Entropy and Dynamics of C^1 Foliations

Steven Hurder*
Department of Mathematics (m/c 249), University of Illinois at Chicago
851 S. Morgan Street, Chicago, IL 60607-7045 USA
hurder@uic.edu, http://www.math.uic.edu/∼hurder/

August 15, 2000

1 Introduction

Let M be a compact Riemannian C^∞–manifold with a C^1-foliation $\mathcal{F}$ of codimension q. The study of the dynamics of $\mathcal{F}$ asks for the "typical" qualitative behavior of the leaves of $\mathcal{F}$ as submanifolds. Typical issues of foliation dynamics are the recurrence properties of the leaves, and the more delicate question of the existence of non–trivial transverse invariant measures. For a foliation with one-dimensional leaves defined by a flow $\{\varphi_t\}$, the problem is to study the dynamical properties of the flow which are independent of the time parametrization. When the leaves have higher dimension, then the geometry of the leaves (for example, their branching towards the ends of the leaves) can make the dynamics of the foliation far more complicated than that encountered in the study of flows.

The geometric entropy $h(\mathcal{G}_{\mathcal{F}})$ of a C^1-foliation $\mathcal{F}$ introduced by Ghys-Langevin-Walczak [5] is a measure of the complexity of the dynamics. This is one of the most important dynamical properties of C^1-foliations, and captures essential information about the global transverse and leaf dynamics:

- Ghys, Langevin and Walczak (Theorem 6.1, [5]) showed that if $\mathcal{F}$ is a C^2–foliation of codimension one with $h(\mathcal{G}_{\mathcal{F}}) > 0$, then $\mathcal{F}$ has a resilient leaf.
- Ghys, Langevin and Walczak (Theorem 5.1, [5]) showed that if $\mathcal{F}$ is a C^1-foliation of codimension $q \geq 1$ with $h(\mathcal{G}_{\mathcal{F}}) = 0$, then $\mathcal{F}$ has a non-trivial holonomy invariant transverse measure.
- Attie and Hurder (Theorem 3, [1]) showed that if $\mathcal{F}$ is a C^1-foliation of codimension $q \geq 1$ with $h(\mathcal{G}_{\mathcal{F}}) = 0$, then every leaf of $\mathcal{F}$ must have zero entropy as complete metric space.

In this paper we establish a variety of results relating the geometric entropy of a C^1–foliation with its dynamics, including a new proof of Theorem 6.1, [5] applicable to C^1-foliations.

THEOREM 1.1 *If $\mathcal{F}$ is a transversally C^1-foliation of codimension one with $h(\mathcal{G}_{\mathcal{F}}) > 0$, then $\mathcal{F}$ has a resilient leaf.*

*Supported in part by NSF grant DMS-9704768. Minor corrections made August 31, 2000

The proof of Theorem 1.1 uses methods similar to techniques of ergodic theory and topological dynamics for flows, invoking counting arguments and properties of the foliation geodesic flow. Our proof is fundamentally different that of Theorem 6.1, [5] for C^2–foliations, which used delicate properties of the structure theory of C^2-foliations of codimension one.

As preliminaries to the proof of Theorem 1.1, we establish a number of useful technical results about the dynamics of C^1-foliations in sections 3, 4 and 5. In particular, Theorem 4.1 relates measure theoretic properties of $\mathcal{F}$ with the geometric entropy, and as an application yields:

THEOREM 1.2 *Suppose $\mathcal{F}$ is a C^1-foliation of codimension one on a compact manifold M. If the foliation geodesic flow $\Phi_t: V \to V$ has an ergodic, t-hyperbolic, Φ_t–invariant measure $\mathbf{m}_*$ on V which is not t-discrete, then $h(\mathcal{G}_{\mathcal{F}}) > 0$.*

In § 7 we give some applications of our results for codimension one to the study of foliation dynamics. For example,we show:

THEOREM 1.3 *Suppose $\mathcal{F}$ is a codimension-one C^1-foliation with $h(\mathcal{G}_{\mathcal{F}}) > 0$, then the relative geometric entropy $h(\mathcal{G}_{\mathcal{F}}, \Omega(\mathcal{F})) > 0$ where $\Omega(\mathcal{F})$ is the non-wandering set for $\mathcal{F}$.*

Our method of proof of Theorem 1.1 uses the codimension one hypothesis in several essential ways. In higher codimension, there are few geometric interpretations of $h(\mathcal{G}_{\mathcal{F}}) > 0$. In the last two sections of this paper, we give two results in this direction.

THEOREM 1.4 *For $\mathcal{F}$ a C^{1+a}-foliation of arbitrary codimension with $h(\mathcal{G}_{\mathcal{F}}) > 0$, then there exists a leafwise geodesic path $\gamma: (-\infty, \infty) \to L \subset M$ so that the transverse holonomy along γ admits a stable transverse manifold which is (transversally) attracted to L at an exponential rate.*

The proof of this result uses the standard Pesin Theory [14, 20, 21, 15], and it is interesting to compare its proof with that of Theorem 1.1. It would be very desirable to extend Theorem 1.4 to conclude from $h(\mathcal{G}_{\mathcal{F}}) > 0$ that there exists (partially) hyperbolic fixed–points for the holonomy of $\mathcal{F}$. Such a higher codimension "closing lemma" would almost certainly require additional hypotheses on the supports of partially hyperbolic Φ_t-invariant measures for the foliation geodesic flow Φ_t.

There is a natural extension of the notion of a distal group action to foliations (cf. § 1 below). Theorem 1.4 implies:

COROLLARY 1.5 *Let $\mathcal{F}$ be C^{1+a}-foliation with $h(\mathcal{G}_{\mathcal{F}}) \neq 0$. Then $\mathcal{F}$ is not distal.*

It was asked in § 7 of [5] whether a foliation with all leaves compact must have zero entropy? A foliation with all leaves compact is distal, so we have:

COROLLARY 1.6 *If $\mathcal{F}$ is a C^{1+a}-foliation with all leaves compact, then $h(\mathcal{G}_{\mathcal{F}}) = 0$.*

Thanks are owed to Larry Conlon, Etienne Ghys, Remi Langevin, Takashi Tsuboi and Pawel Walczak with whom the author has had from numerous technical and philosophical discussions about foliation dynamics and entropy. Our main theorems above can be considered as a partial realization of the program for the study of the dynamics of C^1-foliations outlined in [10]. The reader may find the companion paper [12] useful, where related ideas are applied to study the dynamics of groups of C^1-diffeomorphisms of the circle. The proofs there are often technically much simpler.

2 Basic Foliation Dynamics

Throughout this paper, will assume that M is a compact, orientable, smooth Riemannian manifold. For simplicity of later estimates, we assume that the Riemannian metric on M has been normalized so that M has diameter 1. We also assume that $\mathcal{F}$ is a codimension–q, C^1-foliation with orientable normal bundle, and that the leaves of $\mathcal{F}$ are smoothly immersed submanifolds of dimension p. This is sometimes referred to as a $C^{1,\infty}$-foliation. In this section we introduce a number of standard notions of foliation structure theory and dynamics. More details and discussion of basic foliation theory can be found in the text "Foliations. I" by Alberto Candel and Lawrence Conlon [3].

2.1 Regular foliation atlas

A *regular foliation atlas* for $\mathcal{F}$ is a finite collection $\{(U_\alpha, \phi_\alpha) \mid \alpha \in \mathcal{A}\}$ so that:

1. $\mathcal{U} = \{U_\alpha \mid \alpha \in \mathcal{A}\}$ is a covering of M by coordinate charts $\phi_\alpha : U_\alpha \to (-1,1)^n$
2. Each coordinate chart $\phi_\alpha : U_\alpha \to (-1,1)^n$ admits an extension to a coordinate chart $\widetilde{\phi}_\alpha : \widetilde{U}_\alpha \to (-2,2)^n$ where $\widetilde{U}_\alpha$ is convex in M and contains the closure of the open set U_α
3. For each $z \in (-2,2)^q$, the preimage $\widetilde{\mathcal{P}}_\alpha(z) = \widetilde{\phi}_\alpha^{-1}((-2,2)^p \times \{z\}) \subset \widetilde{U}_\alpha$ is the connected component containing $\widetilde{\phi}_\alpha^{-1}(\{0\}\times\{z\})$ of the intersection of the leaf of $\mathcal{F}$ through $\phi_\alpha^{-1}(\{0\}\times\{z\})$ with the set $\widetilde{U}_\alpha$. Moreover, we assume that $\widetilde{\mathcal{P}}_\alpha(z)$ is convex subset for the induced Riemannian metric, where each pair of points $x, y \in \widetilde{\mathcal{P}}_\alpha(z)$ is joined by a unique geodesic segment in $\mathcal{P}_\alpha(z)$.

Note that the convexity hypotheses (2.1.2) and (2.1.3) imply if $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \neq \emptyset$, then each plaque $\widetilde{\mathcal{P}}_\alpha(z)$ intersects exactly one plaque of $\widetilde{U}_\beta$. The reader interested in the details of the construction of regular coverings and their properties should consult Chapter 1.2 of [3].

The inverse images

$$\mathcal{P}_\alpha(z) = \phi_\alpha^{-1}((-1,1)^p \times \{z\}) \subset U_\alpha$$

are smoothly embedded discs contained in the leaves of $\mathcal{F}$, called the *plaques* associated to the given foliation atlas. One thinks of the collection of all plaques as "tiling stones" which cover the leaves in a regular fashion. The convexity hypotheses in (3) implies that an intersection of plaques $\mathcal{P}_{\alpha_1}(z_1) \cap \cdots \cap \mathcal{P}_{\alpha_d}(z_d)$ is either empty, or a convex set.

For each $\alpha \in \mathcal{A}$, the extended chart $\widetilde{\phi}_\alpha$ defines a smooth embedding

$$t_\alpha = \phi_\alpha^{-1}(\{0\} \times \cdot) : (-2,2)^q \to \widetilde{U}_\alpha \subset M$$

whose image is denoted by $\widetilde{\mathcal{T}}_\alpha$. We will also assume that these images $\widetilde{\mathcal{T}}_\alpha$ are pairwise disjoint; this can be achieved by a small perturbation of the coordinate charts if necessary. We can also assume that each submanifold $\widetilde{\mathcal{T}}_\alpha$ is everywhere perpendicular to the leaves of $\mathcal{F}$ by adjusting the given Riemannian metric on M in an open tubular neighborhood of each $\widetilde{\mathcal{T}}_\alpha$. We may assume that each $\mathcal{T}_\alpha$ has diameter at most 1. Define $\mathcal{T}_\alpha = \phi_\alpha^{-1}(\{0\} \times (-1,1)^q)$. The local coordinate on $\mathcal{T}_\alpha$ is again denoted by $t_\alpha : (-1,1)^q \to \mathcal{T}_\alpha$. We use this coordinate to identify each transversal $\mathcal{T}_\alpha$ with $(-1,1)^q$.

The collection of all plaques for the foliation atlas is indexed by the *complete transversal*

$$\mathcal{T} = \bigcup_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$$

For a point $x \in \mathcal{T}$, by a mild abuse of language we let $\mathcal{P}_\alpha(x)$ denote the plaque containing x.

Given a subset $\mathcal{Z} \subset U_\alpha$ let $\mathcal{Z}_\mathcal{P}$ denote the union of all plaques in U_α having non-empty intersection with $\mathcal{Z}$. We set $\mathcal{Z}_\mathcal{T} = \mathcal{Z}_\mathcal{P} \cap \mathcal{T}_\alpha$. If $\mathcal{Z}$ is an open set, then $\mathcal{Z}_\mathcal{P}$ and $\mathcal{Z}_\mathcal{T}$ are also open.

The Riemannian metric on M induces a Riemannian metric and corresponding distance function $\mathbf{d}_\mathcal{T}$ on each transversal $\widetilde{\mathcal{T}}_\alpha$. For $\alpha \neq \beta$ and $x \in \mathcal{T}_\alpha$, $y \in \mathcal{T}_\beta$ we set $\mathbf{d}_\mathcal{T}(\mathbf{x}, \mathbf{y}) = \infty$. Given $r > 0$ and $x \in \widetilde{\mathcal{T}}_\alpha$ let $\mathbf{B}_\mathcal{T}(x, r) = \{y \in \widetilde{\mathcal{T}}_\alpha | d_\mathcal{T}(x, y) < r\}$.

Let $\epsilon_\mathcal{U} > 0$ be the Lebesgue number for the covering $\mathcal{U}$. That is, for every $x \in M$ there is an index $\alpha \in \mathcal{A}$ so that the ball $\mathbf{B}(x, \epsilon_\mathcal{U}) \subset U_\alpha$.

Let $\epsilon_0 > 0$ be the greatest number so that for all U_α in the regular foliation atlas and for all $x \in U_\alpha$ then $\mathbf{B}_\mathcal{T}(x_\mathcal{T}, \epsilon_0) \subset (B(x, \epsilon_\mathcal{U}) \cap U_\alpha)_\mathcal{T}$ where $x_\mathcal{T} = x_\mathcal{P} \cap \mathcal{T}_\alpha$ (or equivalently, $x \in \mathcal{P}_\alpha(x_\mathcal{T})$.) We call ϵ_0 the t–Lebesgue number of the regular foliation atlas.

2.2 The holonomy pseudogroup

A pair of indices (α, β) is *admissible* if $U_\alpha \cap U_\beta \neq \emptyset$. For each admissible pair (α, β) define

$$\mathcal{T}_{\alpha\beta} = \{x \in \mathcal{T}_\alpha \text{ such that } \mathcal{P}_\alpha(x) \cap U_\beta \neq \emptyset\}.$$

Then there is a well-defined transition function $\mathbf{h}_{\beta\alpha} \colon \mathcal{T}_{\alpha\beta} \to \mathcal{T}_{\beta\alpha}$, which for $x \in \mathcal{T}_{\alpha\beta}$ is given by

$$\mathbf{h}_{\beta\alpha}(x) = y \text{ where } \mathcal{P}_\alpha(x) \cap \mathcal{P}_\beta(y) \neq \emptyset$$

Note that $\mathbf{h}_{\alpha\alpha} : \mathcal{T}_\alpha \to \mathcal{T}_\alpha$ is the identity map for each $\alpha \in \mathcal{A}$.

The *holonomy pseudogroup* $\mathcal{G}_\mathcal{F}$ associated to the regular foliation atlas for $\mathcal{F}$ is the pseudogroup with object space $\mathcal{T}$, and transformations generated by compositions of the local transformations $\{\mathbf{h}_{\beta\alpha} \mid (\alpha, \beta) \text{ admissible}\}$. The holonomy pseudogroup $\mathcal{G}_\mathcal{F}$ depends upon the choice of regular foliation atlas for $\mathcal{F}$, but two such atlases yield Morita equivalent groupoids [6].

The C^1-hypothesis on $\mathcal{F}$ implies that each map $\mathbf{h}_{\beta\alpha}$ is C^1 for the local coordinates

$$t_\alpha : (-1, 1)^q \to \mathcal{T}_\alpha \text{ and } t_\beta : (-1, 1)^q \to \mathcal{T}_\beta$$

Moreover, the hypothesis (2) on regular foliation charts implies that each $\mathbf{h}_{\beta\alpha}$ admits an extension to a C^1-map $\widetilde{\mathbf{h}}_{\beta\alpha} : \widetilde{\mathcal{T}}_{\alpha\beta} \to \widetilde{\mathcal{T}}_{\alpha\beta}$ defined in a similar fashion. Thus, $\mathbf{h}_{\beta\alpha}$ is uniformly C^1 on its domain, and we note that its domain also satisfies a uniformity condition:

LEMMA 2.1 *There exists $\epsilon > 0$ so that for every admissible pair (α, β) and $x \in \mathcal{T}_{\alpha\beta}$ then the closure $\overline{\mathbf{B}_\mathcal{T}(x, \epsilon)} \subset \widetilde{\mathcal{T}}_{\alpha\beta}$.* □

We say the foliation $\mathcal{F}$ is transversally C^{1+a}, for some $0 < a < 1$, if the regular foliation atlas can be chosen so that each of the transition functions $\widetilde{\mathbf{h}}_{\beta\alpha} \colon \widetilde{\mathcal{T}}_{\alpha\beta} \to \widetilde{\mathcal{T}}_{\beta\alpha}$ is C^1 with a uniform a-Hölder estimate on its first derivatives.

Composition of elements in $\mathcal{G}_\mathcal{F}$ will be defined via "'plaque chains". Given $x, y \in \mathcal{T}$ on the same leaf, a plaque chain of length k between them is a collection of plaques

$$\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$$

where $x_0 = x$, $x_k = y$ and for each $0 \leq i < k$ we have $\mathcal{P}_{\alpha_i}(x_i) \cap \mathcal{P}_{\alpha_{i+1}}(x_{i+1}) \neq \emptyset$. A plaque chain $\mathcal{P}$ also defines an "extended" plaque chain for the charts $\{(\widetilde{U}_\alpha, \widetilde{\phi}_\alpha)\}$,

$$\widetilde{\mathcal{P}} = \{\widetilde{\mathcal{P}}_{\alpha_1}(x_0), \ldots, \widetilde{\mathcal{P}}_{\alpha_k}(x_k)\}$$

We say two plaque chains

$$\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\} \text{ and } \mathcal{Q} = \{\mathcal{P}_{\beta_0}(y_0), \ldots, \mathcal{P}_{\beta_\ell}(y_\ell)\}$$

are *composable* if $x_k = y_0$, hence $\alpha_k = \beta_0$ and $\mathcal{P}_{\alpha_k}(x_k) = \mathcal{P}_{\beta_0}(y_0)$). Their composition is defined by

$$\mathcal{Q} \circ \mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k), \mathcal{P}_{\beta_1}(y_1), \ldots, \mathcal{P}_{\beta_\ell}(y_\ell)\}$$

The holonomy transformation defined by a plaque chain is the local diffeomorphism

$$\mathbf{h}_{\mathcal{P}} = \mathbf{h}_{\alpha_k \alpha_{k-1}} \circ \cdots \circ \mathbf{h}_{\alpha_1 \alpha_0}$$

whose domain $\mathcal{D}_{\mathcal{P}} \subset \mathcal{T}_{\alpha_0}$ contains x_0. Note that $\mathcal{D}_{\mathcal{P}}$ is the largest connected open subset of $\mathcal{T}_{\alpha_0}$ containing x_0 on which $\mathbf{h}_{\alpha_\ell \alpha_{\ell-1}} \circ \cdots \circ \mathbf{h}_{\alpha_1 \alpha_0}$ is defined for all $0 < \ell \leq k$. The dependence of the domain of $\mathbf{h}_{\mathcal{P}}$ on the plaque chain $\mathcal{P}$ is a subtle issue, yet is at the heart of the technical difficulties arising in the study of foliation pseudogroups. Given composable plaque chains $\mathcal{P}$ and $\mathcal{Q}$, the local transformations satisfy $\mathbf{h}_{\mathcal{Q}\circ\mathcal{P}} = \mathbf{h}_{\mathcal{Q}} \circ \mathbf{h}_{\mathcal{P}}$ on the domain of the composition $\mathbf{h}_{\mathcal{Q}\circ\mathcal{P}}$ which is typically a proper subset of that of $\mathbf{h}_{\mathcal{P}}$.

We similarly let $\widetilde{\mathbf{h}}_{\widetilde{\mathcal{P}}}$ be the holonomy associated to the chain $\widetilde{\mathcal{P}}$, with domain $\widetilde{\mathcal{D}}_{\widetilde{\mathcal{P}}} \subset \widetilde{\mathcal{T}}_{\alpha_0}$ the largest maximal open subset containing x_0 on which $\widetilde{\mathbf{h}}_{\alpha_\ell \alpha_{\ell-1}} \circ \cdots \circ \widetilde{\mathbf{h}}_{\alpha_1 \alpha_0}$ is defined for all $1 < \ell \leq k$. By the extension property of a regular atlas, the closure $\overline{\mathcal{D}_{\mathcal{P}}} \subset \widetilde{\mathcal{D}}_{\widetilde{\mathcal{P}}}$ and $\widetilde{\mathbf{h}}_{\widetilde{\mathcal{P}}}$ is an extension of $\mathbf{h}_{\mathcal{P}}$.

Given a plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ and a point $y \in \mathcal{D}_{\mathcal{P}}$, there is a "parallel" plaque chain denoted $\mathcal{P}(y) = \{\mathcal{P}_{\alpha_0}(y), \ldots, \mathcal{P}_{\alpha_k}(y_k)\}$ where $\mathbf{h}_{\mathcal{P}}(y) = y_k$.

2.3 Leafwise path holonomy

A *leafwise path* γ is a piecewise C^∞ map $\gamma : [0, T] \to M$ whose image is contained in a leaf of $\mathcal{F}$.

Let $d_{\mathcal{U}}$ denote the maximal diameter in the leafwise metric of all plaques $\mathcal{P}_\alpha(x)$ for all $x \in \mathcal{T}_\alpha$ and $\alpha \in \mathcal{A}$.

Each plaque chain $\mathcal{P}$ defines a leafwise path $\gamma_{\mathcal{P}}$ from x_0 to x_k by concatenating the shortest unit-speed leafwise–geodesic segments joining x_{i-1} to x_i for $1 \leq i \leq k$. Let $0 < d_{min} < d_{max}$ denote the minimum and maximum lengths of all shortest leafwise geodesic segments joining points of distinct transversals $\mathcal{T}_\alpha$ and $\mathcal{T}_\beta$, where (α, β) is admissible. Note that $d_{max} \leq 2d_{\mathcal{U}}$. Given a plaque chain $\mathcal{P}$ of length k, the length of the leafwise path $\gamma_{\mathcal{P}}$ from x_0 to x_k is thus bounded by

$$k \cdot d_{min} \leq \|\gamma_{\mathcal{P}}\| \leq k \cdot d_{max} \tag{1}$$

Conversely, each leafwise path defines a leafwise plaque chain $\mathcal{P}_\gamma$. Choose an index α_0 so that the ball $B(\gamma(0), \epsilon_{\mathcal{U}}) \subset U_{\alpha_0}$, and let x_0 be the point of $\mathcal{T}_{\alpha_0}$ whose plaque contains $\gamma(0)$. Let $t_1 > 0$ be the least time so that $\gamma(t_1) \notin U_{\alpha_0}$, then choose α_1 with $B(\gamma(t_1), \epsilon_{\mathcal{U}}) \subset U_{\alpha_1}$ and let $x_1 \in \mathcal{T}_{\alpha_1}$ be defined by the plaque containing $\gamma(t_1)$. Continue in this way until we obtain $B(\gamma(T), \epsilon_{\mathcal{U}}) \subset U_{\alpha_k}$ with $x_k \in \mathcal{T}_{\alpha_k}$. $\mathcal{P}_\gamma$ is the plaque chain from x_0 to x_k determined by this sequence of foliation charts.

