}$ is the universal group chain in Γ, then χ is surjective by Theorem 5.7. That is, given a commensurator $[\phi] \in \mathrm{Comm}(\Gamma)$ there exists $h_\phi \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ with $\chi(h_\phi) = \phi$.*

(2) *Suppose that Γ satisfies the unique root property in Definition 6.6 and the action $(X_{\mathcal{G}}, \Gamma, \Phi)$ is coherent, then the map ρ defined in Proposition 7.3 is injective. That is, given*

$$\phi \in \mathrm{Image}\{\mathrm{Comm}_{\mathrm{Homeo}(X_{\mathcal{G}}, x)}(\Gamma) \to \mathrm{Comm}(\Gamma)\} \ ,$$

there exists $h_\phi \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ with $\chi(h_\phi) = \phi$.

(3) *Suppose that $(X_{\mathcal{G}}, \Gamma, \Phi)$ is coherent and locally quasi-analytic, then $\chi_\Phi \colon \mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x) \to \mathrm{Comm}(\Gamma)$ is injective by Theorem 7.1. That is, given*

$$\phi \in \mathrm{Image}\{\chi_\Phi \colon \mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x) \to \mathrm{Comm}(\Gamma)\} \ ,$$

there exists $h_\phi \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ with $\chi(h_\phi) = \phi$.

Finally, for a solenoidal manifold not covered by Theorem 1.13, $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ is essentially unknown:

PROBLEM 1.14. *Let $\mathfrak{M}_{\mathcal{P}}$ be a solenoidal manifold with base M. What can be said about the group $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ for the case where the monodromy the action $(X_{\mathcal{G}}, \Gamma, \Phi)$ is not regular?*

Theorems 5.10 and 5.11 in [17] give examples of solenoidal manifolds whose monodromy actions are not regular, and such that the map $\sigma_{\mathcal{P},\widehat{x}} \colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \to \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x)$ is not injective.

2. Solenoidal manifolds

Let M denote a compact connected manifold without boundary, and $\Gamma = \pi_1(M, x)$ for $x \in M$.

Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a descending chain of subgroups of finite index. The group chain $\mathcal{G}$ and $x \in M$ determine a tower of finite index coverings by closed manifolds, for $M_0 = M$:

$$\mathcal{P}_{\mathcal{G}} \equiv \left\{ M_0 \xleftarrow{p_1} M_1 \xleftarrow{p_2} M_2 \xleftarrow{p_3} \cdots \right\} \ , \tag{12}$$

with basepoints $x_0 = x$, and $x_\ell \in M_\ell$ such that $p_{\ell+1}(x_{\ell+1}) = x_\ell$. The inverse limit space

$$\begin{aligned} \mathfrak{M}_{\mathcal{P}} \ &\equiv \ \varprojlim \, \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \\ &= \ \{(x_0, x_1, \ldots) \in \mathfrak{M}_{\mathcal{P}} \mid p_{\ell+1}(x_{\ell+1}) = x_\ell \text{ for all } \ell \geq 0 \ \} \ \subset \prod_{\ell \geq 0} M_\ell \end{aligned} \tag{13}$$

is given the relative topology, induced from the product (Tychonoff) topology, so that $\mathfrak{M}_{\mathcal{P}}$ is itself compact and connected. For each $\ell \geq 0$, there is a fibration $\widehat{q}_\ell \colon \mathfrak{M}_{\mathcal{P}} \to M_\ell$, where each fiber $\mathfrak{X}_\ell = \widehat{q}_\ell^{-1}(x_\ell)$ a Cantor space. The given manifold M is called the base of the tower, and $\widehat{x} = (x_0, x_1, x_2, \ldots) \in \mathfrak{M}_{\mathcal{P}}$ determines a basepoint in the limit.

McCord observed in [44] that $\mathfrak{M}_{\mathcal{P}}$ has a local product structure, and the path-connected components of $\mathfrak{M}_{\mathcal{P}}$ define a foliation denoted by $\mathcal{F}_{\mathcal{P}}$. That is, $\mathfrak{M}_{\mathcal{P}}$ is an n-dimensional foliated space in the sense of [47], for which the leaves of $\mathcal{F}$ are the path connected components of $\mathfrak{M}$, and the local transversals to the foliation are totally disconnected.

We recall an elementary result about solenoids. For $\ell \geq 0$, introduce the truncated presentation

$$\mathcal{P}_\ell \equiv \left\{ M_\ell \overset{p_{\ell+1}}{\longleftarrow} M_{\ell+1} \overset{p_{\ell+2}}{\longleftarrow} M_{\ell+2} \overset{p_{\ell+3}}{\longleftarrow} \cdots \right\} . \tag{14}$$

LEMMA 2.1. *For all $\ell \geq 0$ there is a canonical homeomorphism $\sigma_\ell \colon \mathfrak{M}_{\mathcal{P}_\ell} \cong \mathfrak{M}_{\mathcal{P}}$.*

Proof. Given $\widehat{y} = (y_\ell, y_{\ell+1}, y_{\ell+2}, \ldots) \in \mathfrak{M}_{\mathcal{P}_\ell}$ define the shift map on indices,

$$\sigma_\ell(y_\ell, y_{\ell+1}, y_{\ell+2}, \ldots) = (p_0^\ell(y_\ell), p_1^{\ell+1}(y_{\ell+1}), p_2^{\ell+2}(y_{\ell+2}), \ldots) \in \mathfrak{M}_{\mathcal{P}} \ ,$$

where $p_\ell^{\ell'} = p_{\ell+1} \circ \cdots \circ p_{\ell'-1} \circ p_{\ell'} \colon M_{\ell'} \to M_\ell$. It is standard that σ_ℓ is a homeomorphism onto. □

3. The characteristic map

In this section, we construct the characteristic map $\chi \colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \to \mathrm{Comm}(\Gamma)$ in Theorem 1.3.

Fix a metric $d_{\mathfrak{M}}$ on $\mathfrak{M}_{\mathcal{P}}$ compatible with the inverse limit topology. A map $H \colon \mathfrak{M}_{\mathcal{P}} \times [0,1] \to \mathfrak{M}_{\mathcal{P}}$ is said to be an ε-homotopy if the trace of each point is ε-bounded. That is, for all $\widehat{y} \in \mathfrak{M}_{\mathcal{P}}$ we have $d_{\mathfrak{M}}(H(\widehat{y}, s), H(\widehat{y}, t)) < \varepsilon$ for all $s, t \in [0,1]$. We use a fundamental result of Rogers and Tollefson:

THEOREM 3.1. [52, Corollary 5.2] *Let M be a closed connected manifold, and $\mathfrak{M}_{\mathcal{P}}$ be a solenoidal manifold over M defined by a presentation $\mathcal{P}$. Given a self-homeomorphism $h \colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ and $\varepsilon > 0$, then h is ε-homotopic to a map induced by a* shuffle map *of $\mathcal{P}$. That is, there exists an increasing sequence $i_\ell \geq \ell$, and maps $\varphi_{i_\ell} \colon M_{i_\ell} \to M_\ell$ so that the "zig-zag" diagram* (15) *commutes*

$$\begin{array}{ccccccccc} M_0 & \longleftarrow & M_{i_0} & \longleftarrow & M_{i_1} & \longleftarrow & M_{i_2} & \longleftarrow & \cdots \\ & \swarrow_{\varphi_{i_0}} & & \swarrow_{\varphi_{i_1}} & & \swarrow_{\varphi_{i_2}} & & & \\ M_0 & \longleftarrow & M_1 & \longleftarrow & M_2 & \longleftarrow & M_3 & \longleftarrow & \cdots \end{array} \tag{15}$$

and the self-map $\widehat{\varphi} \colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ induced by the collection $\{\varphi_{i_\ell} \mid \ell \geq 0\}$ is ε-homotopic to h. Moreover, for each $\ell \geq 0$, the map φ_{i_ℓ} induces a monomorphism between the fundamental groups of M_{i_ℓ} and M_ℓ whose image has finite index.

We recall notation for the solenoidal manifold $\mathfrak{M}_{\mathcal{P}}$. For each $\ell \geq 0$, there is the proper covering map $p_0^\ell = p_\ell \circ \cdots p_1 \colon M_\ell \to M_0$, and that $\widehat{q}_\ell \colon \mathfrak{M}_{\mathcal{P}} \to M_\ell$ is the projection onto the factor M_ℓ in (3).

Let $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$ be a choice of basepoint, and set $x_\ell = \widehat{q}_\ell(\widehat{x}) \in M_\ell$ and set $\Gamma_0 = \pi_1(M_0, x_0)$. For each $\ell > 0$, Γ_ℓ is the image of $(p_0^\ell)_\# \colon \pi_1(M_\ell, x_\ell) \to \Gamma_0$, so $\mathcal{G} = \{\Gamma_\ell \mid \ell \geq 0\}$ is a group chain in Γ_0.

Let d_M denote a Riemannian metric on M_0 and let $\delta_0 > 0$ be sufficiently small so that, for all $y \in M_0$, the closed metric ball $B_M(y, \delta_0) \subset M_0$ of radius δ_0 is convex. In particular, δ_0 is less than the injectivity radius of M_0. It follows that for each $y \in M$ and $y' \in B_M(y, \delta_0)$ there is an isomorphism $\iota_{y',y} \colon \pi_1(M_0, y') \cong \pi_1(M_0, y)$ induced by concatenation with a geodesic path in $B_M(y, \delta_0)$ from y' to y.

Finally, let $\varepsilon > 0$ be sufficiently small so that $\widehat{q}_0(B_{\mathfrak{M}}(\widehat{y}, \varepsilon)) \subset B_M(y, \delta_0/3)$ for all $\widehat{y} \in \widehat{q}_0^{-1}(y)$.

Given $h \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$, we first define $\chi(h) \in \mathrm{Comm}(\Gamma)$. We then show that $\chi(h)$ is independent of choices, and depends only on the basepoint fixed homotopy class of h.

Let $h\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ then by Theorem 3.1, there exists an increasing sequence $i_\ell \geq \ell$ and maps $\varphi_{i_\ell}\colon M_{i_\ell} \to M_i$ so that the shuffle diagram (15) commutes, and for the map $\widehat{\varphi}\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ defined by the maps $\{\varphi_{i_\ell} \mid \ell \geq 0\}$, we have that h and $\widehat{\varphi}$ are ε-homotopic. Let $\widehat{y} = \widehat{\varphi}(\widehat{x})$ and set $y_\ell = \widehat{q}_\ell(\widehat{y}) \in M_\ell$. Then we have the diagram:

$$\begin{array}{ccccc} \pi_1(M_{i_0}, x_{i_0}) & \xrightarrow{(\varphi_{i_0})_\#} & \pi_1(M_0, y_0) & \xrightarrow[\cong]{\iota_{y',y}} & \pi_1(M_0, x_0) \\ \Big\downarrow {\scriptstyle (p_0^{i_0})_\#} & & & & \\ \pi_1(M_0, x_0) & & & & \end{array} \tag{16}$$

where $(p_0^{i_0})_\#\colon \pi_1(M_{i_0}, x_{i_0}) \to \Gamma_{i_0} \subset \Gamma_0$ is an isomorphism onto its image Γ_{i_0}. Let $h_\#\colon \Gamma_{i_0} \to \Gamma_0$ be the commensurator defined by the inverse of $(p_0^{i_0})_\#$ followed by $\iota_{y',y} \circ (\varphi_{i_0})_\#$. Define $\chi(h) \in \mathrm{Comm}(\Gamma_0)$ to be the commensurator class of $h_\#$.

