Dynamical Commensurator Groups

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Let $\mathfrak{M}$ be a compact, connected metric space (a continuum).

$\mathrm{Homeo}(\mathfrak{M})$ denotes the homeomorphisms of $\mathfrak{M}$.

$\mathrm{Homeo}_0(\mathfrak{M})$ denotes the closed subgroup of $\mathrm{Homeo}(\mathfrak{M})$ which are isotopic to the identity map, a contractible space.

Definitions:

- [Mapping Class Group] $\mathrm{Mod}(\mathfrak{M}) \equiv \mathrm{Homeo}(\mathfrak{M})/\mathrm{Homeo}_0(\mathfrak{M})$
- [Pointed Mapping Class Group:] $\widehat{x} \in \mathfrak{M}$, set

$$\mathrm{Mod}(\mathfrak{M}, \widehat{x}) \equiv \mathrm{Homeo}(\mathfrak{M}, \widehat{\mathrm{x}})/\mathrm{Homeo}_0(\mathfrak{M}, \widehat{\mathrm{x}})$$

Problem: For $\mathfrak{M}$ a *solenoidal manifold*, calculate

$$\mathrm{Mod}(\mathfrak{M}) \ , \ \mathrm{Mod}(\mathfrak{M}, \widehat{x})$$

Example: For 1-dimensional solenoids:

⋆ Kwapisz, *Homotopy and dynamics for homeomorphisms of solenoids and Knaster continua*, **Fund. Math.**, 168(3):251–278, 2001

Example: Let Σ_g be a closed surface of genus $g \geq 2$ with basepoint $x \in \Sigma_g$. Let $\widehat{\Gamma}$ denote the profinite completion of $\Gamma = \pi_1(\Sigma_g, x)$. Let $\widehat{\Sigma_g}$ be the *universal hyperbolic solenoid*. That is, $\widehat{\Sigma_g}$ is the solenoidal manifold defined by a cofinal collection $\mathcal{P}$ in the set of all finite oriented normal coverings of Σ_g.

Theorem [Odden]:

$$\mathrm{Mod}(\widehat{\Sigma_g}, \widehat{x}) \cong \mathrm{Comm}(\Gamma) \ , \ \mathrm{Mod}(\widehat{\Sigma_g}) \cong \mathrm{Comm}(\Gamma) \times \widehat{\Gamma}$$

⋆ Odden, *The baseleaf preserving mapping class group of the universal hyperbolic solenoid*, **Trans. Amer. Math. Soc.**, 357:1829–1858, 2005.

Theorem: [Bering & Studenmund] Let M be a closed aspherical manifold, and $\widehat{M}$ denote the universal solenoid over M. Let $\mathcal{E}(\widehat{M}, \widehat{x})$ be the group of pointed homotopy self-equivalences of $\widehat{M}$. Then there are isomorphisms

$$\mathcal{E}(\widehat{M}, \widehat{x}) \cong \mathrm{Comm}(\Gamma) \ , \ \mathcal{E}(\widehat{M}) \cong \mathrm{Comm}(\Gamma) \times \widehat{\Gamma}$$

$\star$ Bering & Studenmund, *Topological models of abstract commensurators*, **Groups Geom. Dyn.**, 18:1403–1425, 2024.

Question: What about $\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x})$ for a general solenoid $\mathfrak{M}_{\mathcal{P}}$?

- Let M be a compact connected manifold without boundary.
- Let $\Gamma = \pi_1(M, x)$ for choice of $x \in M$
- Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a subgroup chain.

The chain $\mathcal{G}$ and basepoint $x \in M$ determine a tower of finite index coverings by closed manifolds:

$$\mathcal{P}_{\mathcal{G}} \equiv \qquad \{M_0 \xleftarrow{p_1} M_1 \xleftarrow{p_2} M_2 \xleftarrow{p_3} M_3 \xleftarrow{p_4} \cdots\}$$

where $q_\ell = p_\ell \circ \cdots \circ p_1 \colon M_\ell \to M_0$ induces the map

$$(q_\ell)_\# \colon \pi_1(M_\ell, x_\ell) \to \Gamma_\ell \subset \pi_1(M_0, x) = \Gamma_0$$

with image Γ_ℓ.

$$\mathfrak{M}_{\mathcal{P}} = \varprojlim \{p_{\ell+1} \colon M_{\ell+1} \to M_\ell\} \subset \prod_{\ell \geq 0} M_\ell$$

A point $\widehat{z} \in \mathfrak{M}_{\mathcal{P}}$ is a sequence $\widehat{z} = (z_0, z_1, z_2, \ldots)$ where $z_\ell \in M_\ell$ and $p_{\ell+1}(z_{\ell+1}) = z_\ell$ for $\ell \geq 0$.