Given a leafwise path γ and choices of open sets $\gamma(0) \in U_{\alpha_0}$ and $\gamma(T) \in U_{\alpha_k}$, the germ of the local holonomy map $\mathbf{h}_\gamma$ from $x_0 \in \mathcal{T}_{\alpha_0}$ to $x_k \in \mathcal{T}_{\alpha_k}$ is well-defined. However, the domain of the pseudogroup element $\mathbf{h}_\gamma$ depends upon the choice of the open sets $\{U_{\alpha_0}, \ldots, U_{\alpha_k}\}$ covering γ. Our choices of open sets U_{α_i} so that they contain disks $B(\gamma(t_i), \epsilon_{\mathcal{U}})$ ensures a certain "maximality" of this domain for $\mathbf{h}_\gamma$.

For each $x \in M$ and $B(x, \epsilon_{\mathcal{U}}) \subset U_\alpha$, let $\mathcal{P}_\alpha(z_x)$ be the plaque of U_α containing x, then note that the radius of the "ball" $B(x, \epsilon_{\mathcal{U}}) \cap \mathcal{P}_\alpha(z_x)$ in the leafwise metric is at least $\epsilon_{\mathcal{U}}$. Thus, if we apply the above plaque chain construction to γ, a unit–speed leafwise–geodesic which is length-minimizing in its homotopy class rel $\{x_1, x_k\}$, then $t_i - t_{i-1} \geq \epsilon_{\mathcal{U}}$ and hence

$$\|\gamma\|/d_{\mathcal{U}} \leq k \leq \|\gamma\|/\epsilon_{\mathcal{U}} \tag{2}$$

2.4 The derivative cocycle

Let $\{e_1, \ldots, e_q\}$ denote the standard basis of $\mathbf{R^q}$ corresponding to the coordinate axis $\{x_1, \ldots, x_q\}$. For $x \in \widetilde{\mathcal{T}}_\alpha$, the local coordinate $t_\alpha : (-2, 2)^q \to \widetilde{\mathcal{T}}_\alpha$ induces the *standard basis* of $\mathrm{T_x}\widetilde{\mathcal{T}}_\alpha$ denoted by $\{e_1(x), \ldots, e_q(x)\}$.

Given a plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ from $x = x_0$ to $y = x_k$, the derivative map

$$\mathbf{h}'_{\mathcal{P}}(x) : \mathrm{T_x}\widetilde{\mathcal{T}}_{\alpha_1} \longrightarrow \mathrm{T_y}\widetilde{\mathcal{T}}_{\alpha_k}$$

defines a matrix $D\mathbf{h}_{\mathcal{P}}(x) \in GL(q, R)$ expressing $\mathbf{h}'_{\mathcal{P}}(x)$ in terms of the standard bases. Given composable plaque chains $\mathcal{P}$ and $\mathcal{Q}$, with $x = x_0, y = x_k = y_0, z = y_\ell$ then by the chain rule

$$D\mathbf{h}_{\mathcal{Q}\circ\mathcal{P}}(x) = D\mathbf{h}_{\mathcal{Q}}(y) \cdot D\mathbf{h}_{\mathcal{P}}(x) \tag{3}$$

The map $D\mathbf{h} : \mathcal{G}_{\mathcal{F}} \to GL(q, R)$ defined by $D\mathbf{h}(\mathcal{P}, y) = D\mathbf{h}_{\mathcal{P}(y)}(y)$ is a cocycle over the groupoid, and called naturally enough, the derivative cocycle.

2.5 Resilient leaves and "ping-pong games"

A plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ is *closed* if $x_0 = x_k$. A closed plaque chain $\mathcal{P}$ defines a local diffeomorphism $\mathbf{h}_{\mathcal{P}} : \mathcal{D}_{\mathcal{P}} \to \mathcal{T}_\alpha$ with $\mathbf{h}_{\mathcal{P}}(x) = x$, where $x = x_1 \in \mathcal{T}_\alpha$.

A point $y \in \mathcal{D}_{\mathcal{P}}$ is said to be asymptotic to x if $\mathbf{h}^\ell_{\mathcal{P}}(y) \in \mathcal{D}_{\mathcal{P}}$ for all $\ell > 0$ (where $\mathbf{h}^\ell_{\mathcal{P}}$ denotes the composition of $\mathbf{h}_{\mathcal{P}}$ with itself ℓ times) and the iterates $\lim_{\ell\to\infty} \mathbf{h}^\ell_{\mathcal{P}}(y) \to x$. The map $\mathbf{h}_{\mathcal{P}}$ is said to be a contraction at x if there is some $\delta > 0$ so that every $y \in \mathbf{B}_{\mathcal{T}}(x, \delta)$ is asymptotic to x.

The map $\mathbf{h}_{\mathcal{P}}$ is said to be a hyperbolic contraction at x if the matrix $D\mathbf{h}(\mathcal{P}, x)$ is a linear contraction. By the continuity of $D\mathbf{h}(\mathcal{P}, y)$ for $y \in \mathcal{D}_{\mathcal{P}}$, there exists $\epsilon > 0$ so that $D\mathbf{h}(\mathcal{P}, y)$ is a linear contraction for all $\mathbf{y} \in \mathbf{B}_{\mathcal{T}}(\mathbf{x}, \epsilon)$. It follows that every point of $\mathbf{B}_{\mathcal{T}}(x, \epsilon)$ is asymptotic to x, and moreover, there exists $0 < \delta < \epsilon$ so that the image of the closed δ–ball about x satisfies

$$\mathbf{h}_{\mathcal{P}}(\overline{\mathbf{B}_{\mathcal{T}}(x, \delta)}) \subset \mathbf{B}_{\mathcal{T}}(x, \delta)$$

DEFINITION 2.2 *A point $x \in \mathcal{T}$ is resilient for $\mathcal{G}_{\mathcal{F}}$ if there exists*

1. *a closed plaque chain $\mathcal{P} = \{\mathcal{P}_{\alpha_0}(x_0), \ldots, \mathcal{P}_{\alpha_k}(x_k)\}$ with $x = x_0$*
2. *a point $y \in \mathcal{D}_{\mathcal{P}}$ which is asymptotic to x (and $y \neq x$)*
3. *a plaque chain $\mathcal{Q}$ from x to y.*

If, in addition, $\mathbf{h}_{\mathcal{P}}$ is a hyperbolic contraction at x, then we say x is a hyperbolic resilient point.

Resilient leaves are typically defined only for codimension one foliations, but the definition above makes sense in general. For codimension one, one has the added property that if $y \in \mathcal{D}_{\mathcal{P}}$ is asymptotic to x then $\mathbf{h}_{\mathcal{P}}$ is a one-sided contraction at x. (That is, there is an open half interval $[x, y) \subset \mathcal{T}_\alpha$ consisting of points asymptotic to x.) The existence of a resilient point $x \in \mathcal{T}$ is well-known to imply that the dynamics of $\mathcal{G}_{\mathcal{F}}$ is non-trivial on the closure of the orbit of x. The existence of a hyperbolic resilient point has further consequences, as the hyperbolicity guarantees control of the dynamics of the map $\mathbf{h}_{\mathcal{P}}$ near x.

We next recall a dynamical notion which was been colloquially called a "ping-pong game" by de la Harpe [7], though the concept dates from the work of Klein, and it has many uses in the study of dynamical systems.

DEFINITION 2.3 *The groupoid $\mathcal{G}_{\mathcal{F}}$ has a hyperbolic "ping-pong game" if there exists $x, y \in \mathcal{T}_\alpha$ with $x \neq y$ and*

1. *a closed plaque chain $\mathcal{P}$ such that $\mathbf{h}_{\mathcal{P}}$ is a hyperbolic contraction at $x = x_0$*
2. *a closed plaque chain $\mathcal{Q}$ such that $\mathbf{h}_{\mathcal{Q}}$ is a hyperbolic contraction at $y = y_0$*
3. *$y \in \mathcal{D}_{\mathcal{P}}$ is asymptotic to x by $\mathbf{h}_{\mathcal{P}}$ and $x \in \mathcal{D}_{\mathcal{Q}}$ is asymptotic to y by $\mathbf{h}_{\mathcal{Q}}$*

The hyperbolic contraction hypotheses is stronger than necessary for some applications (cf. [12]) but in this paper we always assume a "ping-pong game" includes the hyperbolic assumption. Let us note the relation between "ping-pong games" and hyperbolic resilient orbits:

PROPOSITION 2.4 *$\mathcal{G}_{\mathcal{F}}$ has a "ping-pong game" if and only if it has a hyperbolic resilient point.*

Proof: Assume that $\mathcal{G}_{\mathcal{F}}$ has a "ping-pong game" with notation as above. Then there exists $\epsilon > 0$ so that every point of $\mathbf{B}_{\mathcal{T}}(x, \epsilon)$ is asymptotic to x. Choose $\mu \gg 0$ so that $\mathbf{h}_{\mathcal{P}}^{\mu}(y) \in \mathbf{B}_{\mathcal{T}}(x, \epsilon)$ and choose $\delta > 0$ so that

$$\mathbf{h}_{\mathcal{P}}^{\mu}(\mathbf{B}_{\mathcal{T}}(y, \delta)) \subset \mathbf{B}_{\mathcal{T}}(x, \epsilon)$$

Next, chose $\nu \gg 0$ so that $\mathbf{h}_{\mathcal{Q}}^{\nu}(x) \in \mathbf{B}_{\mathcal{T}}(y, \delta)$. It follows that $\mathbf{h}_{\mathcal{P}}^{\mu} \circ \mathbf{h}_{\mathcal{Q}}^{\nu}(x) \in \mathbf{B}_{\mathcal{T}}(x, \epsilon)$ and thus x is a hyperbolic resilient point, with the plaque chain $\mathcal{P}^{\mu} \circ \mathcal{Q}^{\nu}$ joining x to a point in the domain of the contraction $\mathbf{h}_{\mathcal{P}}$.

Next, assume that $\mathcal{G}_{\mathcal{F}}$ has a hyperbolic resilient point with notation as above. Let $\mathcal{P}$ be the closed plaque chain at x such that $D\mathbf{h}(\mathcal{P}, x)$ is a linear contraction and $\{\mathbf{h}_{\mathcal{P}}^{\ell}(y) \mid \ell > 0\}$ is a sequence asymptotic to x. Let $\mathcal{R}$ be a plaque chain from x to y. By the continuity of $z \mapsto D\mathbf{h}(\mathcal{P}, z)$, there exists $\epsilon > 0$ so that $D\mathbf{h}(\mathcal{P}, z)$ is a linear contraction for all $z \in \mathbf{B}_{\mathcal{T}}(x, \epsilon)$. It follows that there exists $\delta > 0$ so that for $\ell \gg 0$ the image of the closed δ–ball about y satisfies

$$\mathbf{h}_{\mathcal{P}}^{\ell}(\overline{\mathbf{B}_{\mathcal{T}}(y, \delta)}) \subset \mathbf{B}_{\mathcal{T}}(x, \epsilon)$$

Then for $\ell \gg 0$, the concatenation of plaque chains $\widetilde{\mathcal{Q}} = \mathcal{R} \circ \mathcal{P}^{\ell}$ defines a transformation

$$\mathbf{h}_{\mathcal{R}} : \overline{\mathbf{B}_{\mathcal{T}}(y, \delta)} \longrightarrow \mathbf{B}_{\mathcal{T}}(y, \delta)$$

whose derivative $D\mathbf{h}(\mathcal{R}, z)$ is a contraction for all $z \in \overline{\mathbf{B}_{\mathcal{T}}(y, \delta)}$. It follows that there is a unique hyperbolic fixed-point $y_0 \in \overline{\mathbf{B}_{\mathcal{T}}(y, \delta)}$. Let $\mathcal{Q}$ be the plaque chain parallel to $\widetilde{\mathcal{Q}}$ and starting at y_0. As $x \in \mathbf{B}_{\mathcal{T}}(x, \epsilon)$, we have that x is asymptotic to y_0. By choice, every point of $\overline{\mathbf{B}_{\mathcal{T}}(y, \delta)}$ is asymptotic to x. Thus, we have exhibited a "ping-pong game" for $\mathcal{G}_{\mathcal{F}}$. $\square$

In our proof of Theorem 1.1 we will find it more direct to show there exists a ping-pong table for the dynamics, and thus by Lemma 2.4 is follows there is a hyperbolic resilient orbit.

2.6 Geometric entropy

Given $\epsilon > 0$ and an integer $N > 0$, we say that $x, y \in \mathcal{T}$ are (N, ϵ)–separated if either $x \in \mathcal{T}_\alpha$ and $y \in \mathcal{T}_\beta$ belong to distinct transversals, or there exists a plaque chain $\mathcal{P}$ of length at most N so that $x, y \in D_\mathcal{P}$ and $\mathbf{d}_\mathcal{T}(\mathbf{h}_\mathcal{P}(x), \mathbf{h}_\mathcal{P}(y)) > \epsilon$. If $x, y \in \mathcal{T}_\alpha$ and $\mathbf{d}_\mathcal{T}(\mathbf{x}, \mathbf{y}) > \epsilon$ then we can take $\mathcal{P} = \{\mathcal{P}_\alpha, \mathcal{P}_\alpha\}$ to be the trivial plaque chain with $\mathbf{h}_\mathcal{P} = \mathbf{Id}_{\mathcal{T}_\alpha}$ the identity map, and x, y are (N, ϵ)–separated for all $N \geq 0$. We say that a finite subset $\{x_1, \ldots, x_\nu\} \subset \mathcal{T}$ is (N, ϵ)–separated if for every $k \neq \ell$ the pair of points $x_k, x_\ell \in \mathcal{T}$ is (N, ϵ)–separated.

Let $\mathcal{S}(\mathcal{G}_\mathcal{F}, \epsilon, N)$ denote the maximum cardinality of an (N, ϵ)-separated subset of $\mathcal{T}$. As M is compact and the foliation atlas is regular, this is a finite number. Now define

$$h(\mathcal{G}_\mathcal{F}, \epsilon) = \limsup_{N \to \infty} \frac{\log \mathcal{S}(\mathcal{G}_\mathcal{F}, \epsilon, N)}{N} \tag{4}$$

The *geometric entropy* of Ghys, Langevin and Walczak [5] is the limit

$$h(\mathcal{G}_\mathcal{F}) = \lim_{\epsilon \to 0} \; h(\mathcal{G}_\mathcal{F}, \epsilon)$$

This limit is finite for a transversally C^1-foliation [5]. See Chapter 13, [3] for further properties of this number $h(\mathcal{G}_\mathcal{F})$. In general, $h(\mathcal{G}_\mathcal{F})$ depends upon the choice of the regular foliation atlas, though it is independent of the choice of the Riemannian metric on M. A key point is that the dichotomy $h(\mathcal{G}_\mathcal{F}) > 0$ or $h(\mathcal{G}_\mathcal{F}) = 0$ is independent of the choice of the atlas. We say that $\mathcal{F}$ has positive geometric entropy if $h(\mathcal{G}_\mathcal{F}) > 0$ for some regular foliation atlas.

Given a subset $\mathcal{K} \subset \mathcal{T}$ we can also define the *relative geometric entropy* $h(\mathcal{G}_\mathcal{F}, \mathcal{K})$ where we define $\mathcal{S}(\mathcal{G}_\mathcal{F}, \mathcal{K}, \epsilon, N)$ using subsets $\{x_1, \ldots, x_\nu\} \subset \mathcal{K}$, and the remainder of the definitions follow the same pattern. For example,

$$h(\mathcal{G}_\mathcal{F}) = \sup_{\alpha \in \mathcal{A}} h(\mathcal{G}_\mathcal{F}, \mathcal{T}_\alpha)$$

Thus, given $h(\mathcal{G}_\mathcal{F}) > 0$ there is some transversal $\mathcal{T}_\alpha$ with $h(\mathcal{G}_\mathcal{F}, \mathcal{T}_\alpha) = h(\mathcal{G}_\mathcal{F}) > 0$.

2.7 Distal foliations

The pseudogroup $\mathcal{G}_\mathcal{F}$ is said to be *distal* [9] if for every $\alpha \in \mathcal{A}$ and pair of points $x, y \in \mathcal{T}_\alpha$ with $x \neq y$ then there exists $\epsilon(x, y) > 0$ such that for each holonomy transformation $\mathbf{h}_\mathcal{P}$, if $x, y \in \mathcal{D}_\mathcal{P}$ then $d_\mathcal{T}(\mathbf{h}_\mathcal{P}(x), \mathbf{h}_\mathcal{P}(y)) \geq \epsilon(x, y)$.

If $\mathcal{G}_\mathcal{F}$ has a ping-pong game in its dynamics, then clearly $\mathcal{G}_\mathcal{F}$ is not distal.

A foliation $\mathcal{F}$ is *distal* if $\mathcal{G}_\mathcal{F}$ is distal for some regular foliation atlas.

2.8 Foliation geodesic flow

The compactness of M has been used to obtain a finite regular atlas for $\mathcal{F}$ and thus uniform estimates on the generators of the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$. Analyzing the dynamical consequences of $h(\mathcal{G}_{\mathcal{F}}) > 0$ requires introducing recurrence properties of the plaque chains for $\mathcal{G}_{\mathcal{F}}$. In theory, this can be done using a "coding scheme" from the index set $\mathcal{A}$, but a conceptually and technically easier method is to use the Riemannian manifold structure on M to introduce the *foliation geodesic flow* [10, 11, 23] which will allow us to apply techniques from the ergodic theory of flows.

Let $V = T_1\mathcal{F}_A$ be the $\mathbf{S}^{q-1}$-sphere bundle of unit vectors in TM which are tangent to the leaves of $\mathcal{F}$. A typical point $(x, v) \in V$ consists of a basepoint $x \in M$ and a unit tangent vector $v \in T_1M_x$. The fibration projection $\pi : V \to M$, $\pi(x, v) = x$, pulls $\mathcal{F}$ back to a foliation denoted by $\widehat{\mathcal{F}}$, whose leaves $\widehat{L}$ are the unit tangent bundles to the leaves of $\mathcal{F}$. The Riemannian metric on TM induces a metric on the bundle TV, and we let d_V denote the resulting distance function.

For $x \in M$, let L_x denote the leaf of $\mathcal{F}$ through x, and endow TL_x with the restricted Riemannian metric from TM. Given a unit vector $v \in T_1L_x$, we can form the geodesic $\gamma_{(x,v)}(t)$ in the complete Riemannian manifold L_x, defined for all $t \in \mathbf{R}$. Note that the curve $\gamma_{(x,v)} : \mathbf{R} \to L_x \subset M$ is not necessarily a geodesic for the metric on TM.

The *foliation geodesic flow* , $\Phi : V \times \mathbf{R} \to V$, associated to the geodesic spray vector field on V is characterized by the condition that $\gamma_{(x,v)}(t) = \pi(\Phi(x, v, t))$ is the leafwise geodesic starting at x with initial velocity v, and $(y, w) = \Phi(x, v, t)$ where $\gamma_{(x,v)}(t) = y$ and $\gamma'_{(x,v)}(t) = w$. (cf. [10, 23]). For a C^1-foliation, the flow Φ is C^1. If $\mathcal{F}$ is transversally C^{1+a},then Φ is C^{1+a}. For fixed t, we write $\Phi_t(x, v) = \Phi(x, v, t)$ to emphasize this is a map of V to itself.

Each curve $t \mapsto \Phi(x, v, t)$ is contained in a leaf of $\widehat{\mathcal{F}}$, hence the flow Φ_t maps the leaves of $\widehat{\mathcal{F}}$ into themselves for all t.

Let $\mathbf{Q} \to V$ denote the normal bundle to $T\widehat{\mathcal{F}}$, and identify $\mathbf{Q}$ with $T\widehat{\mathcal{F}}^{\perp}$ using the induced Riemannian metric on V. For a point $x \in \mathcal{T}_\alpha$, the tangent space $T_x\mathcal{T}_\alpha$ inherits a Riemannian metric from TM so that $T_x\mathcal{T}_\alpha$ is naturally isometric to $\mathbf{Q}_z$ for each $z \in V$ with $\pi(z) = x$.

The differential of Φ_t preserves $\mathbf{Q}$ for each t, and we let $D\Phi_t : \mathbf{Q} \to \mathbf{Q}$ denote the induced action on the bundle of normal vectors to $\widehat{\mathcal{F}}$, where

$$D\Phi_t(x, v) \colon T\widehat{\mathcal{F}}^{\perp}_{(x,v)} \to T\widehat{\mathcal{F}}^{\perp}_{\Phi_t(x,v)}$$

Suppose that $x \in \mathcal{T}_\alpha$, $y \in \mathcal{T}_\beta$, and $\Phi(x, v, s) = (y, w)$. We define a leafwise path $\gamma(t) = \pi(\Phi(x, v, t)) : [0, s] \to M$ from x to y, which induces a local diffeomorphism $\mathbf{h}_\gamma$ from its domain $\mathcal{D}_\gamma \subset \mathcal{T}_\alpha$ to an open subset of $\mathcal{T}_\beta$. One then has

$$D\mathbf{h}_\gamma = D\Phi_s$$

This is an exercise in the definitions, as the left hand side is a matrix, while the right-hand-side is a linear transformation on vector spaces of normal vectors to $\mathcal{F}$, but the assumption that x, y lie in $\mathcal{T}$ allows the identification of $D\Phi(x, v, s)$ with a matrix, and the equality follows by the fact that the holonomy $\mathbf{h}_\gamma$ along the path γ has a well-defined germ about x, so can be germinally represented by the flow on Φ on the local leaf spaces, which are identified with the components of the transversal $\mathcal{T}$.

2.9 Flow invariant measures

A probability measure $\mathbf{m}$ on V, viewed as a linear functional on $C^0(V)$, is Φ–invariant if for all $g \in C^0(V)$ and any $t \in \mathbf{R}$, $\mathbf{m}(g) = \mathbf{m}(g \circ \Phi_{-t})$. The measure $\mathbf{m}$ admits an extension to the Borel functions on V, and we define the measure of a Borel subset $K \subset V$ as $\mathbf{m}(K) = \mathbf{m}(\chi_K)$.

We say that $\mathbf{m}$ has support in $K \subset V$ if $\mathbf{m}(g) = 0$ for all g vanishing on K. The support of $\mathbf{m}$, denoted by $|\mathbf{m}|$, is the intersection of all closed sets K such that $\mathbf{m}$ has support in K. The support $|\mathbf{m}|$ is Φ–invariant, and is a non-empty compact set if $\mathbf{m}$ is not the zero measure.

