British Mathematics Colloquium

An Invitation to Matchboxes

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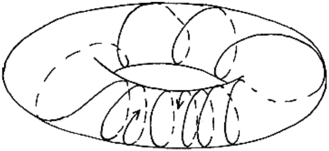
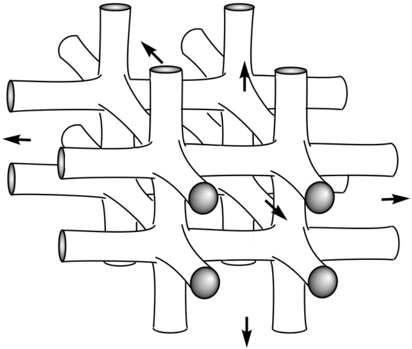
A simple example:



Triply periodic manifold



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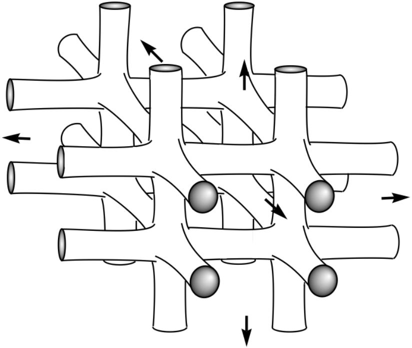
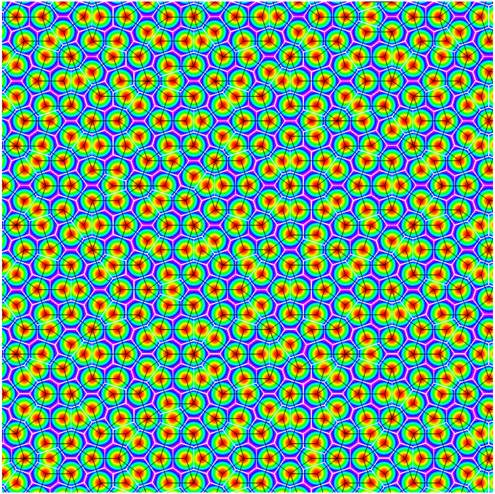


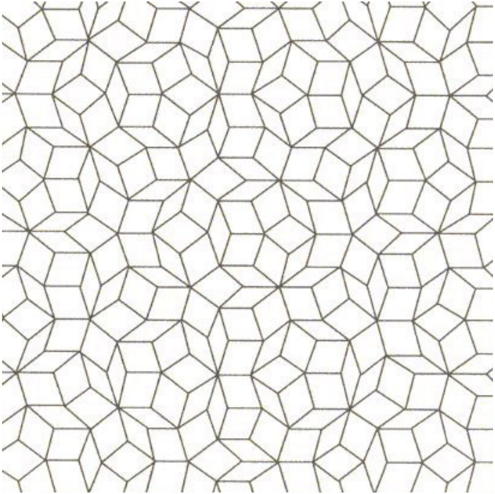
"Compactifying" gives a foliation of a compact 4-manifold

$M = (L \times \mathbb{S}^1)/\mathbb{Z}^3$ with leaf L

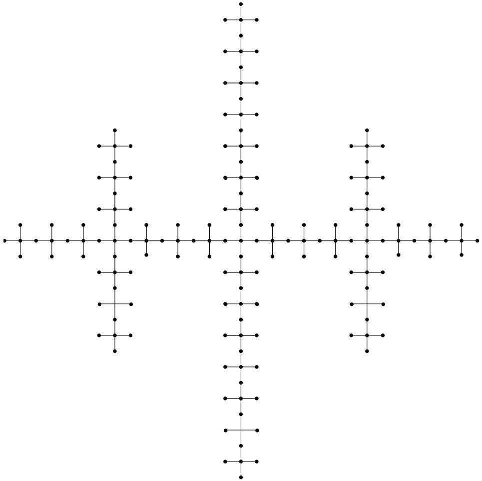
Penrose tiling



Penrose tiling stripped of decorations



Graph closures



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• What do the topological and algebraic properties of $(\mathfrak{M}, \mathcal{F})$ reveal about the original object of study, L_0 & its "marked data"? and vice-versa! Applies to tiling spaces, leaves of foliations, graph constructions, inverse limits, et cetera.

