FIVE LECTURES ON FOLIATION DYNAMICS

STEVEN HURDER

Contents

2000 *Mathematics Subject Classification.* Primary 57R30, 37C55, 37B45; Secondary 53C12 .
Preprint date: June 4, 2010.

1. Introduction

These are preliminary notes, based on a series of five lectures, given May 3–7, 2010, at the *Centre de Recerca Matemàtica, Barcelona*. The goal of the lectures was to present aspects of the theory of foliation dynamics which have particular importance for the classification of foliations of compact manifolds. The lectures emphasized intuitive concepts and informal discussion, as can be seen from the slides [67]. In these notes, we fill in more of the text and discussion; complete details will appear in the final version of these notes [68]. We begin with the most basic question:

What is the subject of "foliation dynamical systems"?

Here are some partial answers to this very broad question; or more precisely, the following are some of the topics we address in these notes:

(1) The asymptotic properties of leaves of $\mathcal{F}$
  - How do the leaves accumulate onto the minimal sets?
  - What are the topological types of minimal sets? Are they "manifold-like"?
  - Invariant measures: can you quantify the rates of recurrence of leaves?

(2) Directions of "stability" and "instability" of leaves
  - Exponents: are there directions of exponential divergence?
  - Stable manifolds: show the existence of dynamically defined transverse invariant manifolds, and how do they influence the global behavior of leaves?

(3) Quantifying chaos
  - Define a measure of transverse chaos – foliation entropy
  - Estimate the entropy using linear approximations

(4) The shape of minimal sets
  - Hyperbolic exotic minimal sets
  - Parabolic exceptional minimal sets

The subject of foliation dynamics is very broad, and includes many other topics to study, such as rigidity of the dynamical system defined by the leaves, the behavior of random walks on leaves and properties of harmonic measures, and the Hausdorff dimension of minimal sets, to name a few additional topics not treated here.

Also, the subject of foliations tends to be quite abstract, as it is difficult to illustrate many of the concepts in higher dimensions. One is typically presented with a few "standard examples" in dimensions two and three, that hopefully yield intuitive insight from which to gain a deeper understanding of the more general cases. For example, many talks with "foliations" in the title start with the following example, the 2-torus $\mathbb{T}^2$ foliated by lines of irrational slope:

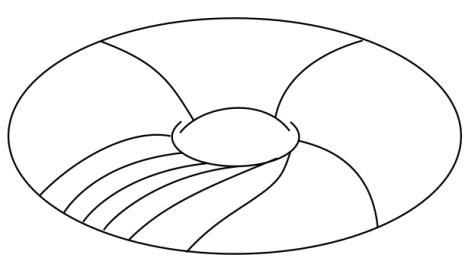

Figure 1. Linear foliation with all leaves dense

Never trust a talk which starts with this example! It is just too simple, in that the leaves are parallel and contractible, hence the foliation has no germinal holonomy. Also, every leaf of $\mathcal{F}$ is uniformly dense in $\mathbb{T}^2$ so the topological nature of the minimal sets for $\mathcal{F}$ is trivial to determine. The key dynamical information about this example is in the rates of returns to open subsets, which is more analytical that topological information.

At the other extreme of examples of foliations defined by flows are those with a compact leaf as the unique minimal set, such as this:

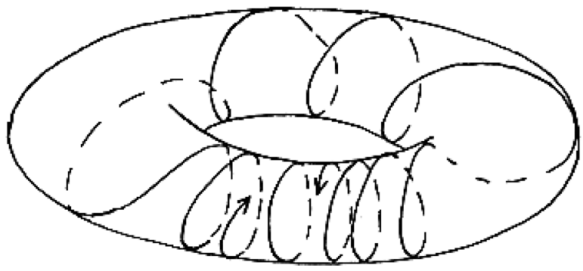

FIGURE 2. Flow with one attracting leaf

Every orbit limits to the circle, which is the forward (and backward) limit set for all leaves.

One other canonical example is that of the Reeb foliation of the solid 3-torus as pictured below, which has a similar dynamical description:

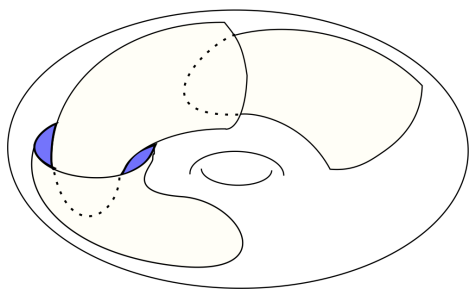

FIGURE 3. Reeb foliation of solid torus

This example illustrates several concepts: the limit sets of leaves, the existence of attracting holonomy for the compact toral leaf, and also the existence of multiple hyperbolic measures for the foliation geodesic flow, as to be discussed later.

We will introduce further examples in the text to follow, which illustrate more advanced dynamical properties of foliations. Although, as mentioned above, it becomes more difficult to illustrate concepts that only arise for foliations of manifolds of more complicated 3-manifolds, or in higher dimensions. The interested reader should view the illustrations in the beautiful article by Étienne Ghys and Jos Ley for flows on 3-manifolds [46] to get a feeling for the complexity that are "normal" for foliation dynamics in higher dimensions.

Many of the illustrations in the following text were drawn by Lawrence Conlon, circa 1994. Our thanks for his permission to use them.

2. Derivatives

We fix the following conventions. M is a compact Riemannian manifold without boundary, and $\mathcal{F}$ is a codimension q-foliation, transversally C^r for $r \in [1, \infty)$. This means that the transition functions for the foliation charts $\varphi_i \colon U_i \to [-1,1]^n \times T_i$ are C^∞ leafwise, and vary C^r with the transverse parameter. The transverse model spaces T_i are usually considered to be open subsets of $\mathbb{R}^q$, although in some cases we allow the T_i to be open subsets of a fixed Polish space.

We recall next the holonomy pseudogroup for a foliation. The point of view we adopt is best illustrated by starting with the classical case of flows.. Recall for a non-singular flow $\varphi_t \colon M \to M$ the orbits define a 1-dimensional foliation $\mathcal{F}$, whose leaves are the orbits of points.

Choose a cross-section $\mathcal{T} \subset M$ which is transversal to the orbits, and intersects each orbit (so $\mathcal{T}$ need not be connected.) Then for each $x \in \mathcal{T}$ there is some least $\tau_x > 0$ so that $\varphi_{\tau_x}(x) \in \mathcal{T}$. The positive constant τ_x is called the *return time* for x. See the illustration Figure 4 below.

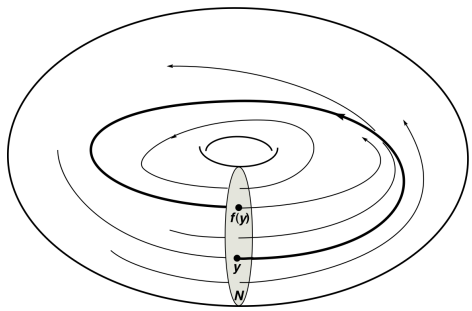

Figure 4. Cross-section to a flow

The induced map $f(x) = \varphi_{\tau_x}(x)$ is a *Borel map* $f \colon \mathcal{T} \to \mathcal{T}$, called the *holonomy* of the flow. The choice of a cross-section for a flow reduces the study of its dynamical properties to that of the discrete dynamical system $f \colon \mathcal{T} \to \mathcal{T}$.

The holonomy for foliations is defined similarly, as local homeomorphisms associated to paths along leaves, starting and ending at a fixed transversal, except that there is a fundamental difference. For the orbit of a flow L_w through a point w, there exists two choices of trajectory along a unit speed path, either forward and backward. However, for a leaf L_w of a foliation $\mathcal{F}$ of dimension at least two, there is no such concept as "forward" or "backward", and all directions yield paths along which one may discover dynamical properties of the foliation.

Choose $z \in L_w$ and a smooth path $\tau_{w,z} \colon [0,1] \to L_w$. Cover path $\tau_{w,z}$ by foliation charts and slide open subset U_w of transverse disk S_w along path to open subset W_z of transverse disk S_z.

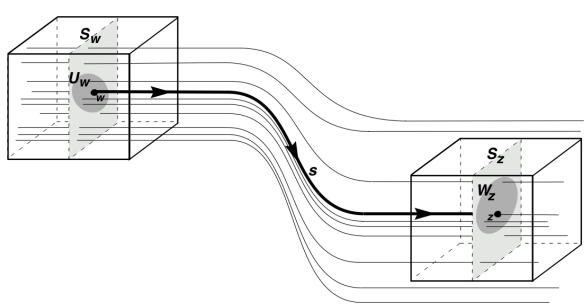

Figure 5. Holonomy along a leafwise path

This defines the local homeomorphism $h_{\tau_{w,z}} \colon U_w \to W_z$ of open subsets of $\mathbb{R}^q$. This most basic concept of foliation theory is developed in detail in the standard texts [17, 18, 48, 53].

The map $h_{\tau_{w,z}}$ depends on the choice of the transversal sections U_w and W_z to the foliation. This ambiguity is removed (in part) by fixing a complete transversal $\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_k \subset M$. For example, choose a finite covering of M by foliation charts, $\{\varphi_i \colon U_i \to [-1,1]^n \times T_i \mid 1 \leq i \leq k\}$, and define transversal sections $\mathcal{T}_i = \varphi_i^{-1}(\{0\} \times T_i) \subset U_i$. Then we obtain the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ for $\mathcal{F}$ modeled on $\mathcal{T}$, which is compactly generated in sense of Haefliger [51].

Given $w \in \mathcal{T}$, $z \in L_w \cap \mathcal{T}$ and leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ from w to z, we associate to it the germ at w of the local homeomorphism $h_{\tau_{w,z}} \colon U_w \to W_z$ which is denoted by $\gamma = [h_{\tau_{w,z}}]_w$. Here are some key points:

PROPOSITION 2.1. *Let $\mathcal{F}$ be a foliation of a manifold M. Then*

(1) *γ depends only on the leafwise homotopy class of the path, relative endpoints*
(2) *the maximal sizes of the domain U_w and range W_z representing an equivalence class γ depends on the path $\tau_{w,z}$;*
(3) *the collection of all such $\{h_{\tau_{w,z}} \colon U_w \to W_z \mid w \in \mathcal{T}, z \in L_w \cap \mathcal{T}\}$ generates $\mathcal{G}_{\mathcal{F}}$*
(4) *the collection of all such germs $\{\gamma = [h_{\tau_{w,z}}]_w \mid w \in \mathcal{T}, z \in L_w \cap \mathcal{T}\}$ generates the* holonomy groupoid*, denoted by $\Gamma_{\mathcal{F}}$.*

Assume that $\mathcal{F}$ is a C^1-foliation of a compact Riemannian manifold, with smoothly immersed leaves. Then for each leaf L_w of $\mathcal{F}$ the induced Riemannian metric on L_w is complete.

COROLLARY 2.2. *For each leafwise path $\tau_{w,z} \colon [0,1] \to L_w$ let $\sigma_{w,z} \colon [0,1] \to L_w$ be the leafwise geodesic segment which is homotopic relative endpoints to $\tau_{w,z}$. Then $[h_{\tau_{w,z}}]_w = [h_{\sigma_{w,z}}]_w$.*

While the germ γ of the holonomy along a leafwise path $\tau_{w,z}$ is well-defined, up to conjugation, the size of the domain of a representative map $h_{\tau_{w,z}} \in \mathcal{G}_{\mathcal{F}}$ need not be. It is a strong assumption on the dynamics of $\mathcal{G}_{\mathcal{F}}$ or the topology of M to assume that a uniform estimate on the sizes of the domains exists. This is a very delicate technical point that arises in many proofs about the dynamics of a foliation. This issue is suppressed in these notes for the purpose of simplicity of exposition.

Next we introduce the *transverse differentials* for the holonomy groupoid.

Consider first the case of a foliation $\mathcal{F}$ defined by a smooth flow $\varphi \colon \mathbb{R} \times M \to M$ defined by a non-vanishing vector field $\vec{X}$. Then $T\mathcal{F} = \langle \vec{X} \rangle \subset TM$.

For $w = \varphi_t(w)$, consider the Jacobian matrix $D\varphi_t \colon T_wM \to T_zM$. The flow satisfies the group law $\varphi_s \circ \varphi_t = \varphi_{s+t}$, which implies the identity $D\varphi_s(\vec{X}_w) = \vec{X}_z$ by the chain rule for derivatives.

Introduce the normal bundle to the flow $Q = TM/T\mathcal{F}$. For each $w \in M$, we identify $Q_w = T_w\mathcal{F}^{\perp}$. Thus, Q can be considered as a subbundle of TM, and thereby the Riemannian metric on TM induces metrics on each fiber $Q_w \subset T_wM$. The derivative transformation preserves the normal bundle $Q \to M$, so defines the *normal derivative cocycle*,

$$D\varphi_t \colon Q_w \longrightarrow Q_z \quad , \quad t \in \mathbb{R}$$

We can then define the norms of the normal derivative maps,

$$\|D\varphi_t\| = \|D\varphi_t \colon Q_w \longrightarrow Q_z\|$$

It is also useful to introduce the symmetric norm

$$\|D\varphi_t \mid_w \|^{\pm} = \max\left\{\|D\varphi_t \colon Q_w \longrightarrow Q_z\|, \|D\varphi_t^{-1} \colon Q_z \longrightarrow Q_w\|\right\}$$

For M compact, the norms $\|D\varphi_t \mid_w \|^{\pm}$ are uniformly bounded for $w \in M$.

The maps $D\varphi_t\| = \|D\varphi_t \colon Q_w \longrightarrow Q_z$ can be thought of as "non-autonomous approximations" to the transverse behavior of the flow φ_t. The actual values of these derivatives is only well-defined up to a global choice of framing of the normal bundle Q, so extracting useful dynamical information from these derivatives presents a challenge. This problem was solved by seminal work in the 1970's. "Pesin Theory" is a collection of results about the dynamical properties of flows, based on defining non-autonomous linear approximations of the normal behavior to the flow. Excellent discussions

and references for this theory are in these references [85, 75, 7]. We use only a small amount of the full Pesin theory in the discussion in these notes.

First, let us recall a basic fact for the dynamics induced by a linear map. Given a matrix $A \in GL(q, \mathbb{R})$, let $L_A \colon \mathbb{R}^n \to \mathbb{R}^n$ be the linear map defined by multiplication by A. We say the action is *hyperbolic* (or more precisely, partially hyperbolic) if A has an eigenvalue of norm not equal to 1. In this case, there is an eigen-space for A which is defined dynamically as the direction of maximum rate of expansion (or minimum contraction) for the action of A. If A is conjugate to an orthogonal matrix, then we say that A is *elliptic*. In this case, the action of L_A preserves ellipses in $\mathbb{R}^n$ and all orbits of L_A and its inverse are bounded. Finally, if all eigenvalues of A have norm 1, but A is not elliptic, then we say that A is *parabolic*. In this case, A is conjugate to an upper triangular matrix with all diagonal entries of norm 1. The dynamics in this case is distal, which is also dynamically defined property. One idea of Pesin theory is that the hyperbolic property is well-defined also for non-autonomous linear approximations to smooth dynamical systems, so we look for this behavior on the level of derivative cocycles. This is the provided by the following concept.