The t-support of $\mathbf{m}$ is

$$|\mathbf{m}|_{\mathcal{T}} = \pi(|\mathbf{m}|)_{\mathcal{P}} \cap \mathcal{T}$$

Given a plaque $\mathcal{P}_\alpha(x)$, we let $\widehat{\mathcal{P}}_\alpha(x) = \pi^{-1}(\mathcal{P}_\alpha(x))$ denote its unit tangent bundle, considered as a subset of V. Then

$$|\mathbf{m}| \subset \pi^{-1}(|\mathbf{m}|_{\mathcal{T}}) = \bigcup_{z \in |\mathbf{m}|_{\mathcal{T}}} \widehat{\mathcal{P}}_{\alpha_z}(z)$$

A Φ–invariant measure $\mathbf{m}$ is said to be t–discrete if there is a finite collection of plaques $\{\mathcal{P}_{\beta_1}(y_1), \ldots, \mathcal{P}_{\beta_k}(y_k)\}$ (not necessarily a plaque chain) so that $|\mathbf{m}|_{\mathcal{T}} = \{y_1, \ldots, y_k\}$. By abuse of notation, we will say that $\mathbf{m}$ is supported on the set $\{\mathcal{P}_{\beta_1}(y_1), \ldots, \mathcal{P}_{\beta_k}(y_k)\}$.

Let $\mathbf{m}$ be a t–discrete measure. For each $x \in \pi(|\mathbf{m}|)$ there is an index α_x for which $\mathbf{B}(x, \epsilon_{\mathcal{U}}/2) \subset U_{\alpha_x}$, and $y_x \in \mathcal{T}_{\alpha_x}$ with $x \in \mathcal{P}_{\alpha_x}(y_x)$. We will assume the plaques $\{\mathcal{P}_{\beta_1}(y_1), \ldots, \mathcal{P}_{\beta_k}(y_k)\}$ covering $\pi(|\mathbf{m}|)$ are chosen so that each point $y_i \in \mathcal{T}_{\beta_i}$ has a neighborhood of diameter at least $\epsilon_{\mathcal{U}}$. Note that the set $\pi(|\mathbf{m}|)_{\mathcal{P}}$ is not necessarily $\mathcal{F}$-saturated.

Given a t-discrete measure $\mathbf{m}$, for each $(x, v) \in |\mathbf{m}|$ the leafwise geodesic $\gamma_{(x,v)}$ determines a closed plaque chain as in § 2.3. As the set $|\mathbf{m}|$ is Φ-invariant, the orbit of $\pi \circ \gamma_{(x,v)}$ is contained in $\pi(|\mathbf{m}|)_{\mathcal{P}}$, so the plaque chain can be chosen a subset of $\{\mathcal{P}_{\beta_1}(y_1), \ldots, \mathcal{P}_{\beta_k}(y_k)\}$. Thus, we can assume $|\mathbf{m}|$ has support in a finite union of closed plaque chains. For example, a closed leafwise geodesic γ gives rise to a t-discrete measure $\mathbf{m}$. However, the t-discrete hypothesis does not imply that the geodesic orbit $\gamma_{(x,v)}(t)$ is periodic. For example, the union of the plaques in the set $\pi(|\mathbf{m}|)_{\mathcal{P}}$ may form a compact leaf L of $\mathcal{F}$, so the plaques $\{\mathcal{P}_{\beta_1}(y_1), \ldots, \mathcal{P}_{\beta_k}(y_k)\}$ form a covering of L by coordinate charts, and the closed plaque chains correspond to elements of $\pi_1(L)$ defined in terms of chains or open neighborhoods of L. The typical geodesic on L is not periodic, but is a covered by compositions of these generating plaque chains.

At the other extreme from t-discrete, if $|\mathbf{m}|_{\mathcal{T}}$ is an uncountable set we say $\mathbf{m}$ has uncountable t-support. A probability measure $\mathbf{m}$ is said to be transversally non-atomic if $\mathbf{m}(\widehat{\mathcal{P}}_\alpha(x)) = 0$ for every plaque $\mathcal{P}_\alpha(x)$. Clearly, such a measure must be t-uncountable. If the set $|\mathbf{m}|_{\mathcal{T}}$ has positive Hausdorff dimension, then again it must be t-uncountable.

Let $\mathcal{M}$ denote the space of Φ–invariant probability measures, and $\mathcal{M}_e \subset \mathcal{M}$ the subspace of ergodic measures. Given a measure $\mathbf{m}$ the ergodic decomposition of $\mathbf{m}$ expresses it as an integral

$$\mathbf{m} = \int_{\mathcal{M}_e} d\mathbf{m}$$

Note that if $\mathbf{m}_*$ is an ergodic measure appearing in the ergodic decomposition of a Φ–invariant measure $\mathbf{m}$, then the support $|\mathbf{m}_*|$ is contained in the support $|\mathbf{m}|$.

3 Stable manifolds and hyperbolic periodic orbits

Assume that $\mathcal{F}$ is a codimension one, C^1-foliation. In this section we establish some of the the basic tools needed for studying the C^1-dynamics of $\mathcal{F}$. We show the existence of "transverse stable manifolds" along t-hyperbolic paths in the leaves, and apply this to prove the existence of hyperbolic contractions for the holonomy group $\mathcal{G}_{\mathcal{F}}$. We define t–hyperbolic Φ_t–invariant measures on V, and show that every such ergodic measure yields a hyperbolic contraction in its support.

The techniques of this section all have counterparts in the Pesin theory for non-uniformly hyperbolic $C^{1+\alpha}$-flows [20, 21, 14]. In our situation, we study the dynamics of the flow Φ_t relative to the invariant foliation $\widehat{\mathcal{F}}$ which complicates the proofs, while the codimension one hypothesis greatly simplifies them. Also, we require only *qualitative* results for this paper, as opposed to the usual quantitative estimates of Pesin theory, which also simplifies the proofs.

Define an additive cocycle $\nu\colon \mathbf{R} \times V \to \mathbf{R}$ over the flow Φ_t by $\nu((x,v),t) = \log\{D\Phi_t(x,v)\}$. By definition, $\exp\{\nu((x,v),s)\}$ is the transverse logarithmic expansion of $\mathcal{F}$ along the geodesic segment $\{\pi(\Phi_t(x,v)) \mid 0 \leq t \leq s\}$. The chain rule implies that ν satisfies the additive cocycle relation over the flow Φ,

$$\nu((x,v), s+t) = \nu(\Phi_t(x,v), s) + \nu((x,v),t) \tag{5}$$

The infinitesimal logarithmic transverse expansion along the geodesic flow is the continuous function $\varphi : V \to \mathbf{R}$ defined by

$$\varphi(x,v) = \frac{d}{dt}\log\{D\Phi_t(x,v)\}\ |_{t=0}$$

Clearly, $\nu((x,v),s) = \int_0^s \varphi(\Phi_t(x,v))\ dt$. By the cocycle property, $\nu(\Phi_t(x,v),-t) = -\nu((x,v),t)$ for all $(x,v) \in V$ and all t. Hence, $\varphi(x,-v) = -\varphi(x,v)$ for all $(x,v) \in V$.

Define $\|\varphi\| = \max\{|\varphi(x,v)| \mid (x,v) \in V\}$.

3.1 t-hyperbolicity and regular values

A continuous leafwise curve $\gamma\colon [a,b] \to M$ is *piecewise-geodesic* if there exist times $a = t_0 < t_1 < \cdots < t_N$ so that for all $0 \leq k < N$ the path $\{\gamma(t) \mid t_k \leq t \leq t_{k+1}\}$ is a leafwise geodesic segment. Thus, there exists points $(x_k, v_k) \in V$ with $x_k = \gamma(t_k)$ and $\pi(\Phi_t(x_k,v_k)) = \gamma(t+t_k)$ for $0 \leq t \leq t_{k+1} - t_k$. We let $\widehat{\gamma}(t)$ denote the curve in V obtained by concatenating the Φ_t flow segments $\{\Phi_t(x_k,v_k) \mid 0 \leq t < t_{k+1} - t_k\}$, so that $\pi(\widehat{\gamma}(t)) = \gamma(t)$. For example, as noted in § 2.3, every plaque chain gives rise to a piecewise geodesic curve. We also consider piecewise geodesic curves $\gamma\colon [a,\infty) \to M$ where now there can exist an infinite number of "corners" $a = t_0 < t_1 < t_2 < \cdots$

The logarithmic expansion along a piecewise geodesic curve $\gamma\colon [a,b] \to M$ is defined by $\nu(\gamma) = \int_a^b \varphi(\widehat{\gamma}(t))\ dt$, and also set $\lambda(\gamma) = \frac{1}{b-a}\int_a^b \varphi(\widehat{\gamma}(t))\ dt$. A piecewise geodesic curve $\gamma\colon [a,\infty) \to M$ is said to be *t–hyperbolic with exponent* $\lambda(\gamma)$ if

$$\lambda(\gamma) = \lim_{s\to\infty} \frac{1}{s}\int_a^s \varphi(\widehat{\gamma}(t))\ dt \neq 0$$

If $\lambda(\gamma) < 0$ then γ is said to be t–stable, and t–unstable if $\lambda(\gamma) > 0$.

Suppose that a piecewise geodesic curve $\gamma\colon [a, \infty) \to M$ is t–stable. For $\epsilon > 0$, we say that $s_0 \geq a$ is *ϵ-regular* if for all $s > s_0$

$$\int_{s_0}^{s} \{\varphi(\widehat{\gamma}(t)) + \epsilon\}\, dt < 0 \tag{6}$$

LEMMA 3.1 *Suppose that a piecewise geodesic curve $\gamma\colon [a, \infty) \to M$ is t–stable with expansion $\lambda(\gamma) < 0$. Then for $0 < \epsilon < -\lambda(\gamma)$ there exists ϵ-regular values $\{s_1, s_2, \ldots\}$ with $s_n \to \infty$.*

Proof : The limit of the integral

$$\frac{1}{s} \int_{a}^{s} \{\varphi(\widehat{\gamma}(t)) + \epsilon\}\; dt \tag{7}$$

equals $\lambda(\gamma) + \epsilon < 0$, so there exists a greatest value $s_1 \geq a$ such that (7) equals 0. Clearly, s_1 is ϵ-regular. Assume that regular values $\{s_1, \ldots, s_n\}$ have been chosen with $s_k \geq k$. Then define s_{n+1} to be the largest value of $s \geq s_n + 1$ such that

$$\frac{1}{s} \int_{s_n+1}^{s} \{\varphi(\widehat{\gamma}(t)) + \epsilon\}\; dt = 0. \quad \square$$

We can ask whether the regular values have a uniform distribution. Such results in general require that the piecewise geodesic curve $\gamma\colon [a, \infty) \to M$ be a generic orbit for an ergodic Φ_t-invariant measure. For our applications, it suffices to consider the periodic case: a ray γ is periodic if there exists a $T_\gamma > 0$ such that $\gamma(t + T_\gamma) = \gamma(t)$ for all $t \geq a$. We also consider a sequence of times $\{s_0, s_1, \ldots, s_N\}$ where $a = s_0 < s_1 < \cdots < s_N = a + T_\gamma$ so that $\gamma(t + s_k) = \pi(\Phi_t(x_k, v_k))$ is a smooth geodesic segment on each interval $0 \leq t \leq s_{k+1} - s_k$. For $1 \leq k \leq N$ and $\epsilon > 0$ set

$$\mu_k = \int_{0}^{s_k - s_{k-1}} \{\varphi(\Phi_t(x_k, v_k)) + \epsilon\}\; dt$$

so that $(\lambda(\gamma) + \epsilon)T_\gamma = (\mu_1 + \cdots + \mu_N)$. Note that each $|\mu_k| \leq (\|\varphi\| + \epsilon)(s_k - s_{k-1})$.

Extend the finite sequence $\{\mu_1, \ldots, \mu_N\}$ to an infinite periodic sequence

$$\{\mu_1, \ldots, \mu_N, \mu_1, \ldots, \mu_N, \ldots\}$$

and let $S_\ell(n)$ denote the sum of the terms from ℓ to n, where $1 \leq \ell \leq N$ and $\ell \leq n < \infty$. The value s_ℓ is ϵ-good if and only if $S_\ell(n) < 0$ for all $n \geq \ell$.

Set $c = \max\{(|\lambda(\gamma)| - \epsilon)/\|\varphi\|(s_k - s_{k-1}) \mid 1 \leq k \leq N\}$.

LEMMA 3.2 *Let $0 < \epsilon < -\lambda(\gamma)$. Then at least cN of the values in $\{s_0, s_1, \ldots, s_{N-1}\}$ are ϵ-good.*

Proof : Since $\lim\limits_{n\to\infty} S_\ell(n)/n = \lambda(\gamma) + \epsilon < 0$ there exists $0 \leq \ell$ such that s_ℓ is ϵ-good. Since γ is periodic, if $\ell > N$ then $s_{\ell-N}$ is also ϵ-good, so we can assume $1 \leq \ell \leq N$. Also, we can assume without loss that it is s_0 which is ϵ-good by changing the starting point on γ to s_ℓ.

Now, if $\mu_N < 0$ then s_{N-1} is also ϵ-good. Let k be the greatest value with $k \leq N$ such that $\mu_k \geq 0$, so that $\{s_{k+1}, \ldots, s_N\}$ are all ϵ-good. A sequence of values $\{s_i, s_{i+1}, \ldots, s_k\}$ is *ϵ-bad* if $\mu_{i+1} + \cdots \mu_{k+1} \geq 0$. Consider the least $i \leq k$ such that $\{s_i, s_{i+1}, \ldots, s_k\}$ is ϵ-bad. Then $\mu_i < 0$ and it is an exercise that s_{i-1} is ϵ-good. We continue in this way to group the values $\{s_0, s_1, \ldots, s_{N-1}\}$ into ϵ-good values and ϵ-bad values. Clearly, the sum of the ϵ-bad values in an ϵ-bad sequence increase the sum $S_\ell(n)$, while the ϵ-good values decrease this sum. Since for an ϵ-good value s_ℓ we have $|\mu_{\ell+1}| \leq (\|\varphi\| - \epsilon)(s_{\ell+1} - s_\ell)$ there must exists at least cN ϵ-good values in order that $\mu_1 + \cdots + \mu_N = (\lambda(\gamma) + \epsilon)T_\gamma$. $\quad \square$

3.2 Stable manifolds

The construction stable manifolds is primarily based on using the derivative of a holonomy map to estimate its action on nearby points. A critical technical step is to establish uniform estimates on the derivative itself. Before giving the construction of stable manifolds, we present in detail the definition of a monotone increasing function $\epsilon_1 \colon (0,\infty) \to (0,\infty)$ which serves as a modulus of continuity in the proofs of this section, and in the next two sections as well. The definition ϵ_1 is technical, so we start with an intuitive interpretation.

Suppose that $\gamma \colon [s_0, s_*] \to M$ is a leafwise piecewise geodesic curve. We require that all of the corners of γ lie in $\mathcal{T}$. Recall the construction of the holonomy transformation $\mathbf{h}_\gamma$ from in § 2.3. Implicit in the definition of $\mathbf{h}_\gamma$ is the use of a collection of short geodesic segments in the plaques $\mathcal{P}_{\alpha_k}(z_k)$ covering γ. The point $z_0 \in \mathcal{T}_{\alpha_0}$ is first connected to the initial point $x_0 = \gamma(s_0)$ via a geodesic segment σ_0 in the plaque $\mathcal{P}_{\alpha_0}(z_0)$. Note that σ_0 has length at most $d_{\mathcal{U}}$. Next, choose s_1 with $\gamma(s_1) \in \mathcal{P}_{\alpha_k}(z_1)$ so that $\{\gamma(t) \mid s_0 \leq t \leq s_1\} \subset \mathcal{P}_{\alpha_0}(z_0) \cup \mathcal{P}_{\alpha_1}(z_1)$. If possible, choose $\gamma(s_1)$ to be a corner. Connect z_1 to x_1 via a geodesic segment σ_1 in the plaque $\mathcal{P}_{\alpha_1}(z_1)$. Note that by the convexity of the plaques, the shortest geodesic segment joining z_0 to z_1, which we call τ_1, is contained in the union $\mathcal{P}_{\alpha_0}(z_0) \cup \mathcal{P}_{\alpha_1}(z_1)$. Repeat this process along the length of γ, until we get to the last plaque where $x_N = \gamma(z_*)$ is connected to z_N by a geodesic segment σ_N. The hypothesis on corners implies that each of the curves $\gamma_k = \{\gamma(t) \mid s_k \leq t \leq s_{k+1}\}$ is a leafwise geodesic segment.

The geodesic segments $\{(\sigma_k, \tau_k, \sigma_{k+1}, \gamma_k) \mid 0 \leq k < N\}$ satisfy $\gamma = \gamma_0 * \gamma_1 * \cdots * \gamma_{N-1}$ and

$$\tau = \tau_0 * \tau_1 * \cdots * \tau_{N-1} = (\sigma_0 * \gamma_0 * \sigma_1^{-1}) * (\sigma_1 * \gamma_1 * \sigma_2^{-1}) * \cdots * (\sigma_{N-1} * \gamma_{N-1} * \sigma_N^{-1}) \tag{8}$$

The holonomy $\mathbf{h}_\gamma$ is by definition the compositions of the holonomies along the paths τ_k which is the composition of the holonomies along the segments appearing on the right hand side of (8). Thus, to estimate the behavior of the map $\mathbf{h}_\gamma$, it will suffice to estimate the holonomies along the curves γ_k. The estimates are all based on the function φ, so we consider the lifts of all the segments in (8) lifted to Φ_t-flow segments in V. The function ϵ_1 is chosen so that given $\delta > 0$ then $\epsilon_1(\delta)$ is chosen so that that the transverse derivative along γ satisfies a uniform estimate up to δ in an $\epsilon_1(\delta)$-neighborhood of γ. The details follow.

Recall that $\epsilon_0 > 0$ is the t-Lebesgue number of the given regular foliation atlas, and $d_{\mathcal{U}}$ denotes the maximal diameter in the leafwise metric of all plaques $\mathcal{P}_\alpha(x)$ for all $x \in \mathcal{T}_\alpha$ and $\alpha \in \mathcal{A}$.

The function φ is continuous on the compact space V, so given $\delta > 0$ there exists $0 < \epsilon_2(\delta)$ such that for $d((y,w),(y',w')) < \epsilon_2(\delta)$ then $|\varphi(y,w) - \varphi(y',w')| < \delta$. Clearly, we can assume $\epsilon_2(\delta)$ is a monotone increasing function of δ.

Choose a monotone increasing function $0 < \epsilon_3(\delta) < \epsilon_0$ so that for all $(y,w),(y',w') \in V$ and all $-4d_{\mathcal{U}} \leq t \leq 4d_{\mathcal{U}}$ then

$$d_V((y,w),(y',w')) < \epsilon_3(\delta) \Rightarrow d_V(\Phi_t(x,v), \Phi_t(y,w)) < \min\{\epsilon_0, \epsilon_2(\delta)\} \tag{9}$$

Finally, the Riemannian metric g_M on M restricts to a Riemannian metric on each foliation chart $\mathcal{U}_\beta$ which is pulled-back via ϕ_β to a Riemannian metric g_β on $(-1,1)^n$. The tensor g_β can be expressed as symmetric matrix-valued function with respect to the standard Euclidean basis of $(-1,1)^n$, and the norm of $g_\beta(u,v)$ is the maximum of the matrix norms of $g_\beta(u,v)$ and $g_\beta(u,v)^{-1}$. Let $\|g_\beta\|$ be the supremum of all such pointwise norms on $(-1,1)^n$, and let $\|g\|$ be the supremum of the norms $\{\|g_\beta\| \mid \beta \in \mathcal{A}\}$. This is finite as the covering of M by foliation charts is finite.

Set $\epsilon_1(\delta) = \max\{\epsilon_3(\delta), \epsilon_3(\delta)/\|g\|^2\}$. Note that for all $(y,z) \in (-1,1)^{n-1} \times (-1,1)$ the curve $t \mapsto \widetilde{\phi}_{\alpha_0}^{-1}(y, z+t) \in V$ for $-\epsilon_1(\delta) < t < \epsilon_1(\delta)$ has length at most $\epsilon_3(\delta)/\|g\| \leq \epsilon_3(\delta)$.

After these technical preliminaries, we give an application.

THEOREM 3.3 *Let $\gamma\colon [a,\infty) \to M$ be a t-stable, piecewise geodesic curve whose corners lie in $\mathcal{T}$ with expansion $\lambda = \lambda(\gamma) < 0$. Set $\epsilon_1 = \epsilon_1(\lambda/10)$, and for $\epsilon = 9\lambda/10$ let $s_0 \geq a$ an ϵ-regular value. Then there exists $z \in \mathcal{T}_\alpha$ with $\gamma(s_0) \in \mathcal{P}_\alpha(z)$ and $\mathcal{I}_x = (z - \epsilon_1, z + \epsilon_1) \subset \mathcal{T}_\alpha$ so that for all $s_* \geq s_0$, the holonomy transformation $\mathbf{h}_*$ defined by the curve $\gamma_* = \{\gamma(t) \mid s_0 \leq t \leq s_*\}$ is defined on $\mathcal{I}_x$. Moreover, for $s_* \gg s_0$ the transformation $\mathbf{h}_*$ is a contraction satisfying*

$$0 < \mathbf{h}_*'(y) < \exp\{(s_1 - s_0) \cdot \lambda(\gamma)/2\} \quad \text{for all} \quad y \in \mathcal{I}_x \tag{10}$$

Proof: Choose a leafwise path chain covering γ_* as in § 2.3, $\mathcal{P}_{\gamma_*} = \{\mathcal{P}_{\alpha_0}(z_0), \ldots, \mathcal{P}_{\alpha_N}(z_N)\}$, which defines the holonomy transformation $\mathbf{h}_*$. We show that $(z_0 - \epsilon_1, z_0 + \epsilon_1)$ is contained in the domain of $\mathbf{h}_*$.

First, $\phi_{\alpha_0}^{-1}(0 \times (z_0 - \epsilon_1, z_0 + \epsilon_1)) \subset \mathcal{T}_{\alpha_0}$ has length at most $\epsilon_3 < \epsilon_0$ so that $\mathbf{h}_{\alpha_1\alpha_0}$ is defined on $(z_0 - \epsilon_1, z_0 + \epsilon_1)$. The image $\mathbf{h}_{\alpha_1\alpha_0}(z_0 - \epsilon_1, z_0 + \epsilon_1) = (z_1 - \epsilon', z_1 + \epsilon'') \subset \mathcal{T}_{\alpha_1}$ for some $\epsilon', \epsilon'' > 0$. As remarked previously, the holonomy $\mathbf{h}_{\alpha_1\alpha_0}$ is the composition of the holonomy along σ_0^{-1}, followed by the holonomy along γ_0, then along σ_1. The image of $(z_0 - \epsilon_1, z_0 + \epsilon_1)$ under the holonomy along σ_0^{-1}, which is just a "coordinate slide" in $\mathcal{U}_{\alpha_0}$, has the form $\mathcal{I}_0 = \{\phi_{\alpha_0}^{-1}(y_0, z_0 + t) \mid -\epsilon_1 < t < \epsilon_1\} \subset \mathcal{U}_{\alpha_0}$ so that its length is again at most $\epsilon_3(\delta)/\|g\| \leq \epsilon_3(\delta)$. The second image has the form $\mathcal{I}_1 = \{\phi_{\alpha_1}^{-1}(y_1, z_1 + t) \mid -\epsilon' < t < \epsilon''\} \subset \mathcal{U}_{\alpha_1}$.

