# Classification of matchbox manifolds

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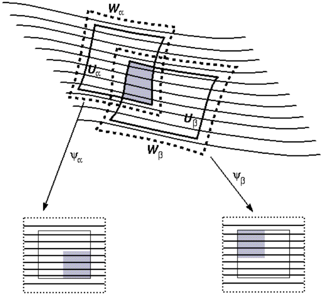
Workshop on Dynamics of Foliations

# Smooth foliated manifolds

**Definition:** $M$ is a $C^r$ *foliated manifold* if the leafwise transition functions for the foliation charts $\varphi_i \colon U_i \to [-1,1]^n \times T_i$ (where $T_i \subset \mathbb{R}^q$ is open) are $C^\infty$ leafwise, and vary $C^r$ with the transverse parameter in the leafwise $C^\infty$-topology.



# What are the minimal sets?

$M$ compact foliated, $\mathcal{F}$ foliation of codimension-$q$.

$\mathfrak{M} \subset M$ is minimal if it is closed, $\mathcal{F}$-saturated, and minimal w.r.t. $\subset$

$\Longrightarrow \mathfrak{M}$ compact and connected, hence an example of:

**Definition:** A *continuum* is a compact and connected metrizable space.

As every leaf of $\mathfrak{M}$ is dense, it is actually an example of something stronger

**Definition:** An *indecomposable continuum* is a continuum that is not the union of two proper subcontinua.

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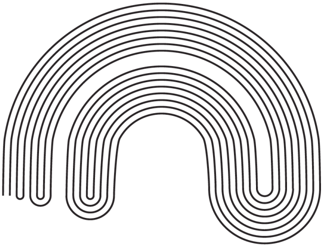
**Definition:** An *indecomposable continuum* is a continuum that is not the union of two proper subcontinua.

**Question:** What indecomposable continua can arise as minimal sets in codimension $q$?.

Can we say something about their types as topological spaces?

# Quiz!

**Question:** Is the Knaster Continuum (or *bucket handle*) realizable as a minimal set?

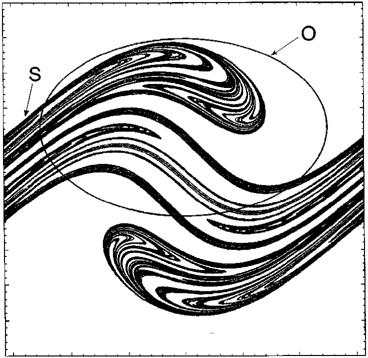


This is one-half of a Smale Horseshoe.

The 2-solenoid over $\mathbb{S}^1$ is a branched double-covering of it.

# Answer

Indecomposable continuum arise naturally as invariant closed sets of dynamical systems; e.g., attractors and minimal sets for diffeomorphisms.




[Picture courtesy Sanjuan, Kennedy, Grebogi & Yorke, "Indecomposable continua in dynamical systems with noise", Chaos 1997]

# Codimension one

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In most cases (for $C^1$-foliations at least) a minimal set is *stable*.

**Definition:** $X$ a Polish space. $\mathfrak{M} \subset X$ is stable if there exists a sequence of open sets

$$X \supset U_1 \supset U_2 \supset \cdots U_k \cdots \supset \mathfrak{M} \quad , \quad \mathfrak{M} = \bigcap_{k=1}^{\infty} U_k$$

with each inclusion $U_{k+1} \subset U_k$ a homotopy equivalence.

Denjoy minimal sets are stable, as are Markov minimal sets.

# Riemannian foliations

**Question:** What are the minimal sets in Riemannian foliations?

**Answer:** Smooth connected submanifolds.

Proof uses Molino structure theory for TP foliations. Key point is that a TP foliation is homogeneous: the group of foliation preserving diffeomorphisms acts transitively on $M$. In particular, it acts transitively on its minimal sets.

**Definition:** A space $\mathfrak{M}$ is *homogeneous* if for every $x, y \in X$ there exists a *homeomorphism* $h\colon X \to X$ such that $h(x) = y$. Equivalently, $X$ is homogeneous if the group $\mathrm{Homeo}(X)$ acts transitively on $X$.