DEFINITION 2.3. *$w \in M$ is a* hyperbolic point *of the flow if*

$$e_\varphi(w) \equiv \lim_{T\to\infty} \sup_{s\geq T} \left\{ \frac{1}{s} \cdot \log\left\{ \|(D\varphi_s \colon Q_w \to Q_z)\|^{\pm} \right\} \right\} \ > \ 0$$

LEMMA 2.4. *The set of hyperbolic points $\mathcal{H}(\varphi) = \{w \in M \mid e_\varphi(w) > 0\}$ is flow-invariant.*

One of the first basic results if that if the set of hyperbolic points is non-empty, then the flow itself has hyperbolic behavior on special subsets where the "lim sup" is replaced by a limit:

PROPOSITION 2.5. *Let φ be a C^1flow. Then the closure $\overline{\mathcal{H}(\varphi)} \subset M$ contains an invariant ergodic probability measure μ_* for φ, for which there exists $\lambda > 0$ such that for μ_*-a.e. w,*

$$e_\varphi(w) = \lim_{s\to\infty} \{ \frac{1}{s} \cdot \log\{\|D\varphi_s \colon Q_w \to Q_z\|\} = \lambda$$

Proof. This follows from the continuity of the derivative and its cocycle property, the definition of the asymptotic Schwartzman cycle associated to a flow [94], plus the usual subadditive techniques of Oseledets Theory [85, 86, 7]. □

We want to apply the ideas behind Proposition 2.5 to the derivatives of the maps in the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$. The difficulty is that the orbits of the pseudogroup are not necessarily ordered into a single direction along which the leaf hyperbolicity is to be found, and hence along which the integrals are defined in obtaining the Schwartzman asymptotic cycle as in the above. The first issue is that we need a method to associate to a foliation $\mathcal{F}$ a flow which captures the dynamics of $\mathcal{F}$. Fortunately, such a flow exists, and has been already suggested.

Let $w \in M$ and consider L_w as a complete Riemannian manifold. For $\vec{v} \in T_w\mathcal{F} = T_wL_w$ with $\|\vec{v}\|_w = 1$, there is unique geodesic $\tau_{w,\vec{v}}(t)$ starting at w with $\tau'_{w,\vec{v}}(0) = \vec{v}$.

Define the flow $\varphi_{w,\vec{v}} \colon \mathbb{R} \to M$ by $\varphi_{w,\vec{v}}(w) = \tau_{w,\vec{v}}(t)$. Let $\widehat{M} = T^1\mathcal{F}$ denote the unit tangent bundle to the leaves, then the maps $\varphi_{w,\vec{v}}$ define the *foliation geodesic flow*

$$\varphi_t^{\mathcal{F}} \colon \mathbb{R} \times \widehat{M} \to \widehat{M}$$

Let $\widehat{\mathcal{F}}$ denote the foliation on $\widehat{M}$ whose leaves are the unit tangent bundles to leaves of $\mathcal{F}$.

LEMMA 2.6. *$\varphi_t^{\mathcal{F}}$ preserves the leaves of the foliation $\widehat{\mathcal{F}}$ on $\widehat{M}$, and hence $D\varphi_t^{\mathcal{F}}$ preserves the normal bundle $\widehat{Q} \to \widehat{M}$ for $\widehat{\mathcal{F}}$.*

Lemma 2.6 permits the direct extension of Definition 2.3 to the case of the normal derivative cocycle for the foliation geodesic flow as follows. We now we consider three possible cases for the asymptotic behavior of the norms of the normal derivative cocycle over the flow.

DEFINITION 2.7. *Let $\varphi_t^{\mathcal{F}}$ be the foliation geodesic flow for a C^1-foliation $\mathcal{F}$. Then $\widehat{w} \in \widehat{M}$ is:*

H: hyperbolic *if*

$$e_{\mathcal{F}}(\widehat{w}) \equiv \lim_{T\to\infty} \sup_{s\geq T} \left\{ \frac{1}{s} \cdot \log \left\{ \|(D\varphi_s^{\mathcal{F}} \colon \widehat{Q}_{\widehat{w}} \to \widehat{Q}_{\widehat{z}})\|^{\pm} \right\} \right\} \ > \ 0$$

E: elliptic *if $e_{\mathcal{F}}(\widehat{w}) = 0$, and there exists $\kappa(\widehat{w})$ such that*

$$\|(D\varphi_t^{\mathcal{F}} \colon \widehat{Q}_{\widehat{w}} \to Q_{\widehat{z}})\|^{\pm} \leq \kappa(\widehat{w}) \text{ for all } t \in \mathbb{R}$$

P: parabolic *if $e_{\mathcal{F}}(\widehat{w}) = 0$, and $\widehat{w}$ is not elliptic.*

It turns out that there is a fundamental variation on this definition which is much more useful. The variation takes into account the fact that for foliation dynamics, one does not necessarily have a preferred direction for the foliation geodesic flow. Thus, we consider all possible directions simultaneously in the generalization of Definition 2.3.

In the definition below, we let $\|\gamma\|$ denote the minimum length of a geodesic σ whose holonomy $h_{\sigma_{w,z}}$ defines the germ $\gamma \in \Gamma_{\mathcal{F}}$. We let $D_w\gamma = D_w h_{\sigma_{w,z}}$ denote the derivative at w, for any map $h_{\sigma_{w,z}}$ whose germ at w represents γ.

DEFINITION 2.8. *The transverse expansion rate function for $\mathcal{G}_{\mathcal{F}}$ at w is*

$$e(\mathcal{G}_{\mathcal{F}}, d, w) = \max_{\|\gamma\|\leq d} \left\{ \frac{\ln\left(\max\{\|D_w\gamma\|^{\pm}\}\right)}{d} \right\} \tag{1}$$

Note that $e(\mathcal{G}_{\mathcal{F}}, d, w)$ is a Borel function of $w \in \mathcal{T}$, as each norm function $\|D_{w'}h_{\sigma_{w,z}}\|$ is continuous for $w' \in D(h_{\sigma_{w,z}})$ and the maximum of Borel functions is Borel.

DEFINITION 2.9. *The asymptotic transverse expansion rate at $w \in \mathcal{T}$ is*

$$e_{\mathcal{F}}(w) = e(\mathcal{G}_{\mathcal{F}}, w) = \limsup_{d\to\infty} \ e(\mathcal{G}_{\mathcal{F}}, d, w) \ \geq \ 0 \tag{2}$$

The limit of Borel functions is Borel, and each $e(\mathcal{G}_{\mathcal{F}}, d, w)$ is a Borel function of w, hence $e(\mathcal{G}_{\mathcal{F}}, w)$ is Borel. The value $e_{\mathcal{F}}(w)$ can be thought of as the "maximal Lyapunov exponent" for the holonomy groupoid at w.

LEMMA 2.10. *$e_{\mathcal{F}}(z) = e_{\mathcal{F}}(w)$ for all $z \in L_w \cap \mathcal{T}$. Hence, the expansion function $e(w)$ is constant along leaves of $\mathcal{F}$.*

Proof. Follows from the chain rule and the definition of $e_{\mathcal{F}}(w)$. □

This trichotomy for the expansion behavior along orbits of the foliation geodesic flow in Definition 2.7 also applies to the expansion rate function $e(\mathcal{G}_{\mathcal{F}}, d, w)$ in Definition 2.8, and this is the basis for a decomposition of the manifold M into leaves which satisfy one of the three types of asymptotic behavior for the normal derivative cocycle. We then have the following decomposition:

THEOREM 2.11 (Dynamical decomposition of foliations)**.** *Let $\mathcal{F}$ be a C^1-foliation on a compact manifold M. Then M has a decomposition into disjoint saturated Borel subsets, $M = \mathbf{E}_{\mathcal{F}} \cup \mathbf{P}_{\mathcal{F}} \cup \mathbf{H}_{\mathcal{F}}$, which are the leaf saturations of the sets defined by:*

1. *Elliptic:* $\mathbf{E}_{\mathcal{F}} = \{w \in \mathcal{T} \mid \forall\ d \geq 0,\ e(\mathcal{G}_{\mathcal{F}}, d, w) \leq \kappa(w)\}$
2. *Parabolic:* $\mathbf{P}_{\mathcal{F}} = \{w \in \mathcal{T} \setminus \mathbf{E}_{\mathcal{F}} \mid e(\mathcal{G}_{\mathcal{F}}, w) = 0\}$
3. *Hyperbolic:* $\mathbf{H}_{\mathcal{F}} = \{w \in \mathcal{T} \mid e(\mathcal{G}_{\mathcal{F}}, w) > 0\}$

Note that $w \in \mathbf{E}_{\mathcal{F}}$ means that the holonomy homomorphism $D_w\gamma$ has bounded image in $GL(q, \mathbb{R})$. The constant $\kappa(w) = \sup\{\|D_w\gamma\| \mid \gamma \in \mathcal{G}_{\mathcal{F}}^w\}$, where $\mathcal{G}_{\mathcal{F}}^w$ denotes the germs of holonomy transport along paths starting at w.

The nomenclature in Theorem 2.11 reflects the trichotonomy for the dynamics of a matrix $A \in GL(q, \mathbb{R})$ acting via the associated linear transformation $L_A \colon \mathbb{R}^q \to \mathbb{R}^q$: The elliptic points are the regions where the infinitesimal holonomy transport "preserves ellipses up to bounded distortion". The parabolic points are where the infinitesimal holonomy acts similarly to that of a parabolic subgroup of $GL(q, \mathbb{R})$; for example, the action is "infinitesimally distal". The hyperbolic points are where the the infinitesimal holonomy has some degree of exponential expansion. Perhaps more properly, the set $\mathbf{H}_{\mathcal{F}}$ should be called "non-uniform, partially hyperbolic leaves", and the study of the dynamical properties of these leaves then parallels (in part) the ideas of [12].

Transversally hyperbolic measures

The decomposition in Theorem 2.11 has many applications to the study of foliation dynamics and classification results. These are discussed in depth, for example, in the author's survey [66]. The interested reader can also consult the papers [57, 59, 63]. In these talks, we illustrate some of these applications with examples and selected results. Here is one important concept:

DEFINITION 2.12. *An invariant probability measure μ_* for the foliation geodesic flow on $\widehat{M}$ is said to be* transversally hyperbolic *if $e_{\mathcal{F}}(\widehat{w}) = \lambda > 0$ for μ_*-a.e. $\widehat{w}$.*

Note that the support of μ_* is contained in the unit tangent bundle $\widehat{M}$, and not M itself. This is because it is specifying both a point in a leaf, and the direction along which to follow a geodesic to find infinitesimal normal hyperbolic behavior.

THEOREM 2.13. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold. If $\mathbf{H}_{\mathcal{F}} \neq \emptyset$, then the foliation geodesic flow has at least one transversally hyperbolic ergodic measure, which is contained in the closure of unit tangent bundle over $\mathbf{H}_{\mathcal{F}}$.*

Proof. The proof is technical, but basically follows from calculus techniques applied to the foliation pseudogroup, as in Oseledets Theory. The key point is that if $L_w \subset \mathbf{H}_{\mathcal{F}}$ then there is a sequence of geodesic segments of lengths going to infinity on the leaf L_w, along which the transverse infinitesimal expansion grows at an exponential rate. Hence, by continuity of the normal derivative cocycle and the cocycle law, these geodesic segments converge to a transversally hyperbolic invariant probability measure μ for the foliation geodesic flow. The existence of an ergodic component μ_* for this measure with positive exponent then follows from the properties of the ergodic decomposition of μ. □

This result has a very useful corollary.

COROLLARY 2.14. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold with $\mathbf{H}_{\mathcal{F}} \neq \emptyset$. Then there exists $w \in \overline{\mathbf{H}_{\mathcal{F}}}$ and a unit vector $\vec{v} \in T_w\mathcal{F}$ such that the forward orbit of the geodesic flow through $(w, \vec{v})$ has a transverse direction which is* uniformly *exponentially contracting.*

Let us return to the examples introduced earlier, and consider what the trichotomy decomposition means in each case. For the linear foliation of the 2-torus in Figure 1, every point is elliptic, as the foliation is Riemannian. However, if $\mathcal{F}$ is a C^1-foliation which is topologically semi-conjugate to a linear foliation, so is a generalized Denjoy example, then $M_{\mathcal{P}}$ is not empty! Shigenori Matsumato has given a new construction of Denjoy-type C^1-foliations on the 2-torus for which the exceptional minimal set consists of elliptic points, and the points in the wandering set are all parabolic [80].

Consider next the Reeb foliation of the solid torus, as in Figure 3. Pick $w \in M$ on an interior parabolic leaf, and a direction $\vec{v} \in T_wL_w$. Follow the geodesic $\sigma_{w,\vec{v}}(t)$ starting from w. It is asymptotic to the boundary torus, so defines a limiting Schwartzman cycle on the boundary torus for some flow. Thus, it limits on either a circle, or a lamination. This will be a hyperbolic measure if the holonomy of the compact leaf is hyperbolic. Note that the exponent of the invariant measure for the foliation geodesic flow depends on the direction of the geodesic used to define it!

One of the basic problems about the foliation geodesic flow is to understand the support of its transversally hyperbolic invariant measures, and if the leaves intersecting the supports of these measures have "chaotic" behavior.

3. Counting

In first lecture, we introduced the decomposition of the foliated manifold $M = \mathbf{E}_{\mathcal{F}} \cup \mathbf{P}_{\mathcal{F}} \cup \mathbf{H}_{\mathcal{F}}$ in terms of the asymptotic properties of the normal "derivative cocycle" $D\colon \mathcal{G}_{\mathcal{F}} \to GL(n, \mathbb{R})$. One of the features of this decomposition is that it allows for the transverse expansion to "develop in all directions" when the leaves are higher dimensional.

A basic question is then, how do you tell whether one of the Borel, $\mathcal{F}$-saturated components, such as the hyperbolic set $\mathbf{H}_{\mathcal{F}}$, is non-empty? Moreover, it is natural to speculate whether the "geometry of the leaves" influences the existence hyperbolic leaves. Let us first consider some examples with more complicated leaf geometry than seen above.

Consider the following three examples of complete 2 manifolds, all of which are realized as leaves of foliations of 3-manifolds by "standard constructions". The first manifold is called the "Infinite Jungle Gym" in the foliation literature [87, 18].

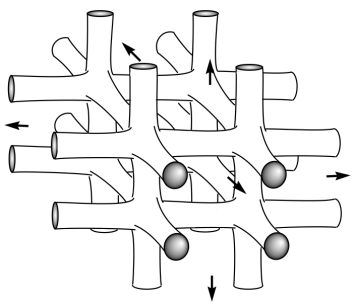

Figure 6. The Infinite Jungle Gym

It can be realized as a leaf of a circle bundle over a surface of genus three, where the holonomy consists of three commuting linearly independent rotations of the circle. Thus, even though this is a surface of infinite genus, the transverse holonomy is just a generalization of that for the Denjoy example, in that it consists of a group of isometries with dense orbits for the circle $\mathbb{S}^1$.

The next manifold L_1 doesn't have a cute name, but has an interesting property:

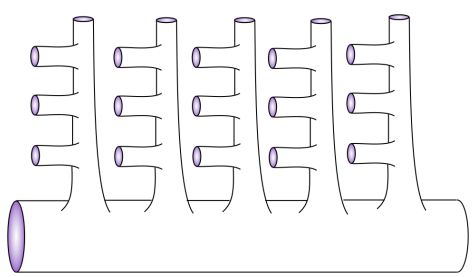

Figure 7. A leaf of "Level 2"

The space of ends $\mathcal{E}(L_1)$ for this manifold has the property that its second derived set is empty. This can be realized as a leaf which is asymptotic to a compact surface of genus two.