The points of $\mathcal{I}_{\alpha_0}$ are joined to the points of $\mathcal{I}_{\alpha_1}$ by geodesic segments in $\mathcal{U}_{\alpha_0} \cup \mathcal{U}_{\alpha_1}$ where the path joining $\phi_{\alpha_0}^{-1}(y_0, z_0) = \gamma(0)$ to $\phi_{\alpha_1}^{-1}(y_1, z_1) = \gamma(s_1)$ is just the geodesic segment γ_0. By the choice of ϵ_3 and (9), all of these segments lie within an ϵ_2 neighborhood of γ_0. By the choice of ϵ_2 and the mean value theorem, the transverse separation of these geodesic segments is estimated by

$$\epsilon_3/\|g\| \cdot \exp\left\{\int_{s_0}^{s_1} \{\varphi(\widehat{\gamma}_0(t)) - \lambda/10\}\, dt\right\} < \epsilon_3/\|g\| \cdot \exp\{8(s_1 - s_0)\lambda/10\} < \epsilon_3/\|g\| \tag{11}$$

hence $(z_1 - \epsilon', z_1 + \epsilon'')$ has length at most ϵ_3. We can thus apply $\mathbf{h}_{\alpha_1\alpha_2}$ to the image of $\mathbf{h}_{\alpha_1\alpha_0}$.

The above argument is repeated almost verbatim for $\mathbf{h}_{\alpha_2\alpha_0} = \mathbf{h}_{\alpha_2\alpha_1} \circ \mathbf{h}_{\alpha_1\alpha_0}$, except that now the estimate (11) involves an integral from $s_0 \leq t \leq s_2$. Continue inductively to obtain that $\mathbf{h}_\gamma$ is defined on $(z_0 - \epsilon_1, z_0 + \epsilon_1)$.

The proof above also shows that $\mathbf{h}_*'$ satisfies an estimate

$$\exp\{(s_* - s_0)\lambda\}/\|g\|^2 \;\leq\; \mathbf{h}_*' \;\leq\; \|g\|^2 \cdot \exp\{8(s_* - s_0)\lambda/10\} \tag{12}$$

so that for $s_* \gg s_0$ the map $\mathbf{h}_\gamma'$ is a hyperbolic contraction satisfying (10). $\square$

3.3 Hyperbolic fixed-points

Theorem 3.3 is applied to show the existence of holonomy transformations with hyperbolic fixed-points when there is suitable recurrence for the flow Φ_t.

Given a Φ_t–invariant measure $\mathbf{m}$ we set $\Lambda(\mathbf{m}) = \mathbf{m}(\varphi) = \int_V \varphi\; d\mathbf{m}$. The diffeomorphism $(x, v) \mapsto (x, -v)$ of V conjugates the flow Φ_t to itself, and transforms a Φ_t–invariant measure $\mathbf{m}$ to another Φ_t–invariant measure $\mathbf{m}^-$ such that $\mathbf{m}^-(\varphi) = -\mathbf{m}(\varphi)$. Thus, if $\Lambda(\mathbf{m}) > 0$ then $\Lambda(\mathbf{m}^-) < 0$. Note that $\pi(|\mathbf{m}^-|) = \pi(|\mathbf{m}|)$.

DEFINITION 3.4 *An ergodic, Φ_t–invariant measure $\mathbf{m}_*$ on V is* t–hyperbolic *if $\Lambda(\mathbf{m}_*) \neq 0$. Given an arbitrary Φ_t–invariant probability measure $\mathbf{m}$ on V, we say $\mathbf{m}$ is* t–hyperbolic *if almost every ergodic measure $\mathbf{m}_*$ in an ergodic decomposition of $\mathbf{m}$ is t–hyperbolic. If there exists $C > 0$ so that $|\Lambda(\mathbf{m}_*)| \geq C$ for all such ergodic measures, then we say $\mathbf{m}$ is* uniformly t–hyperbolic.

THEOREM 3.5 *Let $\mathbf{m}$ be an ergodic, t–hyperbolic, Φ_t–invariant measure on V. Then for $\epsilon_1 = \epsilon_1(|\Lambda(\mathbf{m})|/10)$, there exists $\alpha \in \mathcal{A}$, $z \in \pi(|\mathbf{m}|)_{\mathcal{P}} \cap \mathcal{T}_\alpha$ and a holonomy transformation*

$$\mathbf{h}_{\mathbf{m}}: (z - \epsilon_1, z + \epsilon_1) \to (z - \epsilon_1, z + \epsilon_1)$$

such that $\mathbf{h}_{\mathbf{m}}(z) = z$ and $0 < \mathbf{h}'_{\mathbf{m}_*}(y) < c < 1$ for all $z - \epsilon_1 < y < z + \epsilon_1$.

Proof: By reversing the time if necessary (i.e., using the conjugate measure $\mathbf{m}^-$ in place of $\mathbf{m}$) we can assume that $\Lambda = \Lambda(\mathbf{m}) < 0$.

By the ergodic theorem, $\mathbf{m}$–almost every point $(x, v) \in |\mathbf{m}|$ satisfies

$$\lim_{s \to \infty} \frac{1}{s} \int_0^s f(\Phi_t(x, v))\, dt = \lim_{s \to \infty} \frac{1}{s} \int_{-s}^0 f(\Phi_t(x, v))\, dt = \mathbf{m}(f) \tag{13}$$

for every continuous function $f: V \to \mathbf{R}$. Such points (x, v) are called *generic* for $\mathbf{m}$, and the forward orbit of (x, v) is dense in $|\mathbf{m}|$. Choose (x, v) generic for $\mathbf{m}$ so that

$$\lim_{s \to \infty} \frac{1}{s} \int_0^s \varphi(\Phi_t(x, v))\, dt = \Lambda$$

For $\epsilon = -9\Lambda/10$ let $s_0 \geq 0$ be an ϵ-regular point for the geodesic ray $\gamma(t) = \pi(\Phi_t(x, v))$ where $0 \leq t < \infty$. Choose a sequence of times $s_N \mapsto \infty$ such that $\Phi_{s_N}(x, v) \in B_V(\Phi_{s_0}(x, v), \epsilon_1/10)$.

By Theorem 3.3 there exists $z \in \mathcal{T}_\alpha$ with $\gamma(s_0) \in \mathcal{P}_\alpha(z)$ and $\mathcal{I}_x = (z - \epsilon_1, z + \epsilon_1) \subset \mathcal{T}_\alpha$ so that for all s_N, the holonomy transformation h_N along the curve $\gamma_N = \{\gamma(t) \mid s_0 \leq t \leq s_N\}$ defines a transformation $\mathbf{h}_N: \mathcal{I}_x \to \mathcal{I}_x$. For N sufficiently large, $\mathbf{h}_N$ is a uniform contraction on this interval, hence $\mathbf{h}_N$ has a fixed-point $z \in \mathcal{I}_x \subset \mathcal{T}_\alpha$.

It remains to show that $z \in \pi(|\mathbf{m}_*|)_{\mathcal{P}} \cap \mathcal{T}_\alpha$. The orbit $\{\Phi_t(x, v) \mid t \geq s_0\}$ is contained in the closed set $|\mathbf{m}_*|$, while z is a hyberbolic attractor so the plaques covering the orbit $\{\pi(\Phi_t(x, v)) \mid t \geq s_0\}$ are asymptotic to the plaque containing z, hence this plaque must lie in $\pi(|\mathbf{m}_*|)_{\mathcal{P}}$. □

A key point for the applications of Theorem 3.5 in the next sections is that the size of the domain of the hyperbolic attractor depends only on $|\Lambda(\mathbf{m}_*)|$, and given a collection of ergodic measures $\{\mathbf{m}_k\}$ with $|\Lambda(\mathbf{m}_k)| > C > 0$, the domains of the hyperbolic fixed-points constructed in the proof are bounded below by $\epsilon_1(C/10) > 0$.

4 Hyperbolic measures and resilient leaves

Assume that $\mathcal{F}$ is a codimension one, C^1-foliation with a t–hyperbolic measure $\mathbf{m}$. In the last section, we showed this implies that $\mathcal{F}$ must have elements of holonomy which are hyperbolic contractions. In this section, we show that with a suitable hypotheses on the support $|\mathbf{m}|$ of $\mathbf{m}$, then we can also conclude that $\mathcal{F}$ must have a resilient leaf.

THEOREM 4.1 *Assume there exists an ergodic, t-hyperbolic, Φ_t–invariant measure $\mathbf{m}_*$ on V which is not t-discrete. Then there exists an open connected set $\mathcal{I} \subset \mathcal{T}_\alpha$ and elements of holonomy $\mathbf{h}_1 \colon \mathcal{I} \to \mathcal{I}$ and $\mathbf{h}_2 \colon \mathcal{I} \to \mathcal{I}$ which are hyperbolic contractions, and satisfy $\mathbf{h}_1(\mathcal{I}) \cap \mathbf{h}_2(\mathcal{I}) = \emptyset$.*

Proof: We can assume that $\Lambda(\mathbf{m}_*) < 0$ (otherwise we replace $\mathbf{m}_*$ with $\mathbf{m}_*^-$). By Theorem 3.3, there exists $\epsilon_1 > 0$, a generic point $(x, v) \in |\mathbf{m}_*|$ and $s_0 > 0$ so that for all sufficiently large $s_1 \gg s_0$, the holonomy transformation $\mathbf{h}_\gamma$ defined by the geodesic segment $\gamma = \{\Phi_t(x, v) \mid s_0 \leq t \leq s_1\}$ is defined on the transverse interval $\mathcal{I}_0 = (z - \epsilon_1, z + \epsilon_1) \subset \mathcal{T}_\alpha$ where $\pi(\Phi_{s_0}(x, v)) \in \mathcal{P}_\alpha(z)$. Moreover, $\mathbf{h}_\gamma$ is a contraction with derivative

$$0 < \mathbf{h}'_\gamma(y) < \exp\{(s_1 - s_0) \cdot \Lambda(\mathbf{m})/2\} \quad \text{for all} \quad y \in \mathcal{I}_0 \tag{14}$$

The hypothesis that $|\mathbf{m}_*|_\mathcal{T}$ is not a finite set implies that the orbit $\sigma_\infty = \{\Phi_t(x, v) \mid 0 \leq t\}$ intersects an infinite number of plaques $\widehat{\mathcal{P}}_{\beta_z}(z)$ in V, hence the closure $\overline{\sigma_\infty}$ must be a perfect set transversally. That is, $\pi(\overline{\sigma_\infty})_\mathcal{P} \cap \mathcal{T} \subset |\mathbf{m}_*|_\mathcal{T} \subset \mathcal{T}$ is a perfect set. Thus, for any open connected subinterval $\mathcal{I} \subset \mathcal{T}_\alpha$ with $\pi(\overline{\sigma_\infty})_\mathcal{P} \cap \mathcal{I} \neq \emptyset$, there exists disjoint open connected subintervals $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{I}$ such that $\pi(\overline{\sigma_\infty})_\mathcal{P} \cap \mathcal{I}_1 \neq \emptyset$ and $\pi(\overline{\sigma_\infty})_\mathcal{P} \cap \mathcal{I}_2 \neq \emptyset$. Moreover, as $\pi(\overline{\sigma_\infty})_\mathcal{P}$ is compact, we can assume that the intervals have disjoint closures, $\overline{\mathcal{I}_1} \cap \overline{\mathcal{I}_2} = \emptyset$, and both $\overline{\mathcal{I}_1} \subset \mathcal{I}$ and $\overline{\mathcal{I}_2} \subset \mathcal{I}$.

Apply the above remarks to $\mathcal{I}_0$, which by choice satisfies $\pi(\overline{\sigma_\infty})_\mathcal{P} \cap \mathcal{I}_0 \neq \emptyset$. Choose connected open intervals $\mathcal{I}_1$ and $\mathcal{I}_2$ so that $\overline{\mathcal{I}_1}, \overline{\mathcal{I}_2} \subset \mathcal{I}_0$, $\overline{\mathcal{I}_1} \cap \overline{\mathcal{I}_2} = \emptyset$, and $\pi(|\mathbf{m}|)_\mathcal{P} \cap \mathcal{I}_1 \neq \emptyset$ and $\pi(|\mathbf{m}|)_\mathcal{P} \cap \mathcal{I}_2 \neq \emptyset$. As (x, v) is generic, there is a sequence of times $\{r_k \mid s_0 < r_k \to \infty\}$ with $\pi(\Phi_{r_k}(x, v)) \in (\mathcal{I}_1)_\mathcal{P}$ for all k, and also times $\{s_k \mid s_0 < s_k \to \infty\}$ with $\pi(\Phi_{s_k}(x, v)) \in (\mathcal{I}_2)_\mathcal{P}$ for all k. For k sufficiently large, let $\mathbf{h}_{1,k} \colon \mathcal{I}_0 \to \mathcal{I}_0$ be the holonomy map defined by the geodesic segment $\gamma_{1,k} = \{\pi(\Phi_t(x, v)) \mid s_0 \leq t \leq r_k\}$. Similarly, let $\mathbf{h}_{2,k} \colon \mathcal{I}_0 \to \mathcal{I}_0$ the holonomy map defined by $\gamma_{2,k} = \{\pi(\Phi_t(x, v)) \mid s_0 \leq t \leq s_k\}$. Then observe that by the estimate (14), for k sufficiently large we have $\mathbf{h}_{1,k}\mathcal{I}_0 \cap \mathbf{h}_{2,k}\mathcal{I}_0 = \emptyset$. For such k, set $\mathcal{I} = \mathcal{I}_0$, $\mathbf{h}_1 = \mathbf{h}_{1,k}$ and $\mathbf{h}_2 = \mathbf{h}_{2,k}$. $\square$

The following example shows that the ergodic hypothesis is necessary, and illustrates some of the phenomena which can give rise to t-discrete measures. Consider the foliation $\mathcal{F}$ on a 3-manifold V constructed as follows. Let Σ_2 be a genus 2 compact surface, then $\pi_1(\Sigma_1, x_*)$ maps onto the free group $\mathbf{Z} * \mathbf{Z}$ and by the usual suspension construction, given an action of two diffeomorphisms f, g on $\mathbf{S^1}$, one can construct a foliation $\mathcal{F}$ on a 3-manifold V by suspension of the action of these two maps on $\mathbf{S^1}$. Choose f to be the identity outside of a small interval $\mathcal{I} \subset \mathbf{S^1}$, and to have a hyperbolic fixed-point $x_* \in \mathcal{I}$. Choose g to be a hyperbolic linear fractional transformation with fixed-points $y_*, z_* \notin \mathcal{I}$, and $g\mathcal{I} \cap \mathcal{I} = \emptyset$. (Or, an even simpler example is obtained by taking g to be the identity map!)

For each k and $\ell \neq 0$ we get a hyperbolic attractor in the holonomy of $\mathcal{F}$ given by the composition $\mathbf{h}_{k,\ell} = g^{-k} \circ f^{\ell} \circ g^{k}$. For each element $[\gamma_{k,\ell}] \in \pi_1(\Sigma_1, x_*)$ which maps onto this element of $\mathrm{Diff}(\mathbf{S^1})$, there is a shortest closed geodesic $\gamma_{k,\ell}$ in Σ representing this class. Under the suspension construction, $\gamma_{k,\ell}$ gives rise to a closed leafwise geodesic through the hyperbolic fixed-point for $g^{-k} \circ f^{\ell} \circ g^{k}$ and hence to an ergodic, t-discrete Φ_t-invariant measure $\mathbf{m}_{k,\ell}$ for $\mathcal{F}$. Thus, there are infinitely many t-hyperbolic measures $\mathbf{m}_{k,\ell}$. Note that the expansion rate $\Lambda(\mathbf{m}_{k,\ell})$ is proportional to $\ell/(\ell + 2k)$ so by taking weighted sums we obtain uniformly t-hyperbolic measures for $\mathcal{F}$ whose t-support is countably infinite. However, $\mathcal{F}$ has no resilient leaves as all leaves of $\mathcal{F}$ are proper.

Ergodic, t-discrete Φ_t-invariant measures are an important consideration in the study of the dynamics of foliations with $h(\mathcal{G}_{\mathcal{F}}) > 0$. As the above example suggests, they can arise from a closed leafwise geodesic for which the holonomy $\mathbf{h}_{\gamma}$ has a hyperbolic fixed-point. When the domains of these hyperbolic attractors are disjoint, as in the above example, there does not have to be a ping-pong game associated to the attractors. When these domains sufficiently overlap, $\mathcal{G}_{\mathcal{F}}$ has a ping-pong game and hence $h(\mathcal{G}_{\mathcal{F}}) > 0$.

5 Entropy and t-hyperbolic measures

In this section, we show that $h(\mathcal{G}_\mathcal{F}) > 0$ implies there exists an ergodic, t-hyperbolic Φ–invariant measure $\mathbf{m}_*$ on V, and the exponent of the measure can be estimated from $h(\mathcal{G}_\mathcal{F})$. We start by introducing some conventions which are used in this section and the next.

Set $E = h(\mathcal{G}_\mathcal{F}) > 0$. Let $0 < \epsilon_4 < \epsilon_0$ be such that $0 < \epsilon \leq \epsilon_4$ implies $h(\mathcal{G}_\mathcal{F}, \epsilon) > 3E/4$.

Choose a sequence of integers $\{N_k \mid k > 0\}$ tending to infinity such that

$$e_k = \mathcal{S}(\mathcal{G}_\mathcal{F}, \epsilon_4, N_k) > \exp(N_k \cdot E/2)$$

Recall that $h(\mathcal{G}_\mathcal{F}) = \sup\{h(\mathcal{G}_\mathcal{F}, \mathcal{T}_\alpha) \mid \alpha \in \mathcal{A}\}$, so we can assume there exists a fixed transversal $\mathcal{T}_\alpha$ such that for each $k > 0$ there is an (N_k, ϵ_4)-separated subset $\{x^k_\ell \mid 1 \leq \ell \leq e_k\} \subset \mathcal{T}_\alpha$. Use the coordinate $t_\alpha : (-1, 1) \to \mathcal{T}_\alpha$ to identify $\mathcal{T}_\alpha$ with a subset of the line. We can then assume that the set $\{x^k_\ell \mid 1 \leq \ell \leq e_k\}$ is indexed so that $x^k_\ell < x^k_{\ell+1}$ for all $1 \leq \ell < e_k$.

For each $k > 0$ and $1 \leq \ell < e_k$ there exists a plaque chain $\mathcal{P}_{k,\ell}$ of minimal length, at most N_k, so that $x^k_\ell, x^k_{\ell+1} \in D_{\mathcal{P}_{k,\ell}}$ and $d_\mathcal{T}(\mathbf{h}_{k,\ell}(x^k_\ell), \mathbf{h}_{k,\ell}(x^k_{\ell+1})) > \epsilon_4$. Here, $\mathbf{h}_{k,\ell}$ denotes the holonomy determined by $\mathcal{P}_{k,\ell}$ and $D_{\mathcal{P}_{k,\ell}}$ is the maximal domain of $\mathbf{h}_{k,\ell}$.

By the mean value theorem, for each $k > 0$ and $1 \leq \ell < e_k$ there exists a point $x^k_\ell \leq y^k_\ell \leq x^k_{\ell+1}$ such that $\mathbf{h}'_{k,\ell}(y^k_\ell) \geq \epsilon_4/(x^k_{\ell+1} - x^k_\ell)$. This suggests the definition of the expansiveness of the set $\{x^k_\ell \mid 1 \leq \ell \leq e_k\}$ on a subset $\mathcal{I} \subset \mathcal{T}_\alpha$, given by

$$\mathcal{E}(\mathcal{I}, k) = \max_{x_\ell, x_{\ell+1} \in \mathcal{I}} \left\{ \sup_{x_\ell \leq y \leq x_{\ell+1}} \mathbf{h}'_{k,\ell}(y) \right\} \tag{15}$$

Note that by hypothesis, $\mathcal{T}_\alpha$ has length at most 1, so for each k there must exist an ℓ for which $(x^k_{\ell+1} - x^k_\ell) \leq 1/e_k$ and hence $\mathcal{E}(\mathcal{T}_\alpha, k) \geq \epsilon_4 \cdot \exp(N_k \cdot E/2)$. Introduce the invariant

$$\Lambda(\mathcal{I}) = \limsup_{k \to \infty} \frac{\log\{\mathcal{E}(\mathcal{I}, k)\}}{N_k} \tag{16}$$

which depends on the choices of $\mathcal{G}_\mathcal{F}$, ϵ_4 and the sets of expansive points. For simplicity, we omit this dependence in the notation $\Lambda(\mathcal{I})$. Note that by the above remarks, $\Lambda(\mathcal{T}_\alpha) \geq E/2$.

The proof of the following result introduces a technique used repeatedly for constructing t-hyperbolic measures. The basic idea, first applied to foliations in [10, 11], is that the existence of an exponentially growing number of points in a bounded set, for which there are local diffeomorphisms which expand them to a fixed distance apart, implies (non-uniform) hyperbolicity along increasingly long orbit segments for the flow Φ_t. This data is converted to the existence of t-hyperbolic Φ_t-invariant measures using the continuity of the derivative φ of the transverse expansion cocycle.

PROPOSITION 5.1 *If $h(\mathcal{G}_\mathcal{F}) > 0$, then there exists an ergodic, Φ–invariant measure $\mathbf{m}_*$ on V with $\Lambda(\mathbf{m}_*) \geq \Lambda(\mathcal{T}_\alpha)/d_{max} \geq E/2d_{max} > 0$.*

Proof: For $0 < \epsilon < \Lambda(\mathcal{T}_\alpha)$ set $\lambda = \Lambda(\mathcal{T}_\alpha) - \epsilon$. Choose a sequence $\{\ell_k \mid 1 \leq \ell_k < e_k\}$ and points $y^k_{\ell_k}$ satisfying $x^k_{\ell_k} \leq y^k_{\ell_k} \leq x^k_{\ell_k+1}$ and $\mathbf{h}'_{k,\ell_k}(y^k_{\ell_k}) \geq \exp\{N_k \lambda\}$.