- $\mathfrak{M}_{\mathcal{P}}$ is a compact connected space.
- $\widehat{q}_\ell \colon \mathfrak{M}_{\mathcal{P}} \to M_\ell$ defined by $\widehat{q}_\ell(\widehat{z}) = z_\ell$ is a fibration.
- Each fiber $\mathfrak{X}_\ell = \widehat{q}_\ell^{-1}(x_\ell)$ a Cantor space.
- If each Γ_ℓ is a normal subgroup of Γ, then the fiber $\mathfrak{X}_0$ is a profinite group.

Definition: $\mathfrak{M}_{\mathcal{P}}$ is a *weak solenoid* (McCord, 1985) or a *solenoidal manifold* (Sullivan, 2014)

Theorem: Let $h\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ be a homeomorphism preserving a basepoint $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$. There exists $n(i) \geq i$ and $h_i\colon M_{n(i)} \to M_i$ which induces $h_*\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ that is ϵ-homotopic to h.

Corollary: Let $h\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ be a homeomorphism preserving a basepoint $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$. Then h induces an injection $h_{\#}\colon \Gamma_{n(i)} \to \Gamma_i$ whose image has finite index.

Remark: There is no a priori bound on $n(i)$

$\star$ Rogers & Tollefson, *Homeomorphisms homotopic to induced homeomorphisms of weak solenoidal spaces*, **Colloq. Math.** 25:81–87, 1971/72.

Let Γ be a countable group.

A *commensurator* of Γ is a pair of finite-index subgroups $H, K \subset \Gamma$ and an isomorphism $\phi\colon H \to K$.

Two commensurators $\phi_1\colon H_1 \to K_1$ and $\phi_2\colon H_2 \to K_2$ are equivalent, $\phi_1 \sim \phi_2$, if there exists a finite index subgroup $H_3 \subset H_1 \cap H_2$ such that $\phi_1 | H_3 = \phi_2 | H_3$.

Definition: The *abstract commensurator group* $\mathrm{Comm}(\Gamma)$ is the collection of commensurators $\phi\colon H \to K$, modulo $\sim$.

Remark: Let $\Gamma' \subset \Gamma$ be a finite index subgroup. Given a commensurator $\phi\colon H \to K$ in $\mathrm{Comm}(\Gamma)$, observe that $\phi\colon H \cap \Gamma' \to K \cap \Gamma'$ is a commensurator in $\mathrm{Comm}(\Gamma')$. The converse clearly holds, hence $\mathrm{Comm}(\Gamma') \cong \mathrm{Comm}(\Gamma)$.

$\mathrm{Comm}(\Gamma)$ can be intuitively viewed as the group of "germs" of isomorphisms between finite-index subgroups of G.

Example: $\mathrm{Comm}(\mathbb{Z}) = \mathbf{GL}(\mathbb{Q}) = \mathbb{Q}^*$ the non-zero rationals

Example: $\mathrm{Comm}(\mathbb{Z}^2) = \mathbf{GL}(\mathbb{Q}^2) =$ invertible 2×2 rational matrices

Let $h\colon \mathfrak{M}_{\mathcal{P}} \to \mathfrak{M}_{\mathcal{P}}$ be a homeomorphism preserving a basepoint $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$. Let $\Gamma = \pi_1(M_0, x)$, then by Rogers & Tollefson, the homeomorphism h determines $\phi_h \in \mathrm{Comm}(\Gamma)$. This map does not depend on the isotopy class of h, so we obtain:

Theorem: There is a map $\chi\colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \to \mathrm{Comm}(\Gamma)$.

- The map χ need not be onto, as ϕ_h must preserve the group chain $\mathcal{G}_{\mathcal{P}}$
- The map χ need not be injective.
- Use ideas from dynamical systems to study χ.

Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a group chain in Γ.