As with almost all of the illustrations we are using, the picture credits go to Lawrence Conlon, circa 1992. For details of the construction of the foliation in which this occurs, see their textbook [18].

The dynamics of this foliation give an example of a proper leaf with finite depth [19, 20, 54, 99, 101]. As with the Reeb foliation, the hyperbolic invariant measures for the flow are concentrated on the limiting leaf, so the dynamics, while exhibiting some "hyperbolic" behavior, are still not chaotic.

The next example has endset $\mathcal{E}(L_2)$ which is a Cantor set, and so equal to its own derived set.

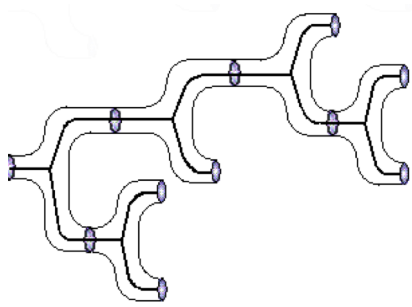

FIGURE 8. A leaf with Cantor endset

This example can be realized in various manners. We present in more detail one particular construction, called the Hirsch foliation", as it illustrates a basic theme of the lectures. The construction is based on a short paper of Morris Hirsch [56]. Generalizations of this construction are given in [10].

Step 1: Choose an analytic embedding of $\mathbb{S}^1$ in the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ so that its image is twice a generator of the fundamental group of the solid torus. Remove an open tubular neighborhood of the embedded $\mathbb{S}^1$.

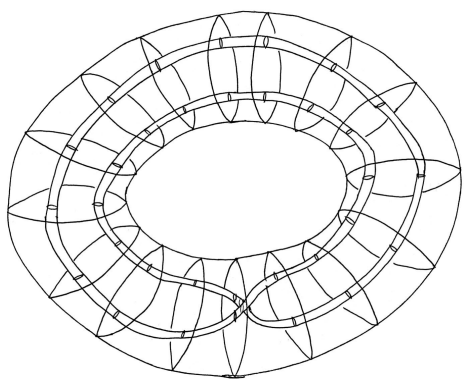

FIGURE 9. Solid torus with tube drilled in it

Step 2: What remains is a three dimensional manifold N_1 whose boundary is two disjoint copies of $\mathbb{T}^2$. $\mathbb{D}^2 \times \mathbb{S}^1$ fibers over $\mathbb{S}^1$ with fibers the 2-disc. This fibration – restricted to N_1 – foliates N_1 with leaves consisting of 2-disks with two open subdisks removed.

Identify the two components of the boundary of N_1 by a diffeomorphism which covers the map $h(z) = z^2$ of S^1 to obtain the manifold N. Endow N with a Riemannian metric; then the punctured 2-disks foliating N_1 can now be viewed as pairs of pants. (See Figure 10 below.)

Step 3: The foliation of N_1 is transverse to the boundary, so the punctured 2-disks assemble to yield a foliation of foliation $\mathcal{F}$ on N, where the leaves without holonomy (corresponding to irrational points for the chosen doubling map of S^1) are infinitely branching surfaces, decomposable into pairs-of-pants which correspond to the punctured disks in N_1.

A basic point is that this works for any covering map $f\colon \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to the doubling map $h(z)$ along a meridian. In particular, as Hirsch remarked in his paper, the proper choice of such a "bonding map" results in a codimension-one, real analytic foliation, such that all leaves accumulate on a unique exceptional minimal set.

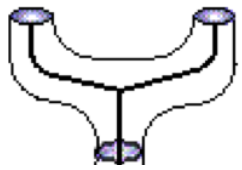

FIGURE 10. "Pair of pants"

The Hirsch foliation always has a leaf L_w as follows, corresponding to a forward periodic orbit of the doubling map $g\colon \mathbb{S}^1 \to \mathbb{S}^1$:

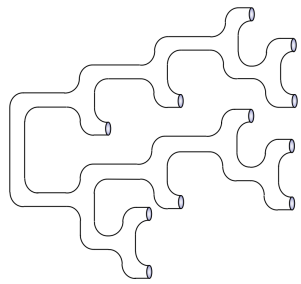

FIGURE 11. Leaf for eventually periodic orbit

Consider the behavior of the geodesic flow, starting at a "bottom point" $w \in L_w$. For a each radius $R \gg 0$, the terminating points of the geodesic rays of length at most R will "jump" between the μ^R ends of this compact subset of the leaf, for some $\mu > 1$. Thus, for these examples, a small variation of the initial vector $\vec{v}$ will result in a large variation of the terminal end of the geodesic $\sigma_{w,\vec{v}}$.

The constant μ appearing in the above example seems to be an "interesting" property of the foliation dynamics, and a key point is that it can be obtained by "counting" to complexity of the leaf at infinity, following a scheme introduced by Joseph Plante for leaves of foliations [88], generalizing a fundamental idea of [82].

Recall the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ constructed in Lecture 1, modeled on a complete transversal $\mathcal{T} = \mathcal{T}_1 \ \cup \cdots \cup \ \mathcal{T}_k$ associated to a finite covering of M by foliations charts. Given $w \in \mathcal{T}$ and $z \in L_w \cap \mathcal{T}$ and a leafwise path $\tau_{w,z}$ joining them, we obtain an element $h_{\tau_{w,z}} \in \mathcal{G}_{\mathcal{F}}$.

The orbit of $w \in \mathcal{T}$ under $\mathcal{G}_{\mathcal{F}}$ is

$$\mathcal{O}(w) = L_w \cap \mathcal{T} = \{z \in \mathcal{T} \mid g(w) = z,\ g \in \mathcal{G}_{\mathcal{F}}, w \in Dom(g)\}$$

The second description allows us to decompose the orbit into "periods". For this, introduce the word norm on elements of $\mathcal{G}_{\mathcal{F}}$. Given open sets $U_i \cap U_j \neq \emptyset$ in the fixed covering of M by foliation charts, they define an element $h_{i,j} \in \mathcal{G}_{\mathcal{F}}$. By the definition of holonomy along a path, for each $\tau_{w,z}\colon [0,1] \to L_w$ there is a sequence of indices $\{i_0, i_1, \ldots, i_\ell\}$ so that

$$[h_{\tau_{w,z}}]_w = [h_{i_{\ell-1},i_\ell} \circ \cdots \circ h_{i_1,i_0}]_w$$

That is, the germ of the holonomy map $h_{\tau_{w,z}}$ at w can be expressed as the composition of ℓ germs of the basic maps $h_{i,j}$. We then say that $\gamma = [h_{\tau_{w,z}}]_w$ has word length at most ℓ. Let $\|\gamma\|$ denote the least such ℓ for which this is possible. The norm of the identity germ is defined to be 0.

Define the "orbit of w of radius ℓ in the groupoid word norm" to be:

$$\mathcal{O}_\ell(w) = \{z \in \mathcal{T} \mid g(w) = z,\ g \in \mathcal{G}_{\mathcal{F}}, w \in Dom(g), \|[g]_w\| \leq \ell\}$$

DEFINITION 3.1. *The growth function of an orbit is defined as* $Gr(w,\ell) = \#\mathcal{O}_\ell(w)$.

Of course, the growth function for w depends upon almost all choices. However, its "growth type function" is independent of choices, as observed by Plante. This follows from one of the basic facts of the theory, that the word norm on $\mathcal{G}_{\mathcal{F}}$ is quasi-isometric to the length of geodesic paths.

PROPOSITION 3.2. [82, 88] *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. Then there exists a constant $C_m > 0$ such that for all $w \in \mathcal{T}$ and $z \in L_w \cap \mathcal{T}$, if $\sigma_{w,z} \colon [0,1] \to L_w$ is a leafwise geodesic segment from w to z of length $\|\sigma_{w,z}\|$, then*

$$C^{-1} \cdot \|\sigma_{w,z}\| \ \leq \ \|[h_{\sigma_{w,z}}]_w\| \ \leq \ C_m \cdot \|\sigma_{w,z}\|$$

In order to obtain a well-defined invariant of growth of an orbit, one introduces the notion the *growth type* of a function. That which we use (there are many - see [52, 36, 6]) is essentially the weakest one. Given given functions $f_1, f_2 \colon [0,\infty) \to [0,\infty)$ say that $f_1 \lesssim f_2$ if there exists constants $A, B, C > 0$ such that for all $r \geq 0$, we have that $f_2(r) \ \leq \ A \cdot f_1(B \cdot r) + C$. Say that $f_1 \sim f_2$ if both $f_1 \lesssim f_2$ and $f_2 \lesssim f_1$ hold. This defines equivalence relation on functions, which defines their *growth class*.

One can consider a variety of special classes of growth types. For example, note that if f_1 is the constant function and $f_2 \sim f_1$ then f_2 is constant also.

We say that f has *exponential growth type* if $f(r) \sim \exp(r)$. Note that $\exp(\lambda \cdot r) \sim \exp(r)$ for any $\lambda > 0$, so there is only one growth class of "exponential type".

A function has *subexponential growth type* if $f \lesssim \exp(r)$, but the converse does not hold. That is, f has subexponential growth type if for any $\lambda > 0$ there exists $A, C > 0$ so that $f(r) \leq A \cdot \exp(\lambda \cdot r) + C$.

Finally, f has *polynomial growth type* if there exists $d \geq 0$ such that $f \lesssim r^d$. The growth type is exactly polynomial of degree d if $f \sim r^d$.

DEFINITION 3.3. *The growth type of an orbit $\mathcal{O}(w)$ is the growth type of $Gr(w, \ell) = \#\mathcal{O}_\ell(w)$.*

A basic result of Plante is that

PROPOSITION 3.4. *Let M be a compact manifold. Then for all $w \in \mathcal{T}$, $Gr(z, \ell) \lesssim \exp(\ell)$. Moreover, for $z \in L_w \cap \mathcal{T}$, then $Gr(z, \ell) \sim Gr(w, \ell)$. Thus, the growth type of a leaf L_w is well-defined, and we say that L_w has the growth type of the function $Gr(w, \ell)$.*

We can thus speak of a leaf L_w of $\mathcal{F}$ having exponential growth type, and so forth. For example, the Infinite Jungle Gym manifold (in Figure 6) has growth type exactly polynomial of degree 3, while the leaves of the Hirsch foliations (in Figures 8 and 11) have exponential growth type.

Before continuing with the discussion of the growth types of leaves, we note the correspondence between these ideas and a basic problem in geometric group theory. Growth functions for finitely generated groups are a basic object of study in geometric group theory in recent years.

Let $\Gamma = \langle \gamma_0 = 1, \gamma_1, \ldots, \gamma_k \rangle$ be a finitely generated group. Then $\gamma \in \Gamma$ has word norm $\|\gamma\| \leq \ell$ if we can express γ as a product of at most ℓ generators, $\gamma = \gamma_{i_1}^{\pm} \cdots \gamma_{i_d}^{\pm}$. Define the ball of radius ℓ about the identity of Γ by

$$\Gamma_\ell \equiv \{\gamma \in \Gamma \mid \|\gamma\| \leq \ell\}$$

The growth function $Gr(\Gamma, \ell) = \#\Gamma_\ell$ depends upon the choice of generating set for Γ, but its growth type does not. The following is a celebrated theorem of Gromov:

THEOREM 3.5. [49] *Suppose Γ has polynomial growth type for some generating set. Then there exists a subgroup of finite index $\Gamma' \subset \Gamma$ such that Γ' is a nilpotent group.*

This seminal result was the basis for the general question of intense study, to what extent does the growth type of a group determine its algebraic structure? Questions of a similar nature can be asked about leaves of foliations, especially, to what extent does the growth function of leaves determine their "dynamics"? However, there is a fundamental difference between this problem for groups and for leaves.

The homogeneity of groups implies that the growth rate is uniformly the same for balls in the word metric about any point $\gamma_0 \in \Gamma$. That is, one can choose the constants $A, B, C > 0$ in the definition of growth type which are independent of the center γ_0. For foliation pseudogroups, there is a basic question about the uniformity of the growth function as the basepoint within an orbit varies:

QUESTION 3.6. *How does the function $w \mapsto Gr(w, d)$ behave, as a Borel function of $w \in \mathcal{T}$?*

Examples of Ana Rechtman [89] show that even for smooth foliations of compact manifolds, this function is not uniform as function of $w \in \mathcal{T}$. Thus, to formulate an analog for foliation pseudogroups of the classification program for finitely generated groups, it is necessary to require uniformity of the growth function $\ell \mapsto Gr(w, \ell)$, as a function of $w \in \mathcal{T}$.

Recall that the equivalence relation on $\mathcal{T}$ defined by $\mathcal{F}$ is the Borel subset

$$\mathcal{R}_{\mathcal{F}} \equiv \{(w, z) \mid w \in \mathcal{T},\ z \in L_w \cap \mathcal{T}\} \subset \mathcal{T} \times \mathcal{T}$$

Two foliations $\mathcal{F}_1, \mathcal{F}_2$ with complete transversals $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively, are said to be *Borel orbit equivalent* if there exists a Borel map $h\colon \mathcal{T}_1 \to \mathcal{T}_2$ which induces a Borel isomorphism $\mathcal{R}_{\mathcal{F}_1} \cong \mathcal{R}_{\mathcal{F}_2}$. The foliations are said to be *measurably orbit equivalent* if there exists a Borel measurable map $h\colon \mathcal{T}_1 \to \mathcal{T}_2$ which induces a Borel equivalence, up to sets of Lebesgue measure zero. See the papers by Jacob Feldman and Calvin Moore [37, 83] for more background on this topic.

Note that if two foliations are diffeomorphic, then they are Borel orbit equivalence, so this is a weaker equivalence notion of equivalence than being differentiably conjugate. Moreover, measurable orbit equivalence is an even weaker notion of equivalence, although of great importance in the measurable classification of dynamical systems defined by a single transformation or flow.

For example, a foliation is said to be (measurably) *hyperfinite* if it is measurably orbit equivalent to an action of the integers $\mathbb{Z}$ on the interval $[0, 1]$. The celebrated result of Dye [34, 35, 71] implies:

THEOREM 3.7 (Dye 1957)**.** *A foliation defined by a non-singular flow is always hyperfinite.*

Uniform polynomial growth estimates were used by Carolyn Series [95] to obtain a "measurable classification" of the equivalence relation on $\mathcal{T}$ defined by the action of $\mathcal{G}_{\mathcal{F}}$.

THEOREM 3.8 (Series 1977)**.** *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. If the growth type of all functions $Gr(w, \ell)$ are uniformly of polynomial type, then the equivalence relation on $\mathcal{T}$ defined by $\mathcal{G}_{\mathcal{F}}$ is hyperfinite.*

The most general form of such results is a celebrated result of Alain Connes, Jack Feldman and Benjamin Weiss [29], which implies:

THEOREM 3.9 (Connes-Feldman-Weiss 1981)**.** *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. If the growth type of all functions $Gr(w, \ell)$ are uniformly of subexponential type, then the equivalence relation on $\mathcal{T}$ it defines is hyperfinite.*

The extension of these results to Borel orbit equivalence is a much more difficult problem - see [4]. The problem seems of fundamental importance to the study of foliations. While the topological classification of foliations is surely an unsolvable problem, in any sense of the word "unsolvable", the Borel problem might be approachable when restricted to special subclasses, such as for foliations with uniformly polynomial growth.

In the late 1970's and early 1980's, Cantwell & Conlon, Hector, Nishimori, Tsuchiya in particular [19, 20, 54, 99, 101], studied the case of codimension-one C^2-foliations with all leaves of polynomial growth type. For real analytic foliations, their results are very satisfying, essentially giving a "classification theory" in terms of levels.