The plaque chain $\mathcal{P}_{k,\ell_k}$ determines a leafwise piecewise-geodesic path τ_k starting at y_{k,ℓ_k} and ending at $w_{k,\ell_k} = \mathbf{h}_{k,\ell_k}(y_{k,\ell_k})$ of length at most $N_k \cdot d_{max}$. By the completeness of the leafwise Riemannian metrics, there exists a length-minimizing, leafwise geodesic γ_k starting at y_{k,ℓ_k} , ending at w_{k,ℓ_k} and homotopic to τ_k rel endpoints. The length of γ_k has the bound $\|\gamma_k\| \leq N_k \cdot d_{max}$.

As τ_k and γ_k are leafwise homotopic, the germs at y_{k,ℓ_k} of the corresponding holonomy transformations $\mathbf{h}_{k,\ell_k}$ and $\mathbf{h}_{\gamma_k}$ are equal, hence $\mathbf{h}'_{\gamma_k}(y_{k,\ell_k}) = \mathbf{h}'_{k,\ell_k}(y_{k,\ell_k}) \geq \exp\{N_k \lambda\}$.

The geodesic segment γ_k is the image of a flow segment, $\widehat{\gamma_k}(t) = \{\Phi_t(y_{k,\ell_k}, v_k) \mid 0 \leq t \leq \|\gamma_k\|\}$ where $v_k = \gamma'_k(0)$. Define a sequence of probability measures $\{\mathbf{m}_k\}$ on V by setting, for g continuous on V,

$$\mathbf{m}_k(g) = \frac{1}{\|\gamma_k\|} \cdot \int_0^{\|\gamma_k\|} g(\widehat{\gamma_k}(t))\, dt \tag{17}$$

Note that from the definitions,

$$\mathbf{m}_k(\varphi) \geq N_k \lambda / \|\gamma_k\| \geq \lambda / d_{max}$$

Choose a weak-* limit $\mathbf{m}^\epsilon$ of the sequence $\{\mathbf{m}_k \mid k = 1, 2, \ldots\}$ which is a Φ_t–invariant probability measure on V such that $\mathbf{m}^\epsilon(\varphi) \geq \lambda/d_{max} = (\Lambda(\mathcal{T}_\alpha) - \epsilon)/d_{max}$.

Consider a weak-* limit $\mathbf{m}$ of the family of Φ_t–invariant probability measures on V $\{\mathbf{m}^\epsilon \mid \epsilon \to 0\}$. We must have $\mathbf{m}(\varphi) \geq (\Lambda(\mathcal{T}_\alpha) - \epsilon)/d_{max}$ for all ϵ, hence $\mathbf{m}(\varphi) \geq \Lambda(\mathcal{T}_\alpha)/d_{max}$.

Finally, take an ergodic decomposition of $\mathbf{m}$ and there must be an ergodic, Φ_t–invariant measure $\mathbf{m}_*$ with $\mathbf{m}_*(\varphi) \geq \mathbf{m}(\varphi) \geq \Lambda(\mathcal{T}_\alpha)/d_{max}$. □

Note that the closure in V of the set of flow segments $\{\widehat{\gamma_k}(t) \mid 0 \leq t \leq \|\gamma_k\|\}$ for $1 \leq k < \infty$ arising in the proof of Proposition 5.1 contains the support of the measures $\{\mathbf{m}^\epsilon \mid \epsilon \to 0\}$. Hence it also contains the supports $|\mathbf{m}|$ and $|\mathbf{m}_*|$ of $\mathbf{m}$ and $\mathbf{m}_*$ respectively.

6 Positive entropy and ping-pong games

In this section we give the proof of Theorem 1.1. The methods used are the most technical of the paper, primarily for two reasons which we discuss to motivate the following proof. Recall the usual approach to establishing dynamical phenomenon for a flow or diffeomorphism with positive topological entropy. For example, Katok's proof of the existence of hyperbolic periodic points for a surface diffeomorphism [14] is the closest analog to the results of this section. First, one starts by showing the measure entropy is positive for some measure of positive Hausdorff dimension, and for which there exist non-zero Lyapunov exponents. Then, the Lyapunov metric is introduced, and at the regular points in the Pesin set one has stable and unstable manifolds which are used to show the existence of hyperbolic phenomenon like homoclinic orbits and hyperbolic periodic points.

The first fundamental difficulty for foliations is that there is no good definition of measure entropy yet, even though the dynamics of Φ_t relative to $\widehat{\mathcal{F}}$ is rich with invariant measures. So the above approach for flows and diffeomrophism, if it is to work at all for foliations, must be based on the core ideas, and not on established theorems. For example, it was shown in the last section that if $h(\mathcal{G}_{\mathcal{F}}) > 0$ then there exists ergodic t-hyperbolic Φ_t-invariant measure. If such a measure is not t-discrete (and hence it has positive transverse Hausdorff dimension) then we can invoke Theorem 4.1, yielding a ping-pong game which is the codimension-one foliation analog of transversally intersecting homoclinic orbits.

The most technical part of the proof of Theorem 1.1 is to analyze the case when all the t-hyperbolic Φ_t-invariant measures are t-discrete. Essentially, we show this cannot happen, using a lengthy case-by-case analysis, in each case showing that there exists a ping-pong game for $\mathcal{G}_{\mathcal{F}}$. For flows, this case does not arise, since by the maximum principle for the measure entropy, positive topological entropy implies there are measures of positive entropy. For foliations, similar ideas as used in the proof of the maximal theorem arise in our analysis of the t-discrete case – except of course, that we cannot use the orbit data to produce invariant measures of positive entropy! Instead, we go directly to the construction of intersecting stable manifolds from the orbit data. However, the second fundamental difficulty with foliation dynamics is then encountered: one can define the Lyapunov metric on the normal bundle to $\mathcal{F}$ lifted to V, viewed as a Φ_t-invariant bundle, but this metric is not necessarily related to a metric on M where the ping-pong game dynamics must be constructed. The point is that we cannot choose the foliations charts on M as in § 2 to respect the resulting Lyapunov metric on V. As a result, we introduce a number of technical devices, all essentially showing some form of non-uniform hyperbolicity has uniform approximations.

We assume that $h(\mathcal{G}_{\mathcal{F}}) > 0$, hence there exists an ergodic t-hyperbolic measure. As discussed above, to prove Theorem 1.1 it suffices to assume every ergodic t-hyperbolic, Φ_t–invariant measure $\mathbf{m}$ on V is t-discrete, and show that there must exist a ping-pong game for the dynamics of $\mathcal{G}_{\mathcal{F}}$. Let $\{\mathbf{m}_b \mid b \in \mathcal{B}\}$ denote the collection of all ergodic Φ_t-invariant probability measures on V with $\lambda_b = \Lambda(\mathbf{m}_b) < 0$. We are going to analyze a sequence of cases, based on the cardinality of $\mathcal{B}$, the values $\{\lambda_b \mid b \in \mathcal{B}\}$ and the nature of the hyperbolic contractions constructed in the proof of Theorem 3.5 applied to each measure $\mathbf{m}_b$.

For each $b \in \mathcal{B}$ let $\pi(\mathbf{m}_b)_{\mathcal{P}}$ be the finite union of plaques $\{\mathcal{P}_{\beta^b_1}(y^b_1), \ldots, \mathcal{P}_{\beta^k_{N_b}}(y^b_{N_b})\}$. Let $\mathcal{J} \subset \mathcal{T}$ denote the union of all plaque centers $\{y^b_\ell \mid b \in \mathcal{B} \ \& \ 1 \leq \ell \leq N_b\}$.

6.1 The case when $\{z_b \mid b \in \mathcal{B}_*\}$ is infinite

Assume that there is $\lambda_* > 0$ such that the set $\mathcal{B}_* = \{b \in \mathcal{B} \mid \lambda_b \leq -\lambda_*\}$ has infinite cardinality. Set $\epsilon_1 = \epsilon_1(\lambda_*/10)$ and $\epsilon = 9\lambda_*/10$ as in § 3.2. Then by Theorem 3.5, for each $b \in \mathcal{B}_*$ there exists $z_b = y^b_\ell \in \pi(|\mathbf{m}_b|)_{\mathcal{P}} \cap \mathcal{T}_{\alpha_b}$ for some $1 \leq \ell \leq N_b$ such that $\mathcal{I}_b = (z_b - \epsilon_1, z_b + \epsilon_1) \subset \mathcal{T}_{\alpha_b}$ and an element of holonomy $\mathbf{h}_b: \mathcal{I}_b \to \mathcal{I}_b$ such that $\mathbf{h}_b(z_b) = z_b$ and $0 < \mathbf{h}'_b(y) < c < 1$ for all $y \in \mathcal{I}_b$.

Suppose that the union of the set $\{z_b \mid b \in \mathcal{B}_*\}$ is infinite. Choose an accumulation point $z_* \in \mathcal{T}_\beta$, and choose distinct points $z_b, z_c \in B_{\mathcal{T}}(z_*, \epsilon_1/10)$. Then $z_b, z_c \in \mathcal{I} = \mathcal{I}_b \cap \mathcal{I}_c$ and we choose $N \gg 0$ so that $\mathbf{h}^N_b \mathcal{I} \cap \mathbf{h}^N_c \mathcal{I} = \emptyset$. Then set $\mathbf{h}_1 = \mathbf{h}^N_b$ and $\mathbf{h}_2 = \mathbf{h}^N_c$ and we are done.

6.2 The case when $\{z_b \mid b \in \mathcal{B}_*\}$ is finite, with bounded periods

With notation as in (6.1), suppose that the union of the set $\{z_b \mid b \in \mathcal{B}_*\}$ is finite, but the set of plaque centers $\mathcal{J}_* = \{y^b_\ell \mid b \in \mathcal{B}_* \text{ \& } 1 \leq \ell \leq N_b\}$ is infinite, and there exists $N_* > 0$ so that $N_b \leq N_*$ for all $b \in \mathcal{B}_*$.

From the proof of Theorem 3.5, each of the holonomy maps $\mathbf{h}_b$ is the holonomy along a plaque chain $\mathcal{P}_b$ of length at most N_*. It follows that we can join each y^b_ℓ to z_b by a plaque chain $\mathcal{P}_{b,\ell}$ of length again at most N_* which defines a holonomy transformation $\mathbf{h}_{b,\ell}$ with $\mathbf{h}_{b,\ell}(y^b_\ell) = z_b$. Thus, there exists $\epsilon_* > 0$ so that for all b, ℓ we have $(z^b_\ell - \epsilon_*, z^b_\ell + \epsilon_*) \subset \mathbf{h}^{-1}_{b,\ell}\mathcal{I}_b$ Then the composition $\mathbf{h}_{b,\ell} \circ \mathbf{h}^n_b \circ \mathbf{h}^{-1}_{b,\ell}$ is a hyperbolic contraction with y^b_ℓ as fixed-point and domain containing $(y^b_\ell - \epsilon_*, y^b_\ell + \epsilon_*)$. The collection of plaque centers $\mathcal{J}_*$ are thus all hyperbolic fixed-points, and must have an accumulation point, so we are done by the method of Case 1.

6.3 The case when $\{z_b \mid b \in \mathcal{B}_*\}$ is finite with unbounded periods

With notation as in (6.1), suppose that the union of the set $\{z_b \mid b \in \mathcal{B}_*\}$ is finite, the set of plaque centers $\mathcal{J}_* = \{y^b_\ell \mid b \in \mathcal{B}_* \text{ \& } 1 \leq \ell \leq N_b\}$ is infinite, but there is no upper bound on the lengths $\{N_b \mid b \in \mathcal{B}_*\}$. We will show that either there is a ping-pong game for $\mathcal{G}_{\mathcal{F}}$ or reduce the problem to Case 4 below, where $\mathcal{J}$ is a finite set.

Fix $b \in \mathcal{B}_*$ and let $\mathcal{P}_b = \{\mathcal{P}_{\alpha_1}(z_0), \ldots, \mathcal{P}_{\alpha_N}(z_N)\}$ be the plaque chain used to define $\mathbf{h}_b$ where $z_b = z_0$. From the proof of Theorem 3.3, form the piecewise geodesic curve $\tau_b: [0, T_b] \to M$ obtained by concatenating the geodesic segments τ_ℓ connecting z_ℓ to $z_{\ell+1}$. Note that $\tau_b(0) = z_0$, and $\{z_0, \ldots, z_N\} \subset \{y^b_1, \ldots, y^b_{N_b}\}$. Let $0 = s_0 < s_1 < \ldots < s_N = T_b$ be such that $\tau_b(s_k) = z_k$.

We say that $\mathbf{h}_b$ is *irreducible* if $\mathcal{P}_{\alpha_i}(z_i) = \mathcal{P}_{\alpha_j}(z_j)$ implies $i = j$, and *reducible* otherwise.

Case 3a. Suppose $\mathbf{h}_b$ is reducible. Choose indices $0 \leq i < j < N$ so that $\tau(s_i) = \tau(s_j)$. Then the plaque chain $\mathcal{P}$ along τ has a subloop starting at $\mathcal{P}_{\alpha_i}(z_i)$ and ending at $\mathcal{P}_{\alpha_i}(z_i)$. We decompose $\mathcal{P}$ at these points, yielding closed plaque chains

$$\mathcal{P}_1 = \{\mathcal{P}_{\alpha_1}(z_1), \ldots, \mathcal{P}_{\alpha_i}(z_i), \mathcal{P}_{\alpha_{j+1}}(z_{j+1}), \ldots \mathcal{P}_{\alpha_N}(z_N)\}$$

$$\mathcal{P}_2 = \{\mathcal{P}_{\alpha_i}(z_i), \ldots, \mathcal{P}_{\alpha_j}(z_j)\}$$

and closed piecewise geodesic curves $\tau_1(t)$ and $\tau_2(t)$. As the logarithmic expansion is additive, $\nu(\tau) = \nu(\tau_1) + \nu(\tau_2) = \lambda_b T_b$ at least one of these shorter loops must be t-hyperbolic with expansion $\lambda^*_b \leq \lambda_b$. We can thus continue to reduce the curve τ until we obtain an irreducible curve τ^*_b which is t-hyperbolic with expansion $\lambda^*_b \leq -\lambda_*$ and supported on a subset of $\mathcal{P}_b$.

For each closed plaque chain constructed from the orbit of each generic point $(x, v) \in |\mathbf{m}_b|$ and each irreducible plaque chain obtained as above, choose one with the longest length, and let τ_b^* be the corresponding piecewise geodesic curve. If the length of τ_b^* is uniformly bounded for all $b \in \mathcal{B}_*$ then we proceed on to Case 4. If the length of τ_b is unbounded, then we proceed to Case 3b.

Before continuing our analysis of cases, a comment on the intuitive motivation may be helpful. In cases 1 and 2, we obtained an infinite set of distinct points in $\mathcal{T}$ which were the fixed-points for hyperbolic attractors in $\mathcal{G}_{\mathcal{F}}$ with domains of uniform size. However, in case 3 we consider the case where the set of fixed-points constructed is finite, even though the set $\mathcal{J}_*$ is infinite. What could go wrong is that there could be a finite set of plaques which "carry" all of the transverse hyperbolicity (as is the case of the map f in the suspension example given previously.) The technique of considering the irreducible plaque chains derived from the measures $\mathbf{m}_b$ isolates this possibility, and when the lengths of the curves τ_b^* are uniformly bounded this is the intuition. This possibility is considered in Case 4, and requires using additional information derived from our hypothesis that $h(\mathcal{G}_{\mathcal{F}}) > 0$ to obtain a ping-pong game.

The case 3b we consider next is where a collection of curves τ_b^* has length tending to infinity. The intuition is that the hyperbolicity along τ_b^* must be distributed over an increasing sequence of plaques dues to the uniform bound $|\varphi(x, v)| \leq \|\varphi\|$. Thus, for each such curve τ_b^* we produce an increasing number of points along its orbit which are ϵ-good, and hence obtain once again an infinite set of hyperbolic fixed points, and by the method of Case 1 produces a ping-pong game.

Case 3b. Let $\tau_b \colon [0, T_b] \to M$ be the piecewise geodesic curve associated to the irreducible plaque chain $\mathcal{P}_b = \{\mathcal{P}_{\beta_1}(z_0), \ldots, \mathcal{P}_{\beta_{n_b}}(z_{n_b})\}$ which is either obtained from the proof of Theorem 3.3 if this is irreducible, or by the reduction in case 3a. We assume that $n_b \to \infty$ and τ_b has exponent $\lambda_b \leq -\lambda_*$. We assume that $\tau_b(0) = z_0$, and let $0 = s_0 < s_1 < \ldots < s_{n_b} = T_b$ be such that $\tau_b(s_k) = z_k$. Extend τ_b to a periodic piecewise geodesic curve $\tau_b \colon [0, \infty) \to M$. Then by Lemma 3.2 there is a constant $c > 0$ depending only on λ_* and ϵ such that at least cN_b of the values in $\{s_0, s_1, \ldots, s_{n_b}\}$ are ϵ-good. It follows that each of the corresponding points in $\{\tau_b(s_0), \ldots, \tau_b(s_{n_b}\}$ is the fixed-point for a hyperbolic attractor in $\mathcal{G}_{\mathcal{F}}$ whose domain has width $2\epsilon_1$. The collection of all such points for $b \in \mathcal{B}_*$ must be infinite, so we can proceed as in Case 1.

6.4 The case when t-hyperbolic measures are t-discrete

In this final case, we use the assumption $h(\mathcal{G}_{\mathcal{F}}) > 0$ and hypotheses on the t-hyperbolic periodic piecewise-geodesic plaques chains (to be made precise later) to show there exists a ping-pong game for the dynamics of $\mathcal{F}$. The idea of the construction of a ping-pong game in this case is simple, although tedious in its implementation. Each t-hyperbolic periodic plaque chain produces an element of holonomy with a hyperbolic fixed-point. Positive entropy implies such paths exists, and also that there exists an exponentially growing number of hyperbolic paths as constructed in the proof of Proposition 5.1. We use these paths to construct an element of holonomy which "translates" one of the hyperbolic fixed-points by an amount less than the size of its domain, which will then produce a ping-pong game. A simple example illustrates this idea: consider the "ax+b" group acting on $\mathbf{R}$ with generators $\mathbf{h}_1(x) = 2x$ and $\mathbf{h}_2(x) = 1$. The composition $\mathbf{h}_2 \circ \mathbf{h}_1(x) = 2x + 1$ has a hyberbolic fixed-point at $x = -1$ so $\{\mathbf{h}_1, \mathbf{h}_2 \circ_1\}$ generate a ping-pong table. The holonomy element analogous to $\mathbf{h}_1$ is obtained from one of the t-hyperbolic plaque chains; the work is to produce the a holonomy translation analogous to $\mathbf{h}_2$. More accurately, the translation is analogous to a map to $\mathbf{h}_1^{-\ell} \circ \mathbf{h}_2 \circ \mathbf{h}_1^{\ell}$ for $\ell \gg 0$. A key point is to show that the map produced actually translates the fixed-point, even though the size of the translation is exponentially small.

We start with some technical preliminaries. Notation will be as in § 5. Recall that for $e_k = \lceil \exp(N_k \cdot E/2) \rceil$ we assume there is chosen an (N_k, ϵ_4)–separated subset $\{x_\ell^k \mid 1 \leq \ell \leq e_k\} \subset \mathcal{I}_\alpha$.

Let $q_k = \lceil \exp(N_k \cdot E/4) \rceil$ be the least integer greater than or equal to $\exp(N_k \cdot E/4)$. Then by the pigeon-hole principle, for each $k > 0$ there exists a closed interval $J_k \subset \mathcal{T}_\alpha$ of length $1/q_k$ so that $J_k \cap \{x_\ell^k \mid 1 \leq \ell \leq e_k\}$ has cardinality at least q_k. Without loss of generality, assume this (N_k, ϵ_4)–separated subset is labeled $\{x_\ell^k \mid 1 \leq \ell \leq q_k\}$ and indexed so that $x_\ell^k < x_{\ell+1}^k$ for all $1 \leq \ell < q_k$.

For each $k > 0$ and $1 \leq \ell < q_k$ there exists a minimal-length plaque chain $\mathcal{P}_{k,\ell}$ of length at most N_k, so that $x_\ell^k, x_{\ell+1}^k \in D_{\mathcal{P}_{k,\ell}}$ and $d_{\mathcal{T}}(\mathbf{h}_{k,\ell}(x_\ell^k), \mathbf{h}_{k,\ell}(x_{\ell+1}^k)) > \epsilon_4$. Here, $\mathbf{h}_{k,\ell}$ denotes the holonomy determined by $\mathcal{P}_{k,\ell}$ and $D_{\mathcal{P}_{k,\ell}}$ is the maximal domain of $\mathbf{h}_{k,\ell}$.

For each $k > 0$ and $1 \leq \ell < q_k$, choose a point y_ℓ^k such that $x_\ell^k \leq y_\ell^k \leq x_{\ell+1}^k$ and $\mathbf{h}'_{k,\ell}(y_\ell^k)$ is maximal. The distance $|x_{k,\ell} - x_{k,\ell+1}| \leq 1/q^k$ so that by the mean value theorem, $\mathbf{h}'_{k,\ell}(y_\ell^k) \geq \epsilon_4 \cdot q_k$.

The plaque chain $\mathcal{P}_{k,\ell}$ determines a leafwise piecewise-geodesic path $\tau_{k,\ell}$ starting at y_ℓ^k and ending at $w_\ell^k = \mathbf{h}_{k,\ell}(y_\ell^k)$ of length at most $N_k \cdot d_{max}$. By the completeness of the leafwise Riemannian metrics, there exists a leafwise length-minimizing geodesic $\gamma_{k,\ell}$ starting at y_ℓ^k, ending at w_ℓ^k and homotopic to $\tau_{k,\ell}$ rel endpoints. The germs at y_ℓ^k of the holonomy transformations for $\mathbf{h}_{k,\ell}$ and $\mathbf{h}_{\gamma_{k,\ell}}$ agree, so $\mathbf{h}'_{\gamma_{k,\ell}}(y_\ell^k) = \mathbf{h}'_{k,\ell}(y_\ell^k)$. Let $T_{k,\ell} = \|\gamma_{k,\ell}\|$ and note that $T_{k,\ell} \leq N_k \cdot d_{max}$.

Let $v_\ell^k = \gamma'_{k,\ell}(0)$ so that $\gamma_{k,\ell}(t) = \pi(\Phi_t(y_\ell^k, v_\ell^k))$. Set $\widehat{\gamma}_{k,\ell}(t) = \Phi_t(y_\ell^k, v_\ell^k)$ for $0 \leq t \leq T_{k,\ell}$.