Matchbox manifolds

Definition: $\mathfrak{M}$ is a C^r-foliated space if it admits a covering by foliated coordinate charts $\mathcal{U} = \{\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i \mid i \in \mathcal{I}\}$ where $\mathfrak{T}_i$ are compact metric spaces.

The transition functions are assumed to be C^r, for $1 \leq r \leq \infty$, along leaves, and the derivatives depend (uniformly) continuously on the transverse parameter.

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Definition: An n-dimensional *matchbox manifold* is a continuum $\mathfrak{M}$ which is a smooth foliated space with codimension zero and leaf dimension n. Essentially, same concept as laminations.

$$\mathfrak{T}_i \quad \text{are totally disconnected} \iff \mathfrak{M} \quad \text{is a matchbox manifold}$$

Why "Matchbox"?

Term "matchbox manifold" was introduced in early 1990's in papers by Aarts, Fokkink, Hagopian and Oversteegen.

Many types of "foliated objects" - look locally:

- *Foliated Manifold*, if $U \cong (-1,1)^n \times \mathbb{D}^q$
- *Foliated Space*, if $U \cong (-1,1)^n \times \mathfrak{X}$ where $\mathfrak{X}$ is Polish
- *Menger Manifold*, if $U \cong$ Menger n-cube
- *Matchbox Manifold*, if $U \cong (-1,1)^n \times \mathfrak{T}$ where $\mathfrak{T}$ is totally disconnected.

Foliated spaces are introduced in Moore & Schochet, and good discussion is in textbook "Foliations, I" by Candel & Conlon.

Remarks

$\mathfrak{M}$ is *transitive* if there exists a dense leaf.

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Proof: Leaves of $\mathcal{F} \iff$ path components of $\mathfrak{M}$

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• A "smooth matchbox manifold" $\mathfrak{M}$ is analogous to a compact manifold, with the pseudogroup dynamics of the foliation $\mathcal{F}$ on the transverse fibers $\mathfrak{T}_i$ representing fundamental groupoid data.

• The category of matchbox manifolds has aspects of both *manifolds & algebraic systems*.

Examples I

- $L_0 = \mathbb{R}^n$ with a quasi-periodic tiling, $\mathfrak{M}$ is the tiling space.

- L_0 is the thickening of some graph $\mathcal{G}$, and $\mathfrak{M}$ is the *graph matchbox manifold* obtained by the Ghys-Kenyon construction applied to $\mathcal{G}$. [“Dynamics of graph matchbox manifolds”, by O. Lukina, 2011.]

- Suspensions of subshifts defined over countable group Γ.

When $\Gamma = \mathbb{Z}$, get classical flows built over a subshift.

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When $\Gamma = \mathbb{Z}^n$, get $\mathbb{R}^n$ action realizing commuting subshifts.

Γ arbitrary, get the sort of mess "that people in dynamics like".

Examples II - foliation minimal sets

• $\mathfrak{M} \subset M$ is a minimal set in a compact foliated manifold M. Assume that $\mathfrak{M}$ is "exceptional type" which means transversally is Cantor-like. Then each leaf $L_0 \subset \mathfrak{M}$ has closure $\mathfrak{M}$.

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Very little is known about the structure of foliation minimal sets

(for codimension $q > 1$.)

Examples III – weak solenoids

Let B_ℓ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving covering maps

$$\xrightarrow{p_{\ell+1}} B_\ell \xrightarrow{p_\ell} B_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} B_1 \xrightarrow{p_1} B_0$$

The p_ℓ are the *bonding maps* for the weak solenoid

$$\mathcal{S} = \varprojlim \; \{p_\ell \colon B_\ell \to B_{\ell-1}\} \; \subset \; \prod_{\ell=0}^{\infty} B_\ell$$

Proposition: $\mathcal{S}$ has natural structure of a matchbox manifold, with every leaf dense.