Their results for the general case of codimension-one C^2-foliations constitute part of the much more general "theory of levels" for foliations, a form of a generalized Poincaré-Bendixson Theory for leaves. However, without the assumption of polynomial growth type, or that the elements of the holonomy

pseudogroup act via local real analytic maps, the classification becomes much more complicated, as there are numerous counter-examples which have been constructed to show that the conclusions in the analytic case do not extend so easily.

The classification problem is even more problematic for C^1-foliations of codimension-one, for example. This is one reason for the focus in these notes on using other invariants to at least characterize classes of foliations based on their transverse hyperbolicity, the complexity of the orbit growth functions, and topological dynamics. This theme is discussed much more extensively in the paper [66].

One approach to classification if to impose restrictions on their "topological dynamics". We introduce three basic concepts, dating from 1930's, and extensively studied for topological group actions in 1950's and 1960's. First, we have the dichotomy:

DEFINITION 3.10. *A pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$ is*

- proximal *if there exists $\delta > 0$ such that for all $w, w' \in \mathcal{T}$ with $d_{\mathcal{T}}(w, w') < \delta$, then for all $\epsilon > 0$ there exists $h_{\tau w,z} \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in Dom(h_{\tau w,z})$ and $d_{\mathcal{T}}(h_{\tau w,z}(w), h_{\tau w,z}(w')) < \epsilon$*
- distal *if it is not proximal.*

DEFINITION 3.11. *A pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on $\mathcal{T}$ is* equicontinuous *if there exists a metric $d'_{\mathcal{T}}$ on $\mathcal{T}$ equivalent to the Riemannian distance function, such that for all $w, w' \in \mathcal{T}$ and $h_{\tau w,z} \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in Dom(h_{\tau w,z})$,*

$$d'_{\mathcal{T}}(h_{\tau w,z}(w), h_{\tau w,z}(w')) = d'_{\mathcal{T}}(w, w')$$

Let us consider this notion for the case of exceptional minimal sets, which are *transversally* zero-dimensional. In the case of flows, exceptional minimal sets have been called "matchbox manifolds" in the topological dynamics literature [1, 38]. The author, in the works with Alex Clark and Olga Lukina [27, 28], use the nomenclature matchbox manifolds for foliated spaces which are transversally zero-dimensional, as it is very descriptive.

Minimal matchbox manifolds appear to be an excellent test case for the study of classification problems, as the transverse Cantor structure implies the difference between continuous, Borel and measurable equivalence is manageable in many cases. For example, here is a recent result:

THEOREM 3.12 (Clark-Hurder 2009)**.** *If $\mathfrak{M} \subset M$ is an* exceptional minimal set *for a foliation, and the dynamics of $\mathcal{F}$ restricted to $\mathfrak{M}$ are equicontinuous, then $\mathfrak{M}$ is homeomorphic as a foliated space to a generalized (McCord) solenoid.*

Recall that an *n-dimensional solenoid* is an inverse limit space

$$\mathcal{S} = \lim_{\leftarrow} \{p_{\ell+1} \colon L_{\ell+1} \to L_\ell\}$$

where for $\ell \geq 0$, L_ℓ is a closed, oriented, n-dimensional manifold, and $p_{\ell+1} \colon L_{\ell+1} \to L_\ell$ are smooth, orientation-preserving proper covering maps. These were introduced by McCord [81], and are classified up to foliated homeomorphism by the algebraic structure of the inverse limit of the fundamental groups of the spaces appearing in their definition.

The results of the paper [27] and the techniques developed to prove them suggest the following:

PROBLEM 3.13. *Give an algebraic classification for minimal matchbox manifolds.*

Miguel Bermudez and Gilbert Hector study two-dimensional Borel laminations in [9], and obtain partial classification results up to Borel equivalence.

4. Exponential Complexity

Lecture 1 introduced exponential growth criteria for the normal derivative cocycle of the pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on the transverse space $\mathcal{T}$, and Lecture 2 discussed the growth types for the orbits of the groupoid. In both cases, exponential behavior represents a type of exponential complexity for the dynamics of $\mathcal{G}_{\mathcal{F}}$ which are part of a larger theme, that when studying classification problems, *Complexity is Simplicity.* In this third lecture, we develop this theme further. First, we give an aside, presenting a well-known phenomenon for map germs.

Recall a simple example from advanced calculus. Let $f(x) = x/2$, and let $g\colon (-\epsilon, \epsilon) \to \mathbb{R}$ be a smooth map with $g(0) = 0$, $g'(0) = 1/2$. Then $g \sim f$ near $x = 0$. That is, for $\delta > 0$ sufficiently small, there is a smooth map $h\colon (-\epsilon, \epsilon) \to \mathbb{R}$ such that $h^{-1} \circ g \circ h = f(x)$ for all $|x| < \delta$.

This illustrates the principle that exponentially contracting maps, or more generally hyperbolic maps in higher dimensions, the derivative is a complete invariant for their germinal conjugacy class at the fixed-point. For maps which are "completely flat" at the origin, where $g(0) = 0$, $g'(0) = 1$, $g^k(0) = 0$ for all $k > 1$, no such classification exists; their "classification" is much more difficult.

Analogously, for foliation dynamics, and the related problem of studying the dynamics of a finitely generated group acting smoothly on a compact manifold, exponential complexity in the dynamics often gives rise to hyperbolic behavior for the holonomy pseudogroup. Hyperbolic maps can be put into a standard form, and so one obtains a fundamental tool for studying the dynamics of the pseudogroup. The problem is thus, how does one pass from exponential complexity to hyperbolicity?

The rate of growth of the function $\ell \mapsto Gr(w, \ell)$ is one measure of the complexity of the leaf L_w. We saw previously that if the action of $\mathcal{G}_{\mathcal{F}}$ on $\mathcal{T}$ has uniformly subexponential growth functions, then Theorem 3.9 of Connes, Feldman and Weiss implies the equivalence relation it defines on the transversal space $\mathcal{T}$ is hyperfinite, and the classification stops, as no general further results are known. In this case, we say that *subexponential complexity often leads to ambiguity.* Of course, this is just an intuitive statement, but is indicative of the state of our understanding of the classification problem for such foliations.

One issue with the "counting argument" for the growth of leaves, as seen in for example in the example of the Hirsch foliations, is that just counting the growth rate of an orbit ignores fundamental information about the dynamics. The orbit growth rate counts the number of times the leaf crosses a transversal $\mathcal{T}$ in fixed distance within the leaf, but does not take into account whether these crossings are "nearby" or "far apart".

As an example of this, there are Riemannian foliations with all leaves of exponential growth type. Ken Richardson discusses constructions of such examples in [90], for example. Thus, exponential orbit growth rate does not imply (exponential) transversally hyperbolic behavior. On the other hand, in the Hirsch examples, the handles at the end of each ball of radius ℓ in a leaf appear to be widely separated transversally, so somehow this is different. The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of the Hirsch example is topologically semi-conjugate to the pseudogroup generated by the doubling map $z \mapsto z^2$ on $\mathbb{S}^1$. After ℓ-iterations, the inverse map to $h(z) = z^{2^\ell}$ has derivative of norm 2^ℓ, and so for a Hirsch foliation modeled on this map, every leaf is transversally hyperbolic.

We next discuss a measure of the exponential complexity for pseudogroup C^1-actions, their (local) geometric entropy, following Bowen [14] and Ghys, Langevin & Walczak [45]. These invariants provide a bridge between the two types of complexity introduced previously, and have found many applications in the study of foliation dynamical systems. One example of this is the surprising role of these invariants in the question of whether the secondary classes of C^2-foliations are non-trivial in cohomology, or not [22, 58, 57, 64].

We begin with the basic notion of ϵ-separated sets, due to Bowen [14] for diffeomorphisms, and extended to groupoids in [45]. Let $\epsilon > 0$ and $\ell > 0$. A subset $\mathcal{E} \subset \mathcal{T}$ is said to be (ϵ, ℓ)-separated if for all $w, w' \in \mathcal{E} \cap \mathcal{T}_i$ there exists $g \in \mathcal{G}_{\mathcal{F}}$ with $w, w' \in Dom(g) \subset \mathcal{T}_i$, and $\|g\|_w \leq \ell$ so that $d_{\mathcal{T}}(g(w), g(w')) \geq \epsilon$. If $w \in \mathcal{T}_i$ and $w' \in \mathcal{T}_j$ for $i \neq j$ then they are (ϵ, ℓ)-separated by default.

The "expansion growth function" counts the maximum of this quantity:

$$h(\mathcal{G}_{\mathcal{F}}, \epsilon, \ell) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset \mathcal{T} \text{ is } (\epsilon, \ell)\text{-separated}\}$$

If the pseudogroup $\mathcal{G}_{\mathcal{F}}$ consists of isometries, for example, then applying elements of $\mathcal{G}_{\mathcal{F}}$ does not increase the separation between points, so the growth functions $h(\mathcal{G}_{\mathcal{F}}, \epsilon, \ell)$ have polynomial growth of degree equal the dimension of $\mathcal{T}$, as functions of ℓ. Thus, if the functions $h(\mathcal{G}_{\mathcal{F}}, \epsilon, \ell)$ have greater than polynomial growth type, then the action of the pseudogroup cannot be elliptic, for example.

Introduce the measure of the exponential growth type of the expansion growth function:

$$h(\mathcal{G}_{\mathcal{F}}, \epsilon) = \limsup_{d \to \infty} \frac{\ln\{\max\{\#\mathcal{E} \mid \mathcal{E} \text{ is } (\epsilon, d)\text{-separated}\}\}}{d} \tag{3}$$

$$h(\mathcal{G}_{\mathcal{F}}) = \lim_{\epsilon \to 0} h(\mathcal{G}_{\mathcal{F}}, \epsilon) \tag{4}$$

Then we have the fundamental result of Ghys, Langevin & Walczak [45]:

THEOREM 4.1. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M. Then $h(\mathcal{G}_{\mathcal{F}})$ is finite.*

Moreover, the property $h(\mathcal{G}_{\mathcal{F}}) > 0$ is independent of all choices.

For example, if $\mathcal{F}$ is defined by a C^1-flow $\phi_t \colon M \to M$, then $h(\mathcal{G}_{\mathcal{F}}) > 0$ if and only if $h_{top}(\phi_1) > 0$. Note that $h(\mathcal{G}_{\mathcal{F}})$ is defined using the word growth function for orbits, while the topological entropy of the map ϕ_1 is defined using the geodesic length function (the time parameter) along leaves. These two notions of "distance along orbits" are comparable, which can be used to give estimates, but not necessarily any more precise relations. This point is discussed in detail in [45].

In any case, the essential information contained in the invariant $h(\mathcal{G}_{\mathcal{F}})$ is simply whether the foliation $\mathcal{F}$ exhibits exponential complexity for its orbit dynamics, or not. Exploiting further the information contained in this basic invariant of C^1-foliations has been one of the fundamental problems in the study of foliation dynamics since the introduction of the concept of geometric entropy in 1988.

One aspect of the geometric entropy $h(\mathcal{G}_{\mathcal{F}})$ is that it is a "global invariant", which does not indicate "where" the chaotic dynamics is happening. The author introduced a variant of $h(\mathcal{G}_{\mathcal{F}})$ in [66], the *local geometric entropy* $h(\mathcal{G}_{\mathcal{F}}, w)$ of $\mathcal{G}_{\mathcal{F}}$ which is a refinement of the global entropy. The local geometric entropy is analogous to the local measure-theoretic entropy for maps introduced by Brin and Katok [16]. The concept of local entropy, as adapted to pseudogroups, is very useful for the study of foliation dynamics.

In the definition of (ϵ, ℓ)-separated sets above, the separated points can be restricted to a given subset $X \subset \mathcal{T}$, where the set X is not assumed to be saturated. Introduce the relative expansion growth function:

$$h(\mathcal{G}_{\mathcal{F}}, X, \epsilon, \ell) = \max\{\#\mathcal{E} \mid \mathcal{E} \subset X \text{ is } (\epsilon, \ell)\text{-separated}\}$$

Form the corresponding limits as in (3) and (4), to obtain the *relative geometric entropy* $h(\mathcal{G}_{\mathcal{F}}, X)$.

Now, fix $w \in \mathcal{T}$ and let $X = B(w, \delta) \subset \mathcal{T}$ be the open δ–ball about $w \in \mathcal{T}$. Perform the same double limit process as used to define $h(\mathcal{G}_{\mathcal{F}})$ for the sets $B(w, \delta)$, but then also let the radius of the balls tend to zero, to obtain:

DEFINITION 4.2. *The* local geometric entropy *of $\mathcal{G}_{\mathcal{F}}$ at w is*

$$h_{loc}(\mathcal{G}_{\mathcal{F}}, w) = \lim_{\delta \to 0} \Big\{ \lim_{\epsilon \to 0} \Big\{ \limsup_{\ell \to \infty} \frac{\ln\{h(\mathcal{G}_{\mathcal{F}}, B(w, \delta), \ell, \epsilon)\}}{\ell} \Big\} \Big\} \tag{5}$$

The quantity $h_{loc}(\mathcal{G}_{\mathcal{F}}, w)$ measures of the amount of "expansion" by the pseudogroup in an open neighborhood of w. The local entropy has some very useful properties, which are elementary to show. Here is one:

PROPOSITION 4.3 (Hurder, [66])**.** *Let $\mathcal{G}_{\mathcal{F}}$ a C^1-pseudogroup. Then $h_{loc}(\mathcal{G}_{\mathcal{F}}, w)$ is a Borel function of $w \in \mathcal{T}$, and $h_{loc}(\mathcal{G}_{\mathcal{F}}, w) = h_{loc}(\mathcal{G}_{\mathcal{F}}, z)$ for $z \in L_w \cap \mathcal{T}$. Moreover,*

$$h(\mathcal{G}_{\mathcal{F}}) = \sup_{w \in \mathcal{T}} h_{loc}(\mathcal{G}_{\mathcal{F}}, w) \tag{6}$$

It follows that there is a disjoint Borel decomposition of $\mathcal{T}$ into $\mathcal{G}_{\mathcal{F}}$-saturated subsets $\mathcal{T} = \mathbf{Z}_{\mathcal{F}} \cap \mathbf{C}_{\mathcal{F}}$, where $\mathbf{C}_{\mathcal{F}} = \{w \in \mathcal{T} \mid h(\mathcal{G}_{\mathcal{F}}, w) > 0\}$ consists of the "chaotic" points for the groupoid action, and $\mathbf{Z}_{\mathcal{F}} = \{w \in \mathcal{T} \mid h(\mathcal{G}_{\mathcal{F}}, w) = 0\}$ are the "tame" points. Here is a corollary of Proposition 4.3:

COROLLARY 4.4. *$h(\mathcal{G}_{\mathcal{F}}) > 0$ if and only if $\mathbf{C}_{\mathcal{F}} \neq \emptyset$.*

We will discuss in the next section, the relationship between local entropy $h(\mathcal{G}_{\mathcal{F}}, w) > 0$ and the transverse Lyapunov spectrum of ergodic invariant measures for the leafwise geodesic flow on the closure $\overline{L_w}$.

Next, we consider some examples where $h(\mathcal{G}_{\mathcal{F}}) > 0$.

PROPOSITION 4.5. *The Hirsch foliations always have positive geometric entropy.*

Proof. The idea of the proof is as follows. The holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of the Hirsch examples is topologically semi-conjugate to the pseudogroup generated by the doubling map $z \mapsto z^2$ on $\mathbb{S}^1$.