Let $\gamma_{k,\ell}^-(t) = \gamma_{k,\ell}(T_{k,\ell} - t)$ be the time-reversed geodesic segment, and $\widehat{\gamma}_{k,\ell}^-(t) = \widehat{\gamma}_{k,\ell}(T_{k,\ell} - t)$ the time-reversed orbit segment projecting to $\gamma_{k,\ell}^-(t)$. The holonomy defined by $\gamma_{k,\ell}^-$ is just $\mathbf{h}_{k,\ell}^{-1}$.

For each $k > 0$ and $1 \leq \ell < q_k$, set

$$\lambda_{k,\ell} = \frac{\log\{\mathbf{h}'_{k,\ell}(y_\ell^k)\}}{T_{k,\ell}} \tag{18}$$

Note that $\lambda_{k,\ell} \leq \|\varphi\|$, and combined with the above estimates we have

$$\|\varphi\| \geq \lambda_{k,\ell} \geq \frac{\log\{\epsilon_4 \cdot q^k\}}{T_{k,\ell}} \geq \frac{N_k E + \log(\epsilon_4)}{4T_{k,\ell}} \geq \frac{N_k E + \log(\epsilon_4)}{4N_k d_{max}} \to \frac{E}{4d_{max}} \tag{19}$$

For each $k > 0$, choose an index ℓ_k with $1 \leq \ell_k < q_k$ so that

$$\mathbf{h}'_{k,\ell}(y_{\ell_k}^k) = \max\left\{\mathbf{h}'_{k,\ell}(y_\ell^k) \mid 1 \leq \ell < q_k\right\}$$

Note that by hypothesis, J_k has length $1/q_k$ and contains q_k distinct points, so for each k there must exist an $1 \leq \ell < q_k$ for which $(x_{\ell+1}^k - x_\ell^k) \leq 1/(q_k)^2$ and hence $\mathbf{h}'_{k,\ell_k}(y_{\ell_k}^k) \geq \epsilon_4 \cdot \exp(N_k \cdot E/2)$.

For notational convenience, set $\mathbf{h}_k = \mathbf{h}_{k,\ell_k}$, $y_k = y_{\ell_k}^k$ and $T_k = T_{k,\ell_k}$. Introduce the invariants

$$\lambda_k = \log\{\mathbf{h}'_k(y_k)\}/T_k \quad \& \quad \lambda_* = \limsup_{k \to \infty} \lambda_k \tag{20}$$

From previous remarks, we have the estimate $E/2d_{max} \leq \lambda_* \leq \|\varphi\|$. Passing to a subsequence of the k if necessary, we can assume that for each k and $1 \leq \ell < q_k$

$$\lambda_k \geq \lambda_*(1 - 1/k) \quad \& \quad \lambda_{k,\ell} \geq E/5d_{max} \tag{21}$$

Set $\Lambda_* = \min\{1/2\ ,\ E/(96 d_{max})\}$. The factoring $96 = 2 \cdot 4 \cdot 12$ reflect the role of Λ_* – the key to the proof of Theorem 1.1 will turn out to be the factor "2"! The estimate $\Lambda_* \leq 1/2$ is included in the definition to simplify a later estimate.

We assume that every ergodic t-hyperbolic, Φ_t–invariant measure $\mathbf{m}$ on V is t-discrete, and that there exists a finite collection of plaques $\mathcal{P}_* = \{\mathcal{P}_{\beta_1}(z_1), \ldots, \mathcal{P}_{\beta_N}(z_N)\}$ so that for every irreducible plaque chain τ_* with expansion $\lambda(\tau_*) \leq -\Lambda_*$ is composed of plaques in $\mathcal{P}_*$. We also assume that $\mathcal{P}_*$ is minimal, in the sense that every plaque in $\mathcal{P}_*$ intersects at least one hyperbolic plaque chain. The integer N depends on the choice of Λ_* but to simplify notation, this dependence is not indicated.

Set $\epsilon_1 = \epsilon_1(\Lambda_*/10)$. As in case (6.2) we can choose $0 < \epsilon_5 \leq \epsilon_1$ so that for each $1 \leq \xi \leq N$, set $\mathcal{I}_\xi = (z_\xi - \epsilon_5, z_\xi + \epsilon_5)$ then $\mathcal{I}_\xi \cap \mathcal{I}_\eta = \emptyset$ for $\xi \neq \eta$, and there exists a holonomy map $\mathbf{h}_\xi \colon \mathcal{I}_\xi \to \mathcal{I}_\xi$ with $0 < \mathbf{h}'_\xi(y) < c < 1$ for all $y \in \mathcal{I}_\xi$. Let L_ξ denote the leaf of $\mathcal{F}$ through z_ξ and note that if $L_\xi \cap \mathcal{I}_\xi \neq \{z_\xi\}$ then L_ξ is a resilient leaf and we are done. Otherwise, we can assume each leaf L_ξ is proper.

For each $1 \leq \xi \leq N$ and $\epsilon > 0$, let

$$\mathcal{S}(z_\xi, \epsilon) = \bigcup_{z \in \mathbf{B}_\mathcal{T}(z_\xi, \epsilon)} \mathcal{P}_\alpha(z)$$

denote the open subset of M given by the union of the plaques in $\mathbf{B}_\mathcal{T}(z_\xi, \epsilon)$, and let $\widehat{\mathcal{S}}(z_\xi, \epsilon) = \pi^{-1}(\mathcal{S}(z_\xi, \epsilon)) \subset V$. Also set $\mathcal{S}(\epsilon) = \bigcup_{\xi=1}^{N} \mathcal{S}(z_\xi, \epsilon)$ and $\widehat{\mathcal{S}}(\epsilon) = \pi^{-1}(\mathcal{S}(\epsilon)) \subset V$.

We next prove a technical result providing a uniform estimate on the transverse expansion outside of the set $\widehat{\mathcal{S}}(\epsilon_5)$.

LEMMA 6.1 *For each $\epsilon > 0$ there exists $T(\epsilon) > 0$ so that if $(x_k, v_k) \in V$,$s > 0$ and $T > 0$ are such that $\{\Phi_t(x, v) \mid s \leq t \leq s + T\} \cap \widehat{\mathcal{S}}(\epsilon) = \emptyset$, then*

$$\left| \int_s^{s+T} \varphi(\Phi_t(x, v))\ dt \right| \ <\ T(\epsilon)\ \|\varphi\| + T\ \Lambda_* \tag{22}$$

Proof: Define a continuous function $\Delta \colon [0, \infty) \to \mathbf{R}$ where $\Delta(T)$ is the maximum of

$$\frac{1}{T} \cdot \left| \int_s^{s+T} \varphi(\Phi_t(x, v))\ dt \right| - \Lambda_* \tag{23}$$

for all $(x, v) \in V$ and $s \geq 0$ such that $\{\Phi_t(x, v) \mid s \leq t \leq s + T\} \cap \widehat{\mathcal{S}}(\epsilon) = \emptyset$. If $\limsup_{T \to \infty} \Delta(T) \geq 0$, then there exists $\{(x_k, v_k) \in V, s_k, T_k \mid k = 1, 2, \ldots\}$ such that

$$\lim_{T_k \to \infty} \frac{1}{T_k} \cdot \int_{s_k}^{s_k + T_k} \varphi(\Phi_t(x_k, v_k))\ dt\ \leq\ -\Lambda_*$$

(Recall that we can assume the integral is negative by simply reversing time along an orbit where the integral is positive.) The orbit segments $\{\Phi_t(x_k, v_k) \mid s_k \leq t \leq s_k + T_k\}$ define a sequence of probability measures $\{\mathbf{m}_k\}$ as in equation (17), and let $\mathbf{m}$ be a weak-* limit. Then $\mathbf{m}(\varphi) \leq -\Lambda_*$ so there exists an ergodic Φ–invariant measure $\mathbf{m}_*$ in the ergodic decomposition of $\mathbf{m}$ for which $\mathbf{m}_*(\varphi) \leq -\Lambda_*$ whose support is contained in the closure

$$|\mathbf{m}_*| \subset \overline{\bigcup_{k=1}^{\infty} \{\Phi_t(x_k, v_k) \mid s_k \leq t \leq s_k + T_k\}}$$

which is disjoint from $\widehat{\mathcal{S}}(\epsilon)$ by hypothesis. The measure $|\mathbf{m}_*|$ is t-hyperbolic, so must be t-discrete. Associated to $\mathbf{m}_*$ is an irreducible plaque chain and piecewise geodesic curve τ^* with exponent $\lambda(\tau^*) \leq -\Lambda_*$. The plaques used to define τ^* are contained in $|\mathbf{m}_*|_{\mathcal{P}}$ so this contradicts our assumptions. Thus, $\limsup_{T\to\infty} \Delta(T) < 0$.

Since $\Delta(0) = 0$ there is a greatest value $T = T(\epsilon)$ such that $\Delta(T) = 0$. Since $\Delta(T) \leq \|\varphi\|$ for all T, the conclusion follows. $\square$

COROLLARY 6.2 *For all $T > T(\epsilon)$ and for all $(x,v) \in V - \widehat{\mathcal{S}}(\epsilon)$, $s \geq 0$, if*

$$\left| \int_s^{s+T} \varphi(\Phi_t(x,v))\ dt \right| \ \geq\ T\,\Lambda_* \tag{24}$$

then $\{\Phi_t(x,v) \mid s \leq t \leq s+T\} \cap \widehat{\mathcal{S}}(\epsilon) \neq \emptyset$.

Proof: If the intersection is empty, then as $T > T(\epsilon)$ we have $\Delta(T) < 0$ contradicting (24). $\square$

We return now to the analysis of the dynamics of $\mathcal{G}_{\mathcal{F}}$. It should be remarked that by (19) the geodesic lengths $T_{k,\ell} \to \infty$ uniformly in k. By Corollary 6.2, for $k \gg 0$ and each $1 \leq \ell < q_k$ there exists a value $0 \leq F_{k,\ell} < T_{k,\ell}$ such that $\{\gamma_{k,\ell}(t) \mid 0 \leq t \leq F_{k,\ell}\}$ intersects $\mathcal{S}(\epsilon_5)$ in a connected, half-open interval $F_{k,\ell} - \delta_1 < t \leq F_{k,\ell}$ for some $\delta_1 > 0$. That is, $F_{k,\ell}$ is the approximate time of first entry in $\mathcal{S}(\epsilon_5)$ for the geodesic segment $\gamma_{k,\ell}$.

Let $\rho_{k,\ell}$ denote the piecewise-geodesic curve formed from the segment $\{\gamma_{k,\ell}(t) \mid 0 \leq t \leq F_{k,\ell}\}$ followed by a geodesic segment $\sigma^0_{k,\ell}$ contained in a plaque from $\gamma_{k,\ell}(F_{k,\ell})$ to $u_{k,\ell} \in \mathcal{T}$. Let $\mathbf{f}_{k,\ell}$ denote the holonomy map determined by $\rho_{k,\ell}$ so that $\mathbf{f}_{k,\ell}(y_{k,\ell}) = u_{k,\ell}$ and by Lemma 6.1 for $k \gg 0$,

$$\left|\log\{\mathbf{f}'_{k,\ell}(y_{k,\ell})\}\right| \leq \|\varphi\|(T(\epsilon_1) + d_{\mathcal{U}}) + \Lambda_* F_{k,\ell} \leq 2\Lambda_* T_{k,\ell}$$

$$\exp\{-2\Lambda_* T_{k,\ell}\} \leq \mathbf{f}'_{k,\ell}(y_{k,\ell}) \leq \exp\{2\Lambda_* T_{k,\ell}\} \tag{25}$$

We next estimate the distance from each point $u_{k,\ell}$ to the set $\{z_1, \ldots, z_N\}$.

Let $|\mathcal{A}|$ denote the cardinality of the set $\mathcal{A}$. By the "pigeon-hole principle", for any collection of points $\{x_1, \ldots, x_a\} \subset \mathcal{T}$ with $a \geq \lceil 2|\mathcal{A}|/\epsilon_1 \rceil + 1$ there exists $1 \leq i < j \leq a$ so that $d_{\mathcal{T}}(x_i, x_j) \leq \epsilon_1/2$. More generally, given an integer $n > 1$, if $a \geq n\lceil 2|\mathcal{A}|/\epsilon_1 \rceil + 1$ then there exists $z \in \mathcal{T}$ and integers $1 \leq i_1 < \ldots < i_n \leq a$ so that $d_{\mathcal{T}}(z, x_{i_j}) \leq \epsilon_1/4$ for $j = 1, \ldots, n$.

Choose an integer $K_2 > 2\|\varphi\| d_{\mathcal{U}} + 2$. We want to ensure that there are at least K_2 "close points", so we set

$$K_3 = K_2 \lceil 2|\mathcal{A}|/\epsilon_1 \rceil + 1 \tag{26}$$

Set $\epsilon_6 = 10\Lambda_*$ and assume $k \gg 0$. We exhibit a sequence of points which are ϵ_6-regular for $\gamma^-_{k,\ell}$. First, there exists $0 \leq s_{k,\ell,1} < T_{k,\ell}$ which is the greatest value of s such that

$$\int_0^s \left\{\varphi(\widehat{\gamma}^-_{k,\ell}(t)) + 10\Lambda_*\right\}\ dt \ =\ 0$$

and then

$$\int_{s_{k,\ell,1}}^{T_{k,\ell}} \varphi(\widehat{\gamma}^-_{k,\ell}(t))\ dt \ = \int_0^{T_{k,\ell}} \left\{\varphi(\widehat{\gamma}^-_{k,\ell}(t)) + 10\Lambda_*\right\}\ dt \ = -\lambda_{k,\ell}\, T_{k,\ell} + 10\Lambda_*\, s_{k,\ell,1}$$

Then choose the least $t_{k,\ell,1} > s_{k,\ell,1}$ so that

$$\int_{s_{k,\ell,1}}^{t_{k,\ell,1}} \varphi(\widehat{\gamma}^-_{k,\ell}(t))\ dt \ = \ -1 - \Lambda_*$$

Continuing, let $s_{k,\ell,2} \geq t_{k,\ell,1}$ be the greatest value such that

$$\int_{t_{k,\ell,1}}^{s_{k,\ell,2}} \left\{\varphi(\widehat{\gamma}^-_{k,\ell}(t)) + 10\Lambda_*\right\}\ dt \ = \ 0$$

and then

$$\int_{s_{k,\ell,2}}^{T_{k,\ell}} \varphi(\widehat{\gamma}^-_{k,\ell}(t))\ dt \ = -\lambda_{k,\ell}\ T_{k,\ell} + 10\Lambda_*\ s_{k,\ell,2} + 1 + \Lambda_*$$

Iterate this procedure K_3 steps to obtain values $\{s_{k,\ell,1}, \ldots, s_{k,\ell,K_3}\}$ all of which are ϵ_6-regular, and for $s_{k,\ell,i}$ satisfies

$$\int_{s_{k,\ell,i}}^{T_{k,\ell}} \varphi(\widehat{\gamma}^-_{k,\ell}(t))\ dt \ = -\lambda_{k,\ell}\ T_{k,\ell} + 10\Lambda_*\ s_{k,\ell,i} + (i-1)(1 + \Lambda_*) \tag{27}$$

For each $1 \leq i \leq K_3$, let $\sigma^i_{k,\ell}$ be the geodesic segment contained in a plaque from some point $w_{k,\ell,i} \in \mathcal{T}$ to $\gamma^-_{k,\ell}(s_{k,\ell,i})$. We note that $\sigma^i_{k,\ell}$ has length at most $d_{\mathcal{U}}$.

By the choice of K_3 there must exist $1 \leq i_1 < \cdots < i_{K_2} \leq K_3$ and some $x \in \mathcal{T}$ so that $d_{\mathcal{T}}(x, w_{k,\ell,i_\ell}) < \epsilon_1/4$ for $1 \leq \ell \leq K_2$. Set $i = i_1$ and $j = i_{K_2}$ then $d_{\mathcal{T}}(w_{k,\ell,i}, w_{k,\ell,j}) < \epsilon_1/2$.

Let $\tau_{k,\ell}$ denote the piecewise-geodesic curve formed from the segment $\sigma^i_{k,\ell}$ followed by the segment $\{\gamma^-_{k,\ell}(t) \mid s_{k,\ell,i} \leq t \leq s_{k,\ell,j}\}$, then followed by the reverse of $\sigma^j_{k,\ell}$. (This is a new definition of τ from before.)

Let $\mathbf{g}_{k,\ell}$ denote the holonomy map determined by $\tau_{k,\ell}$. Then as $s_{k,\ell,i}$ is an ϵ_6-regular value, the domain of $\mathbf{g}_{k,\ell}$ contains the interval $\mathcal{I}_{k,\ell,i} = (w_{k,\ell,i} - \epsilon_1, w_{k,\ell,i} + \epsilon_1)$ and satisfies the uniform estimate for all $y \in \mathcal{I}_{k,\ell,i}$

$$\mathbf{g}'_{k,\ell}(y) \leq \exp\{-(K_2 - 1)(1 + \Lambda_* - \Lambda_*/10) + 2\|\varphi\| d_{\mathcal{U}}\} \leq \exp\{-1\} < 1/2 \tag{28}$$

Hence,

$$\mathbf{g}_{k,\ell}(\mathcal{I}_{k,\ell,i}) \subset (w_{k,\ell,j} - \epsilon_1/2, w_{k,\ell,j} + \epsilon_1/2) \subset \mathcal{I}_{k,\ell,i}$$

It follows that $\mathbf{g}_{k,\ell}$ has a hyperbolic contracting fixed point $z_{k,\ell,i} \in \mathcal{I}_{k,\ell,i}$. Moreover, $\Lambda_* < 1$ so (28) implies $\log\{\mathbf{g}'_{k,\ell}(z_{k,\ell,i})\} < -\Lambda_*$. It follows that $\tau_{k,\ell}$ determines an irreducible plaque chain and corresponding piecewise-geodesic τ_* with expansion $\lambda(\tau_*) \leq -\Lambda_*$. It follows that τ_* must be composed of plaques from $\mathcal{P}_*$ and hence the hyperbolic fixed point $z_{k,\ell,i}$ of $\mathbf{g}_{k,\ell}$ lies on a leaf L_ξ for some $1 \leq \xi \leq N$.

We have shown that if $k \gg 0$ then $z_{k,\ell,i} \in \mathcal{I}_{k,\ell,i} \cap L_\xi$. Thus, the holonomy $\mathbf{g}_{k,\ell}$ along the geodesic segment $\{\gamma^-_{k,\ell}(t) \mid s_{k,\ell,i} \leq t \leq T_{k,\ell} - F_{k,\ell}\}$ maps $z_{k,\ell,i}$ to a point of $L_\xi \cap \mathcal{S}(\epsilon_1)$. By our choice of ϵ_6 so that $L_\xi \cap \mathcal{T}_{\beta_\xi} = \{z_\xi\}$ we must have $\mathbf{g}_{k,\ell}(z_{k,\ell,i}) \in \mathcal{P}_{\beta_\xi}(z_\xi)$.

Recall the goal is to estimate the distance from $u_{k,\ell}$ to z_ξ. Our eventual goal is actually to estimate the distance from $y_{k,\ell}$ to the image $w_\xi \in \mathcal{T}_\alpha$ of z_ξ in $\mathcal{I}_k$ (which exists by the above!) so we do that directly using the estimate (27) and and the definition of Λ_*. The estimate for the distance from $u_{k,\ell}$ to z_ξ then follows from (25).

Let $\mathbf{h}^-_{k,\ell,i}: \mathcal{I}_{k,\ell,i} \to \mathcal{T}_\alpha$ be the holonomy along the geodesic segment $\{\widehat{\gamma}^-_{k,\ell}(t) \mid s_{k,\ell,i} \leq t \leq T_{k,\ell}\}$. From the definitions we have

$$\mathbf{h}^-_{k,\ell,i}(\widehat{\gamma}^-_{k,\ell}(s_{k,\ell,i})) = y_{k,\ell} \ \ \& \ \ \mathbf{h}^-_{k,\ell,i}(u_{k,\ell,i})) = w_\xi$$

By the estimate (27) and the definition $\epsilon_1 = \epsilon_1(\Lambda_*/10)$, for all $y \in \mathcal{I}_{k,\ell,i}$ we have

$$\begin{aligned}
\log\left\{\mathbf{h}^-_{k,\ell,i}(y)\right\} &\leq \int_{s_{k,\ell,K_3}}^{T_{k,\ell}} \varphi(\widehat{\gamma}^-_{k,\ell}(t))\ dt + \Lambda_*\ T_{k,\ell}/10 \\
&\leq -\lambda_{k,\ell}\ T_{k,\ell} + 10\Lambda_*\ s_{k,\ell,i} + (i-1)(1+\Lambda_*) + \Lambda_*\ T_{k,\ell}/10 \\
&\leq -\lambda_{k,\ell}\ T_{k,\ell} + 10\Lambda_*\ T_{k,\ell} + (K_3-1)(1+\Lambda_*) + \Lambda_*\ T_{k,\ell} \\
&\leq (12\Lambda_* - \lambda_{k,\ell})\ T_{k,\ell} \qquad (29)
\end{aligned}$$

where (29) assumes that $(T_{k,\ell} - K_3)\Lambda_* \geq K_3$. We note that the definition of K_3 in (26) is independent of k, ℓ and $T_{k,\ell}$. Thus, the asymptotic estimate of $\mathbf{h}^-_{k,\ell,i}(y)$ is completely determined by $\lambda_{k,\ell} - 12\Lambda_*$ which we now estimate.

$$12\Lambda_* T_{k,\ell} \leq \frac{ET_{k,\ell}}{8d_{max}} \leq \frac{EN_k}{8} \leq \frac{1}{2}\log(q_k) \qquad (30)$$

The patient reader will now be rewarded. The following completes the proof of Theorem 1.1.

LEMMA 6.3 *For $k \gg 0$, let z_ξ correspond to the hyperbolic fixed-point constructed using the holonomy of the path γ_k. There exists $0 < \epsilon_7 < \epsilon_5$ and an element of holonomy*

$$\mathbf{k}_\xi: (z_\xi - \epsilon_7, z_\xi + \epsilon_7) \to (z_\xi - \epsilon_5, z_\xi + \epsilon_5) = \mathcal{I}_\xi$$

such that $\mathbf{k}_\xi(z_\xi) \neq z_\xi$.

Proof: The first step is to set up the notation. Choose $k \gg 0$ so the previous estimates hold, and consider the geodesic segment $\gamma_k(t) = \gamma_{k,\ell_k}(t)$ defined for $0 \leq t \leq T_{k,\ell_k} = T_k$.

Let $z_k = z_{k,\ell_k,i} \in \mathcal{I}_{k,\ell_k,i} = \mathcal{I}_k$ be the hyperbolic fixed-point constructed as above.