For $\ell \geq 0$ let $X_\ell = \Gamma/\Gamma_\ell$ as a left Γ-space.

- Γ acts transitively on finite set $X_\ell = \Gamma/\Gamma_\ell$

$$X_{\mathcal{G}} \equiv \varprojlim \, \{p_{\ell+1} \colon X_{\ell+1} \to X_\ell \mid \ell \geq 0\} \subset \prod_{\ell \geq 0} X_\ell \ .$$

By the definition of the inverse limit,

$$x = (x_0, x_1, \ldots) \in X_{\mathcal{G}} \iff p_{\ell+1}(x_{\ell+1}) = x_\ell \text{ for all } \ell \geq 0 \ .$$

Γ acts on $X_{\mathcal{G}}$ by acting on each factor X_ℓ.

The action $\Phi \colon \Gamma \times X_{\mathcal{G}} \to X_{\mathcal{G}}$ is a (generalized) *odometer*. That is, it is a minimal, equicontinuous action on a Cantor space.

Clopen set $U \subset X_{\mathcal{G}}$ is *adapted* if $U \neq \emptyset$, and for all $\gamma \in \Gamma$, $\gamma \cdot U \cap U \neq \emptyset$ then $\gamma \cdot U = U$

$\Gamma_U = \{\gamma \in \Gamma \mid \gamma \cdot U = U\}$ is a subgroup of finite index in Γ.

Definition. Odometers $\Phi\colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ and $\Phi'\colon \Gamma' \times \mathfrak{X}' \to \mathfrak{X}'$ are *return equivalent* if there exists adapted sets $U \subset \mathfrak{X}$ and $U' \subset \mathfrak{X}'$ and homeomorphism $h_U\colon U \to U'$ that conjugates the subgroups

$$\begin{aligned} \mathcal{H}_U &= \mathrm{Image}\{\Phi_U\colon \Gamma_U \to \mathrm{Homeo(U)}\} \\ \mathcal{H}'_{U'} &= \mathrm{Image}\{\Phi'_{U'}\colon \Gamma'_{U'} \to \mathrm{Homeo(U')}\} \end{aligned}$$

The map $\Phi_U\colon \Gamma_U \to \mathcal{H}_U$ may have kernel, and likewise for $\Phi'_{U'}$.

Thus, a homeomorphism $h_U\colon U \to U'$ which conjugates $\mathcal{H}_U$ with $\mathcal{H}'_{U'}$ need not induce an isomorphism $\phi_U\colon \Gamma_U \to \Gamma'_{U'}$.

Definition: Let $(\mathfrak{X}, \Gamma, \Phi)$ be an odometer action, and $x \in \mathfrak{X}$.

- A *dynamical commensurator* is a homeomorphism $h_U \colon U \to V$, where U and V are adapted subsets with $x \in U \cap V$, such that $h_U(x) = x$, and h_U induces an isomorphism $\Theta_U \colon \mathcal{H}_U \to \mathcal{H}_V$.

- Commensurators $h_U \colon U \to V$ and $h_{U'} \colon U' \to V'$ are equivalent if there exists and adapted set U'' with $x \in U'' \subset U \cap U'$ such that $h_U | U'' = h'_{U'} | U''$. We write $(h_U, U, V) \overset{c}{\sim} (h'_{U'}, U', V')$.

Definition: The *dynamical commensurator group* of $(\mathfrak{X}, \Gamma, \Phi)$ at x is the set of germs of dynamical commensurators,

$$\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x) = \{h \colon U \to V \mid x \in U \cap V\} / \overset{c}{\sim}$$

Problem: How is $\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x)$ related to $\mathrm{Comm}(\Gamma)$?

The action $\Phi\colon \Gamma \times X_{\mathcal{G}} \to X_{\mathcal{G}}$ is *effective* if the action map $\Phi\colon \Gamma \to \mathrm{Homeo}(\mathfrak{X})$ has trivial kernel.