Next, we show that $\chi(h)$ is independent of the choice of an approximation to h by an induced map. Suppose there are given two sequences $\{\varphi_{i_\ell}\colon M_{i_\ell} \to M_\ell \mid \ell \geq 0\}$ and $\{\psi_{j_\ell}\colon M_{j_\ell} \to M_\ell \mid \ell \geq 0\}$, so that the shuffle diagram (15) commutes for both. Moreover, assume that the induced maps $\widehat{\varphi}\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ and $\widehat{\psi}\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ are both ε-homotopic to h. Thus $\widehat{\varphi}$ and $\widehat{\psi}$ are 2ε-homotopic.

Let $\widehat{y} = \widehat{\varphi}(\widehat{x})$ then set $y_k = \widehat{q}_k(\widehat{y}) \in M_k$ for $k \geq 0$. Likewise, let $\widehat{z} = \widehat{\psi}(\widehat{x})$ and set $z_k = \widehat{q}_k(\widehat{z}) \in M_k$ for $k \geq 0$. Choose an index $k_0 > \max\{i_0, j_0\}$. Then we have two maps:

$$\Psi_{\widehat{\varphi}}\colon \quad \mathfrak{M}_{\mathcal{P}} \xrightarrow{\widehat{q}_{k_0}} M_{k_0} \xrightarrow{p_{i_0}^{k_0}} M_{i_0} \xrightarrow{\varphi_{i_0}} M_0$$

$$\Psi_{\widehat{\psi}}\colon \quad \mathfrak{M}_{\mathcal{P}} \xrightarrow{\widehat{q}_{k_0}} M_{k_0} \xrightarrow{p_{j_0}^{k_0}} M_{j_0} \xrightarrow{\psi_{j_0}} M_0 \ .$$

Now, given $\alpha \in \pi_1(M_{k_0}, x_{k_0})$ choose a closed curve γ in M_{k_0} with basepoint x_{k_0} representing α.

Let $\widetilde{\gamma}$ be the lift of γ to a curve in the leaf $L_{\widehat{x}}$ in $\mathfrak{X}_{\mathcal{P}}$ starting at $\widehat{x}$ and which covers γ.

The images $\Psi_{\widehat{\varphi}}(\widetilde{\gamma})$ and $\Psi_{\widehat{\psi}}(\widetilde{\gamma})$ in M_0 of the curve $\widetilde{\gamma}$ in $\mathfrak{M}_{\mathcal{P}}$ are then homotopic and lie within $2\delta_0/3$ of each other. Moreover, by our choice of δ_0, the images of the curves lie in a convex neighborhood of each other, and hence are homotopic. Thus, they determine the same class in $\pi_1(M_0, x_0)$. This implies that the definition of $\chi(h)$ on elements of Γ_{k_0} agrees using either approximations $\widehat{\varphi}$ or $\widehat{\psi}$ and hence χ is well-defined. This completes the proof of Theorem 1.3.

The conclusion of Theorem 1.3 can be reformulated in terms of pro-categories. Theorem 3.1 implies that a homeomorphism $h\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ induces a map $\{\varphi_{i_\ell}\}$ in the pro-category of spaces. By the local contractibility of the base M_0 the pro-map induces a pro-map in the category of groups, which induces the map χ. This is the approach followed in [7] for the universal solenoidal manifold $\widehat{M}$.

4. Generalized odometers

Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a group chain in Γ. We associate to $\mathcal{G}$ a left odometer action of Γ on a Cantor space $X_{\mathcal{G}}$, and recall some of its properties as a dynamical system. The action $(X_{\mathcal{G}}, \Gamma, \Phi)$ is identified with the monodromy action of the foliation $\mathcal{F}_{\mathcal{P}}$ of the solenoidal manifold $\mathfrak{M}_{\mathcal{P}}$ associated to a presentation $\mathcal{P}$ defined by the group chain $\mathcal{G}$, where $\Gamma = \pi_1(M_0, x_0)$. This correspondence is discussed in detail in [31, Section 2].

For $\ell \geq 0$ let $X_\ell = \Gamma/\Gamma_\ell$ as a left Γ-space. Then Γ acts transitively on the left on the finite coset $X_\ell = \Gamma/\Gamma_\ell$. The projection map of cosets $p_{\ell+1}\colon X_{\ell+1} \to X_\ell$ commutes with the left action of Γ. Introduce the inverse limit space

$$X_{\mathcal{G}} \equiv \varprojlim \, \{p_{\ell+1}\colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} X_\ell \ . \tag{17}$$

A point $x = (x_0, x_1, \ldots) \in X_{\mathcal{G}}$ if $p_{\ell+1}(x_{\ell+1}) = x_\ell$ for all $\ell \geq 0$. The Γ-action on each factor X_ℓ induces the Γ-action on $X_{\mathcal{G}}$. That is, $g \in \Gamma$ acts coordinate-wise, so $g \cdot x = (g \cdot x_0, g \cdot x_1, \ldots)$.

Let d_ℓ denote the indicator metric on X_ℓ, so $d_\ell(y_\ell, z_\ell) = 0$ if $y_\ell = z_\ell$, and $d_\ell(y_\ell, z_\ell) = 1$ for $y_\ell \neq z_\ell \in X_\ell$. This metric is invariant under the Γ-action on X_ℓ. Define a Γ-invariant metric d_X on $X_{\mathcal{G}}$ by setting, for $y, z \in X_{\mathcal{G}}$,

$$d_X(y, z) = \sum_{\ell \geq 0} 2^{-\ell} d_\ell(y_\ell, z_\ell) \ .$$

Let $U_\ell = \{x \in X_{\mathcal{G}} \mid x_0 = e, \ldots, x_{\ell-1} = e\}$ where $e \in \Gamma$ is the identity. The collection $\{U_\ell \mid \ell \geq 0\}$ form a neighborhood basis of the basepoint $x_0 = (e, e, \ldots) \in X_{\mathcal{G}}$, and the collection of Γ-translates of this neighborhood basis forms a basis for the topology on $X_{\mathcal{G}}$, which is a Cantor space.

A clopen set $U \subset X_{\mathcal{G}}$ is *adapted* if $U \neq \emptyset$, and for all $\gamma \in \Gamma$, if $\gamma \cdot U \cap U \neq \emptyset$ then $\gamma \cdot U = U$. It follows that $\Gamma_U = \{\gamma \in \Gamma \mid \gamma \cdot U = U\}$ is a subgroup of finite index in Γ. Observe that $\Gamma_{U_\ell} = \Gamma_\ell$.

The action of Γ on each set X_ℓ is transitive, so the action of Γ on $X_{\mathcal{G}}$ is minimal.

Thus, the action $\Phi \colon \Gamma \times X_{\mathcal{G}} \to X_{\mathcal{G}}$ is minimal, transitive and isometric, so is a (generalized) *odometer* in the sense of [19, 20].

Associated to an odometer $\Phi \colon \Gamma \times \mathfrak{X} \to X_{\mathcal{G}}$ is a profinite group $\mathfrak{G}_\Phi$ and odometer action $\widehat{\Phi}$ of $\mathfrak{G}_\Phi$ on $\mathfrak{X}$. The group $\mathfrak{G}_\Phi$ can be defined as the closure of the image of $\Phi \colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ in the uniform topology, which is called the Ellis semi-group for the action [25, 3]. In this case, $\mathfrak{G}_\Phi$ is a group since the action is equicontinuous. There is an alternate, and more useful description as follows.

DEFINITION 4.1. *The* normal core *of $H \subset \Gamma$ is the largest normal subgroup $C(H) \subset H$.*

For each subgroup Γ_ℓ its normal core is given by $C_\ell = C(\Gamma_\ell) = \bigcap_{g \in \Gamma} g \Gamma_\ell g^{-1}$ which is identified with the kernel of the action map $\Phi_\ell \colon \Gamma_\ell \to \mathrm{Aut}(\Gamma/\Gamma_\ell)$. Let $G_\ell = \Gamma/C_\ell$ then the inverse limit

$$\mathfrak{G}_{\mathcal{G}} \equiv \varprojlim \ \{p_{\ell+1} \colon G_{\ell+1} \to G_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} \ G_\ell \ , \tag{18}$$

is a profinite group with a left isometric $\mathfrak{G}_{\mathcal{G}}$-action on $X_{\mathcal{G}}$ denoted by $\widehat{\Phi} \colon \mathfrak{G}_{\mathcal{G}} \times X_{\mathcal{G}} \to X_{\mathcal{G}}$. As the action of Γ on $X_{\mathcal{G}}$ is minimal, the action $\widehat{\Phi}$ of $\mathfrak{G}_{\mathcal{G}}$ is transitive.

Let $\mathfrak{D}_{\mathcal{G}} = \{\widehat{g} \in \mathfrak{G}_{\mathcal{G}} \mid \widehat{\Phi}(\widehat{g})(x_0) = x_0\}$ be the isotropy subgroup of the basepoint $x_0 \in X_{\mathcal{G}}$. Then there is a $\mathfrak{G}_{\mathcal{G}}$-equivariant isomorphism $X_{\mathcal{G}} = \mathfrak{G}_{\mathcal{G}}/\mathfrak{D}_{\mathcal{G}}$.

For a subgroup $H \subset \Gamma$, let $\widehat{H} \subset \mathfrak{G}_{\mathcal{P}}$ denote the closure of its image in $\mathfrak{G}_{\mathcal{G}}$. In particular, the closure $\widehat{\Gamma}_\ell$ acts transitively on U_ℓ, hence $U_\ell = \widehat{\Gamma}_\ell/\mathfrak{D}_{\mathcal{G}}$ for all ℓ, and so $\bigcap_\ell \widehat{\Gamma}_\ell = \mathfrak{D}_{\mathcal{G}}$.

There is an alternate definition of $\mathfrak{D}_{\mathcal{G}}$ as an inverse limit. Consider the quotient $D_\ell = G_\ell/C_\ell$. Since C_ℓ is a normal subgroup in G, and hence also normal in G_ℓ, the quotient D_ℓ is a subgroup of $G_\ell = G/C_\ell$. For $\ell' > \ell$ the quotient map $G_{\ell'} \to \Gamma_\ell$ induces a homomorphism $\theta^\ell_{\ell'} \colon G_\ell/C_\ell \to G_{\ell'}/C_{\ell'}$. By [23, Theorem 4.4] we then have

$$\mathcal{D}_{\mathcal{G}} \cong \varprojlim \ \left\{\theta^\ell_{\ell'} \colon G_\ell/C_\ell \to G_{\ell'}/C_{\ell'}\right\} \ . \tag{19}$$

The subgroup $\mathfrak{D}_{\mathcal{G}}$ is *totally not normal* in $\mathfrak{G}_{\mathcal{G}}$. That is, the largest subgroup of $\mathfrak{D}_{\mathcal{G}}$ which is normal in $\mathfrak{G}_{\mathcal{G}}$ is the trivial group by [23, Corollary 5.4]. Consequently, the action $\mathrm{Adj} \colon \mathfrak{D}_{\mathcal{G}} \to \mathrm{Homeo}(X_{\mathcal{G}}, x)$ induced from the adjoint action of $\mathfrak{D}_{\mathcal{G}}$ on $\mathfrak{G}_{\mathcal{G}}$ is an injection.