After ℓ-iterations, the inverse map to $z \mapsto z^{2^\ell}$ has derivative of norm 2^ℓ so we have a rough estimate $h(\mathcal{G}_{\mathcal{F}}, \epsilon, \ell) \sim (2\pi/\epsilon) \cdot 2^\ell$. Thus, $h(\mathcal{G}_{\mathcal{F}}) \sim \ln 2$. □

For these examples, the relationship between "orbit growth type" and expansion growth type is transparent. Observe that in the Hirsch example, as we wander out the tree-like leaf, the exponential growth of the ends of the typical leaf yield an exponential growth for the number of ϵ-separated points, as we are also wandering around the transversal space $\mathcal{T}$, as represented by a core circle. This is suggested by comparing the two illustrations below:

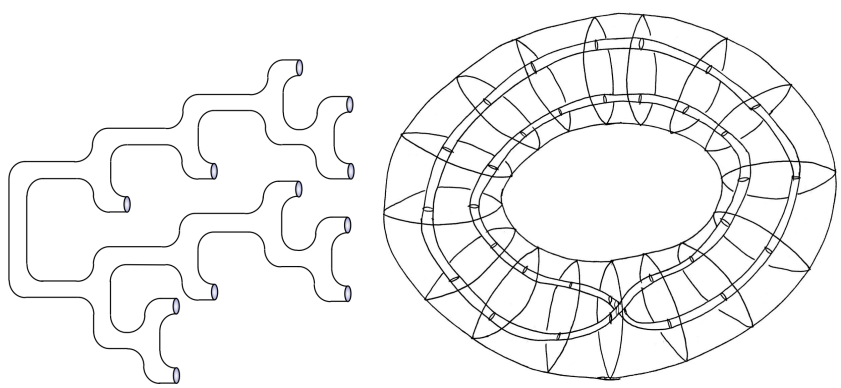

FIGURE 12. Comparing orbit with endset

It is natural to ask whether there are other classes of foliations for which this phenomenon occurs, that exponential growth type of the leaves is equivalent to positive foliation geometric entropy? It turns out that for the weak stable foliations of Anosov flows, this is also the case in general. First, let us recall a result of Anthony Manning [76]:

THEOREM 4.6. *Let B be a compact manifold of negative curvature, let $M = T^1B$ denote the unit tangent bundle to B, and let $\phi_t \colon M \to M$ denote the geodesic flow of B. Then $h_{top}(\phi) = Gr(\pi_1(B, b_0))$. That is, the entropy for the geodesic flow of B equals the growth rate of the volume of balls in the universal covering of B.*

Proof. The idea of proof for this result is conveyed by the illustration Figure 13, representing the fundamental domains for the universal covering. The assumption that B has non-positive curvature implies that its universal covering $\widetilde{B}$ is a disk, and we can "tile" it with fundamental domains.

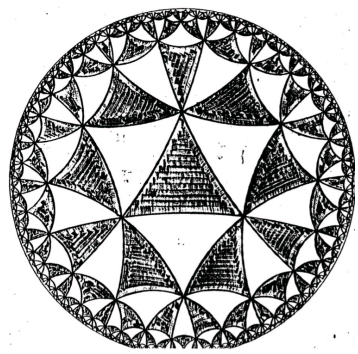

FIGURE 13. Tiling by fundamental domains for hyperbolic manifold cover

From the center basepoint, there is a unique geodesic segment to the corresponding basepoint in each translate. The number of such in a given radius is precisely the growth function for the fundamental group $\pi_1(B, b_0)$. On the other hand, the negative curvature hypothesis implies that these geodesics separate points for the geodesic flow as well. $\square$

We include this example, because it is actually a result about foliation entropy! The assumption that B has uniformly negative sectional curvatures implies that the geodesic flow $\phi_t \colon M \to M$ defines a foliation on M, its weak-stable foliation. Then by a result of Pugh and Shub, the weak-stable manifolds L_w form the leaves of a C^1-foliation of M, called the *weak-stable foliation* for ϕ_t. Moreover, the orbits of the geodesic flow $\phi_t(w)$ are contained in the leaves of $\mathcal{F}$. Then again one has $h(\mathcal{G}_{\mathcal{F}}) \sim h_{top}(\phi_1)$ which equals the growth type of the leaves.

Besides special cases such as for the Hirsch foliations and their generalizations in [10] where there is uniformly expanding holonomy groups, and the weak stable foliations for Anosov flows, how does one determine when a foliation $\mathcal{F}$ has positive entropy?

There is a third case where $h(\mathcal{G}_{\mathcal{F}}) > 0$ can be concluded, as noted in [45], when the dynamics of $\mathcal{G}_{\mathcal{F}}$ admits a "ping-pong game". The notation "ping-pong game" is adopted from the paper [31] which gives a more geometric proof of Tits Theorem [98] for the dichotomy of the growth types of countable subgroups of linear groups. To say that the dynamics of $\mathcal{G}_{\mathcal{F}}$ admits a ping-pong game, means that there are disjoint open sets $U_0, U_1 \subset V \subset \mathcal{T}$ and maps $g_0, g_1 \in \mathcal{G}_{\mathcal{F}}$ such that for $i = 0, 1$:

- the closure $\overline{V} \subset Dom(g_i)$ for $i = 0, 1$
- $g_i(\overline{V}) \subset U_i$

It follows that for each $w \in V$ the forward orbit

$$\mathcal{O}^+_{g_0,g_i}(w) = \{g_I(w) \mid I = (i_1, \ldots, i_k)\ , i_\ell \in \{0, 1\}\ ,\ g_I = g_{i_k} \circ \cdot \circ g_{i_1}\}$$

consists of distinct points, and so the full orbit $\mathcal{O}(w)$ has exponential growth type. Moreover, if $\epsilon > 0$ is less than the distance between the disjoint closed subsets $g_0(\overline{V})$ and $g_1(\overline{V})$, then the points in $\mathcal{O}^+_{g_0,g_i}(w)$ are all (ϵ, ℓ)-separated for appropriate $\ell > 0$, and hence $h(\mathcal{G}_{\mathcal{F}}, K) > 0$.

For codimension-one foliations, the existence of ping-pong game dynamics for its pseudogroup is equivalent to the existence of a "resilient leaf", which in turn is analogous to the existence of homoclinic orbits for a diffeomorphism. It is a well-known principle that the existence of homoclinic orbits for a diffeomorphism implies positive topological entropy.

We conclude with a general question:

QUESTION 4.7. *Are there other canonical classes of C^1-foliations where positive entropy is to be "expected"? For example, if $\mathcal{F}$ has leaves of exponential growth, when does there exist a C^1-close perturbation of $\mathcal{F}$ with positive entropy?*

5. Entropy and Exponent

In the first three lectures, we introduced aspects of "exponential complexity" for foliation dynamics: Lyapunov spectrum for the foliation geodesic flow, exponential growth of orbits, and transverse exponential expansion and geometric entropy. In Lecture 4, we discuss the known relationships between these invariants. The theme is summarized by:

$$\textit{Positive Entropy} \leftrightarrow \textit{Chaotic Dynamics} \leftrightarrow \textit{??}$$

Here are three questions that we address in part here. As always, we assume that $\mathcal{F}$ is a C^r-foliation of a compact manifold M, for $r \geq 1$.

PROBLEM 5.1. *If $h(\mathcal{G}_{\mathcal{F}}) > 0$, what conclusions can we reach about the dynamics of $\mathcal{F}$?*

PROBLEM 5.2. *What hypotheses on the dynamics of $\mathcal{F}$ are sufficient to imply that $h(\mathcal{G}_{\mathcal{F}}) > 0$?*

PROBLEM 5.3. *Are there cohomology hypotheses on $\mathcal{F}$ which would "improve" our understanding of its dynamics? How does leafwise cohomology $H^*(\mathcal{F})$ influence dynamics? How are the secondary cohomology invariants for $\mathcal{F}$ related to entropy?*

The difficulty with addressing these questions, is that there are only limited sets of techniques applicable to foliation dynamics, relating transverse expansion growth with the transverse Lyapunov spectrum of the foliation geodesic flow. The principle difficulty, as noted previously, is that there is no good groupoid replacement for the notion of "uniform recurrence" in the support of an invariant measure, which we have for flows. Thus, given asymptotic data about either the transverse derivative cocycle, or the transverse expansion growth function, one has to develop new techniques to extract from this data dynamical conclusions.

We begin by recalling a result of Ghys, Langevin and Walczak [45] which gives a straightforward conclusion valid in all codimension, and which is especially potent for codimension one foliations (see Corollary 5.5 below.)

THEOREM 5.4 (G-L-W 1988)**.** *Let M be compact with a C^1-foliation $\mathcal{F}$ of codimension $q \geq 1$, and $X \subset \mathcal{T}$ a closed subset. If $h(\mathcal{G}_{\mathcal{F}}, X) = 0$, then the restricted action of $\mathcal{G}_{\mathcal{F}}$ on X admits an invariant probability measure.*

The idea of the proof is to interpret the condition $h(\mathcal{G}_{\mathcal{F}}) = 0$ as a type of *equidistribution* result, and then form averaging sequences over the orbits, which consequently yield $\mathcal{G}_{\mathcal{F}}$-invariant probability measures on X.

COROLLARY 5.5. *Let M be compact with a C^1-foliation $\mathcal{F}$ of codimension one, and suppose that $\mathfrak{M} \subset M$ is a minimal set for which the local entropy $h(\mathcal{G}_{\mathcal{F}}, \mathfrak{M}) = 0$. Then the dynamics of $\mathcal{G}_{\mathcal{F}}$ on $X = \mathcal{T} \cap \mathfrak{M}$ is semi-conjugate to the pseudogroup of an isometric dense action on $\mathbb{S}^1$. If $\mathcal{F}$ is C^2, and M is connected, then $\mathfrak{M} = M$ and the action is conjugate to a rotation group.*

Proof. By Theorem 5.4, there exists an invariant probability measure for the action of $\mathcal{G}_{\mathcal{F}}$ on X. The conclusions then follow from Sacksteder [91]. □

In the remainder of this section, we discuss three more recent results of the author concerning geometric entropy.

THEOREM 5.6. [63] *Let M be compact with a C^r-foliation $\mathcal{F}$ of codimension-q. If $q = 1$ and $r \geq 1$, or $q \geq 2$ and $r > 1$, then*

$$\mathcal{G}_{\mathcal{F}} \ \textit{distal} \quad \Longrightarrow \quad h(\mathcal{G}_{\mathcal{F}}) = 0$$

THEOREM 5.7. [63] *Let M be compact with a codimension one, C^1-foliation $\mathcal{F}$. Then*

$$h(\mathcal{G}_{\mathcal{F}}) > 0 \quad \Longrightarrow \quad \mathcal{F} \ \textit{has a resilient leaf}$$

THEOREM 5.8. [64] *Let M be compact with a codimension one, C^2-foliation $\mathcal{F}$. Then*

$$0 \neq GV(\mathcal{F}) \in H^3(M, \mathbb{R}) \quad \Longrightarrow \quad h(\mathcal{G}_{\mathcal{F}}) > 0$$

where $GV(\mathcal{F}) \in H^3(M, \mathbb{R})$ is the Godbillon-Vey class of $\mathcal{F}$.

The proofs of all three results are based on the existence of *stable transverse manifolds* for hyperbolic measures for the foliation geodesic flow. The first step is the following:

PROPOSITION 5.9. *Let M be compact with a C^1-foliation $\mathcal{F}$, and suppose that $\mathfrak{M} \subset M$ is a minimal set for which the relative entropy $h(\mathcal{G}_{\mathcal{F}}, \mathfrak{M}) > 0$. Then there exists a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow, with support the support of μ_* contained in the unit leafwise tangent bundle to $\mathfrak{M}$.*

Proof. We give a sketch of the proof. Let $X = \mathcal{T} \cap \mathfrak{M}$. The assumption $\lambda = h(\mathcal{G}_{\mathcal{F}}, X) > 0$ implies there exists $\epsilon > 0$ so that $\lambda_\epsilon = h(\mathcal{G}_{\mathcal{F}}, X, \epsilon) > \frac{3}{4}\lambda > 0$. Thus, there exists a sequence of subsets $\{\mathcal{E}_\ell \subset X \mid \ell \to \infty\}$ such that $\mathcal{E}_\ell$ is (ϵ_ℓ, r_ℓ)-separated, where $\epsilon_\ell \to 0$ and $r_\ell \geq \ell$, and $\#\mathcal{E}_\ell \geq \exp\{3r_\ell\lambda/4\}$.

We can assume without loss that $\mathcal{E}_\ell$ is contained in the transversal for a single coordinate chart, say $\mathcal{E}_\ell \subset \mathcal{T}_1$ as the number of charts is fixed. As $\mathcal{T}_1$ has bounded diameter, this implies there exists pairs $\{x_\ell, y_\ell\} \subset \mathcal{E}_\ell$ so that

$$d_{\mathcal{T}}(x_\ell, y_\ell) \lesssim \exp\{-3r_\ell\lambda/4\} \cdot \mathrm{diam}(\mathcal{T}_1)$$

and leafwise geodesic segments $\sigma_\ell \colon [0,1] \to L_{x_\ell}$ with $\|\sigma_\ell\| \leq r_\ell$ such that $d_{\mathcal{T}}(h_{\sigma_\ell}(x_\ell), h_{\sigma_\ell}(y_\ell)) \geq \epsilon$.

It follows by the Mean Value Theorem that there exists $z_\ell \in B_{\mathcal{T}}(x_\ell, \exp\{-3r_\ell\lambda/4\} \cdot \mathrm{diam}(\mathcal{T}_1))$ such that $\|D_{z_\ell} h_{\sigma_\ell}\| \gtrsim \exp\{3r_\ell\lambda/4\}$.

Noting that $\epsilon_\ell \to 0$ and choosing appropriate subsequences, the resulting geodesic segments σ_ℓ define an invariant probability measure μ_* for the geodesic flow, with support in $\mathfrak{M}$. Moreover, by the cocycle equation and continuity of the derivative, the measure μ_* will be hyperbolic. In fact, with careful choices above, the exponent can be made arbitrarily close to $h(\mathcal{G}_{\mathcal{F}}, X)$, modulo the adjustment for the relation between geodesic and word lengths. See [63] for details. □

The construction sketched in the proof of Proposition 5.9 is very "lossy" - at each stage, information about the transverse expansion due to the assumption that $h(\mathcal{G}_{\mathcal{F}}, X) > 0$ gets discarded, especially in that for each n we only consider a pair of points (x_ℓ, y_ℓ) to obtain a geodesic segment σ_ℓ along which the transverse derivative has exponentially increasing norm. We will return to this point later.

The next step in the construction of stable manifolds, is to assume we are given a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow. Then for a typical point $\widehat{x} = (x, \vec{v}) \in \widehat{M}$ in the support of μ_* the geodesic ray at $(x, \vec{v})$ has an exponentially expanding norm of its transverse derivative, and hence its Lyapunov spectrum acting as a flow on $\widehat{M}$ contains a non-trivial expanding eigenspace. By reversing the time flow (via the inversion $\vec{v} \mapsto -\vec{v}$ of $\widehat{M}$) we obtain an invariant probability measure μ_*^- for the foliation geodesic flow for which the Lyapunov spectrum of the flow contains a non-trivial contracting eigenspace.