The literature contains a notion of the commensurator group relative to $\mathrm{Homeo}(\mathfrak{X})$:

$$\mathrm{Comm}_{\mathrm{Homeo}(\mathfrak{X})}(\Gamma) = \{h \in \mathrm{Homeo}(\mathfrak{X}) \mid \mathrm{h}\colon \Phi(\Gamma) \cong \Phi(\mathrm{K}); \mathrm{H}, \mathrm{K} \subset_{\mathrm{f}} \Gamma\}$$

Problem: How are these groups related?

$$\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x)\ ,\ \mathrm{Comm}_{\mathrm{Homeo}(\mathfrak{X})}(\Gamma)\ ,\ \mathrm{Comm}(\Gamma)$$

For an effective action Φ, clearly $\mathrm{Comm}_{\mathrm{Homeo}(\mathfrak{X})}(\Gamma) \subset \mathrm{Comm}(\Gamma)$.

The distinction between $\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x)$ and $\mathrm{Comm}_{\mathrm{Homeo}(\mathfrak{X})}(\Gamma)$ is in the domains of the conjugating maps. Also, for $\mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x)$ we need not assume the action is effective.

Let $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an odometer. The action is:

- *quasi-analytic* if for each clopen set $U \subset \mathfrak{X}$, if the action of $g \in \Gamma$ satisfies $\Phi(g)(U) = U$ and the restriction $\Phi(g)|U$ is the identity map on U, then $\Phi(g)$ acts as the identity on $\mathfrak{X}$.

(The dynamical analog of the *Unique Root Property*.)

- *topologically free* if it is effective, and the quasi-analytic condition holds for $U = \mathfrak{X}$.

- *locally quasi-analytic* if there exists $\epsilon > 0$ such that for any non-empty open set $U \subset \mathfrak{X}$ with $\mathrm{diam}(U) < \epsilon$, and for any non-empty open subset $V \subset U$, if the action of $g \in \Gamma$ satisfies $\Phi(g)(V) = V$ and the restriction $\Phi(g)|V$ is the identity map on V, then $\Phi(g)$ acts as the identity on U.

(A localized *Unique Root Property*.)

Definition: An odometer $(\mathfrak{X}, \Gamma, \Phi)$ is *coherent* if for every adapted set $U \subset \mathfrak{X}$, the restricted holonomy action map $\Phi_U \colon \Gamma_U \to Homeo(U)$ has finite kernel.

Example: If Γ is virtually nilpotent, that is, Γ contains a nilpotent subgroup $\Gamma' \subset \Gamma$ with finite index, then an effective odometer action of Γ is coherent and locally quasi-analytic.

Example: If Γ is a weakly branch group, and $\mathfrak{X}$ is the boundary of the tree on which the group acts, then the odometer action of Γ on $\mathfrak{X}$ is inherently not coherent, and not locally quasi-analytic.

Theorem: Let $\Phi \colon \Gamma \times \mathfrak{X} \to \mathfrak{X}$ be an effective, coherent and locally quasi-analytic odometer. Let $x \in \mathfrak{X}$. Then there exists an injection

$$\chi_\Phi \colon \mathrm{Comm}(\mathfrak{X}, \Gamma, \Phi, x) \to \mathrm{Comm}(\Gamma)$$

Theorem. Suppose that $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are weak solenoids. If $\mathfrak{M}_{\mathcal{P}}$ and $\mathfrak{M}'_{\mathcal{P}'}$ are homeomorphic, then their monodromy odometers $\Phi\colon \Gamma \times X_{\mathcal{P}} \to X_{\mathcal{P}}$ and $\Phi'\colon \Gamma' \times X'_{\mathcal{P}'} \to X'_{\mathcal{P}'}$ are *return equivalent*.

$\star$ Clark, Hurder & Lukina, *Classifying matchbox manifolds*, *Geom. Topol.*, 23(1):1–27, 2019.

Corollary: For $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$ there is a well-defined map

$$\sigma_{\mathcal{P},\widehat{x}}\colon \mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \longrightarrow \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, x)$$

Theorem: Let $\mathcal{P}$ be a presentation such that M_0 is strongly Borel, and the associated odometer $(X_{\mathcal{P}}, \Gamma, \Phi)$ is effective, coherent and locally quasi-analytic. Then for $\widehat{x} \in \mathfrak{M}_{\mathcal{P}}$ we have the composition

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \xrightarrow{\sigma_{\mathcal{P},\widehat{x}}} \mathrm{Comm}(X_{\mathcal{P}}, \Gamma, \Phi, x) \xrightarrow{\chi_\Phi} \mathrm{Comm}(\Gamma)$$

where $\sigma_{\mathcal{P},\widehat{x}}$ is surjective, and χ_Φ is injective.