From Vietoris solenoids to McCord solenoids

Basepoints $x_\ell \in B_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$, set $G_\ell = \pi_1(B_\ell, x_\ell)$.

There is a descending chain of groups and injective maps

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0$$

Set $q_\ell = p_\ell \circ \cdots \circ p_1 \colon B_\ell \longrightarrow B_0$.

Definition: $\mathcal{S}$ is a *McCord solenoid* for some fixed $\ell_0 \geq 0$, for all $\ell \geq \ell_0$ the image $G_\ell \to H_\ell \subset G_{\ell_0}$ is a normal subgroup of G_{ℓ_0}.

Theorem [McCord 1965] Let B_0 be an oriented smooth closed manifold. Then a McCord solenoid $\mathcal{S}$ is an orientable, homogeneous, equicontinuous smooth matchbox manifold.

Classifying weak solenoids

A weak solenoid is determined by the base manifold B_0 and the tower equivalence of the descending chain

$$\mathcal{P} \equiv \left\{ \xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0 \right\}$$

Theorem:[Pontragin 1934; Baer 1937] For $G_0 \cong \mathbb{Z}$, the homeomorphism types of McCord solenoids is uncountable.

Theorem:[Kechris 2000; Thomas2001] For $G_0 \cong \mathbb{Z}^k$ with $k \geq 2$, the homeomorphism types of McCord solenoids is not classifiable, *in the sense of Descriptive Set Theory*.

The number of such is not just huge, but indescribably large.

Some open problems

Problem: *When does a matchbox manifold* $(\mathfrak{M}, \mathcal{F})$ *embed as an invariant set, for a* C^r*-foliation* $\mathcal{F}_0$ *of a compact manifold* M*?*

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The embedding of $\mathfrak{M}$ into M is up to homeomorphism of $\mathfrak{M}$, which is a subtle point in realizing such examples in general, for $r \geq 2$.

["Embedding matchbox manifolds", by A. Clark & S. Hurder, submitted 2009.]

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• Find relations between particular types of topological dynamics and the C^r-embedding problem, to obtain a new approach to non-realization results of Ghys [1985]; Inaba, Nishimori, Takamura and Tsuchiya [1985]; and Attie & Hurder [1996].

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- $\mathbf{Homeo}(\mathfrak{M}) = \mathbf{Homeo}(\mathfrak{M}, \mathcal{F})$ – all homeomorphisms
- $\mathbf{Inner}(\mathfrak{M}, \mathcal{F}) = \mathbf{Homeo}(\mathcal{F})$ – leaf-preserving homeomorphisms
- $\mathbf{Out}(\mathfrak{M}) = \mathbf{Homeo}(\mathfrak{M}, \mathcal{F})/\mathbf{Inner}(\mathfrak{M}, \mathcal{F})$ - outer automorphisms

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Problem: Study $\mathbf{Out}(\mathfrak{M})$.

$\mathbf{Out}(\mathfrak{M})$ captures many aspects of the space $\mathfrak{M}$ – its topological, dynamical and algebraic properties. We discuss this more.

Pseudogroups

Covering of $\mathfrak{M}$ by foliation charts $\Longrightarrow$ transversal $\mathcal{T} \subset \mathfrak{M}$ for $\mathcal{F}$

Holonomy of $\mathcal{F}$ on $\mathcal{T}$ $\Longrightarrow$ compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$:

- relatively compact open subset $\mathcal{T}_0 \subset \mathcal{T}$ meeting all leaves of $\mathcal{F}$
- a finite set $\Gamma = \{g_1, \ldots, g_k\} \subset \mathcal{G}_{\mathcal{F}}$ such that $\langle \Gamma \rangle = \mathcal{G}_{\mathcal{F}} | \mathcal{T}_0$;
- $g_i \colon D(g_i) \to R(g_i)$ is the restriction of $\widetilde{g}_i \in \mathcal{G}_{\mathcal{F}}$, $\overline{D(g)} \subset D(\widetilde{g}_i)$.