DEFINITION 4.2. *Cantor actions $(\mathfrak{X}_1, \Gamma_1, \Phi_1)$ and $(\mathfrak{X}_2, \Gamma_2, \Phi_2)$ are said to be* isomorphic *if there is a homeomorphism $h \colon \mathfrak{X}_1 \to \mathfrak{X}_2$ and group isomorphism $\Theta \colon \Gamma_1 \to \Gamma_2$ so that*

$$\Phi_1(g) = h^{-1} \circ \Phi_2(\Theta(g)) \circ h \in \mathrm{Homeo}(\mathfrak{X}_1) \ \textit{for all} \ g \in \Gamma_1 \ . \tag{20}$$

Note that when $\mathfrak{X}_1 = \mathfrak{X}_2$ and the map $\Theta \colon \Gamma_1 \to \Gamma_2 = \Gamma_1$ is the identity, then the notion of an isomorphism reduces to the notion of an automorphism defined previously.

Next, we recall the notion of *return equivalence* of odometers, which is the analog of *Morita equivalence* for the holonomy groupoid of the foliation $\mathcal{F}_{\mathcal{G}}$ of $\mathfrak{M}_{\mathcal{P}}$.

DEFINITION 4.3. *Odometers $\Phi\colon \Gamma\times\mathfrak{X}\to\mathfrak{X}$ and $\Phi'\colon \Gamma'\times\mathfrak{X}'\to\mathfrak{X}'$ are* return equivalent *if there exists adapted sets $U\subset\mathfrak{X}$ and $U'\subset\mathfrak{X}'$ and homeomorphism $h_U\colon U\to U'$ that induces an isomorphism between the actions of the groups*

$$\begin{aligned}\mathcal{H}_U &= \mathrm{Image}\{\Phi_U\colon \Gamma_U\to\mathrm{Homeo}(U)\}\\ \mathcal{H}'_{U'} &= \mathrm{Image}\{\Phi'_{U'}\colon \Gamma'_{U'}\to\mathrm{Homeo}(U')\}\end{aligned}$$

Note that the map $\Phi_U\colon\Gamma_U\to\mathcal{H}_U$ may have kernel, and likewise for $\Phi'_{U'}$. Thus, a homeomorphism $h_U\colon U\to U'$ which conjugates $\mathcal{H}_U$ with $\mathcal{H}'_{U'}$ need not induce an isomorphism $\phi_U\colon\Gamma_U\to\Gamma'_{U'}$.

Let us recall the following key notions from the introduction.

DEFINITION 4.4. *Let $(\mathfrak{X},\Gamma,\Phi)$ be an odometer action, and $x\in\mathfrak{X}$.*

(1) A dynamical commensurator *at x is a homeomorphism $h_U\colon U\to V$, where U and V are adapted subsets with $x\in U\cap V$, such that $h_U(x)=x$, and h_U induces an isomorphism $\Theta_U\colon\mathcal{H}_U\to\mathcal{H}_V$.*
(2) Commensurators $h_U\colon U\to V$ and $h_{U'}\colon U'\to V'$ are equivalent if there exists an adapted set U'' with $x\in U''\subset U\cap U'$ such that $h_U|U''=h'_{U'}|U''$. We write $(h_U,U,V)\overset{c}{\sim}(h'_{U'},U',V')$.
(3) The dynamical commensurator group *of $(\mathfrak{X},\Gamma,\Phi)$ at x is the set of germs of dynamical commensurators,*

$$\mathrm{Comm}(\mathfrak{X},\Gamma,\Phi,x)=\{h_U\colon U\to V\mid x\in U\cap V,\ h_U \text{ commensurates } \mathcal{H}_U,\mathcal{H}_V\}/\overset{c}{\sim}$$

The motivation for this definition comes from the following result and its corollary:

THEOREM 4.5. [17] *Suppose that $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are weak solenoids. If $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are homeomorphic, then their monodromy odometers $\Phi\colon\Gamma\times X_{\mathcal{P}}\to X_{\mathcal{P}}$ and $\Phi'\colon\Gamma'\times X'_{\mathcal{P}'}\to X'_{\mathcal{P}'}$ are* return equivalent.

For a homeomorphism $h\in\mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}},\widehat{x})$, the return equivalence induced by h is defined using the holonomy along leafwise paths that start and return in a foliation chart at $\widehat{x}$. It follows that the germ of the return equivalence is unchanged by a leafwise homotopy which fixes $\widehat{x}$. We thus obtain:

COROLLARY 4.6. *Let $\mathfrak{M}_{\mathcal{P}}$ be a solenoidal manifold, with monodromy action $(X_{\mathcal{G}},\Gamma,\Phi)$. For $\widehat{x}\in\mathfrak{M}_{\mathcal{P}}$, there there is a well-defined map $\sigma_{\mathcal{P},\widehat{x}}\colon\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}},\widehat{x})\to\mathrm{Comm}(X_{\mathcal{G}},\Gamma,\Phi,x)$.*

5. The Borel Property

Let $\mathfrak{M}_{\mathcal{P}}$ be the solenoidal manifold constructed from a presentation $\mathcal{P}_{\mathcal{G}}$ associated to the group chain $\mathcal{G}$ in $\Gamma=\pi_1(M,x)$. We investigate the image of the map $\chi\colon\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}},\widehat{x})\to\mathrm{Comm}(\Gamma)$, where $\widehat{x}\in\mathfrak{M}_{\mathcal{P}}$ is a choice of basepoint over x.

PROBLEM 5.1. *For a commensurator $\phi\colon H\to K$ for Γ, give conditions such that $[\phi]\in\mathrm{Comm}(\Gamma)$ is induced from a homeomorphism $h\colon\mathfrak{M}_{\mathcal{P}}\to\mathfrak{M}_{\mathcal{P}}$. That is, so that $[\phi]=\chi([h])$.*

First, recall the suspension construction from the theory of foliations. This is discussed for solenoidal manifolds in [17, Theorem 3.2], and more generally for foliated spaces in [15, Section 3.1].

Let $(X_{\mathcal{G}},\Gamma,\Phi)$ denote the odometer associated to $\mathcal{G}$ in (17), and let $\mathfrak{M}_{\mathcal{P}}$ denote the solenoid associated to $\mathcal{P}_{\mathcal{G}}$ in (3). For each $\ell\geq 0$, the coordinate projection $\widehat{q}_\ell\colon\mathfrak{M}_{\mathcal{P}}\to M_\ell$ is a fibration, where the fiber $\mathfrak{X}_\ell=\widehat{q_\ell}^{-1}(x_\ell)$ a Cantor space, and is identified with the closure $U_\ell\subset X_{\mathcal{G}}$ of $\Gamma_\ell\subset\Gamma$. Thus the restriction of Φ yields an odometer action $\Phi_\ell\colon\Gamma_\ell\times U_\ell\to U_\ell$.

Let $\widetilde{M}$ denote the universal covering of M with the right action of Γ by deck translations. Define

$$\mathfrak{M}_{\Phi_\ell}=\widetilde{M}\times U_\ell/(z\cdot g^{-1},y)\sim(z,\Phi_\ell(g)(y)) \text{ for } z\in\widetilde{M},\ g\in\Gamma_\ell,\ y\in U_\ell \tag{21}$$

which is a minimal matchbox manifold.

THEOREM 5.2. *For each $\ell\geq 0$, there is a canonical homeomorphism $\tau_\ell\colon\mathfrak{M}_{\Phi_\ell}\to\mathfrak{M}_{\mathcal{P}}$.*

Proof. For all $\ell \geq 0$, there is a natural homeomorphism $\rho_\ell \colon \mathfrak{M}_{\Phi_\ell} \to \mathfrak{M}_{\mathcal{P}_\ell}$, where $\mathfrak{M}_{\mathcal{P}_\ell}$ is the the truncated solenoid as defined in (14). This is a consequence of the lifting property for maps between coverings, as shown in [16]. By Lemma 2.1, there is a canonical homeomorphism $\sigma_\ell \colon \mathfrak{M}_{\mathcal{P}_\ell} \cong \mathfrak{M}_{\mathcal{P}}$. Then set $\tau_\ell = \sigma_\ell \circ \rho_\ell$. □

Next suppose we are given $[\phi] \in \mathrm{Comm}(\Gamma)$, then without loss of generality we can assume that $[\phi]$ is represented by an isomorphism $\phi \colon \Gamma_\ell \to \Gamma_{\ell'}$. We want to construct a homeomorphism $h_\phi \colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ such that $h_\phi(x) = x$, and $\chi([h_\phi]) = [\phi]$. By Theorem 3.1 it is necessary to construct homeomorphisms $\overline{\phi} \colon M_i \to M_{i'}$ whose maps on fundamental groups are induced by the map ϕ.

Recall that a finite CW-complex Y is *aspherical* if it is connected and its universal covering space is contractible. If Y and Y' are aspherical, then an isomorphism $\alpha \colon \pi_1(Y, y) \to \pi_1(Y', y')$ induces a homotopy equivalence $h_\alpha \colon Y \to Y'$. Then also recall:

DEFINITION 5.3. *A closed manifold N is said to be* topologically rigid *if any homotopy equivalence $h \colon M \to N$, where M is a closed manifold, is homotopic to a homeomorphism.*

CONJECTURE 5.4 (Borel Conjecture)**.** *Every aspherical closed manifold is topologically rigid.*

DEFINITION 5.5. *A closed aspherical manifold M is said to be* Borel *if it satisfies the Borel Conjecture. It is said to be* strongly Borel *if every finite covering of M satisfies the Borel Conjecture.*

REMARK 5.6. *The Borel Conjecture has been proven for many classes of aspherical manifolds:*

- *the torus $\mathbb{T}^n$ for all $n \geq 1$,*
- *all infra-nilmanifolds,*
- *closed Riemannian manifolds Y with negative sectional curvatures,*
- *closed Riemannian manifolds Y of dimension $n \neq 3, 4$ with non-positive sectional curvatures,*
- *all closed 3-manifolds.*

The above list is not exhaustive, see [41, Theorem 6.8] and also [40]. Each class of manifolds in Remark 5.6 is closed under taking covers, so all of these are also strongly Borel.

We now assume that M is a strongly Borel manifold, then the choice of commensurator ϕ induces a homeomorphism $\overline{\phi} \colon M_\ell \to M_{\ell'}$. This in turn, induces a map $\widehat{\phi} \colon \mathfrak{M}_{\mathcal{P}_\ell} \to \mathfrak{M}_{\mathcal{P}^\phi_{\ell'}}$ where $\mathcal{P}^\phi_{\ell'}$ is the tower associated to the group chain

$$\mathcal{G}^\phi_\ell = \{\Gamma_{\ell'} = \phi(\Gamma_\ell) \supset \phi(\Gamma_{\ell+1}) \supset \phi(\Gamma_{\ell+2}) \supset \cdots\} \ .$$

If the group chain $\mathcal{G}^\phi_\ell$ is zig-zag equivalent to $\mathcal{G}_{\ell'}$, then we have

$$h_\phi \colon \mathfrak{M}_{\mathcal{P}} \cong \mathfrak{M}_{\mathcal{P}_\ell} \cong \mathfrak{M}_{\mathcal{P}^\phi_{\ell'}} \cong \mathfrak{M}_{\mathcal{P}_{\ell'}} \cong \mathfrak{M}_{\mathcal{P}} \ .$$

Thus, $h_\phi \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{G}}, \widehat{x})$ and $\chi([h_\phi]) = [\phi]$. We summarize these remarks as follows:

THEOREM 5.7. *Let M be a strongly Borel manifold, with $\Gamma = \pi_1(M, x)$. Let $\mathcal{G}$ be a group chain, and let $\mathcal{P}_{\mathcal{G}}$ be the associated presentation, with basepoint $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$. Given a commensurator $[\phi] \in \mathrm{Comm}(\Gamma)$ suppose that the conjugate chain $\mathcal{G}^\phi_\ell$ is zig-zag equivalent to $\mathcal{G}_{\ell'}$, then there exists $h_\phi \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{G}}, \widehat{x})$ such that $\chi(h_\phi) = \phi$.*

REMARK 5.8. Definition 5.3 does not assert that the homeomorphism h_ϕ is unique up to isotopy. That is, the isotopy class of h_ϕ in Theorem 5.7 need not be unique. However, for the special cases of manifolds in the list in Remark 5.6, uniqueness follows from results in [21].