The following classes of closed manifolds are strongly Borel:

- infra-nilmanifolds,
- closed Riemannian manifolds M with negative sectional curvatures,
- closed Riemannian manifolds M of dimension $n \neq 3, 4$ with non-positive sectional curvatures.

Example: Σ_g a closed surface of genus $g \geq 2$, $\Gamma = \pi_1(\Sigma_g, x)$.

Let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ with $\cap \Gamma_\ell = \{1\}$.

Let $\mathfrak{M}_{\mathcal{P}}$ be the solenoidal manifold over Σ_g defined by $\mathcal{G}$.

Theorem [H & Lukina, 2025]: If the odometer $(\Gamma, X_{\mathcal{G}}, \Phi)$ is coherent and locally quasi-analytic, then

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{P}}, \widehat{x}) \cong \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x) \subset \mathrm{Comm}(\Gamma)$$

⋆ H & Lukina, *Mapping class groups of solenoidal manifolds*, *preprint*, 2025.

Let $\Gamma = \pi_1(M, x)$ let $\mathcal{G} = \{\Gamma = \Gamma_0 \supset \Gamma_1 \supset \Gamma_2 \supset \cdots\}$ be a group chain with $\cap \Gamma_\ell = \{1\}$.

Theorem: [H & Lukina, 2025] Assume that M is strongly Borel. If the odometer $(\Gamma, X_{\mathcal{G}}, \Phi)$ is coherent and locally quasi-analytic, then

$$\mathrm{Mod}(\mathfrak{M}_{\mathcal{G}}, \widehat{x}) \twoheadrightarrow \mathrm{Comm}(X_{\mathcal{G}}, \Gamma, \Phi, x) \subset \mathrm{Comm}(\Gamma)$$

The first map is onto, but may have kernel.

$\star$ These results reduce the calculation of $\mathrm{Mod}(\mathfrak{M}_{\mathcal{G}}, \widehat{x})$ to calculating the image $Comm(X_{\mathcal{G}}, \Gamma, \Phi, x) \subset \mathrm{Comm}(\Gamma)$.

$\star$ That is, we look for subgroups $H, K \subset \Gamma$ and isomorphisms $\phi \colon H \to K$ which map the group chain $\mathcal{G}$ into itself.

Example: The integer Heisenberg group:

$$\Gamma_{\mathbb{Z}} = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \mid a, b, c \in \mathbb{Z} \right\}.$$

We denote a 3×3 matrix in $\Gamma_{\mathbb{Z}}$ by the coordinates as (a, b, c). Let $\mathcal{H}$ denote the real Heisenberg group, a, b, c are real numbers.

For distinct primes $p, q \geq 2$, define the self-embedding $\varphi_{p,q} \colon \Gamma \to \Gamma$ by $\varphi(a, b, c) = (pa, qb, pqc)$. Define

$$\Gamma_\ell = \varphi^\ell_{p,q}(\Gamma) = \{(p^\ell a, q^\ell b, (pq)^\ell c) \mid a, b, c \in \mathbb{Z}\}$$

which yields a group chain $\mathcal{G}_{p,q}$ in $\Gamma_{\mathbb{Z}}$. The subgroups Γ_ℓ are not normal, and the limit Cantor space $X_{p,q}$ is not a group. The order of the subgroups generated by a and b are invariants.

Let $\mathcal{H}_{p,q}$ be the solenoidal manifold defined by the tower of coverings $\mathcal{H}/\Gamma_\ell$ of $\mathcal{H}/\Gamma_\mathbb{Z}$. Then:

$$\mathrm{Mod}(\mathcal{H}_{p,q}, \widehat{x}) \cong \mathbb{Z} \times \mathbb{Z} \times \{\pm 1\} \times \{\pm 1\}$$

Many more examples of nilpotent odometer actions to which the theorem applies are given in the papers

⋆ H & Lukina, *Type invariants for non-abelian odometers*, **Ergodic Theory Dynamical Systems**, to appear, 2025.

⋆ H & Lukina, *Prime spectrum and dynamics for nilpotent Cantor actions*, **Pacific J. Math.**, 327:107–128, 2023.