If we assume that the flow is $C^{1+\alpha}$ for some Hölder exponent $\alpha > 0$, then there exists non-trivial stable manifolds in $\widehat{M}$ for almost every $(x, \vec{v})$ in the support of μ_*^-. Denote this stable manifold by $\mathcal{S}(x, \vec{v})$ and note that its tangent space projects non-trivially onto $\widehat{Q}$. Moreover, for points $\widehat{y}, \widehat{z} \in \mathcal{S}(x, \vec{v})$ the distance $d(\varphi_t(\widehat{y}), \varphi_t(\widehat{z}))$ converges to 0 exponentially fast, as $t \to \infty$. Thus, the images $y, z \in M$ of these points converge exponentially fast together under the holonomy of $\mathcal{F}$.

Combining these results we obtain

THEOREM 5.10. *Let $\mathcal{F}$ be $C^{1+\alpha}$ and suppose that $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then there exists a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow. Moreover, for a typical point $\widehat{x} = (x, \vec{v})$ in the support of μ_*^- there is a transverse stable manifold $\mathcal{S}(x, \vec{v})$ for the geodesic ray starting at $\widehat{x}$.*

If the codimension of $\mathcal{F}$ is one, then the differentiability is just required to be C^1, as the stable manifold for φ_t consists simply of the full transversal to $\widehat{\mathcal{F}}$.

Observe that Theorem 5.10 implies Theorem 5.6.

The assumption that $h(\mathcal{G}_{\mathcal{F}}) > 0$ has much stronger implications that simply implying that the dynamics of $\mathcal{G}_{\mathcal{F}}$ is not distal, but obtaining these results requires much more care. We sketch next some ideas for analyzing these dynamics in the case of codimension-one foliations.

In the proof of Proposition 5.9, instead of choosing only a single pair of points (x_ℓ, y_ℓ) at each stage, one can also use the Pigeon Hole Principle to choose a subset $\mathcal{E}'_\ell \subset \mathcal{E}$ contained in a fixed ball $B_{\mathcal{T}}(w, \delta_\ell)$ where $\#\mathcal{E}'_\ell$ grows exponentially fast as a function of ℓ, and the diameter δ_ℓ of the ball decreases exponentially fast, although at a rates less that λ_ϵ. This leads to the following notion.

DEFINITION 5.11. *An* $(\epsilon_\ell, \delta_\ell, \ell)$-quiver *is a subset* $\mathcal{Q}_\ell = \{(x_i, \vec{v}_i) \mid 1 \leq i \leq k_\ell\} \subset \widehat{M}$ *such that* $x_j \in B_{\mathcal{T}}(x_i, \delta_\ell)$ *for all* $1 \leq j \leq k_\ell$*, and for the unit-speed geodesic segment* $\sigma_i \colon [0, s_i] \to L_{x_i}$ *of length* $s_i \leq d$*, we have*

$$d_{\mathcal{T}}(h_{\sigma_i}(x_i), h_{\sigma_i}(x_j)) \geq \epsilon \ , \quad \textit{for all } j \neq i$$

An exponential quiver *is a collection of quivers* $\{\mathcal{Q}_\ell \mid \ell = 1, 2, \ldots\}$ *such that the function* $\ell \mapsto \#\mathcal{Q}_\ell$ *has exponential growth rate.*

The idea is that one has a collection of points $\{x_i \mid 1 \leq i \leq k_\ell\}$ contained in a ball of radius δ_ℓ and a geodesic segment based at each point, along whose flows the transverse holonomy separates points. The nomenclature "quiver" is based on the intuitive notion that the collection of geodesic segments emanating from the δ_ℓ-clustered set of basepoints $\{x_i\}$ is like a collection of arrows in a quiver. It is immediate that $h(\mathcal{G}_{\mathcal{F}}, \epsilon, d) \geq \#\mathcal{Q}_\ell$.

PROPOSITION 5.12. *If* $\mathcal{F}$ *admits an exponential quiver, then* $h(\mathcal{G}_{\mathcal{F}}) > 0$.

For codimension-one foliations, the results of [59] and [73] yields the converse estimate:

PROPOSITION 5.13. *Let* $\mathcal{F}$ *be a* C^1*-foliation of codimension-one on a compact manifold* M*. If* $h(\mathcal{G}_{\mathcal{F}}) > 0$ *then there exists an exponential quiver.*

It is an unresolved question whether a similar result holds for higher codimension. The point is that if so, then $h(\mathcal{G}_{\mathcal{F}})$ is estimated by the entropy of the foliation geodesic flow, and most of the problems we address here can be resolved using a form of Pesin Theory for flows relative to the foliation $\mathcal{F}$. (See [59] for further discussion of this point.)

The existence of an exponential quiver for a codimension-one foliation of a compact manifold M has strong implications for its dynamics. The basic idea is that the basepoints of the geodesic rays in the quiver are tightly clustered, and because the ranges of the endpoints of the geodesic rays are lie in a compact set, one can pass to a subsequence for which the endpoints are also tightly clustered. From this observation, one can show:

THEOREM 5.14. [63] *Let* $\mathcal{F}$ *be a* C^1*-foliation with codimension-one foliation of a compact manifold* M*. If* $h(\mathcal{G}_{\mathcal{F}}) > 0$*, then* $\mathcal{G}_{\mathcal{F}}$ *acting on* $\mathcal{T}$ *admits a "ping-pong game", which implies the existence of a resilient leaf for* $\mathcal{F}$*.*

This result is a C^1-version of one of the main results concerning the dynamical meaning of positive entropy given in [45]. In their paper, the authors require the foliations be C^2 as they invoke the Poincaré-Bendixson Theory for codimension-one foliations which is only valid for C^2-pseudogroups.

There is another approach to obtaining exponential quivers for a foliation $\mathcal{F}$, which is based on cohomology assumptions about $\mathcal{F}$. For a C^1-foliation $\mathcal{F}$, there exists a leafwise closed, continuous 1-form η on M whose cohomology class $[\eta] \in H^1(M, \mathcal{F})$ in the leafwise foliated cohomology group is well-defined. The form η has the property that its integral along a leafwise path gives the logarithmic

infinitesimal expansion of the determinant of the linear holonomy defined by the path. Thus, for codimension-one foliations, this integral is the expansion exponent of the holonomy.

For a C^2-foliation $\mathcal{F}$ of codimension-q the form η can be chosen to be C^1, and thus the exterior form $\eta \wedge d\eta^q$ is well-defined. As observed by Godbillon and Vey [47], the form $\eta \wedge d\eta^q$ is closed and yields a well-defined cohomology class $GV(\mathcal{F}) \in H^{2q+1}(M, \mathbb{R})$. One of the basic problems of foliation theory has been to understand the "dynamical meaning" of this class. A fundamental breakthrough was obtained by Gerard Duminy in the unpublished manuscripts [32, 33], where the study of this problem "entered its modern phase". (See also the reformulation of these results by Cantwell and Conlon in [22].) Based on this breakthrough, the papers [55, 58] showed that if $GV(\mathcal{F}) \neq 0$ then there is a saturated set of positive measure on which η is non-zero, and hence the set of hyperbolic leaves $\mathbf{H}_{\mathcal{F}}$ has *positive* Lebesgue measure. This study culminated in the following result of the author with Remi Langevin from [64]:

THEOREM 5.15. *Let $\mathcal{F}$ be a C^2-foliation of codimension-one on a compact manifold. If $\mathbf{H}_{\mathcal{F}}$ has positive Lebesgue measure, then $\mathcal{F}$ admits exponential quivers, and in particular the dynamics of $\mathcal{G}_{\mathcal{F}}$ admits ping-pong games. Thus, $h(\mathcal{G}_{\mathcal{F}}) > 0$.*

Combining Theorem 5.15 with the previous remarks yields Theorem 5.8.

Theorem 5.15 is the basis for the somewhat-cryptic Problem 5.3 given at the start of this section. The assumption that the class $[\eta] \in H^1(M, \mathcal{F})$ is non-trivial on a set of positive Lebesgue measure leads to positive entropy, suggests the question whether there are other such classes, and to what extent their values in cohomology are related to foliation dynamics.

In general, the results of this section are just part of a more general "Pesin Theory for foliations" as sketched in the author's overview paper [59], whose study continues to yield new insights into the dynamical properties of foliations for which $\mathbf{H}_{\mathcal{F}}$ is non-empty.

6. Classification of Minimal Sets

In this last lecture, we discuss a special case of the study of foliation dynamics, the properties of minimal sets – their dynamical special properties and approaches towards their classification.

Recall that a *minimal set* for $\mathcal{F}$ is a closed, saturated subset $\mathfrak{M} \subset M$ for which every leaf $L \subset \mathfrak{M}$ is dense. Thus, in spirit at least, what is true for one leaf in $\mathfrak{M}$ must be "true for all leaves" of $\mathfrak{M}$. This is, of course, false in general; but the study of dynamical properties for which this fails to be true is of particular interest as well. As usual, we note that M compact implies that there always exists at least one minimal set for $\mathcal{F}$ in M.

A related notion is that of a *transitive set* for $\mathcal{F}$, which is a closed saturated subset $\mathcal{Z} \subset M$ such that there exists at least one dense leaf $L_0 \subset \mathcal{Z}$. In this case, one can also ask what properties of this dense leaf are propagated to the other leaves in $\mathcal{Z}$. Very little is known about the transitive sets for foliations. In general, understanding the properties of foliation minimal sets is a sufficiently challenging problem.

Note that if $\mathfrak{M} \subset M$ is any closed saturated subset, then it is an example of a foliated space, as studied for example in [84] or [18, Chapter 11]. In particular, the minimal sets of a foliation $\mathcal{F}$ can be studied "independently" as foliated spaces.

Traditionally, the minimal sets are divided into three classes. A compact leaf of $\mathcal{F}$ is a minimal set. If every leaf of $\mathcal{F}$ is dense, then M itself is a minimal set. The third possibility is that the minimal set $\mathfrak{M}$ has no interior, but contains more than one leaf, hence the intersection $\mathfrak{M} \cap \mathcal{T}$ is always a perfect set. This third case can be subdivided into further cases: if the intersection $\mathfrak{M} \cap \mathcal{T}$ is a Cantor set, then $\mathfrak{M}$ is said to be an exceptional minimal set, and otherwise if $\mathfrak{M} \cap \mathcal{T}$ has no interior but is not totally disconnected, then it is said to be an *exotic minimal set*. For codimension one foliations, the case of exotic minimal sets cannot occur, but for foliations with codimension greater than one there are various types of constructions of exotic minimal sets [11].

The shape of a minimal set is one of the most basic properties, for which there is surprisingly little discussion in the foliation literature. The concept of shape for a compact metric space was introduced by Borsuk [13] and "modern shape theory" develops algebraic topology of the shape approximations of spaces [77, 78]. The Conley Index of invariant sets for flows is one traditional application of shape theory to the dynamics of flows. We consider a few basic questions about the shape of minimal sets.

DEFINITION 6.1. *Let $\mathcal{Z} \subset M$ be a saturated compact subset. The* shape *of $\mathcal{Z}$ is the equivalence class of any descending chain of open subsets $M \supset V_1 \supset \cdots \supset V_k \supset \cdots \supset \mathcal{Z}$ with $\mathcal{Z} = \bigcap_{k=1}^{\infty} V_k$*

The notion of equivalence referred to in the definition is defined by a "tower of equivalences" between such approximating neighborhood systems. The reader is referred to [77, 78] for details and especially the subtleties of this definition. Here is one basic property of shape:

DEFINITION 6.2. *Let $\mathcal{Z} \subset M$ be a saturated compact subset, and $w_0 \in \mathcal{Z}$ a fixed basepoint. Then $\mathcal{Z}$ has* stable *shape if the pointed inclusions $(V_{k+1}, w_0) \subset (V_k, w_0)$ are homotopy equivalences for all $k \gg 0$.*

The *shape fundamental group* of $\mathcal{Z}$ is defined by

$$\widehat{\pi}_1(\mathcal{Z}, w_0) = \text{inv lim}\{\pi_1(V_{k+1}, w_0) \to \pi_1(V_k, w_0)\} \tag{7}$$

If $\mathcal{Z} = M$ then clearly $\mathcal{Z}$ has stable shape.

Note that if $\mathcal{Z}$ has stable shape, then for $k \gg 0$ we have $\widehat{\pi}_1(\mathcal{Z}, w_0) \cong \pi_1(V_k, w_0)$. The following example from [25] shows that even for minimal sets with stable shape, the isomorphisms between stages of the shape tower exhibit subtleties.

Consider a Denjoy flow on the 2-torus $\mathbb{T}^2$, obtained by applying inflation to an orbit of the flow, as illustrated in Figure 14 below. These minimal sets are stable: $\widehat{\pi}_1(\mathcal{Z}, w_0) = \pi_1(\mathbb{T}^2 - \{w_1\}) \cong \mathbb{Z} * \mathbb{Z}$.

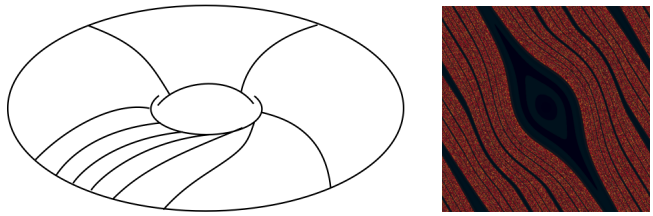

FIGURE 14. Inflating an orbit to obtain a Denjoy flow

As another example, let $\mathcal{F}$ be a codimension-one foliation with an exceptional minimal set $\mathfrak{M} \subset M$. Then $\mathfrak{M}$ has stable shape if and only if the complement $M - \mathfrak{M}$ consists of a *finite* union of connected open saturated subsets. In the shape framework, one of the long-standing open problems for codimension-one foliation theory is then:

PROBLEM 6.3. *Let $\mathcal{F}$ be a codimension-one, C^2foliation of a compact manifold M. Show that an exceptional minimal set $\mathfrak{M}$ for $\mathcal{F}$ must have stable shape.*

More generally, one can ask whether there are other classes of foliations with codimension greater than one, for which the minimal sets are "expected" to have stable shape?

Associated to a shape approximation of a transitive set $\mathcal{Z}$ is another group-like object, defined using the foliate structure of $\mathcal{Z}$. Let $L_0 \subset \mathcal{Z}$ be a dense leaf, and choose a basepoint $w_0 \in L_0$. Let $\epsilon_0 > 0$ be a Lebesgue number for an open cover of M by foliation charts. Define a system of approximating open neighborhoods,

$$V_k = \{x \in M \mid d_M(x, \mathcal{Z}) < \epsilon_0/k\}$$

For $k > 0$, let $\tau_{w_0,z} \colon [0,1] \to L_0$ be a leafwise path such that $d_M(w_0, z) < \epsilon_0/k$. Then τ defines closed path $\widehat{\tau}$ in V_k, obtained by concatenating τ with a short path contained in $B_M(w_0, \epsilon_0/k)$ that joins the endpoints of τ. Define an equivalence relation on such loops by $\widehat{\tau}_0 \sim \widehat{\tau}_1$ if there is a leafwise homotopy τ_t from τ_0 to τ_1 such that $\tau_t(0) = w_0$ and $\tau_t(1) \in B_M(w_0, \epsilon_0/k)$ for all $0 \leq t \leq 1$. The collection of all such loops up to equivalence is denoted by

$$\pi_1^{\mathcal{F}}(V_k, w_0) = \{\widehat{\tau} \mid \widehat{\tau} \sim \widehat{\tau'}\} \tag{8}$$

The sets $\pi_1^{\mathcal{F}}(V_k, w_0)$ do not have a group structure, as concatenation of paths is not necessarily well-defined, unless the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ is equicontinuous. In any case, there are always inclusions $\pi_1^{\mathcal{F}}(V_{k'}, w_0) \subset \pi_1^{\mathcal{F}}(V_k, w_0)$ for $k' > k$.