The assumption in Theorem 5.7 that the conjugate chain $\mathcal{G}^\phi_\ell$ is zig-zag equivalent to $\mathcal{G}_{\ell'}$ is actually quite strong, and is the point of introducing the dynamical commensurator groups. However, there is one case where this condition is always satisfied.

Let $\mathcal{G}_u(\Gamma)$ denote the collection of all finite-index subgroups, partially ordered by inclusions. Let $[\Gamma : H]$ denote the index of H in Γ. Then the intersection $N_k = \bigcap_{[\Gamma:H] \leq k} H$ is a normal subgroup which is characteristic. Define a normal group chain $\mathcal{G}_\nu(\Gamma) = \{\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ where $\Gamma_i = N_{k_i}$ is a strictly decreasing subchain. Then for any commensurator ϕ, we have that $\mathcal{G}_\nu^\phi(\Gamma)$ is zig-zag equivalent to $\mathcal{G}_\nu(\Gamma)$. Thus, assuming that M is a strongly Borel manifold, then we obtain $h_\phi \in \mathrm{Homeo}(\widehat{M}, \widehat{x})$ for the universal solenoid $\widehat{M}$ over M, such that $\chi(h_\phi) = \phi \in \mathrm{Comm}(\Gamma)$.

Now consider the case where the presentation $\mathcal{P}$ is associated to a group chain $\mathcal{G}$ in Γ. Given $h_{\phi,U} \in \mathrm{Comm}(X_\mathcal{G}, \Gamma, \Phi, x)$ by Definition 4.4 there are adapted sets $x \in U \cap V$ such that $h_{\phi,U}$ induces a map Θ_U that conjugates the action $\mathcal{H}_U = \Phi_U(\Gamma_U)$ with the action $\mathcal{H}_V = \Phi_V(\Gamma_V)$. It is given that $h_{\phi,U}(x) = x$, so if we can show that the map Θ_U lifts to a commensurator between Γ_U and Γ_V, and hence to an element $[\phi] \in \mathrm{Comm}(\Gamma)$, then it follows from [23, Theorem 1.4] that the group chain $\mathcal{G}_\ell^\phi$ is zig-zag equivalent to $\mathcal{G}_{\ell'}$. Thus, by the constructions above, if M_0 is strongly Borel, then there exists $h_\phi \in \mathrm{Homeo}(\mathfrak{M}_\mathcal{G}, \widehat{x})$ such that $\chi(h_\phi) = \chi_\Phi(h_{\phi,U})$ in the diagram below:

$$\begin{array}{ccc} & & \phi \in \mathrm{Comm}(\Gamma) \\ & \overset{\chi}{\nearrow} & \\ h_\phi \in \mathrm{Homeo}(\mathfrak{M}_\mathcal{P}, \widehat{x}) & & \uparrow \chi_\Phi \\ & \underset{\sigma_{\mathcal{P},\widehat{x}}}{\searrow} & \\ & & h_{\phi,U} \in \mathrm{Comm}(X_\mathcal{G}, \Gamma, \Phi, x) \end{array} \tag{22}$$

The next step in the analysis of $\mathrm{Mod}(\mathfrak{M}_\mathcal{P})$ is to develop conditions on the group chain $\mathcal{G}$ which are sufficient to imply that an induced map $\Theta_U \colon \mathcal{H}_U \to \mathcal{H}_V$ lifts to a commensurator between Γ_U and Γ_V and hence to an element $\phi \in \mathrm{Comm}(\Gamma)$. This requires further examinations of the dynamical properties of odometers, their relations to the algebraic properties of Γ, and various consequences.

6. Odometer dynamics

Let $(X_\mathcal{G}, \Gamma, \Phi)$ be an odometer action. The action is *effective* if the action map $\Phi \colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ is injective. For an action defined by a group chain $\mathcal{G}$, the action is effective if and only if $\bigcap_{\ell \geq 0} C_\ell = \{e\}$ is the trivial group , where $\{C_\ell \mid \ell \geq 0\}$ is the normal group chain as defined in Definition 4.1. In particular, if the action is effective then Γ is residually finite.

An action $(\mathfrak{X}, \Gamma, \Phi)$ is said to be *free* if for all $x \in \mathfrak{X}$ and $g \in \Gamma$, $g \cdot x = x$ implies that $g = e$, the identity of the group. Let $\mathrm{Fix}(g) = \{x \in \mathfrak{X} \mid g \cdot x = x\}$, and define the *isotropy set*

$$\mathrm{Iso}(\Phi) = \{x \in \mathfrak{X} \mid \exists\, g \in \Gamma \ ,\ g \neq e \ ,\ g \cdot x = x\} = \bigcup_{e \neq g \in \Gamma} \mathrm{Fix}(g)\ . \tag{23}$$

DEFINITION 6.1. [12, 13] $(\mathfrak{X}, \Gamma, \Phi)$ *is said to be* topologically free *if* $\mathrm{Iso}(\Phi)$ *is meager in* $\mathfrak{X}$.

Note that if $\mathrm{Iso}(\Phi)$ is meager, then $\mathrm{Iso}(\Phi)$ has empty interior. That is, if there exists a non-identity element $g \in \Gamma$ such that $\mathrm{Fix}(g)$ has interior, then the action is not topologically free.

The notion of a *quasi-analytic* action, introduced in the works [1, 2], is an alternative formulation of the topologically free property.

DEFINITION 6.2. *An action* $\Phi \colon H \times \mathfrak{X} \to \mathfrak{X}$*, where* H *is a topological group and* $\mathfrak{X}$ *a Cantor space, is said to be* quasi-analytic *if for each clopen set* $U \subset \mathfrak{X}$ *and* $g \in H$ *such that* $\Phi(g)(U) = U$ *and the restriction* $\Phi(g)|U$ *is the identity map on* U*, then* $\Phi(g)$ *acts as the identity on* $\mathfrak{X}$*.*

A topologically free action is quasi-analytic. Conversely, the Baire Category Theorem implies that a quasi-analytic effective action of a *countable* group is topologically free [51, Section 3].

A local formulation of the quasi-analytic property for odometers was introduced in the works [24, 31].

DEFINITION 6.3. *An odometer action $\Phi\colon H \times \mathfrak{X} \to \mathfrak{X}$, where H is a topological group and $\mathfrak{X}$ is a Cantor metric space with metric $d_{\mathfrak{X}}$, is* locally quasi-analytic *(or LQA) if there exists $\varepsilon > 0$ such that for any non-empty clopen set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \varepsilon$, and for any non-empty clopen subset $V \subset U$, if the action of $g \in H$ satisfies $\Phi(g)(V) = V$ and the restriction $\Phi(g)|V$ is the identity map on V, then $\Phi(g)$ acts as the identity on all of U.*

The LQA property for odometer actions of profinite groups is used to define *stable* and *wild* actions:

DEFINITION 6.4. *$\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an odometer. We say that the action is* stable *if the associated profinite action $\widehat{\Phi}\colon \mathfrak{G}_{\Phi} \times \mathfrak{X} \to \mathfrak{X}$ is LQA. The odometer is said to be* wild *otherwise.*

There are examples where the odometer action $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ is LQA, but the associated profinite action $\widehat{\Phi}\colon \mathfrak{G}_{\Phi} \times \mathfrak{X} \to \mathfrak{X}$ is wild (see [31]). Also, there exist wild actions of odometers with Γ nilpotent (see [33, Section 5.3]).

The third property of odometer actions is intermediate between the quasi-analytic (topologically free) and locally quasi-analytic conditions.

DEFINITION 6.5. *An odometer $(\mathfrak{X}, \Gamma, \Phi)$ is* coherent *if for every adapted set $U \subset \mathfrak{X}$, the restricted holonomy action map $\Phi_U : \Gamma_U \to Homeo(U)$ has finite kernel.*

Note that if the action Φ is quasi-analytic, then the kernel of the map Φ_U is trivial, so the action is coherent. On the other hand, if Φ is a locally quasi-analytic action, then for U sufficiently small, the image group $\Phi_U(\Gamma_u) \subset \mathrm{Homeo}(U)$ defines a quasi-analytic action on U.

For example, if Γ is a nilpotent group, then an effective Γ-odometer is quasi-analytic by Proposition 6.11 below, hence the action is coherent. On the other hand, if Γ is a weakly branch group (see for example [38, 48]), and $\mathfrak{X}$ is the boundary of the tree on which the group acts, then the odometer action of Γ on $\mathfrak{X}$ is neither locally quasi-analytic nor coherent.

Next, consider two algebraic properties of a group Γ and their relation to the dynamical properties above. First, recall that $g \in \Gamma$ is a *torsion* element if there exists some $n > 0$ such that $\gamma^n = e$, where $e \in \Gamma$ is the identity element. The least such $n > 0$ is called the order of Γ. Let $\Gamma_{tor} \subset \Gamma$ denote the collection of torsion elements. The group Γ is *torsion free* if the identity is the only torsion element. In general, Γ_{tor} is not a subgroup. For example, there is the well-known example when $\Gamma = \mathrm{SL}(2, \mathbb{Z})$ then Γ is generated by two torsion elements, of order 2 and 3. On the other hand, when Γ is nilpotent, then Γ_{tor} is a subgroup.

A profinite completion of a finitely generated, torsion free group Γ need not be torsion free, as shown by Lubotzky in [39]. This is the basis for the construction in [31, Section 9] of free odometer actions of finitely generated groups which are wild. Further examples of effective odometer actions which are wild are given in [33, Example 3.3].

Next, recall a definition from group theory:

DEFINITION 6.6 (Unique root property)**.** *A group Γ satisfies the* unique root property *if for any $n \geq 2$ and $g, h \in \Gamma$, $g^n = h^n$ implies $g = h$.*

This is related to the notion of $\mathcal{D}$-group, or a *divisible* group, which assumes that in addition, every element of Γ has a unique n^{th} root for all $n > 0$. Section 2 of [4] gives a series of consequences of the unique root property. For example, if Γ has the unique root property, it is immediate that Γ is torsion-free. Free groups, torsion-free nilpotent groups, and fundamental groups of closed orientable surfaces have the unique root property (see [4, Lemma 2.2]). The unique root property has consequences for the study of $\mathrm{Comm}(\Gamma)$:

LEMMA 6.7. [4, Lemma 2.4] *Suppose Γ satisfies the unique root property, and $H \subset \Gamma$ is of finite index. Then the natural map $\mathrm{Aut}(H) \to \mathrm{Comm}(\Gamma)$ is injective.*

Next we give two elementary consequences for the dynamics of odometers.