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Dynamical properties of $\mathcal{F}$ formulated in terms of $\mathcal{G}_{\mathcal{F}}$; e.g.,

$\mathcal{F}$ has no leafwise holonomy if for $g \in \mathcal{G}_{\mathcal{F}}$, $x \in Dom(g)$, $g(x) = x$ implies $g|V = Id$ for some open neighborhood $x \in V \subset \mathcal{T}$.

Topological dynamics

Definition: $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts as above, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, x' \in D(h_{\mathcal{I}}) \text{ with } d_{\mathcal{T}}(x, x') < \delta \quad \Longrightarrow \quad d_{\mathcal{T}}(h_{\mathcal{I}}(x), h_{\mathcal{I}}(c')) < \epsilon$$

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Theorem: [Clark-Hurder 2010] Let $\mathfrak{M}$ be an equicontinuous matchbox manifold. Then $\mathfrak{M}$ is minimal.

Shape theory

The *shape* of a set $\mathfrak{M} \subset \mathcal{B}$ is defined by a co-final descending chain $\{U_\ell \mid \ell \geq 1\}$ of open neighborhoods in Banach space $\mathcal{B}$,

$$U_1 \supset U_2 \supset \cdots \supset U_\ell \supset \cdots \supset \mathfrak{M} \quad ; \quad \bigcap_{\ell=1}^{\infty} U_\ell = \mathfrak{M}$$

Such a tower is called a shape approximation to $\mathfrak{M}$.

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Such a tower is called a shape approximation to $\mathfrak{M}$.

Homeomorphism $h \colon \mathfrak{M} \to \mathfrak{M}'$ induces maps $h_{\ell,\ell'} \colon U_\ell \to U'_{\ell'}$ of shape approximations.

Main technical result

Theorem: Let $\mathfrak{M}$ be a transitive matchbox manifold with no leafwise holonomy. Then $\mathfrak{M}$ has a shape approximation such that each U_ℓ admits a quotient map $\pi_\ell \colon U_\ell \to B_\ell$ for $\ell \geq 0$ where B_ℓ is a "branched n-manifold", covered by a leaf of $\mathcal{F}$.

Moreover, the system of induced maps $p_\ell \colon B_\ell \to B_{\ell-1}$ yields an inverse limit space homeomorphic to $\mathfrak{M}$.

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• For $\mathfrak{M}$ a tiling space on $\mathbb{R}^n$, this is just the presentation of $\mathfrak{M}$ as inverse limit in usual methods.

• For $\mathfrak{M}$ with foliation defined by free G-action and tiling on orbits, as in Benedetti & Gambaudo, same as their result.

Remarks on general case

For general $\mathfrak{M}$, the problem is to find good local product structures, which are stable under transverse perturbation. The leaves are not assumed to have flat structures, so this adds an extra level of difficulty, as compared to the methods in paper of Giordano, Matui, Hiroki, Putnam, & Skau: "Orbit equivalence for Cantor minimal $\mathbb{Z}^d$-systems", Invent. Math. 179 (2010)

The difficulties depends on the dimension:

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The difficulties depends on the dimension:

- For $n = 1$, it is trivial.

- For $n = 2$, given a uniformly spaced net in L_0, the volumes of triangles in the associated Delaunay triangulation in the plane are *a priori* bounded by the net spacing estimates.

• For $n \geq 3$, there are no *a priori* estimates on simplicial volumes, and the method becomes much more involved.

All solutions require some form of positivity restriction in the choice of the leafwise nets formed by a refined transversal, to control lack of transversality arguments due to the transversal geometry being "totally disconnected".

In terms of *leaf dimensions*, we have the fundamental observation:

$$1 \ll 2 \ll 3 < n$$

"Homogeneous matchbox manifolds" , by A. Clark & S. Hurder, 2010.

"Shape of matchbox manifolds" , by A. Clark, S. Hurder & O. Lukina, 2011.

Coding

Final method, motivated by technique of E. Thomas in 1970 paper for 1-dimensional matchbox manifolds.