Associated to each such $\widehat{\tau}$ is the holonomy map h_τ defined by τ which contains w_0 in its domain. The collection of all such holonomy maps define the *shape dynamics* of $\mathcal{Z}$. The idea is simply to consider the holonomy along "almost closed leafwise paths", which has a long tradition in foliation folklore. Shape theory simply adds some additional formal structure to their consideration. This notion is closely associated to the concept of "germinal holonomy" introduced by Timothy Gendron [42, 43]. A related construction has been used by André Haefliger in his study of the isometry groups associated to the holonomy along a fixed leaf of a Riemannian foliation [50].

PROBLEM 6.4. *Given a minimal set $\mathfrak{M}$, what can we saw about the "shape dynamics" of $\mathfrak{M}$?*

For example, if $\mathcal{G}_{\mathcal{F}}$ is equicontinuous, then the shape dynamics are associated to a local isometric group action on $\mathcal{T}$. As another example, if $\mathcal{F}$ is defined by a suspension of a finitely-generated group Γ acting on a compact space N, then the shape dynamics is a form of associated inverse limit action, and the problem asks what aspects of the dynamics of the foliation are captured by this action.

We mention one other aspect of the shape theory for minimal sets, which has precedents in the study of codimension-one foliations and Riemannian foliations. Let $\{V_k \mid \mathfrak{M} \subset V_k\}$ be a system of open shape approximations to a minimal set $\mathfrak{M}$, then there are maps $\iota_k \colon \pi_\ell(V_k, w_0) \to \pi_\ell(M, w_0)$ for each $\ell > 0$, induced by the inclusions $V_k \subset M$. Thus, there are induced maps

$$\iota_* \colon \widehat{\pi}_\ell(\mathcal{Z}, w_0) \to \pi_\ell^{\mathfrak{M}}(M, w_0) \subset \pi_\ell(M, w_0)$$

We formulate another general question:

PROBLEM 6.5. *Given a minimal set $\mathfrak{M}$, how is the image group $\pi_\ell^{\mathfrak{M}}(M, w_0)$ related to the dynamics of $\mathcal{F}$ and the topology of M?*

For the remainder of the section, we discuss the dynamical properties of minimal sets and how these are related to their classification. First, we recall a result mentioned previously:

THEOREM 6.6 (G-L-W 1988)**.** *Let $\mathcal{F}$ be a C^1-foliation $\mathcal{F}$ of codimension $q \geq 1$ on a compact manifold M, and let $\mathfrak{M}$ be a minimal set for $\mathcal{F}$. If the restricted entropy $h(\mathcal{G}_\mathcal{F}, \mathfrak{M}) = 0$, then the action of $\mathcal{G}_\mathcal{F}$ restricted to $\mathfrak{M} \cap \mathcal{T}$ admits an invariant probability measure.*

Thus, if $\mathfrak{M}$ does not admit a transverse invariant measure, then the restricted entropy $h(\mathcal{G}_\mathcal{F}, \mathfrak{M}) > 0$.

Moreover, if $h(\mathcal{G}_\mathcal{F}, \mathfrak{M}) > 0$ then by Proposition 5.9 there exists a transversally hyperbolic invariant probability measure μ_* for the foliation geodesic flow restricted to $\mathfrak{M}$, and hence $\mathfrak{M} \cap \mathbf{H}_\mathcal{F} \neq \emptyset$. One of the main open problems in foliation dynamics is to obtain a partial converse to this.

DEFINITION 6.7. *A minimal set $\mathfrak{M}$ is said to be* hyperbolic *if $\mathfrak{M} \cap \mathbf{H}_\mathcal{F} \neq \emptyset$.*

PROBLEM 6.8. *Let $\mathcal{F}$ be a C^r-foliation $\mathcal{F}$ of codimension $q \geq 1$ on a compact manifold M, and let $\mathfrak{M}$ be a hyperbolic minimal set. Find conditions on $r \geq 1$, the topology of $\mathfrak{M}$, and/or the Hausdorff dimension of $\mathfrak{M} \cap \mathbf{H}_\mathcal{F} \cap \mathcal{T}$ which are sufficient to imply that $h(\mathcal{G}_\mathcal{F}, \mathfrak{M}) > 0$.*

In general, one might expect that generically, the typical minimal set of $\mathcal{F}$ is hyperbolic. There is a family of constructions of foliations for which this is always the case. Let N be a Riemannian manifold of dimension q with metric d_N. Let $X \subset N$ be a convex subset for the metric. A diffeomorphism $f \colon N \to N$ is said to be *contracting on* X if $f(X) \subset X$ and for all $x, y \in X$, we have $d_N(f(x), f(y)) < d_N(x, y)$. Then define

DEFINITION 6.9. *An iterated function system (IFS) on N is a collection of diffeomorphisms $\{f_1, \ldots, f_k\}$ of N and a compact convex subset $X \subset N$ such that each f_ℓ is contracting on X.*

Note that since X is assumed compact, the contracting assumption implies that for each map f_ℓ the norm of its differential Df_ℓ is uniformly less than 1. That is, they are infinitesimal contractions.

The suspension construction [17, 18] yields a foliation $\mathcal{F}$ on a fiber bundle M over a surface of genus k with fiber N, for which the maps $\{f_1, \ldots, f_k\}$ define the holonomy of $\mathcal{F}$. If the manifold N is compact then M will also be compact.

The relevance of this construction is that such a system admits a minimal set $K \subset X$, which is necessarily hyperbolic. In fact, K is the unique minimal set for the restriction of the action to X. Let $\Sigma = \langle f_1, \cdots, f_k \rangle$ be the positive semigroup generated by the generators. Then we have

$$K = K\{f_1, \ldots, f_k\} \;=\; \bigcap_{f \in \Sigma} f(X)$$

The set K is called the *Markov Minimal Set* associated to the IFS (see [11]).

The traditional construction of an IFS is for $N = \mathbb{R}^q$ and the maps f_ℓ are assumed to be affine contractions. The compact convex set X can then be chosen to be any sufficiently large closed ball about the origin in $\mathbb{R}^q$. There is a vast literature on affine IFS's as well as illustrations of the possible invariant sets. We include in Figure 15 below just one random example to give an idea of the complexity of the sets so obtained.

Note that every affine map of $\mathbb{R}^q$ extends to a conformal map of $\mathbb{S}^q$, so these constructions also provide examples of hyperbolic minimal sets for smooth foliations of compact manifolds. However, the construction of affine minimal sets has many generalizations, and leads to a variety of interesting examples, which have not been examined from the foliation point of view.

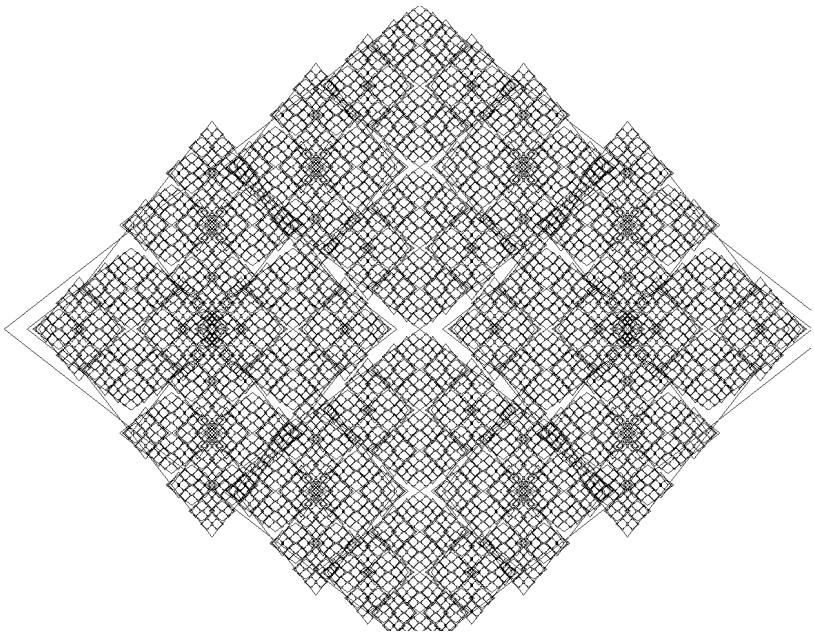

FIGURE 15. Minimal set for an IFS

Finally, we consider the case of minimal sets $\mathfrak{M}$ for which the restricted entropy $h(\mathcal{G}_{\mathcal{F}}, \mathfrak{M}) = 0$. This "zero entropy" case is most interesting, as it includes the solenoidal minimal sets mentioned previously.

DEFINITION 6.10. *A minimal set $\mathfrak{M}$ is said to be* parabolic *if $\mathfrak{M} \cap \mathbf{H}_{\mathcal{F}} = \emptyset$.*

We have seen previously that $h(\mathcal{G}_{\mathcal{F}}, \mathfrak{M}) > 0$ implies $\mathfrak{M} \cap \mathbf{H}_{\mathcal{F}} \neq \emptyset$ so the parabolic minimal sets include the zero entropy case. Also, a minimal set for a foliation for which $\mathcal{G}_{\mathcal{F}}$ acts distally will be parabolic.

Recall that a compact foliation is one for which every leaf is compact.

PROPOSITION 6.11. *Let $\mathcal{F}$ be a C^1-foliation of a compact manifold M, with all leaves of $\mathcal{F}$ compact. Then every leaf of $\mathcal{F}$ is a parabolic minimal set.*

Proof. A compact foliation is clearly distal, so by the proof of Theorem 5.6, we have $\mathbf{H}_{\mathcal{F}} = \emptyset$. □

In general, one can ask for other constructions of parabolic minimal sets of C^1-foliations, about which little is known. We conclude this section with one such construction, which yields minimal sets with the special structure of a generalized solenoid.

An *n-dimensional solenoid* is an inverse limit space

$$\mathcal{S} = \lim_{\leftarrow} \{p_{\ell+1} \colon L_{\ell+1} \to L_\ell\}$$

where for $\ell \geq 0$, L_ℓ is a closed, oriented, n-dimensional manifold, and $p_{\ell+1} \colon L_{\ell+1} \to L_\ell$ are smooth, orientation-preserving proper covering maps. These were introduced by McCord [81]as generalizations of the classical notion of a solenoid defined by covering maps of the circle, where all $L_\ell = \mathbb{S}^1$.

To the author's knowledge, the following problem is completely open, with the exception of the results cited afterwards.

PROBLEM 6.12. *Let $\mathcal{S} = \lim_{\leftarrow} \{p_{\ell+1} \colon L_{\ell+1} \to L_\ell\}$ be an n-dimensional solenoid. When does there exists a C^r-foliation of a compact manifold M for which $\mathcal{S}$ is homeomorphic to a parabolic minimal set of $\mathcal{F}$?*

Theorem 7 of Clark and Fokkink [24] give a partial solution to this question in the case of $r = 0$.

Another partial result is provided by a recent result of Clark and the author [26], which constructs such minimal sets for C^r-foliations, but the construction severely restricts the bonding maps due to the differentiability requirement.

THEOREM 6.13 (Clark-H 2008)**.** *Let $\mathcal{F}_0$ be a C^r-foliation of codimension $q \geq 2$ on a manifold M. Let L_0 be a compact leaf with $H^1(L_0; \mathbb{R}) \neq 0$, and suppose that $\mathcal{F}_0$ is a product foliation in some open neighborhood U of L_0. Then there exists a foliation $\mathcal{F}$ on M which is C^r-close to $\mathcal{F}_0$, and $\mathcal{F}$ has a solenoidal minimal set contained in U with base L_0. If $\mathcal{F}_0$ is a distal foliation, then $\mathcal{F}$ is also distal.*

The method of proof is to build a "solenoidal plug" for the foliation in a tubular foliated neighborhood of the compact leaf L_0. The construction of such plugs follows from a general procedure:

THEOREM 6.14. *Let L_0 be a closed oriented manifold of dimension n, with $H^1(L_0, \mathbb{R}) \neq 0$. Let $q \geq 2$, $r \geq 1$, and $\mathcal{F}_0$ denote the product foliation of $M = L_0 \times \mathbb{D}^q$. Then there exists a C^r-foliation $\mathcal{F}$ of M which is C^r-close to $\mathcal{F}_0$, such that $\mathcal{F}$ is a volume-preserving, distal foliation, and satisfies*

(1) *L_0 is a leaf of $\mathcal{F}$*
(2) *$\mathcal{F} = \mathcal{F}_0$ near the boundary of M*
(3) *$\mathcal{F}$ has a minimal set $\mathcal{S}$ which is a generalized solenoid with base L_0*
(4) *each leaf $L \subset \mathcal{S}$ is a covering of L_0.*

We conclude with the general:

PROBLEM 6.15. *Find constructions and a classification for the parabolic) or even just distal) minimal sets of C^r-foliations when $r \geq 1$.*

When considered as a homework exercise, one should allow several years for its completion. Hints may be contained in the references [28, 39, 74].

7. Some Open Problems

Monday: Characterize the transversally hyperbolic invariant probability measures μ_* for the foliation geodesic flow of a given foliation.

Tuesday: Classify the foliations with subexponential orbit complexity and distal transverse structure.

Wednesday: Find conditions on the geometry of a foliation such that exponential orbit growth implies positive entropy.

Thursday: Find conditions on the Lyapunov spectrum and invariant measures for the geodesic flow which implies positive entropy.

Friday: Characterize the exceptional minimal sets of zero entropy.

A solution (or even a reasonable partial solution) to any of these problems would constitute a fundamental advance in our understanding of foliation dynamical systems.

References

[1] J.M. Aarts and R. Fokkink, *The classification of solenoids*, **Proc. Amer. Math. Soc.**, 111:1161–1163, 1991.

[2] E. Akin, J. Auslander and E. Glasner, **The topological dynamics of Ellis actions**, Mem. Amer. Math. Soc., Vol. 195, 2008.

[3] J. Álvarez López and A. Candel, *Equicontinuous foliated spaces*, **preprint** , 2001.

[4] C. Anantharaman-Delaroche and J. Renault, **Amenable groupoids**, Monographies de L'Enseignement Mathématique, Vol. 36, 2000.

[5] J. Anderson and I. Putnam, *Topological invariants for substitution tilings and their associated C^*-algebras*, **Ergodic Theory Dyn. Syst.**, 18:509–537, 1998.

[6] M. Badura, *Realizations of growth types*, **Ergodic Theory Dynam. Systems**, 25:353–363, 2005.

[7] L. Barreira and Ya.B. Pesin, **Lyapunov exponents and smooth ergodic theory**, University Lecture Series, Vol. 23, American Mathematical Society, Providence, RI, 2002.

[8] R. Benedetti and J.-M. Gambaudo, *On the dynamics of $\mathbb{G}$-solenoids. Applications to Delone sets*, **Ergodic Theory Dyn. Syst.**, 23:673–691, 2003.

[9] M. Bermudez and G. Hector, *Laminations hyperfinies et revêtements*, **Ergodic Theory Dynam. Systems**, 26:305–339, 2006.

[10] A. Bís, S. Hurder, and J. Shive, *Hirsch foliations in codimension greater than one*, In **Foliations 2005**, World Scientific Publishing Co. Inc., River Edge, N.J., 2006: 71–108.