LEMMA 6.8. *Let Γ have the unique root property, and suppose that the odometer $(\mathfrak{X}, \Gamma, \Phi)$ is coherent. Then for every adapted set $U \subset \mathfrak{X}$, the restricted action map $\Phi_U \colon \Gamma_U \to \mathrm{Homeo}(U)$ is injective.*

Proof. The kernel of Φ_U is a finite subgroup of Γ, so consists of torsion elements. But Γ has the unique root property, so it is torsion free, hence Φ_U is injective. □

LEMMA 6.9. *Suppose $(\mathfrak{X}, \Gamma, \Phi)$ is an odometer where $\mathfrak{X} = \mathfrak{G}_{\mathcal{G}}$ is a profinite completion of Γ, and the action is by left multiplication. Then the action is free.*

Proof. Let $g \in \Gamma$ with $\Phi(g)(x) = x$ for some $x \in \mathfrak{X}$. Then for any $y \in \mathfrak{X}$ there exists $h \in \mathfrak{G}_{\mathcal{G}}$ such that $y = x \cdot h$. Then $\Phi(g)(y) = \Phi(g)(x \cdot h) = \Phi(g)(x) \cdot h = x \cdot h = y$. Thus $\Phi(g)$ is the identity. □

Finally, much stronger properties hold for nilpotent odometer actions. Recall that a group Γ is *virtually nilpotent* if it contains a finitely-generated nilpotent subgroup $\Gamma' \subset \Gamma$ with finite index. The following result is a consequence of the Noetherian property of Γ.

THEOREM 6.10. [32, Theorem 1.6] *Suppose $(\mathfrak{X}, \Gamma, \Phi)$ is an odometer with Γ virtually nilpotent, then the action is locally quasi-analytic.*

In the case where Γ is nilpotent, there is a stronger conclusion:

PROPOSITION 6.11. *If Γ is nilpotent, then an odometer $(\mathfrak{X}, \Gamma, \Phi)$ is quasi-analytic.*

Proof. Let $U \subset \mathfrak{X}$ be an adapted set and let $h \in \Gamma_U$ such that $\Phi_U(h)$ is the identity on U. We claim that $\Phi(h)$ acts as the identity on $X_{\mathcal{G}}$. We proceed by induction on the upper central series

$$\{e\} = \Gamma_{(0)} \subset \Gamma_{(1)} \subset \cdots \subset \Gamma_{(k)} = \Gamma$$

So $\Gamma_{(1)}$ is the center of Γ, and $\Gamma_{(i+1)} = \{g \in \Gamma \mid [g, h] \in \Gamma_{(i)} \ \forall h \in \Gamma\}$.

Let $g \in \Gamma_{(1)}$, then for $x \in U$, we have $h \cdot g \cdot x = g \cdot h \cdot x = g \cdot x$. Thus $\Phi(h)$ acts as the identity on the union of the clopen sets $\{g \cdot U \mid g \in \Gamma_{(1)}\}$.

For $1 \leq i < k$, assume that $\Phi(h)$ acts as the identity on the union of the clopen sets $\{g \cdot U \mid g \in \Gamma_{(i)}\}$. Let $g \in \Gamma_{(i+1)}$ then for $x \in U$, we have $[g, h] = g^{-1}h^{-1}gh \in \Gamma_{(i)}$ and

$$h \cdot g \cdot x = h \cdot g \cdot [g, h] \cdot x = h \cdot g \cdot g^{-1}h^{-1}gh \cdot x = g \cdot h \cdot x = g \cdot x$$

Thus, $\Phi(h)$ acts as the identity on the union of the clopen sets $\{g \cdot U \mid g \in \Gamma_{(i+1)}\}$. It follows by finite induction that $\Phi(h)$ acts as the identity on the union of the clopen sets $\{g \cdot U \mid g \in \Gamma\}$, which is all of $\mathfrak{X}$. □

Then from the definition of coherent actions in Definition 6.5 we obtain:

COROLLARY 6.12. *If Γ is nilpotent, then an effective odometer $(\mathfrak{X}, \Gamma, \Phi)$ is coherent.*

By Corollary 6.12, for Γ nilpotent, an effective odometer $(X_{\mathcal{G}}, \Gamma, \Phi)$ is quasi-analytic and hence coherent.

7. Realizing commensurators

We return to the problem of determining when a commensurator $[\phi] \in \mathrm{Comm}(\Gamma)$ is in the image of the characteristic map $\chi \colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \to \mathrm{Comm}(\Gamma)$. The first approach to this problem uses dynamical properties of the odometer action $(X_{\mathcal{G}}, \Gamma, \Phi)$ to construct the image of χ. The conclusion is that with sufficient regularity of the odometer action, the calculation of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ reduces to the analysis of the germinal groups $\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x)$. This will be used in the calculations of examples in later sections.

The second approach, discussed afterwards, essentially assumes that such a lift exists. The second method appears in the literature, and we contrast it with the following result, which is a restatement of Theorem 1.10.

THEOREM 7.1. *Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an effective, coherent and locally quasi-analytic odometer. Let $x \in \mathfrak{X}$. Then there exists an injection*

$$\chi_\Phi\colon \mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x) \to \mathrm{Comm}(\Gamma)$$

such that diagram (8) *commutes.*

Proof. Let $h\colon U \to V$ with $h(\widehat{x}) = \widehat{x}$, and h_* conjugates the images $\mathcal{H}_U = \Phi_U(\Gamma_U) \subset \mathrm{Homeo}(U)$ with $\mathcal{H}_V = \Phi_V(\Gamma_V) \subset \mathrm{Homeo}(V)$. We require the following:

LEMMA 7.2. *Let $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an effective and coherent odometer action, and let $U \subset \mathfrak{X}$ be an adapted set. Then there exists a finite index subgroup $\Gamma'_U \subset \Gamma_U$ such that the restriction $\Phi_U : \Gamma'_U \to \mathrm{Homeo}(U)$ is injective.*

Proof. First note that the action map $\Phi\colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ is injective, and $\Gamma_U \subset \Gamma$ has finite index. The coherent assumption implies that $K_U = \ker\{\Phi_U : \Gamma_U \to \mathrm{Homeo}(U)\} \subset \Gamma$ is a finite group.

Let $U' \subset U$ be adapted and sufficiently small such that K_U acts effectively on the finite quotient space $\Gamma/\Gamma_{U'}$. Let $C_{U'} = \ker\{\Phi_{U'} : \Gamma \to \mathrm{Homeo}(\Gamma/\Gamma_{U'})\}$ which is a normal subgroup with finite index, and $K_U \cap C_{U'} = \{1\}$.

Then $\Gamma'_U = C_{U'} \cap \Gamma_U$ has finite index in Γ, and the restriction $\Phi_U : \Gamma'_U \to \mathrm{Homeo}(U)$ is injective. □

So by Lemma 7.2, there exists $\Gamma'_U \subset \Gamma_U$ with finite index such that $\Phi_U : \Gamma'_U \to \mathrm{Homeo}(U)$ is injective.

Similarly, the map $\Phi_V : \Gamma'_V \to \mathrm{Homeo}(V)$ has finite kernel, and there exists $\Gamma'_V \subset \Gamma_V$ with finite index such that $\Phi_V : \Gamma'_V \to \mathrm{Homeo}(V)$ is injective. Now set

$$\Gamma''_U = \Phi_U^{-1}\{\Phi_U(\Gamma'_U) \cap h_*^{-1}(\Phi_V(\Gamma'_V))\} \tag{24}$$

which has finite index in Γ. Then $\varphi = \Phi_V^{-1} \circ h_* \circ \Phi_U : \Gamma''_U \to \Gamma''_V$ is a commensurator. Define $\chi_\Phi[h, U, V] = [\varphi]$ where $[\varphi]$ is the equivalence class of the map φ. The class $[\varphi]$ is independent of the germinal class of h by the construction. □

There is an alternate approach to analyzing the image of χ. Introduce the *relative commensurator* group of $\Phi(\Gamma)$ in $\mathrm{Homeo}(X_\mathcal{G})$. For $x \in X_\mathcal{G}$:

$$\mathrm{Comm}_{\mathrm{Homeo}(X_\mathcal{G},x)}(\Gamma) = \{h \in \mathrm{Homeo}(X_\mathcal{G}, x) \mid h\colon \Phi(H) \cong \Phi(K); H, K \subset_f \Gamma\} \ . \tag{25}$$

That is, require that the isomorphism ϕ between $H, K \subset \Gamma$ is induced by a homeomorphism of $X_\mathcal{G}$.

If $(X_\mathcal{G}, \Gamma, \Phi)$ is effective, then given $h \in \mathrm{Comm}_{\mathrm{Homeo}(X_\mathcal{G},x)}(\Gamma)$, the induced map between the images $\Theta_h\colon \Phi(H) \to \Phi(K)$ lifts to a commensurator $\phi_h\colon H \to K$. In this case, we then have $\mathrm{Comm}_{\mathrm{Homeo}(X_\mathcal{G})}(\Gamma) \subset \mathrm{Comm}(\Gamma)$, and every commensurator in the image of this inclusion induces a zig-zag map on the group chain $\mathcal{G}$.

PROPOSITION 7.3. *There is a natural map:*

$$\rho\colon \mathrm{Comm}_{\mathrm{Homeo}(X_\mathcal{G},x)}(\Gamma) \to \mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x) \ . \tag{26}$$

If Γ has the unique root property, and the action on $X_\mathcal{G}$ is coherent, then ρ is injective.

Proof. The map ρ is defined by restricting a homeomorphism $h\colon X_\mathcal{G} \to X_\mathcal{G}$ to its germ at x.

Assume that Γ has the unique root property, and the action on $X_\mathcal{G}$ is coherent, then by Lemma 6.8 the map Φ is injective. It follows that the image $\Phi(\Gamma) \subset \mathrm{Homeo}(X_\mathcal{G})$ also has the unique root property.

Now suppose that $h, h' \in \mathrm{Comm}_{\mathrm{Homeo}(X_{\mathcal{G}},x)}(\Gamma)$ have the same germ at x, say for an adapted clopen set $x \in U$. Recall that $\Theta_h \colon \mathcal{H}_U \to \mathcal{H}_V$ denotes the map induced by h in Definition 4.3, and similarly for $\Theta_{h'}$. Then they induce the same map on the image of Γ_U in $\mathrm{Homeo}(U)$, so $\Theta_h = \Theta_{h'}$.

Again by Lemma 6.8, the map is Φ_U is injective, so they induce the same maps on Γ_U which we again denote by Θ_h and $\Theta_{h'}$. Given $g \in \Gamma$ there is some $n > 0$ such that $g^n \in \Gamma_U$. Thus

$$h(g)^n = h(g^n) = h'(g^n) = h'(g)^n$$

and thus $h(g) = h'(g)$. As Φ is injective, h and h' induce the same map on Γ, and as the action of Γ on $X_{\mathcal{G}}$ is minimal, this implies that $h = h'$. □

Röver showed in [53, Theorem 1.2] that $\mathrm{Comm}(\Gamma) \cong \mathrm{Comm}_{\mathrm{Homeo}(\mathfrak{X},x)}(\Gamma)$, when Γ is weakly branch and $\mathfrak{X}$ is the boundary of a tree on which Γ acts. This identification is used to calculate $\mathrm{Comm}(\Gamma)$ in various cases. However, weakly branch groups are not torsion free, so the proof of Proposition 7.3 does not apply.