Theorem: Suppose that $\mathcal{F}$ is equicontinuous without leafwise holonomy. Then for all $\epsilon > 0$, there is a decomposition into disjoint clopen sets, for $k_\epsilon \gg 0$,

$$\mathcal{T} = \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_{k_\epsilon}$$

such that $diam(\mathcal{T}_i) < \epsilon$, and the sets $\mathcal{T}_i$ are permuted by the action of $\mathcal{G}_{\mathcal{F}}$. Thus, we obtain a "good coding" of the orbits of the pseudogroup $\mathcal{G}_{\mathcal{F}}$.

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Theorem: [C & H, 2010] Let $\mathfrak{M}$ be a equicontinuous matchbox manifold without holonomy. Then $\mathfrak{M}$ is minimal, and homeomorphic to a weak solenoid.

Corollary: Let $\mathfrak{M}$ be a equicontinuous matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to the suspension of an minimal action of a countable group on a Cantor space $\mathbb{K}$.

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Theorem: [C & H, 2010] Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then $\mathfrak{M}$ is equicontinuous, minimal, without holonomy; and $\mathfrak{M}$ is homeomorphic to a McCord solenoid.

Solves generalization of conjecture by Bing [1960], previous cases: Thomas [1973]; Aarts, Hagopian, Oversteegen [1991]; Clark [2002].

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Question': Let $\mathfrak{M}, \mathfrak{M}'$ be matchbox manifolds of leaf dimension n, with transversals $\mathcal{T}, \mathcal{T}'$ and associated pseudogroups $\mathcal{G}_{\mathcal{F}}$ and $\mathcal{G}'_{\mathcal{F}'}$. Given a homeomorphism $h\colon \mathcal{T} \to \mathcal{T}'$ which intertwines actions of $\mathcal{G}_{\mathcal{F}}$ and $\mathcal{G}'_{\mathcal{F}'}$, when does there exists a homeomorphism $H\colon \mathfrak{M} \to \mathfrak{M}'$ which induces h?

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Theorem: True for $n = 1$, i.e., for oriented flows.

J.M. Aarts and M. Martens, "Flows on one-dimensional spaces", Fund. Math., 131:3958, 1988.

co-Hopfian

Example of Alex Clark shows this is false for $n = 2$!

False even for solenoids built over a surface B_0 of higher genus.

The problem comes up from the fact that covers of the base B_0 need not be homeomorphic to the base.

Definition: A group Γ is *co-Hopfian* if every injective map $j\colon \Gamma \to \Gamma$ is surjective.

Definition: A compact manifold B is *co-Hopfian* if every self-covering $\pi\colon B \to B$ is a diffeomorphism.

For example, the torus $\mathbb{T}^n$ is not co-Hopfian, while a surface B with genus at least 2 is co-Hopfian.

ψ-expansive matchbox manifolds

We want a version of this for solenoids and matchbox manifolds.

Definition: $\mathfrak{M}$ is ψ-*expansive* if there exists a transversal $\mathcal{T} \subset \mathfrak{M}$, such that for any $\epsilon > 0$, there exists a homeomorphism $\psi\colon \mathfrak{M} \to \mathfrak{M}$ such that $\psi(\mathcal{T}) \subset \mathcal{T}$ with diameter at most ϵ.

For example, if $\mathfrak{M}$ is the tiling space of a substitution dynamical system, then it is ψ-expansive.

Theorem: [C,H & L, 2011] Let $\mathfrak{M}, \mathfrak{M}'$ be equicontinuous without holonomy, and assume that both $\mathfrak{M}$ and $\mathfrak{M}'$ are ψ-expansive.

Suppose that we are given fibrations $\pi \colon \mathfrak{M} \to B_0$ and $\pi' \colon \mathfrak{M}' \to B_0'$, and a homeomorphism $h_0 \colon B_0 \to B_0'$, and there exists a conjugation of their holonomy pseudogroups as above.

Then h_0 extends to a homeomorphism $H \colon \mathfrak{M} \to \mathfrak{M}'$ inducing h_0.

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Problem: Understand equivalence between matchbox manifolds in terms of their holonomy pseudogroups, and other invariants of their dynamics and geometry.

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Long way to go...