Let $y_k = y_{k,\ell_k} = \gamma_k(0)$ be the initial point, and $v_k = v_{k,\ell_k,i} \in \mathcal{T}$ be the plaque center containing $\gamma_k(s_{k,\ell_k,i})$. The holonomy along γ^-_k for $s_{k,\ell_k,i} \leq t \leq T_k$ is denoted $\mathbf{g}_k: \mathcal{I}_k \to \mathcal{I}_\alpha$ so we have $\mathbf{g}_k(v_k) = y_k$ and $\mathbf{g}_k(z_k) = w_\xi$.

Recall that $\mathbf{h}_k$ denote the holonomy along the full path $\gamma_k(t) \mid 0 \leq t \leq T_k\}$ and this was chosen so that $\mathbf{h}'_k(y_k)$ is maximum for the maps $\mathbf{h}_{k,\ell}$ defined by the plaque chains $\mathcal{P}_{k,\ell}$ constructed from the points $x_{k,\ell}$ and $x_{k,\ell+1}$ for all $1 \leq \ell < q_k$.

From (29) and (30) and $k \gg 0$ we get

$$\log\{\mathbf{g}'_k(y)\} \leq (12\Lambda_* - \lambda_k)\ T_k \leq -T_k\lambda_k + \log(q_k)/2 = -\log\{\mathbf{h}'_k(y_k)\} + \log(q_k)/2 \qquad (31)$$

We conclude that

$$d_{\mathcal{T}}(y_k, w_\xi) \leq 2\epsilon_1\sqrt{q_k}/\mathbf{h}'_k(y_k)$$

Let $\Xi_k = (y_k - \delta_1, y_k + \delta_2) \subset \mathcal{T}_\alpha$ be the open interval with length $4\epsilon_1\sqrt{q_k}/\mathbf{h}'_k(y_k)$ on either side of y_k, so that $w_\xi \in \Xi_k$.

Let $\Psi_k > 0$ denote the number of elements in the intersection $\Xi_k \cap \{x^k_\ell \mid 1 \leq \ell \leq e_k\}$. By the pigeon-hole principle applied to the this subset of Ξ_k, if $\Psi_k > 2$ there must be some pair $(x_{k,\ell}, x_{k,\ell+1})$ with

$$d_{\mathcal{T}}(x_{k,\ell}, x_{k,\ell+1}) \leq \frac{8\epsilon_1\sqrt{q_k}}{(\Psi_k - 2)hh'_k(y_k)}$$

Then the holonomy map $\mathbf{h}_{k,\ell}$ satisfies

$$\mathbf{h}'_{k,\ell}(y_{k,\ell}) \geq \frac{\epsilon_4(\Psi_k - 2)hh'_k(y_k)}{8\epsilon_1\sqrt{q_k}}$$

But it is given that $\mathbf{h}'_k(y_k) \geq \mathbf{h}'_{k,\ell}(y_{k,\ell})$ so that

$$\Psi_k \leq \frac{8\epsilon_1}{\epsilon_4}\sqrt{q_k} + 2$$

Thus, the set $\mathcal{S}^c_k = (J_k - \Xi_k) \cap \{x^k_\ell \mid 1 \leq \ell \leq e_k\}$ has cardinality $\Gamma_k \geq q_k - 2 - 8\epsilon_1\sqrt{q_k}/\epsilon_4$ which satisfies $\lim\limits_{k\to\infty} \Gamma_k/q_k = 1$.

The complement of Ξ_k in J_k can be written as as a disjoint union of intervals, $(J_k - \Xi_k) = J^1_k \cup J^2_k$ where possibly one of J^1_k or J^2_k is empty. The above estimate on the number of points in Ξ_k was independent of $\lambda_k \lambda_{k,\ell_k}$. Hence, we can then repeat the above arguments for the set $\mathcal{S}^c_k$ to obtain a pair of points $x_{k,\ell}$ and $x_{k,\ell+1}$ and holonomy map (with analogous notation) $\tilde{\mathbf{g}}_k$, and points $\tilde{y}_k$ and $\tilde{w}_\zeta$ such that $\tilde{\mathbf{g}}_k(\tilde{w}_\zeta) = z_\zeta$ where $\tilde{w}_\zeta \notin \Xi$.

This argument can be repeated at most $N+1$ times to obtain the case where $\tilde{w}_\zeta = w_\xi$, so we assume this is the case.

Let $\mathbf{f}_k$ denote the holonomy along the initial segment with $\mathbf{f}_k(y_k) = u_{k,\ell_k} = u_k$. By (25) there is the estimate

$$\exp\{-2\Lambda_* T_k\} \leq \mathbf{f}'_k(y_k) \leq \exp\{2\Lambda_* T_k\}$$

Similarly, we have

$$\exp\{-2\Lambda_* T_k\} \leq \tilde{\mathbf{f}}'_k(\tilde{y}_k) \leq \exp\{2\Lambda_* T_k\}$$

Note that we estimate $\log\{|\mathbf{f}_k(J_k)|\}$ by

$$\begin{aligned} 2\Lambda_* T_k + \log\{q_k\} &\leq T_k E/48d_{max} - N_k E/4 \\ &\leq (T_k - 12N_k d_{max} E)/48d_{max} \\ &\leq -11N_k E/48 \\ &\leq \epsilon_6 \quad \text{for k} \gg 0 \end{aligned}$$

Thus, both $\mathbf{f}_k(J_k)$ and $\tilde{\mathbf{f}}_k(J_k)$ are defined, with $\mathbf{k}_\xi = \tilde{\mathbf{f}} \circ \mathbf{f}^{-1}$ defined on some ϵ_7-neighborhood of z_ξ and by construction, $\mathbf{k}_\xi(z_\xi) = \tilde{\mathbf{f}}(w_\xi) \neq z_\xi$. $\square$

Case 4 of the proof of Theorem 1.1 is the hardest part of showing $h(\mathcal{G}_{\mathcal{F}}) > 0$ implies there exists a ping-pong game for $\mathcal{G}_{\mathcal{F}}$. The first three cases all just use information about the Φ_t-invariant measures – that they are uniformly t-discrete with an assumption on the supports – so philosophically belong to § 5. In Case 4, the dynamical implications of $h(\mathcal{G}_{\mathcal{F}}) > 0$ are used fully. As suggested in the introduction to this section, there is a conjectural proof of Theorem 1.1 using the measure entropy for foliations. There is no definition of the measure entropy for foliations, but considering this possibility still offers some insight to the use of the "closing lemma" type arguments and the need for working with the maximal exponent λ_* in the proof of Case 4.

For each $k > 0$, the collection of geodesic segments $\{\gamma_{k,\ell}(t) \mid 1 \leq \ell < q_k\}$ on M can be used to define a Φ_t-invariant probability measure μ_k on V. Given a continuous function g we set

$$\mu_k(g) = \frac{1}{q_k} \sum_{1}^{q_k} \frac{1}{T_{k,\ell}} \int_0^{T_{k,\ell}} g(\widehat{\gamma}_{k,\ell}(t)) \, dt \tag{32}$$

Let μ_* be a weak-* limit of the collection $\{\mu_k \mid k > 0\}$, with support $|\mu_*|$. One conjectures that the "measure entropy" associated to μ_* is positive. In any case, one can study the putative approximate measure entropies of the approximating measures $\mathbf{m}_k$, even if the limiting behavior has not been well-defined, especially when only qualitative dynamical information is needed.

Consider the proof in Case 4 where $\mathcal{F}$ has an exceptional minimal set K. The intervals I_k can then be chosen to be co-gaps for the Cantor set $K_\alpha = K \cap \mathcal{T}_\alpha$ corresponding to basic open sets in a coding of K_α, and we choose the points $\{x_{k,\ell} \mid 1 \leq \ell < q_k\}$ so that after N_k iterations of the holonomy on K_α the endpoints are separated by a gap of size at least ϵ_4. The maximal exponent λ_* provides a lower bound $1/\lambda_*$ on the local Hausdorff dimension of K_α, and the choice of the special paths γ_k starting at $y_k \in K_\alpha$ are obtained by selecting a generic point in K_α where this dimension is locally minimized. This choice ensures that that the iterations of y_k have maximal density in the coding of the set, so that the code for the orbit γ_k is well-approximated by t-hyperbolic periodic orbits with respect to the function φ. Thus, the analogy suggests that $h(\mathcal{G}_{\mathcal{F}}) > 0$ implies the approximating measure μ_k is supported on enough codes to obtain good periodic approximations to generic orbits. This is exactly the intuitive idea behind tthe proof that positive measure entropy implies positive topological entropy for diffeomorphisms, at least in the case of hyperbolic measures.

For a general codimension-one foliation with $h(\mathcal{G}_{\mathcal{F}}) > 0$ it is not possible to restrict the dynamics to (exceptional) minimal sets where some form of good coding is available, so the above analogy is limited. For example, the Cases 1 to 3 of the proof corespond to pathologies that are not typical of the dynamics on minimal sets. Still, the technical nature of the proof given here supports the hope for a more conceptual approach.

7 Entropy and the non-wandering set

We give two applications to the topological dynamics of codimension one C^1-foliations.

A point $x \in M$ is *wandering* (§ 7, [5]) if there exists an open neighborhood U_x such that for every leaf L of $\mathcal{F}$, either $L \cap U_x$ is empty, or is a connected set. The wandering points form an open saturated set whose complement is called the *non-wandering set* , and denoted by $\Omega(\mathcal{F})$. Clearly, the wandering set $M - \Omega(\mathcal{F})$ is open and $\Omega(\mathcal{F})$ is closed.

A leaf L in the wandering set must be proper, although if L is proper this is not sufficient for it to be in the wandering set.

Every resilient leaf is non-wandering. If $\mathcal{K}$ is a minimal set for $\mathcal{F}$ and does not consist of a single compact leaf, then every leaf of $\mathcal{K}$ is non-wandering.

THEOREM 7.1 *Let $\mathcal{F}$ be a codimension-one C^1-foliation. If $\mathbf{m}$ is an ergodic, t–hyperbolic, Φ_t–invariant measure on V, then $\pi(|\mathbf{m}|) \subset \Omega(\mathcal{F})$.*

Proof: From the proof of Theorem 3.5, given a generic point $(x, v) \in |\mathbf{m}|$ and $\epsilon > 0$ there is a -hyperbolic periodic orbit (z, w) such that $d_{\mathcal{T}}(x, z) < \epsilon$. Moreover, there exists $\epsilon_1 > 0$, $\alpha \in \mathcal{A}$, $z \in \pi(|\mathbf{m}|)_{\mathcal{P}} \cap \mathcal{T}_\alpha$ and a holonomy map

$$\mathbf{h_m}: (z - \epsilon_1, z + \epsilon_1) \to (z - \epsilon_1, z + \epsilon_1)$$

such that $\mathbf{h_m}(z) = z$ and $0 < \mathbf{h}'_{\mathbf{m}_*}(y) < c < 1$ for all $z - \epsilon_1 < y < z + \epsilon_1$. It follows that for every $z - \epsilon_1 < y < z + \epsilon_1$ with $y \neq z$ the leaf L_y through y is asymptotic to the leaf L_z through z. Thus, given any open set $U_x \subset U_\alpha$ the intersection $L_y \cap U_x$ will have an infinite number of connected components. Hence, $z \in \Omega(\mathcal{F})$.

As $\Omega(\mathcal{F})$ is closed and saturated, and z can be chosen arbitrarily close to x, $L_z \subset \Omega(\mathcal{F})$ for all such z implies $L_x \subset \Omega(\mathcal{F})$. But the Φ_t-orbit of (x, v) is dense in $|\mathbf{m}|$ so L_x is dense in $\pi(|\mathbf{m}|)$ and the claim follows. □

THEOREM 7.2 *Suppose $\mathcal{F}$ is a codimension-one C^1-foliation with $h(\mathcal{G}_{\mathcal{F}}) > 0$, then the relative geometric entropy $h(\mathcal{G}_{\mathcal{F}}, \Omega(\mathcal{F})) > 0$.*

Proof: We proved in § 6 that $h(\mathcal{G}_{\mathcal{F}}) > 0$ implies $\mathcal{G}_{\mathcal{F}}$ must have a ping-pong game. That is, there exists $\alpha \in \mathcal{A}$, an open connected set $\mathcal{I} \subset \mathcal{T}_\alpha$ and elements of holonomy $\mathbf{h}_1: \mathcal{I} \to \mathcal{I}$ and $\mathbf{h}_2: \mathcal{I} \to \mathcal{I}$ which are hyperbolic contractions, and satisfy $\mathbf{h}_1(\mathcal{I}) \cap \mathbf{h}_2(\mathcal{I}) = \emptyset$.

For $\ell = 1, 2$ let z_ℓ be the hyperbolic fixed-point for $\mathbf{h}_\ell$. Then the leaves L_1 and L_2 through these points must lie in $\Omega(\mathcal{F})$ and hence their closure K also. Thus, all the forward orbits of $\{z_1, z_2\}$ by the holonomy sub-semigroup generated by $\{\mathbf{h}_1, \mathbf{h}_2\}$ must lie in K. This implies $h(\mathcal{G}_{\mathcal{F}}, K) > 0$ and the claim follows. □

The Hirsch example [8] is a real analytic codimension one foliation of a compact 3-manifold with an exceptional minimal set K so that the complement $M - K$ is wandering. The entropy $h(\mathcal{G}_{\mathcal{F}}, K) > 0$ but also $h(\mathcal{G}_{\mathcal{F}}, M - K) > 0$, as is easily seen. Thus, unlike the case of dynamics for a single diffeomorphism, it is possible for the wandering set of a foliation to have positive entropy.

8 Entropy and partially t-expansive measures

In these last two sections, we consider the ergodic theory for C^1-foliations of codimension $q > 1$ whose leaves are smooth submanifolds. In this section, we prove the higher codimension version of Proposition 5.1. We first establish the notation and conventions needed.

Fix a Riemannian metric on TM and identify the normal bundle $Q \to M$ to $\mathcal{F}$ with the subbundle $T\mathcal{F}^\perp$ of vectors perpendicular to $T\mathcal{F}$. Endow Q with the subbundle Riemannian metric. The definition of the foliation geodesic flow $\Phi_t: V \to V$ remains unchanged, and preserves the foliation $\widehat{\mathcal{F}}$ on V whose leaves cover those of $\mathcal{F}$. The tangent space TV inherits a Riemannian metric from TM, and let $\mathbf{Q} \to V$ denote the normal bundle to $T\widehat{\mathcal{F}}$, which is identified with $T\widehat{\mathcal{F}}^\perp$. Let $\pi_\perp: T\widehat{\mathcal{F}} \to T\widehat{\mathcal{F}}^\perp$ be the fiberwise orthogonal projection.

The differential of Φ_t preserves $\mathbf{Q}$ for each t, and we let $D\Phi_t : \mathbf{Q} \to \mathbf{Q}$ denote the induced action on the bundle of normal vectors to $\widehat{\mathcal{F}}$, so induces a map

$$H_t(x,v) = D_\perp \Phi_t(x,v): T\widehat{\mathcal{F}}^\perp_{(x,v)} \to T\widehat{\mathcal{F}}^\perp_{\Phi_t(x,v)} \tag{33}$$

Note that while $D\Phi_t(x,v): TV \to TV$ must preserve the tangent distribution $T\widehat{\mathcal{F}}$, it need not preserve the orthogonal complements, so that we must compose $D\Phi_t(x,v)$ with orthogonal projection onto $T\widehat{\mathcal{F}}^\perp_{\Phi_t(x,v)}$. That is, $H_t(x,v) = \pi_\perp \circ D\Phi_t(x,v)$. Because each map $D\Phi_t(x,v)$ preserves $T\widehat{\mathcal{F}}$, the chain rule implies there is a cocycle identity

$$H_{t+s}(x,v) = H_s(\Phi_t(x,v)) \circ H_t(x,v) \tag{34}$$

Each vector space $\mathbf{Q}_{(x,v)}$ has a norm $\|\cdot\|_{(x,v)}$ obtained from the Riemannian metric, so we can define the norm of each transformation,

$$\|H_t(x,v)\| = \sup_{0 \neq w \in \mathbf{Q}_{(x,v)}} \frac{\|H_t(x,v)(w)\|_{\Phi_t(x,v)}}{\|w\|_{(x,v)}} \tag{35}$$

Define $G(x,v,t) = \log\{\|H_t(x,v)\|\}$. The norm of linear transformations is submultiplicative, $\|AB\| \leq \|A\| \cdot \|B\|$, so by (3) the function G satisfies the subadditive cocycle identity,

$$G(x,v,t+s) \leq G(\Phi_t(x,v),s) + G(x,v,t) \tag{36}$$

DEFINITION 8.1 *Assume that $\mathcal{F}$ is a C^1 foliation. For each $(x,v) \in V$ set*

$$\lambda_*(\mathcal{F})(x,v) = \limsup_{t\to\infty} \frac{G(x,v,t)}{t} \tag{37}$$

Define the maximal t-exponent *of $\mathcal{F}$ on a Φ_t-invariant set $K \subset V$ to be*

$$\lambda_*(\mathcal{F},K) = \limsup_{t\to\infty} \limsup_{(x,v)\in K} \frac{G(x,v,t)}{t} \tag{38}$$

We set $\lambda_*(\mathcal{F}) = \lambda_*(\mathcal{F},V)$.

PROPOSITION 8.2 *If $\mathcal{F}$ is a C^1-foliation with $h(\mathcal{G}_{\mathcal{F}}) > 0$, then $\lambda_*(\mathcal{F}) > 0$.*

Proof: We proceed as in the codimension one case, with some nuances due to the higher codimension. Set $E = h(\mathcal{G}_{\mathcal{F}}) > 0$. Let $0 < \epsilon_4 < \epsilon_0$ be such that $0 < \epsilon \leq \epsilon_4$ implies $h(\mathcal{G}_{\mathcal{F}}, \epsilon) > 3E/4$. Choose a sequence of integers $\{N_k \mid k > 0\}$ tending to infinity such that

$$e_k = \mathcal{S}(\mathcal{G}_{\mathcal{F}}, \epsilon_4, N_k) > \exp(N_k \cdot E/2)$$

Recall that $h(\mathcal{G}_{\mathcal{F}}) = \sup\{h(\mathcal{G}_{\mathcal{F}}, \mathcal{T}_\alpha) \mid \alpha \in \mathcal{A}\}$, so we can assume there exists a fixed transversal $\mathcal{T}_\alpha$ such that for each $k > 0$ there is an (N_k, ϵ_4)-separated subset $\{x^k_\ell \mid 1 \leq \ell \leq e_k\} \subset \mathcal{T}_\alpha$.

For each $k > 0$ and pair of indices $1 \leq i < j < e_k$ there exists a minimal length plaque chain $\mathcal{P}^k_{i,j}$ of length at most N_k, so that $x^k_i, x^k_j \in D_{\mathcal{P}^k_{i,j}}$ and $d_{\mathcal{T}}(\mathbf{h}^k_{i,j}(x^k_i), \mathbf{h}^k_{i,j}(x^k_j)) > \epsilon_4$. Here, $\mathbf{h}^k_{i,j}$ denotes the holonomy determined by $\mathcal{P}^k_{i,j}$ and $D_{\mathcal{P}^k_{i,j}}$ is the maximal domain of $\mathbf{h}^k_{i,j}$.

Recall that for $\alpha \in \mathcal{A}$, the local coordinate on $\mathcal{T}_\alpha$ is denoted by $t_\alpha : (-1,1)^q \to \mathcal{T}_\alpha$ so that we can identify transversal $\mathcal{T}_\alpha$ with $(-1,1)^q$. The Riemannian metric on Q induces a metric of the tangent space to each transversal $\mathcal{T}_\alpha$ which is pulled back via t_α to a metric on $(-1,1)^q$. We let $\|\cdot\|_\alpha$ denote the associated norm on the tangent space to $(-1,1)^q$. Let $C_{\mathcal{U}} \geq 1$ denote the maximum of all these norms:

$$C_{\mathcal{U}} = \max_{\alpha \in \mathcal{A}} \ \limsup_{\|v\|=1} \ \left\{\|v\|_\alpha, \|v\|_\alpha^{-1}\right\}$$

where the supremum is taken over all points $(x,v) \in T_x(-1,1)^q$ where v has length one in the Euclidean metric, and is finite as the foliation atlas is regular.

Given a linear map $L\colon T_x(-1,1)^q \to T_y(-1,1)^q$ viewed as a map between the normed spaces with norm $\|\cdot\|_\alpha$ on the domain and $\|\cdot\|_\beta$ on the range, we let $\|L\|_{\alpha\beta}$ denote the operator norm.

Choose a pair of points x^k_i, x^k_j so that $d_{\mathcal{T}_\alpha}(x^k_i, x^k_j)$ is minimal for $i \neq j$. As $\mathcal{T}_\alpha$ has dimension q and diameter at most 1, the pigeon-hole principle applied to the domain $(-1,1)^q$ implies that

$$d_{\mathcal{T}_\alpha}(x^k_i, x^k_j) \leq C_{\mathcal{U}}\sqrt{q} \cdot e_k^{-1/q} \leq C_{\mathcal{U}}\sqrt{q} \cdot \exp(-N_k \cdot E/2q)$$

By the mean value theorem applied to the composition $\mathbf{g}^k_{i,j} = t_\beta^{-1} \circ \mathbf{h}^k_{i,j} \circ t_\alpha$ there exists a point $y^k_{i,j} \in \mathcal{T}_\alpha$ such that

$$\|D\mathbf{h}^k_{i,j}(y^k_{i,j})\| = \|D\mathbf{g}^k_{i,j}(t_\alpha^{-1}(y^k_{i,j}))\|_{\alpha\beta} \geq \frac{\epsilon_4 \cdot \exp(N_k \cdot E/2q)}{C_{\mathcal{U}}^{2q} \cdot \sqrt{q}}$$

The plaque chain $\mathcal{P}^k_{i,j}$ determines a leafwise piecewise-geodesic path τ_k starting at $y^k_{i,j}$ and ending at $w^k_{i,j} = \mathbf{h}^k_{i,j}(y^k_{i,j})$ of length at most $N_k \cdot d_{max}$. By the completeness of the leafwise Riemannian metrics, there exists a length-minimizing, leafwise geodesic γ_k starting at $y^k_{i,j}$, ending at $w^k_{i,j}$ and homotopic to τ_k rel endpoints. The length of γ_k has the bound $\|\gamma_k\| \leq N_k \cdot d_{max}$. As τ_k and γ_k are leafwise homotopic, the germs at $y^k_{i,j}$ of the corresponding holonomy transformations $\mathbf{h}^k_{i,j}$ and $\mathbf{h}_{\gamma_k}$ are equal, hence $D\mathbf{h}_{\gamma_k}(y^k_{i,j}) = D\mathbf{h}^k_{i,j}(y^k_{i,j}) \geq \exp\{N_k\lambda\}$. It follows that

$$\frac{\log\{\|D\mathbf{h}^k_{i,j}(y^k_{i,j})\|\}}{\|\gamma_k\|} \geq \frac{\log(\epsilon_4) - 2q\log(C_{\mathcal{U}}) - \log(q)/2 + N_k \cdot E/2q}{N_k \cdot d_{max}} \to \frac{E}{2q \cdot d_{max}}$$

so that $\lambda_*(\mathcal{F}) \geq E/2qd_{max} > 0$. $\square$

A Φ_t-invariant probability measure $\mathbf{m}$ on V is said to be *partially t-expansive* if for $\mathbf{m}$-almost every $(x,v) \in |\mathbf{m}|$ we have $\lambda_*(\mathcal{F})(x,v) > 0$.