8. Normalizer actions

In this section, we develop the relation between the groups $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ and $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$. The action of $\mathrm{Homeo}_0(\mathfrak{M}_{\mathcal{P}})$ is transitive on the leaf of $\mathcal{F}_{\mathcal{P}}$ containing $\widehat{x}$, so determining this relation reduces to an algebraic problem as discussed below.

Recall that $\mathfrak{G}_{\mathcal{G}}$ is the profinite group associated to the group chain $\mathcal{G}$ in Γ as given by in (18), and $\mathfrak{D}_{\mathcal{G}} = \{\widehat{g} \in \mathfrak{G}_{\mathcal{G}} \mid \widehat{\Phi}(\widehat{g})(x_0) = x_0\}$ is the isotropy subgroup of the basepoint $x_0 \in X_{\mathcal{G}}$, so that there is a $\mathfrak{G}_{\mathcal{G}}$-equivariant isomorphism $X_{\mathcal{G}} = \mathfrak{G}_{\mathcal{G}}/\mathfrak{D}_{\mathcal{G}}$. The fiber $\mathfrak{X}_0 = \widehat{q}^{-1}(x_0)$ is identified with $X_{\mathcal{G}}$ and $\widehat{x}$ is identified with x_0.

Recall the seminal result of Fokkink and Oversteegen [27], as reformulated in [16].

THEOREM 8.1. *The solenoidal manifold $\mathfrak{M}_{\mathcal{P}}$ is homogeneous if and only if $\mathfrak{D}_{\mathcal{G}} = \{\widehat{e}\}$.*

If $\mathfrak{D}_{\mathcal{G}}$ is trivial, then $X_{\mathcal{G}} = \mathfrak{G}_{\mathcal{G}}$ is a profinite group, and acts transitively of the leaf space of $\mathcal{F}_{\mathcal{P}}$ via the identification in (21) for $\ell = 0$, so $\mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}})$ acts transitively on $\mathfrak{M}_{\mathcal{P}}$. The converse direction is more subtle to prove, but in essence it follows from a well-known result for finite covering spaces [43, Corollary 7.3]. In the following, we extend this classical result to solenoidal manifolds.

We first consider the special case when $h \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}})$ maps the basepoint $\widehat{x}$ into the fiber $\mathfrak{X}_0$. Introduce the group $\mathfrak{N}_{\mathcal{G}} = \mathfrak{N}_{\mathfrak{G}_{\mathcal{G}}}(\mathfrak{D}_{\mathcal{G}})$ which is the normalizer of $\mathfrak{D}_{\mathcal{G}} \subset \mathfrak{G}_{\mathcal{G}}$.

PROPOSITION 8.2. *Let $h \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}})$ with $\widehat{y} = h(\widehat{x}) \in \mathfrak{X}_0$. Then there exists $\widehat{g} \in \mathfrak{N}_{\mathcal{G}}$ with $\widehat{g} \cdot \widehat{x} = \widehat{y}$.*

Proof. Let $h \in \mathrm{Homeo}(\mathfrak{M}_{\mathcal{P}})$ with $\widehat{y} = h(\widehat{x}) \in \mathfrak{X}_0$. Then by a small isotopy of h supported in a foliated chart about $\widehat{x}$, we can assume there is an open neighborhood $\widehat{x} \in U \subset \mathfrak{X}_0$ with $h(U) \subset \mathfrak{X}_0$. Identify $\mathfrak{X}_0 \cong X_{\mathcal{G}} \cong \mathfrak{G}_{\mathcal{G}}/\mathfrak{D}_{\mathcal{G}}$ so that U is an open neighborhood of x_0 and $h(U) \subset X_{\mathcal{G}}$ is an open neighborhood of $y_0 = h(x_0)$, and $\mathfrak{G}_{\mathcal{G}}$ acts transitively on the left on $X_{\mathcal{G}}$. Let $\widehat{g} \in \mathfrak{G}_{\mathcal{G}}$ be such that $\widehat{g} \cdot x_0 = y_0$. We show that $\widehat{g}^{-1} \cdot \widehat{k} \cdot \widehat{g} \in \mathfrak{D}_{\mathcal{G}}$ for all $\widehat{k} \in \mathfrak{D}_{\mathcal{G}}$, and hence $\widehat{g} \in \mathfrak{N}_{\mathcal{G}}$.

Recall that U_ℓ is the adapted neighborhood of x_0 with $\Gamma_{U_\ell} = \Gamma_\ell$, and the collection $\{U_\ell \mid \ell > 0\}$ is a neighborhood basis at x_0. Then $\{\widehat{g} \cdot U_\ell \mid \ell > 0\}$ is a neighborhood basis of y_0. Recall that $\widehat{\Gamma}_\ell$ is the closure of Γ_ℓ in $\mathfrak{G}_{\mathcal{G}}$, and acts transitively on U_ℓ, hence $U_\ell = \widehat{\Gamma}_\ell/\mathfrak{D}_{\mathcal{G}}$ for all ℓ, and so $\bigcap_\ell \widehat{\Gamma}_\ell = \mathfrak{D}_{\mathcal{G}}$.

Let $\widehat{k} \in \mathfrak{D}_{\mathcal{G}}$. Then there exists a sequence $\{k_\ell \in \Gamma_\ell \mid \ell > 0\}$ such that $\lim_{\ell \to \infty} \overline{k}_\ell = \widehat{k}^{-1}$ where $\overline{k}_\ell \in \widehat{\Gamma}_\ell$ is the image of k_ℓ.

Choose $j_0 > 0$ such that $\widehat{g} \cdot U_{j_0} \subset h(U)$. Then choose i_0 such that $h(U_{i_0}) \subset \widehat{g} \cdot U_{j_0}$. Set $U'_{j_0} = h(U_{i_0})$ which is an adapted set. The map h induces a return equivalence as in Definition 4.3, so there is an induced isomorphism $\Theta\colon \mathcal{H}_{U_{i_0}} \to \mathcal{H}_{U'_{j_0}}$, where

$$\begin{aligned}\mathcal{H}_{U_{i_0}} &= \text{Image}\{\Phi_{U_{i_0}} : \Gamma_{U_{i_0}} \to \text{Homeo}(U_{i_0})\} \\ \mathcal{H}'_{U'_{j_0}} &= \text{Image}\{\Phi_{U'_{j_0}} : \Gamma_{U'_{j_0}} \to \text{Homeo}(U'_{j_0})\} \ .\end{aligned}$$

Let $j_1 \geq j_0$ so that for $\ell \geq j_1$ we have $k_\ell \in \Gamma_{U'_{j_1}}$.

Then there exists $k'_\ell \in \Gamma_{U_{i_0}}$ such that $\Theta(\Phi_{U_{i_0}}(k'_\ell)) = \Phi_{U'_{j_0}}(k_\ell)$. Then we have

$$h(k'_\ell \cdot x_0) = k_\ell \cdot h(x_0) = k_\ell \cdot \widehat{g} \cdot x_0 \ . \tag{27}$$

Now let $\ell \to \infty$, so in the limit we get

$$\widehat{g} \cdot x_0 = h(x_0) = \widehat{k}^{-1} \cdot h(x_0) = \widehat{k}^{-1} \cdot \widehat{g} \cdot x_0 \tag{28}$$

and so $\widehat{g}^{-1} \cdot \widehat{k} \cdot \widehat{g} \cdot x_0 = x_0$. Since $\mathfrak{D}_\mathcal{G}$ is to isotropy group at x_0 we obtain that $\widehat{g}^{-1} \cdot \widehat{k} \cdot \widehat{g} \in \mathfrak{D}_\mathcal{G}$ for all $\widehat{k} \in \mathfrak{D}_\mathcal{G}$, as was to be shown. □

Note that while the adjoint induced map $\text{Adj}\colon \mathfrak{D}_\mathcal{G} \to \text{Homeo}(X_\mathcal{G}, x_0)$ is always injective, its restriction to a map $\text{Adj}_\ell\colon \mathfrak{D}_\mathcal{G} \to \text{Homeo}(U_\ell, x_0)$ may have kernel. This behavior underlies the definition of wild actions in [31], and necessitates the choices of the pre-images k'_ℓ in the proof above.

Now consider the general case when $h \in \text{Homeo}(\mathfrak{M}_\mathcal{P})$ with $\widehat{y} = h(\widehat{x})$. Let $L_{\widehat{y}} \subset \mathfrak{M}_\mathcal{P}$ denote the leaf containing $\widehat{y}$. Then there exists a leafwise path $\sigma\colon [0,1] \to L_{\widehat{y}}$ with $\sigma(0) = \widehat{y}$ and $\sigma(1) = \widehat{y}_\sigma \in \mathfrak{X}_0$. Then by Proposition 8.2 there exists $\widehat{g}_\sigma \in \mathfrak{N}_\mathcal{G}$ with $\widehat{y}_\sigma = \widehat{g}_\sigma \cdot \widehat{x}$.

Suppose that $\tau\colon [0,1] \to L_{\widehat{y}}$ with $\tau(0) = \widehat{y}$ and $\tau(1) = \widehat{y}_\tau \in \mathfrak{X}_0$. Then again by Proposition 8.2, there exists $\widehat{g}_\tau \in \mathfrak{N}_\mathcal{G}$ with $\widehat{y}_\tau = \widehat{g}_\tau \cdot \widehat{x}$.

Observe that the concatenation of the paths, $\omega = \tau \star \sigma^{-1}\colon [0,1] \to L_{\widehat{y}}$ satisfies $\omega(0), \omega(1) \in \mathfrak{X}_0$ and so descends to a closed path in M_0 hence defines a class $[\omega] \in \pi_1(M_0, x) = \Gamma$. Moreover, we have that for the left action $\widehat{\Phi}\colon \Gamma \times \mathfrak{G}_\mathcal{G} \to \mathfrak{G}_\mathcal{G}$, then $[\omega] \cdot \widehat{g}_\sigma = \widehat{g}_\tau$. Thus the loop ω defines an element of the normalizer, $\overline{[\omega]} = \widehat{g}_\tau \widehat{g}_\sigma^{-1} \in \mathfrak{N}_\mathcal{G}$. In other words, given $h \in \text{Homeo}(\mathfrak{M}_\mathcal{P})$, the fiber translation on $X_\mathcal{G}$ induced by h is only well-defined up to the left action on $\mathfrak{G}_\mathcal{G}$ of

$$\Gamma_\mathfrak{N} = \{g \in \Gamma \mid \overline{g} \in \mathfrak{N}_\mathcal{G}\} \tag{29}$$

where $\overline{g}$ denotes the image of $g \in \Gamma$ in $\mathfrak{G}_\mathcal{G}$. Introduce the coset space

$$\mathfrak{Z}_\mathcal{G} = \Gamma_\mathfrak{N} \backslash \mathfrak{G}_\mathcal{G} \ . \tag{30}$$

Note that $\Gamma_\mathfrak{N}$ need not be normal in $\mathfrak{N}_\mathcal{G}$, so $\mathfrak{Z}_\mathcal{G}$ need not have a group structure.