[11] A. Bís and S. Hurder, *Markov minimal sets of foliations*, in preparation, 2010.

[12] C. Bonatti, L. Díaz and M. Viana **Dynamics beyond uniform hyperbolicity**, Encyclopaedia of Mathematical Sciences, Vol. 102, Springer-Verlag, Berlin, 2005.

[13] K. Borsuk, *Concerning homotopy properties of compacta*, **Fund. Math.**, 62:223–254, 1968.

[14] R. Bowen, *Entropy for group endomorphisms and homogeneous spaces*, **Trans. Amer. Math. Soc.**, 153:401–414, 1971.

[15] R. Bowen, *Anosov foliations are hyperfinite*, **Ann. of Math. (2)**, 106:549–565, 1977.

[16] M. Brin and A. Katok, *On local entropy*, In **Geometric dynamics (Rio de Janeiro, 1981)**, Lecture Notes in Math., Vol. 1007, Springer, Berlin, 1983:30–38.

[17] C. Camacho and A. Lins Neto, **Geometric Theory of Foliations**, Translated from the Portuguese by Sue E. Goodman, Progress in Mathematics, Birkhäuser Boston, MA, 1985.

[18] A. Candel and L. Conlon, **Foliations I**, Amer. Math. Soc., Providence, RI, 2000.

[19] J. Cantwell and L. Conlon, *Growth of leaves*, **Comment. Math. Helv.**, 53:93–111, 1978.

[20] J. Cantwell and L. Conlon, *Poincaré-Bendixson theory for leaves of codimension one*, **Transactions Amer. Math. Soc.**, 265:181–209, 1981.

[21] J. Cantwell and L. Conlon, *Analytic foliations and the theory of levels*, **Math. Annalen**, 265:253–261, 1983.

[22] J. Cantwell and L. Conlon, *The dynamics of open, foliated manifolds and a vanishing theorem for the Godbillon-Vey class*, **Advances in Math.**, 53:1–27, 1984.

[23] J. Cantwell and L. Conlon, *Foliations and subshifts*, **Tohoku Math. J.**, 40:165–187, 1988.

[24] A. Clark and R. Fokkink, *Embedding solenoids*, **Fund. Math.**, 181:111–124, 2004.

[25] A. Clark and M. Sullivan, *The linking homomorphism of one-dimensional minimal sets*, **Topology Appl.**, 141:125–145, 2004.

[26] A. Clark and S. Hurder, *Embedding matchbox manifolds*, preprint 2008.

[27] A. Clark and S. Hurder, *Homogeneous matchbox manifolds*, preprint 2010.

[28] A. Clark, S. Hurder and O. Lukina, *Classifying matchbox manifolds*, in preparation, 2010.

[29] A. Connes, J. Feldman and B. Weiss, *An amenable equivalence relation is generated by a single transformation*, **Ergodic Theory Dynamical Systems**, 1:431–450, 1981.

[30] A. Connes, **Noncommutative Geometry**, Academic Press, San Diego, CA, 1994.

[31] P. de la Harpe, *Free groups in linear groups*, **Enseign. Math. (2)**, 29:129–144, 1983.

[32] G. Duminy, *L'invariant de Godbillon–Vey d'un feuilletage se localise dans les feuilles ressort*, **unpublished preprint**, Université de Lille, I, 1982.

[33] G. Duminy, *Sur les cycles feuilletés de codimension un*, **unpublished manuscript**, Université de Lille, I, 1982.

[34] H.A. Dye, *On groups of measure preserving transformation. I*, **Amer. J. Math.**, 81:119–159, 1959.

[35] H.A. Dye, *On groups of measure preserving transformation. II*, **Amer. J. Math.**, 85:551–576, 1963.

[36] S. Egashira, *Expansion growth of foliations*, **Ann. Fac. Sci. Toulouse Math. (6)**, 2:15–52, 1993.

[37] J. Feldman and C.C. Moore, *Ergodic equivalence relations, cohomology, and von Neumann algebras*, **Bull. Amer. Math. Soc.**, 81:921–924, 1975.

[38] F. Fokkink and L. Oversteegen, *Homogeneous weak solenoids*, **Trans. Amer. Math. Soc.**, 354:3743–3755, 2002.

[39] H. Furstenberg, *The structure of distal flows*, **Amer. J. Math.**, 85:477–515, 1963.

[40] J.-M. Gambaudo and C. Tresser, *Diffeomorphisms with infinitely many strange attractors*, **J. Complexity**, 6:409–416, 1990.

[41] J.-M. Gambaudo, D. Sullivan and C. Tresser, *Infinite cascades of braids and smooth dynamical systems*, **Topology**, 33:85–94, 1994.

[42] T.M. Gendron, *The algebraic theory of the fundamental germ*, **Bull. Braz. Math. Soc. (N.S.)**, 37:49–87, 2006.

[43] T.M. Gendron, *The geometric theory of the fundamental germ*, **Nagoya Math. J.**, 190:1–34, 2008.

[44] É. Ghys, *Topologie des feuilles génériques*, **Ann. of Math. (2)**, 141:387–422, 1995.

[45] É. Ghys, R. Langevin, and P. Walczak. Entropie geometrique des feuilletages. *Acta Math.*, 168:105–142, 1988.

[46] É. Ghys and J. Ley, *Lorenz and Modular Flows: A Visual Introduction*, **Feature Column, Notices AMS**, Novembre 2006. Best viewed on the web at *http://www.ams.org/samplings/feature-column/fcarc-lorenz*.

[47] C. Godbillon and J. Vey, *Un invariant des feuilletages de codimension* 1, **C. R. Acad. Sci. Paris Sér. A-B**, 273:A92-A95, 1971.

[48] C. Godbillon, **Feuilletages: Etudes géométriques I, II**, Publ. IRMA Strasbourg (1985-86); Progress in Math., Vol. 98, Birkhäuser, Boston, Mass., 1991.

[49] M. Gromov, *Groups of polynomial growth and expanding maps*, **Inst. Hautes Études Sci. Publ. Math.**, no. 53:53–73, 1981.

[50] A. Haefliger, *Leaf closures in Riemannian foliations* In **A fête of topology**, Academic Press, Boston, MA, 1988:3–32.

[51] A. Haefliger, *Foliations and compactly generated pseudogroups* In **Foliations: geometry and dynamics (Warsaw, 2000)**, World Sci. Publ., River Edge, NJ, 2002:275–295.

[52] G. Hector, *Leaves whose growth is neither exponential nor polynomial*, **Topology**, 16:451–459, 1977.

[53] G. Hector and U. Hirsch, **Introduction to the Geometry of Foliations, Parts A,B**, Vieweg, Braunschweig, 1981.

[54] G. Hector, *Architecture des feuilletages de classe C^2*, **Third Schnepfenried geometry conference, Vol. 1 (Schnepfenried, 1982)**, Astérisque, 107, Société Mathématique de France 1983, 243–258.

[55] J. Heitsch and S. Hurder, *Secondary classes, Weil measures and the geometry of foliations*, **Jour. Differential Geom.**, 20:291–309, 1984.

[56] M. Hirsch, *A stable analytic foliation with only exceptional minimal sets.* in **Dynamical Systems, Warwick, 1974**, Lect. Notes in Math. vol. 468, , Springer-Verlag, 1975, 9–10.

[57] S. Hurder and A. Katok. Ergodic Theory and Weil measures for foliations. *Annals of Math.*, 126:221–275, 1987.

[58] S. Hurder, *The Godbillon measure of amenable foliations*, **Jour. Differential Geom.**, 23:347–365, 1986.

[59] S. Hurder. Ergodic theory of foliations and a theorem of Sacksteder. In *Dynamical Systems: Proceedings, University of Maryland 1986-87*. Lect. Notes in Math. Vol. 1342, pages 291–328, Springer-Verlag, New York and Berlin, 1988.

[60] S. Hurder and A. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows*, **Publ. Math. Inst. Hautes Etudes Sci.**, 72:5–64, 1990.

[61] S. Hurder, *Exceptional minimal sets of $C^{1+\alpha}$ actions on the circle*, **Ergodic Theory and Dynamical Systems**, 11:455-467, 1991.

[62] S. Hurder, *Rigidity for Anosov actions of higher rank lattices*, **Annals of Math.**, 135:361–410, 1992.

[63] S. Hurder, *Entropy and Dynamics of C^1 Foliations*, **preprint**, August 2000. Available at http://www.math.uic.edu/∼hurder/publications/

[64] S. Hurder and R. Langevin, *Dynamics and the Godbillon-Vey Class of C^1 Foliations*, **preprint**, September 2000. Available at http://www.math.uic.edu/∼hurder/publications/

[65] S. Hurder, *Dynamics and the Godbillon-Vey class: a History and Survey*, In **Foliations: Geometry and Dynamics (Warsaw, 2000)**, World Scientific Publishing Co. Inc., River Edge, N.J., 2002:29–60.

[66] S. Hurder, *Classifying foliations*, **Foliations, Geometry and Topology. Paul Schweitzer Festschrift**, (eds. Nicoalu Saldanha et al), Contemp Math. Vol. 498, American Math. Soc., Providence, RI, 2009.

[67] S. Hurder, *Dynamics of Foliations*, Advanced Course on Foliations Centre de Recerca Matemàtica, Barcelona, May 3–7, 2010. Slides posted at *http://www.math.uic.edu/∼hurder/talks/*

[68] S. Hurder, *Dynamics of Foliations*, in preparation, 2010.

[69] V. Kaimanovich, *Equivalence relations with amenable leaves need not be amenable*, In **Topology, ergodic theory, real algebraic geometry**, Amer. Math. Soc. Transl. Ser. 2 202:151–166, 2001.

[70] J. Kennedy and J. Yorke, *Bizarre topology is natural in dynamical systems*, **Bull. Amer. Math. Soc. (N.S.)**, 32:309–316, 1995.

[71] W. Krieger, *On ergodic flows and the isomorphism of factors*, **Math. Ann.** 223:19–70, 1976.

[72] H.B. Lawson, Jr., **The Quantitative Theory of Foliations**, NSF Regional Conf. Board Math. Sci., Vol. 27, 1975.

[73] R. Langevin and P. Walczak, *Entropy, transverse entropy and partitions of unity*, **Ergodic Theory Dynamical Systems**, 14:551–563, 1994.

[74] E. Lindenstrauss, *Measurable distal and topological distal systems*, **Ergodic Theory Dynam. Systems**, 19:1063–1076, 1999.

[75] R. Mañé, **Ergodic theory and differentiable dynamics**, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Vol. 8, Springer-Verlag, Berlin, 1987.

[76] A. Manning, *Topological entropy for geodesic flows*, **Annals of Math.**, 110:567–573, 1979.

[77] S. Mardešić, **Strong Shape and Homology**, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2000.

[78] S. Mardešić and J. Segal, *History of shape theory and its application to general topology*, In **Handbook of the history of general topology, Vol. 3**, Kluwer Acad. Publ., Dordrecht, 2001, 1145–1177.

[79] S. Matsumoto, *Measure of exceptional minimal sets of codimension one foliations*, In **A Fête of Topology**, Academic Press, Boston, 1988, 81–94.

[80] S. Matsumoto, *Minimal C^1-diffeomorphisms of the circle which admit measurable fundamental domains*, preprint, arXiv:1005.0585, May 2010.
[81] C. McCord, *Inverse limit sequences with covering maps*, **Trans. Amer. Math. Soc.**, 114:197–209, 1965.
[82] J. Milnor, *Curvature and growth of the fundamental group*, **Jour. Differential Geom.**, 2:1–7, 1968.
[83] C.C. Moore, *Ergodic theory and von Neumann algebras*, In **Proc. Symp. in Pure Math., Vol 38 Pt. 2** No. 9, American Mathematical Society, Providence, RI, 1982, 179–226.
[84] C.C. Moore and C. Schochet, **Analysis on Foliated Spaces**, Math. Sci. Res. Inst. Publ. vol. 9, Second Edition, Cambridge University Press, New York, 2006.
[85] V.I. Oseledets, *A multiplicative ergodic theorem. Lyanpunov characteristic numbers for dynamical systems*, **Transl. Moscow Math. Soc.**, 19:197–221, 1968.
[86] Ya.B. Pesin, *Characteristic Ljapunov exponents, and smooth ergodic theory*, **Uspehi Mat. Nauk**, 32:55–112, 1977. Transl. **Russian Math. Surveys**, 32(4):55–114, 1977.
[87] A. Phillips and D. Sullivan, *Geometry of leaves*, **Topology** 20:209–218, 1981.
[88] J. Plante, *Foliations with measure-preserving holonomy*, **Ann. of Math.**, 102:327–361, 1975.
[89] A. Rechtman, *Use and disuse of plugs in foliations*, **Thesis**, École Normale Supérieure de Lyon, U.M.P.A. , Lyon, France, February 2009.
[90] K. Richardson, *The transverse geometry of G-manifolds and Riemannian foliations*, **Illinois J. Math.**, 45:517–535, 2001.
[91] R. Sacksteder, *Foliations and pseudogroups*, **Amer. J. Math.**, 87:79–102, 1965.
[92] R. Sacksteder and J. Schwartz, *On the existence of exceptional leaves in foliations of co-dimension one*, **Ann. Inst. Fourier (Grenoble)**, 14:221–225, 1964.
[93] R. Sacksteder, *Limit sets of foliations*, **Ann. Inst. Fourier (Grenoble)**, 15:201–213, 1965.
[94] S. Schwartzman, *Asymptotic cycles*, **Ann. of Math. (2)**, 66:270–284, 1957.
[95] C. Series, *Foliations of polynomial growth are hyperfinite*, **Israel J. Math.**, 34:245–258, 1980.
[96] S. Smale, *Differentiable Dynamical Systems*, **Bull. Amer. Math. Soc.**, 73:747-817, 1967.
[97] D. Sullivan, *A Counterexample to the Periodic Orbit Conjecture*, **Publ. Math. IHES**, 46:5-14, 1976.
[98] J. Tits, *Free subgroups in linear groups*, **J. of Algebra**, 20:250-270, 1972.
[99] N. Tsuchiya, *Growth and depth of leaves*, **J. Fac. Sci. Univ. Tokyo Sect. IA Math.**, 26:465–471, 1979.
[100] N. Tsuchiya, *Lower semicontinuity of growth of leaves*, **J. Fac. Sci. Univ. Tokyo Sect. IA Math.**, 26:473–500, 1979.
[101] N. Tsuchiya, *Leaves of finite depth*, **Japan. J. Math. (N.S.)**, 6:343–364, 1980.
[102] W. Veech, *Point-distal flows*, **Amer. J. Math.**, 92:205–242, 1970.
[103] P. Walczak, *Dynamics of the geodesic flow of a foliation*, **Ergodic Theory Dynamical Systems**, 8:637–650, 1988.
[104] P. Walczak, *Hausdorff dimension of Markov invariant sets*, **Journal Math. Society of Japan**, 48:125–133, 1996.
[105] P. Walczak, **Dynamics of foliations, groups and pseudogroups**, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)], Vol. 64. Birkhäuser Verlag, Basel, 2004.
[106] R.F. Williams, *One-dimensional non-wandering sets*, **Topology**, 6:473–487, 1967.
[107] R.F. Williams, *Expanding attractors*, **Inst. Hautes Études Sci. Publ. Math.**, 43:169–203, 1974.

Steven Hurder, Department of Mathematics, University of Illinois at Chicago, 322 SEO (m/c 249), 851 S. Morgan Street, Chicago, IL 60607-7045

E-mail address: hurder@uic.edu