THEOREM 8.3 *If $\mathcal{F}$ is a C^1-foliation with $h(\mathcal{G}_{\mathcal{F}}) > 0$, then there exists an ergodic, partially t-expansive Φ_t-invariant measure $\mathbf{m}_*$ on V with $\lambda_*(\mathcal{F})(x,v) = \lambda_*(\mathcal{F})$ for $\mathbf{m}_*$-almost every (x,v).*

Proof: Note that for codimension one, this is just Proposition 5.1. Unfortunately, because the norm on matrix products is only subadditive, the same method of proof, using continuity of the cocycle derivative followed by an application of the ergodic theorem, does not work for codimension $q > 1$. Instead, we obtain an asymptotic average from the subadditive cocycle (36) using a standard orbit averaging technique, followed by an application of Kingman's subadditive ergodic theorem.

First, we establish a uniform expansion criterion.

LEMMA 8.4 *There exists a sequence $\{(x_k, v_k) \in V \mid k = 1, 2, \ldots\}$ such that*

$$\frac{G(x_k, v_k, k)}{k} \geq \lambda_*(\mathcal{F}) - 1/k \tag{39}$$

Proof: For each $k > 0$, using that $G(x,v,t)$ is uniformly bounded on $V \times [0,k]$ and the subadditive property (36), there exists $(y_k, w_k) \in V$ and $N_k > 0$ such that for $T_k = kN_k$ we have

$$\frac{G(y_k, w_k, T_k)}{T_k} > \lambda_*(\mathcal{F}) - 1/k$$

Again by the subadditive property (36) there is the estimate

$$G(y_k^1, w_k^1, k) + G(y_k^2, w_k^2, k) + \cdots + G(y_k^{N_k}, w_k^{N_k}, k) \geq G(y_k, w_k, T_k) \geq (\lambda_*(\mathcal{F}) - 1/k)T_k \tag{40}$$

where $(y_k^\ell, w_k^\ell) = \Phi_{\ell-1}(y_k, w_k)$. Then at least one term on the left-hand-side of (40) must satisfy $G(y_k^\ell, w_k^\ell, k) \geq (\lambda_*(\mathcal{F}) - 1/k)$. Set $(x_k, v_k) = (y_k^\ell, w_k^\ell)$. $\square$

For each $k > 0$ let $\sigma_k = \Phi_t(x_k, v_k)$ and define a probability measure μ_k on V by

$$\mu_k(g) = \frac{1}{k} \int_0^k g(\sigma_k(t))\, dt$$

Let μ_* be a weak-* limit of the sequence of measures $\{\mu_k \mid k = 1, 2, \ldots\}$.

Let $G_N \colon V \to \mathbf{R}$ be defined by $G_N(x,v) = G(x,v,N)$. Note that (36) can be rewritten

$$G_{K+N} \leq G_K \circ \Phi_N + G_N \tag{41}$$

LEMMA 8.5 *For each integer $N > 0$*

$$\frac{1}{N} \int_V G_N \, d\mu_* \ \geq \ \lambda_*(\mathcal{F}) \tag{42}$$

Proof: Fix $N > 0$ and consider sequences $k = m \cdot N + \ell$ for fixed $0 \le \ell < N$ and $m \to \infty$. Calculate

$$\begin{aligned}
\frac{1}{N} \int_V G_N \, d\mu_* &= \lim_{k\to\infty} \frac{1}{kN} \int_0^k G_N(\sigma_k(t)) \, dt \\
&= \lim_{m\to\infty} \frac{1}{kN} \int_0^{mN+\ell} G_N(\sigma_k(t)) \, dt \\
&= \lim_{m\to\infty} \frac{1}{mN^2} \int_0^{mN} G_N(\sigma_k(t)) \, dt \\
&= \lim_{m\to\infty} \frac{1}{mN^2} \sum_{i=1}^{m} \int_{(i-1)N}^{iN} G_N(\sigma_k(t)) \, dt \\
&= \lim_{m\to\infty} \frac{1}{mN^2} \sum_{i=1}^{m} \int_0^{N} G_N(\Phi_t(\Phi_{(i-1)N}(x_k, v_k))) \, dt \\
&= \lim_{m\to\infty} \frac{1}{mN^2} \int_0^{N} \sum_{i=1}^{m} G_N(\Phi_{t+(i-1)N}(x_k, v_k))) \, dt \\
&= \lim_{m\to\infty} \frac{1}{mN^2} \int_0^{N} \sum_{i=1}^{m} \log\left\{ \left\| D_\perp \Phi_N(\Phi_{t+(i-1)N}(x_k, v_k)) \right\| \right\} \, dt \\
&= \lim_{m\to\infty} \frac{1}{mN^2} \int_0^{N} \log\left\{ \prod_{i=1}^{m} \left\| D_\perp \Phi_N(\Phi_{t+(i-1)N}(x_k, v_k)) \right\| \right\} \, dt \\
&\ge \lim_{m\to\infty} \frac{1}{mN^2} \int_0^{N} \log\left\{ \left\| \prod_{i=1}^{m} D_\perp \Phi_N(\Phi_{t+(i-1)N}(x_k, v_k)) \right\| \right\} \, dt \\
&= \lim_{m\to\infty} \frac{1}{mN^2} \int_0^{N} \log\left\{ \| D_\perp \Phi_{mN}(\Phi_t(x_k, v_k)) \| \right\} \, dt && (43) \\
&= \lim_{m\to\infty} \frac{1}{mN^2} \, N \, \log\left\{ \| D_\perp \Phi_{mN}(x_k, v_k) \| \right\} && (44) \\
&= \lim_{m\to\infty} \frac{1}{mN} \log\left\{ \| D_\perp \Phi_{mN}(x_k, v_k) \| \right\} && (45) \\
&= \lim_{mN\to\infty} \lambda_*(\mathcal{F}) - 1/mN && (46) \\
&= \lambda_*(\mathcal{F})
\end{aligned}$$

where (46) follows from (45) by Lemma 8.4, and (44) follows from (43) by noting that the cocycle rule (34) implies

$$\begin{aligned}
D_\perp \Phi_{mN}(\Phi_t(x_k, v_k)) &= D_\perp \Phi_{mN+t}(x_k, v_k) \cdot D_\perp \Phi_{-t}(\Phi_t(x_k, v_k)) \\
&= D_\perp \Phi_t(\Phi_{mN}(x_k, v_k)) \cdot D_\perp \Phi_{mN}(x_k, v_k) \cdot D_\perp \Phi_{-t}(\Phi_t(x_k, v_k))
\end{aligned}$$

and thus

$$\begin{aligned}
&| \log\{\| D_\perp \Phi_{mN}(\Phi_t(x_k, v_k)) \|\} - \log\{\| D_\perp \Phi_{mN}(x_k, v_k) \|\} | \\
&\quad \le (\log\{\| D_\perp \Phi_t(\Phi_{mN}(x_k, v_k)) \|\} + \log\{\| D_\perp \Phi_{-t}(\Phi_t(x_k, v_k)) \|\})
\end{aligned}$$

Observe that $(\log\{\| D_\perp \Phi_t(\Phi_{mN}(x_k, v_k)) \|\} + \log\{\| D_\perp \Phi_{-t}(\Phi_t(x_k, v_k)) \|\})$ is uniformly bounded in m for $0 \le t \le N$, so taking the limit of the integrals in (43) yields (44). $\square$

Kingman's subadditive ergodic theorem ([16, 17]; page 231 [24]) applied to the sequence of functions $\{G_N \mid N = 1, 2, \ldots\}$ yields an $\mathbf{m}$-measurable function $\lambda_{\mathbf{m}}: V \to \mathbf{R}$ such that $\lambda_{\mathbf{m}} \circ \Phi_1 = \lambda_{\mathbf{m}}$ and $\lim\limits_{N\to\infty} G_N(x, v)/N = \lambda_{\mathbf{m}}(x, v)$ for $\mathbf{m}$-almost every $(x, v) \in V$. Moreover,

$$\int_V \lambda_{\mathbf{m}} \, d\mathbf{m} = \lim_{N\to\infty} \frac{1}{N} \int_V G_N \, d\mathbf{m} \geq \lambda_*(\mathcal{F})$$

Let $\mathbf{m}_*$ be an ergodic, Φ_t–invariant measure in the ergodic decomposition of $\mathbf{m}$ such that $\int_V \lambda_{\mathbf{m}} \, d\mathbf{m}_* \geq \lambda_*(\mathcal{F})$. Then for $\mathbf{m}_*$-almost every $(x, v) \in V$,

$$\lambda(\mathcal{F})(x, v) = \lim_{N\to\infty} G_N(x, v)/N \geq \lambda_*(\mathcal{F})$$

Since $\lambda(\mathcal{F})(x, v) \leq \lambda(\mathcal{F})$ this completes the proof Theorem 8.3. $\square$

The C^1-diffeomorphism $\Phi_1: V \to V$ preserves the measure $\mathbf{m}_*$ constructed in Theorem 8.3 so we can apply the Oseledets Theorem [19, 22] to $D\Phi_1(x, v): TV \to TV$ to obtain the *Lyapunov decomposition* of TV with respect to $\mathbf{m}_*$

- real numbers $\lambda_1 > \cdots > \lambda_k$ for $k \leq \dim V$
- positive integers $n_1, \ldots, n_k$ such that $n_1 + \cdots + n_k = \dim V$
- for $\mathbf{m}_*$-almost every $z \in V$ a measurable splitting $T_zV = E_z^1 \oplus \cdots E_z^k$, with $\dim E_z^\ell = n_\ell$ and $D\Phi_1(z): E_z^\ell \cong E_{\Phi_1(z)}^\ell$ such that

$$\lim_{N\to\infty} \frac{1}{N} \log\{\|D\Phi_N(z)w\|\} = \lambda_\ell \tag{47}$$

  whenever $w \in E_z^1 \oplus \cdots \oplus E_z^\ell$ but $w \notin E_z^1 \oplus \cdots \oplus E_z^{\ell-1}$.

Let $\Omega(\mathbf{m}_*) \subset V$ denote the set of points of full $\mathbf{m}_*$-measure for which (47) holds. Replacing $\Omega(\mathbf{m}_*)$ with $\bigcap\limits_{N=-\infty}^{\infty} \Phi_N(\Omega(\mathbf{m}_*))$, we can assume that $\Omega(\mathbf{m}_*)$ is a Φ_1-invariant set of full $\mathbf{m}_*$-measure. A point $z \in \Omega(\mathbf{m}_*)$ is said to be *regular* for $\mathbf{m}_*$.

The numbers $\{\lambda_1, \ldots, \lambda_k\}$ are called the *Lyapunov exponents* of $\mathbf{m}_*$. Since Φ_1 is the time 1 map of a flow, the flow direction in TV is an invariant subbundle of exponent 0. Thus, there exists some $\lambda_\ell = 0$. The positive exponents are $\{\lambda_1, \ldots, \lambda_{\ell-1}\}$ and the negative exponents are $\{\lambda_{\ell+1}, \ldots, \lambda_k\}$.

For $z \in \Omega(\mathbf{m}_*)$ let

$$E_z^+ = \bigoplus_{\lambda_i > 0} E_z^\ell \quad , \quad E_z^0 = E_z^\ell \quad , \quad E_z^- = \bigoplus_{\lambda_i < 0} E_z^\ell \tag{48}$$

E_z^+ is called the *unstable* subspace, E_z^0 the *neutral* subspace, and E_z^- the *stable* subspace.

Let $\Omega(\mathcal{F}, \mathbf{m}_*)$ denote the points of $\Omega(\mathbf{m}_*)$ such that $\lambda_*(\mathcal{F})(z) = \lambda_*(\mathcal{F})$ for all $z \in \Omega(\mathcal{F}, \mathbf{m}_*)$. Again, we can assume that $\Omega(\mathcal{F}, \mathbf{m}_*)$ is a Φ_1-invariant set of full $\mathbf{m}_*$-measure.

Let $z \in \Omega(\mathcal{F}, \mathbf{m}_*)$ and $w \in \mathbf{Q}_z = T\widehat{\mathcal{F}}_z^{\perp}$. Using the direct sum $T_zV = E_z^+ \oplus E_z^0 \oplus E_z^-$ we can uniquely decompose $w = w^+ + w^0 + w^-$. By (47),

$$\lim_{N\to\infty} \frac{1}{N} \log \left\{ \left\| D\Phi_N(z)(w^0 + w^-) \right\| \right\} \leq 0$$

so that $\lambda_*(\mathcal{F})(z) = \lambda_*(\mathcal{F}) > 0$ implies there exists a sequence of unit vectors $\{w_N \in \mathbf{Q}_z \mid N = 1, 2, \ldots\}$ such that

$$\lim_{N\to\infty} \frac{1}{N} \log \left\{ \left\| \pi_{\perp}(D\Phi_N(z)(w_N^+)) \right\| \right\} = \lambda_*(\mathcal{F})$$

Thus, $\mathbf{Q}_z^+ = \{w^+ \mid w \in \mathbf{Q}_z\}$ is a non-trivial vector space for all $z \in \Omega(\mathcal{F}, \mathbf{m}_*)$.

As a technical aside, note that for $\epsilon > 0$ and $N \gg 0$,

$$\begin{aligned} \left\| \pi_{\perp}(D\Phi_N(z)(w_N^+)) \right\| &\leq \left\| D\Phi_N(z)(w_N^+) \right\| \\ &\leq \exp\{N(\lambda_1 + \epsilon)\} \end{aligned}$$

so that

$$\exp\{-N(\lambda_1 - \lambda_*(\mathcal{F}) - \epsilon)\} \;\leq\; \|w_N^+\| \;\leq\; 1$$

In particular, as $\epsilon > 0$ was arbitrary, $\lambda_*(\mathcal{F}) \leq \lambda_1$.

Define $\mathbf{Q}_z^- = \{w^- \mid w \in \mathbf{Q}_z\}$.

If we replace the diffeomorphism Φ_1 with Φ_{-1} and the ergodic invariant measure $\mathbf{m}_*$ with the flow-reversed conjugate measure $\mathbf{m}_*^-$ then in the Oseledets decomposition (48) the stable and unstable bundles are reversed. Thus, the above arguments have shown

PROPOSITION 8.6 *Suppose that $\mathcal{F}$ is a C^1-foliation with $h(\mathcal{G}_{\mathcal{F}}) > 0$. Then there exists an ergodic, Φ_t-invariant probability measure $\mathbf{m}_*$ on V such that for all $z \in \Omega(\mathcal{F}, \mathbf{m}_*)$ we have E_z^- is non-trivial, and $\mathbf{Q}_z^-$ is a non-trivial subspace of $\mathbf{Q}_z$.* □

9 Transverse Pesin theory

We combine the results of the last section with the stable manifold theory of Pesin to obtain:

THEOREM 9.1 *Assume that $\mathcal{F}$ is a C^{1+a}-foliation, for some $a > 0$, with an ergodic, partially t-hyperbolic Φ_t-invariant measure $\mathbf{m}$ on V. Then there exists a leaf L and a leafwise geodesic path*

$$\gamma\colon [0, \infty) \to L \subset M$$

such that the transverse holonomy along γ admits a stable transverse manifold which is (transversally) attracted to L at an exponential rate.

Proof: For references on Pesin sets and stable manifolds see [14, 20, 21, 15].

By Proposition 8.6, there exists an ergodic, Φ_t-invariant probability measure $\mathbf{m}_*$ on V such that for all $z \in \Omega(\mathcal{F}, \mathbf{m}_*)$ we have E^-_z is non-trivial, and $\mathbf{Q}^-_z$ is a non-trivial subspace of $\mathbf{Q}_z$.

Let λ_ℓ be the neutral exponent of Φ_1 so that $\lambda_i < 0$ for $i > \ell$. Choose $0 < \nu < -\lambda_{\ell+1}$, $0 < \mu < \lambda_{\ell-1}$ and $\nu, \mu \gg \epsilon > 0$. Let $\Lambda(\nu, \mu, \epsilon) \subset \Omega(\mathcal{F}, \mathbf{m}_*)$ be the Pesin set, which is closed and non-empty for ϵ sufficiently small. Then for $z \in \Lambda(\nu, \mu, \epsilon)$, define the stable manifold through z for $\delta > 0$ by

$$W^s_\delta(z) = \{y \in V \mid d_V(\Phi_N(z), \Phi_N(y)) \leq \delta \exp\{-(\nu - \epsilon)N\} \text{ for all } N > 0\} \tag{49}$$

Then there exists $\epsilon, \delta > 0$ sufficiently small so that there exists $z \in \Lambda(\nu, \mu, \epsilon)$ and $W^s_\delta(z)$ is a C^1-submanifold of V with $T_z W^s_\delta(z) = E^-_z$.

Since $\mathbf{Q}^-_z$ is non-trivial, there is a curve $\sigma\colon (-\epsilon_8, \epsilon_8) \to W^s_\delta(z)$ with $\sigma(0) = z$ and $\pi(\sigma)$ is a transverse curve to $\mathcal{F}$ on M.

Let $\gamma(t) = \pi(\Phi_t(z))$ for $t \geq 0$ which is a leafwise geodesic path in M. Define

$$\Gamma\colon (-\epsilon_8, \epsilon_8) \times [0, \infty) \to M \quad \text{by} \quad \Gamma(s, t) = \pi(\Phi_t(\sigma(s))$$

Then $\Gamma(0, t) = \gamma(t)$, $\Gamma(s, 0) = \pi(\sigma(s))$ is a curve transverse to $\mathcal{F}$, and $\Gamma(s, t)$ is a leafwise geodesic curve for each $(-\epsilon_8, s < \epsilon_8)$.

By the definition of $W^s_\delta(z)$ and of the various metrics on TV, TM and $\mathbf{Q}$, we have

$$d_M(\gamma(t), \Gamma(s, t)) \leq C \exp\{-(\nu - \epsilon)t\} \tag{50}$$

for a suitable constant $C > 0$ and all $t \geq 0$. □

Combining Theorems 8.3 and 9.1 we obtain the proof of Theorem 1.4.

References

[1] O. Attie and S. Hurder. Manifolds which cannot be leaves of foliations. *Topology*, 35:335–353, 1996.

[2] A. Biś and P. Walczak. Pseudo-orbits, pseudo-leaves and geometric entropy of foliations. *Ergodic Theory Dynamical Systems*, 18:1335–1348, 1998.

[3] Alberto Candell and Lawrence Conlon. *Foliations I.* Amer. Math. Soc., Providence, RI, 2000.

[4] P. Dippolito. Codimension one foliations of closed manifolds. *Annals of Math.*,107:403–457, 1978.

[5] E. Ghys, R. Langevin, and P . Walczak. Entropie geometrique des feuilletages. *Acta Mathematica*, 168:105–142, 1988.

[6] A. Haefliger. Groupoïdes d'holonomie et classifiants. *Astérisque*, 116:70–97, Société Mathématique de France 1984.

[7] P. de la Harpe. Free groups in linear groups. *Ensign. Math.*, 29:129–144, 1983.

[8] M. Hirsch. A stable analytic foliation with only exceptional minimal sets. *Lecture Notes in Math.*, 468:9–10, 1975. Springer–Verlag, New York and Berlin.

[9] S. Hurder. The Jacobian Cocycle of a distal group action. *Math. Sci. Res. Inst. Preprint*, MSRI 12811-85, September 1985.

[10] S. Hurder. Ergodic theory of foliations and a theorem of Sacksteder. In *Dynamical Systems: Proceedings, University of Maryland 1986-87. Lect. Notes in Math. volume 1342*, pages 291–328, 1988. Springer–Verlag, New York and Berlin.

[11] S. Hurder. Exceptional minimal sets for $C^{1+\alpha}$-group actions on the circle. *Ergodic Theory Dynamical Systems*, 11:455–467, 1991.

[12] S. Hurder. Dynamics of expansive group actions on the circle. *preprint*, August 2000. Available at hhtp://www.math.uic.edu/∼hurder/publications.

[13] S. Hurder and R. Langevin. Dynamics and the Godbillon-Vey Class of C^1 Foliations. *in preparation*, to appear Fall 2000.

[14] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publ. Math. Inst. Hautes Etudes Sci.*, 51:137–173, 1980.

[15] A. Katok and L. Mendoza. Dynamical systems with nonuniformly hyperbolic behavior. Supplement to *Introduction to the Modern Theory of Dynamical Systems* by Anatole Katok and Boris Hasselblatt, Cambridge Univeristy Press, 1995.

[16] J. F. C. Kingman. The ergodic theory of subadditive ergodic stochastic processes. *J. Royal Stat. Soc.*, B30:499–510, 1968.

[17] J. F. C. Kingman. Subadditive processes. In *Lecture Notes in Math. Volume 539*, 1976. Springer–Verlag, New York and Berlin.

[18] R. Langevin and P. Walczak. Entropy, transverse entropy and partitions of unity. *Ergodic Theory Dynamical Systems*, 14:551–563, 1994.

[19] V. I. Oseledets. A multiplicative ergodic theorem. Lyanpunov characteristic numbers for dynamical systems. *Trans. Moscow Math. Soc.*, 19:197–221, 1968.

[20] Ya. B. Pesin. Characteristic Lyapunov exponents and smooth ergodic theory. *Russian Math. Surveys*, 32(4):55–114, 1977.

[21] Mark Pollicott. *Lectures on ergodic theory and Pesin theory on compact manifolds.* LMS Lecture Note Series vol. **180**, Cambridge University Press, 1993.

[22] D. Ruelle. Ergodic theory and differentiable dynamical systems. *Publ. Math. Inst. Hautes Etudes Sci.*, 50:275–305, 1979.

[23] P. Walczak. Dynamics of the geodesic flow of a foliation. *Ergodic Theory Dynamical Systems*, 8:637–650, 1988.

[24] P. Walters. An Introduction to Ergodic Theory. *Graduate Texts in Math. Volume 79*, 1982. Springer–Verlag, New York and Berlin.