Note that the path ω can be realized as the trace of an isotopy $\Omega_t \in \text{Homeo}_0(\mathfrak{M}_\mathcal{P})$, where Ω_t is defined by the usual construction of sliding along the path $\omega(t)$ in a tubular neighborhood of ω. That is, $\Omega_t(\widehat{g}_\sigma) = \omega(t)$ for $0 \leq t \leq 1$. It follows that if $h, h' \in \text{Homeo}(\mathfrak{M}_\mathcal{G})$ are isotopic, then the translational element in $\mathfrak{N}_\mathcal{G}$ they define is well-defined as a coset in $\mathfrak{Z}_\mathcal{G} = \Gamma_\mathfrak{N} \backslash \mathfrak{G}_\mathcal{G}$. Thus we obtain an analog of Theorem 4.13 and related results by Odden in [50, Section 4.3]:

THEOREM 8.3. *Let $\mathfrak{M}_\mathcal{P}$ be a solenoidal manifold with basepoint $\widehat{x} \in \mathfrak{M}_\mathcal{P}$. Then there is a well-defined surjection of sets*

$$\tau_\mathcal{P}\colon \text{Mod}(\mathfrak{M}_\mathcal{P}) \to \mathfrak{Z}_\mathcal{G} \ . \tag{31}$$

Proof. Given $h \in \text{Homeo}(\mathfrak{M}_\mathcal{P})$ let $\widehat{g}_\sigma \in \mathfrak{N}_\mathcal{G}$ be the translation element defined by $\widehat{y} = h(\widehat{x}) = \widehat{g}_\sigma \cdot \widehat{x}$ as above. Let $\tau_\mathcal{P}(h) = [\widehat{g}_\sigma] \in \mathfrak{Z}_\mathcal{G}$ be the class it determines. Then by the above, $\tau_\mathcal{P}(h)$ is well-defined, and is an invariant of the isotopy class of h. The fact that $\tau_\mathcal{P}$ is a homomorphism follows from the definition of $\widehat{g}_\sigma$ using the path lifting property of the projection $\widehat{q}_0\colon \mathfrak{M}_\mathcal{P} \to M_0$ restricted to the leaves of $\mathcal{F}_\mathcal{P}$. □

COROLLARY 8.4. *Let $\mathfrak{M}_\mathcal{P}$ be a solenoidal manifold with basepoint $\widehat{x} \in \mathfrak{M}_\mathcal{P}$. Suppose that the normalizer $\mathfrak{N}_\mathcal{G} = \mathfrak{D}_\mathcal{G}$, then* $\text{Mod}(\mathfrak{M}_\mathcal{P}) = \text{Mod}(\mathfrak{M}_\mathcal{P}, \widehat{x})$.

9. Examples

The abstract commensurator group $\mathrm{Comm}(\Gamma)$ has been calculated for a wide variety of groups. (See for example [56].) For such Γ, the calculation of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ is reduced to determining which commensurators $\phi \in \mathrm{Comm}(\Gamma)$ asymptotically preserve the group chain $\mathcal{G}$ defining $\mathcal{P}$. Then ϕ defines an element $h_{\phi,U} \in \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x)$ with $\chi_{\Phi}(h_{\phi,U}) = \phi$ in Diagram 22.

If the action $(X_{\mathcal{G}}, \Gamma, \Phi)$ satisfies the hypotheses of Theorem 7.1 and the manifold M_0 satisfies the strong Borel Property, as in Remark 5.6, then the class $h_{\phi,U}$ is in the image of $\sigma_{\mathcal{P},\widehat{x}}$ in Diagram (22).

9.1. Vietoris Solenoids.

A 1-dimensional solenoid has base manifold $\mathbb{S}^1$, and so we observe that $\mathrm{Comm}(\mathbb{Z}) \cong \mathbb{Q}^*$ the group of invertible rational numbers. Given a group chain in $\mathbb{Z}$, the problem is then to determine when multiplication by $\xi = a/b \in \mathbb{Q}^*$ asymptotically preserves the group chain. We briefly recall the results, which are folklore, and described by Kwapisz in [37].

Choose an infinite collection of integers $\vec{m} = \{m_\ell > 1 \mid 1 \leq \ell < \infty\}$. For each ℓ, let $p_\ell \colon \mathbb{S}^1 \to \mathbb{S}^1$ be a covering map of degree m_ℓ. Let $\mathfrak{S}_{\vec{m}}$ be the inverse limit of these maps. The Steinitz number associated to $\vec{m}$ is the formal infinite product

$$\xi(\vec{m}) = \mathrm{lcm}\{m_1 m_2 \cdots m_\ell \mid \ell > 0\} \ , \tag{32}$$

where lcm denotes the least common multiple of the collection of integers. A Steinitz number ξ can be uniquely written as the formal product over the set of primes Π,

$$\xi = \prod_{p \in \Pi} p^{\chi_\xi(p)} \ , \tag{33}$$

where the *characteristic function* $\chi_\xi \colon \Pi \to \{0, 1, \ldots, \infty\}$ counts the multiplicity with which a prime p appears in the infinite product ξ.

DEFINITION 9.1. *Two Steinitz numbers ξ and ξ' are said to be* asymptotically equivalent *if there exists finite integers $m, m' \geq 1$ such that $m \cdot \xi = m' \cdot \xi'$, and we then write $\xi \overset{a}{\sim} \xi'$. The equivalence class of ξ is called its* type*, and denoted by $\tau[\xi]$.*

Then we have:

THEOREM 9.2 (Bing [8], McCord [44], Fokkink and Oversteegen [27])**.** *Let $\mathfrak{M}$ be an orientable 1-dimensional solenoid. Then there exists $\vec{m}$ such that there is a homeomorphism $\mathfrak{M} \cong \mathfrak{S}_{\vec{m}}$. Moreover, solenoids $\mathfrak{S}_{\vec{m}}$ and $\mathfrak{S}_{\vec{m}'}$ are homeomorphic if and only if their types are equal, $\tau[\vec{m}] = \tau[\vec{m}']$.*

Kwapisz calculated the group $\mathrm{Mod}(\mathfrak{S}_{\vec{m}})$ for arbitrary type $\tau(\vec{m})$ in [37]. We recall three examples.

Let $\mathcal{G}_u(\mathbb{Z})$ be the universal group chain, and $\widehat{\mathbb{S}^1}$ the resulting universal solenoid over $\mathbb{S}^1$. Then for every non-zero $a \in \mathbb{Z}$, multiplication by a maps the chain into itself, so we obtain a homeomorphism $h_a \in \mathrm{Homeo}(\widehat{\mathbb{S}^1}, \widehat{x})$. Thus, $\mathrm{Mod}(\widehat{\mathbb{S}^1}, \widehat{x}) \cong \mathbb{Q}^*$. The profinite completion $\widehat{\mathbb{Z}}$ acts on the right on $\widehat{\mathbb{S}^1}$, so we obtain $\mathrm{Mod}(\widehat{\mathbb{S}^1}) \cong \mathbb{Q}^* \times \widehat{\mathbb{Z}}$.

For the second example, let $\vec{p} = \{1 < p_1 < p_2 < \cdots\}$ be an infinite sequence of distinct primes, and $\mathcal{G}_{\vec{p}}$ the corresponding group chain in $\mathbb{Z}$. Then there is no $a \in \mathbb{Z}^*$ other than ± 1 which preserves $\mathcal{G}_{\vec{p}}$, hence the group chain is rigid. Thus, $\mathrm{Mod}(\mathfrak{S}_{\vec{p}}, \widehat{x}) \cong \{\pm 1\}$ and so $\mathrm{Mod}(\mathfrak{S}_{\vec{p}}) \cong \mathbb{Z}_{\vec{p}}$.

Finally, choose a prime $p > 1$ and let $\mathcal{G}_{\widehat{p}}$ be the group chain formed by the subgroups $\Gamma_\ell = p^\ell \mathbb{Z} \subset \mathbb{Z}$. Then only multiplication by a power of p maps the group chain into itself, hence $\mathrm{Mod}(\mathfrak{S}_{\widehat{p}}, \widehat{x}) \cong \mathbb{Z}$ and $\mathrm{Mod}(\mathfrak{S}_{\widehat{p}}) \cong \mathbb{Z} \times \mathbb{Z}_{\widehat{p}}$.

The calculation of $\mathrm{Mod}(\mathfrak{S}_{\vec{m}})$ for all other collections of integer sequences $\vec{m}$ gives similar results which are analogous to these calculations. See [37] for detailed calculations, and [33, 34] for more discussion of type invariants for solenoidal manifolds.

9.2. **Toroidal Solenoids.** Consider the case of a solenoid $\mathfrak{M}_{\mathcal{P}}$ with base manifold the n-dimensional torus, and group $\Gamma \cong \mathbb{Z}^n$. Then $\mathrm{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}(\mathbb{Q}^n)$. The manifold $\mathbb{T}^n$ satisfies the Borel Conjecture, so we can apply the results above to odometer actions of $\mathbb{Z}^n$.

Let $\mathcal{G} = \{\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \cdots\}$ where $\Gamma_0 = \mathbb{Z}^n$, and each Γ_ℓ is a free subgroup of rank n. Then for each $\ell > 0$, there exists an invertible $n \times n$ matrix A_ℓ with integer coefficients and determinant $m_\ell = |A_\ell| > 1$, such that the map $\phi_{A_\ell} \colon \mathbb{Z}^n \to \mathbb{Z}^n$ defined by multiplication with A_ℓ maps the subgroup $\Gamma_{\ell-1}$ isomorphically onto Γ_ℓ. Thus, $\Gamma_\ell = \phi_{A_\ell} \circ \cdots \phi_{A_1}(\mathbb{Z}^n)$. Consider first the simplest case, where A_ℓ is a diagonal matrix for all ℓ.

The universal toroidal solenoid $\widehat{\mathbb{T}^n}$ is defined by the group chain with $\Gamma_\ell = (\ell+1)! \cdot \mathbb{Z}^n$. That is, the matrices A_ℓ above are given by $(\ell+1)$ times the identity, and $\widehat{\Gamma}_{\mathcal{P}} \cong \widehat{\mathbb{Z}^n}$. Every element of $\mathrm{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}(\mathbb{Q}^n)$ preserves this group chain, so we have $\mathrm{Mod}(\widehat{\mathbb{T}^n}) \cong \mathrm{GL}(\mathbb{Q}^n) \times \widehat{\mathbb{Z}^n}$.

For a prime $p \geq 2$, the p-adic toroidal solenoid $\mathbb{T}^n_{\widehat{p}}$ is defined by the group chain with $\Gamma_\ell = p^\ell \cdot \mathbb{Z}^n$. That is, the matrices A_ℓ above are given by p times the identity, and $\widehat{\Gamma}_{\mathcal{P}} \cong \mathbb{Z}^n_{\widehat{p}}$ This has Steinitz order p^∞. Every element of the commensurator group $\mathrm{Comm}(\mathbb{Z}^n) \cong \mathrm{GL}(\mathbb{Q}^n)$ then preserves the group chain, so we have $\mathrm{Mod}(\mathbb{T}^n_{\widehat{p}}) \cong \mathrm{GL}(\mathbb{Q}^n) \times \mathbb{Z}^n_{\widehat{p}}$.

For a disjoint collection of primes $S = \{p_1, p_2, \ldots, p_n\}$ let $\mathcal{G}_S$ be the group chain defined by the matrices A_ℓ whose diagonal entries are the primes in S, and are 0 off the diagonal. The associated toroidal solenoid $\mathbb{T}^n_S$ is the product of 1-dimensional solenoids, $\mathbb{T}^n_S \cong \mathfrak{S}_{\widehat{p_1}} \times \cdots \times \mathfrak{S}_{\widehat{p_n}}$. A map which is a return equivalence of the action of $\mathbb{Z}^n$ on $X_{\mathcal{G}_P}$ must preserve the factors for distinct primes, so

$$\mathrm{Comm}(X_{\mathcal{G}_S}, \mathbb{Z}^n, \Phi, x) \cong \mathrm{GL}(\mathbb{Q}) \times \cdots \times \mathrm{GL}(\mathbb{Q}) \ . \tag{34}$$

One can modify this construction by choosing a multiplicity n_i for each prime, then taking the product of solenoids $\mathbb{T}^n_S = \mathbb{T}^{n_1}_{p_1} \times \cdots \times \mathbb{T}^{n_k}_{p_k}$ where $n = n_1 + \cdots n_k$. We then obtain

$$\mathrm{Comm}(X_{\mathcal{G}_S}, \mathbb{Z}^n, \Phi, x) \cong \mathrm{GL}(\mathbb{Q}^{k_1}) \times \cdots \times \mathrm{GL}(\mathbb{Q}^{k_n}) \ , \tag{35}$$

and there is the corresponding formula to (34) for $\mathrm{Mod}(\mathbb{T}^n_S, \widehat{x})$.

The next class of examples to consider are when $A_\ell = A$ for all $\ell > 0$, and all eigenvalues of the integer $n \times n$ matrix A are larger than 1, and $n \geq 2$. Then A induces a proper self-covering map $\phi_A \colon \mathbb{T}^n \to \mathbb{T}^n$ and let $\mathbb{T}^n_A$ denote the inverse limit solenoid. The map ϕ_A induces a self-map $\widehat{\phi}_A \in \mathrm{Homeo}(\mathbb{T}^n_A, \widehat{x})$, so that $\mathrm{Mod}(\mathbb{T}^n_A, \widehat{x})$ always contains a factor of $\mathbb{Z}$. The surprising fact is that the group is much richer than this, and its calculation depends on depends on the class field theory of the number field generated by the eigenvalues of A.

The calculation of $\mathrm{Comm}(X_{\mathcal{G}_A}, \mathbb{Z}^n, \Phi, x)$ is equivalent to calculating the endomorphisms of the subgroup $G_{A^t} = \{(A^t)^k \vec{x} \mid \vec{x} \in \mathbb{Z}^k \ , \ k \in \mathbb{Z}\} \subset \mathbb{Q}^n$ determined by the adjoint matrix A^t. For details of these calculations, see Cabezas and Petite [14], Sabitova [54], or Ha and Lee [30] for example.

For $n = 2$, Cabezas and Petite give in Theorem 3.3 and Example 3.4 [14] a complete description of the group of endomorphisms. Note that the calculation depends on the algebraic number theory properties of the roots of the characteristic polynomial $p_A(\lambda)$ of A. When $n > 2$, in most cases the complete calculation is unknown, even in the case where $p_A(\lambda)$ is separable with distinct roots. The most complete results to date have been obtained by Sabitova in [54].

9.3. **Nilpotent solenoids.** A closed connected manifold M is a *nilmanifold*, if $\Gamma = \pi_1(M, x)$ is a nilpotent group, and the universal cover $\widetilde{M}$ is contractible. Note that such Γ is torsion-free. Given a group chain $\mathcal{G}$ in Γ, we form the associated *nilpotent solenoid* $\widehat{M}_{\mathcal{P}}$.

A nilmanifold satisfies the strong Borel conjecture, see Remarks 5.6 and 5.8, so by Corollary 1.11 we can calculate $\mathrm{Mod}(\widehat{M}_{\mathcal{P}})$ in terms of its dynamical commensurator group $\mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x)$.

A finitely generated nilpotent group Γ is residually finite, so the universal odometer action of Γ on $\widehat{\Gamma}$ is effective. Let $\widehat{M}$ be the corresponding universal solenoid over M. Then we have:

THEOREM 9.3. *Let M be a nilmanifold with $\Gamma = \pi_1(M, x)$. Then* $\mathrm{Mod}(\widehat{M}) \cong \mathrm{Comm}(\Gamma) \times \widehat{\Gamma}$.

Proof. For the universal odometer action, we have $\mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x) = \mathrm{Comm}(\Gamma)$. □

For a finitely generated nilpotent group Γ, the calculation of $\mathrm{Comm}(\Gamma)$ reduces to the calculation of the automorphisms of the rational Lie algebra $\mathfrak{g}_\Gamma$ associated to the Mal'cev completion of Γ (see for example [5, Chapter 4]). Thus, the calculation is analogous to the abelian case discussed above, although the calculation of $\mathrm{Aut}(\mathfrak{g}_\Gamma)$ can be much more subtle.

The next class of examples uses the fact that a finitely generated torsion-free nilpotent group admits an embedding $\sigma \colon \Gamma \to \mathrm{SL}(\mathbb{Z}^m)$ for some $m > 0$ sufficiently large, as shown by Jennings [36] (see also [28].) Fix an embedding σ, and choose a prime $p > 0$. Then define, for each $\ell \geq 0$, the normal subgroup of finite index

$$\Gamma_{\sigma,p,\ell} = \ker \left\{ \mathrm{mod}_p \circ \sigma \colon \Gamma \to \mathrm{SL}(\mathbb{Z}^m) \to \mathrm{SL}(\mathbb{Z}^m / p^\ell \mathbb{Z}^m) \right\} \ .$$

This defines a normal group chain $\mathcal{G}_{\sigma,p}$ with $\cap_{\ell > 0} \Gamma_{\sigma,p,\ell} = \{e\}$. Let $\widehat{\Gamma}_{\mathcal{G}_{\sigma,p}}$ denote the profinite group associated to the inverse limit of the quotients $\Gamma_{\Gamma_{\sigma,p,\ell}}$. The associated tower of coverings $\mathcal{P}_{\sigma,p}$ of M induces the solenoidal manifold $\widehat{M}_{\sigma,p}$. Then as before, we have:

THEOREM 9.4. *Let M be a nilmanifold with $\Gamma = \pi_1(M, x)$, $\sigma \colon \Gamma \to \mathrm{SL}(\mathbb{Z}^n)$ an embedding, and $p > 1$ a prime. Then* $\mathrm{Mod}(\widehat{M}_{\sigma,p}) \cong \mathrm{Comm}(X_{\mathcal{G}_{\sigma,p}}, \Gamma, \Phi, x) \times \widehat{\Gamma}_{\mathcal{G}_{\sigma,p}}$.

The calculation of $\mathrm{Comm}(X_{\mathcal{G}_{\sigma,p}}, \Gamma, \Phi, x)$ is related to the representation theory associated with the action of Γ on the p-localization $\widehat{\Gamma}_{\mathcal{G}_{\sigma,p}}$ of the profinite completion, as studied for example in [46]. Some examples are calculated in [34].

Recall that a group Γ is said to be (finitely) *non-coHopfian* if there exists a proper self-embedding $\varphi \colon \Gamma \to \Gamma$ (with finite index) [35]. Such a map φ is called a *renormalization* of Γ, and we say that G is renormalizable. For example, homogeneous nilpotent groups are renormalizable [18, 22], though there are many examples of renormalizable groups which are not nilpotent, as discussed in [35].

Suppose that Γ admits a proper self-embedding $\varphi \colon \Gamma \to \Gamma$. For $\ell \geq 0$, let $\Gamma_\ell = \varphi^\ell(\Gamma) = \varphi \circ \cdots \circ \varphi(\Gamma)$ be the image of the ℓ-th iteration of φ. Also assume that $\cap_{\ell > 0} \Gamma_\ell = \{e\}$. Then the group chain $\mathcal{G}_\varphi = \{\Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ is a *strong scale* in the sense of [49].

Let $(X_\varphi, \Gamma, \Phi)$ denote the odometer associated to the chain $\mathcal{G}_\varphi$. If the groups Γ_ℓ are normal in Γ, then van Limbeek showed in [61] that Γ must be abelian. Thus, if Γ is non-abelian, then the subgroups Γ_ℓ are never normal, and so the inverse limit space X_φ is not a pro-finite group. None-the-less, Γ is renormalizable implies the action of Γ on X_φ is quasi-analytic by [35, Theorem 4.4]. Let $\widehat{M}_\varphi$ denote the nilpotent solenoid associated to the coverings defined by the group chain $\mathcal{G}_\varphi$. Then as above,

$$\mathrm{Mod}(\widehat{M}_\varphi, \widehat{x}) \cong \mathrm{Comm}(X_\varphi, \Gamma, \Phi, x) \ . \tag{36}$$

The map φ induces a self map $\widetilde{\varphi} \colon M \to M$, which in turn defines $\widehat{\varphi} \in \mathrm{Homeo}(\widehat{M}_\varphi, \widehat{x})$ so that the group $\mathrm{Comm}(X_\varphi, \Gamma, \Phi, x)$ always contains at least a copy of $\mathbb{Z}$. However, as with the abelian case analyzed by Sabitova [54], we show in [34] that there are many more "exotic" classes in $\mathrm{Mod}(\widehat{M}_\varphi)$ which arise from number theoretic properties of the map φ.

The calculation of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}})$ from $\mathrm{Mod}(\widehat{M}_\varphi, \widehat{x})$ using Theorem 8.3 depends on understanding the factor $\mathfrak{Z}_{\mathcal{G}}$, which is not a straightforward problem in general.

9.4. **Hyperbolic solenoids.** Let Σ_g denote the closed orientable surface of genus $g \geq 2$, $x \in \Sigma_g$ a basepoint, and $\Gamma_g = \pi_1(\Sigma_g, x)$. Theorem 1.5 by Odden states that the characteristic map χ yields an isomorphism $\mathrm{Mod}(\widehat{\Sigma_g}) \cong \mathrm{Comm}(\Gamma_g) \times \widehat{\Gamma}$, where $\widehat{\Sigma_g}$ is the universal hyperbolic solenoid. This result can be generalized in two directions.

THEOREM 9.5. *Let $\mathcal{G}$ be a group chain in Γ_g such that the associated odometer $(X_{\mathcal{G}}, \Gamma, \Phi)$ is effective, coherent and locally quasi-analytic. Then* $\mathrm{Mod}(\widehat{\Sigma_g}, \widehat{x}) \cong \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x)$.

Proof. This follows from Theorem 7.1 and Remarks 5.6 and 5.8. □

The second generalization extends the results of Bering and Studenmund in [7]. Let M be a closed connected n-manifold with a metric of negative sectional curvatures, for $n \geq 2$. Then M satisfies the strong Borel property. Let $x \in M$ and $\Gamma = \pi_1(M, x)$

If the metric on M has constant curvature, then Γ is linear; that is, Γ admits an embedding into a connected Lie group, and so Γ is residually finite. For more general hyperbolic groups, Tholozan and Tsouvalas have constructed in [59] examples which are Gromov hyperbolic and not residually finite. Thus, we restrict consideration to the case where Γ is linear.

Let $\widehat{\Gamma}$ be the profinite completion of Γ, then the universal odometer action of Γ on $\widehat{\Gamma}$ is effective as Γ is residually finite. We then obtain the corresponding result to Theorem 5.6 in [7]:

THEOREM 9.6. *Let M be a connected manifold with constant negative curvature, and let $\Gamma = \pi_1(M, x)$. Then* $\mathrm{Mod}(\widehat{M}) \cong \mathrm{Comm}(\Gamma) \times \widehat{\Gamma}$.

Proof. For the universal odometer action, we have $\mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x) = \mathrm{Comm}(\Gamma)$, and the map χ is then an isomorphism by Remarks 5.6 and 5.8. □

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