# Bing Conjecture

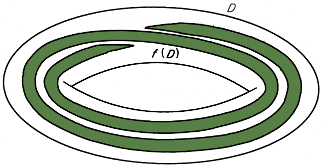
**Question:** [Bing1960] Let $\mathfrak{M}$ be a homogeneous continuum, and suppose that every proper subcontinuum of $\mathfrak{M}$ is an arc, hence $\mathfrak{M}$ is a foliated space with one-dimensional leaves.

Must $X$ then be a circle or a solenoid?

**Question:** [Bing1960] Let $\mathfrak{M}$ be a homogeneous continuum, and suppose that every proper subcontinuum of $\mathfrak{M}$ is an arc, hence $\mathfrak{M}$ is a foliated space with one-dimensional leaves.

Must $X$ then be a circle or a solenoid?

**Theorem:** [Hagopian 1977] Let $\mathfrak{M}$ be a homogeneous continuum such that every proper subcontinuum of $\mathfrak{M}$ is an arc, then $\mathfrak{M}$ is an inverse limit over the circle $\mathbb{S}^1$.



# Matchbox manifolds (*a.k.a.* laminations)

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We rephrase the context:

**Definition:** An $n$-dimensional *matchbox manifold* is a continuum $\mathfrak{M}$ which is a foliated space with leaf dimension $n$, and codimension zero.

$\mathfrak{M}$ is a foliated space if it admits a covering $\mathcal{U} = \{\varphi_i \mid 1 \leq i \leq \nu\}$ with foliated coordinate charts $\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i$. The compact metric spaces $\mathfrak{T}_i$ are totally disconnected $\Longleftrightarrow$ $\mathfrak{M}$ is a matchbox manifold.

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$$\text{Leaves of } \mathcal{F} \iff \text{path components of } \mathfrak{M} \iff \text{proper subcontinua}$$

# Automorphisms of matchbox manifolds

A "smooth matchbox manifold" $\mathfrak{M}$ is analogous to a compact manifold, with the transverse dynamics of the foliation $\mathcal{F}$ on the Cantor-like fibers $\mathfrak{T}_i$ representing fundamental groupoid data. They naturally appear in:

- dynamical systems, as minimal sets & attractors
- geometry, as laminations
- complex dynamics, as universal Riemann surfaces
- algebraic geometry, as models for "stacks".

*F*or more examples, see Chapter 11 of *Foliations I*, by Candel & Conlon.

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**"Zimmer":** What countable groups $\Lambda$ act effectively on $\mathfrak{M}$?

**"Haefliger":** What are the topological invariants associated to matchbox manifolds, and do they characterize them in some fashion?

# A solution to the Bing Question

**Theorem** [Clark & Hurder 2009] Let $\mathfrak{M}$ be an orientable homogeneous smooth matchbox manifold. Then $\mathfrak{M}$ is homeomorphic to a generalized solenoid. In particular, the dynamics of the foliation $\mathcal{F}$ on $\mathfrak{M}$ is equicontinuous and minimal – every leaf is dense.

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The key to the proof in the general case is the extensive use of *pseudogroup dynamics*, and actions of totally discrete groups, in place of Lie group actions as in Molino Structure Theory.

# Two applications

Here are two consequences of the Main Theorem:

**Corollary:** Let $\mathfrak{M}$ be an orientable homogeneous $n$-dimensional smooth matchbox manifold, which is embedded in a closed $(n+1)$-dimensional manifold. Then $\mathfrak{M}$ is itself a manifold.

For $\mathfrak{M}$ a homogeneous continuum with a non-singular flow, this was a question/conjecture of Bing, solved by [Thomas 1971]. Non-embedding for solenoids of dimension $n \geq 2$ was solved by [Clark & Fokkink, 2002]. Proofs use shape theory and Alexander-Spanier duality for cohomology.

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**Corollary:** An exceptional minimal set $\mathfrak{M}$ for a codimension-1 foliation is never homogeneous.

**Remark:** For codimension-$q$ (with $q \geq 2$), there are many examples of foliations with homogeneous minimal sets that are not submanifolds.

# Generalized solenoids

Let $M_\ell$ be compact, orientable manifolds of dimension $n \geq 1$ for $\ell \geq 0$, with orientation-preserving covering maps

$$\xrightarrow{p_{\ell+1}} M_\ell \xrightarrow{p_\ell} M_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} M_1 \xrightarrow{p_1} M_0$$

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The $p_\ell$ are called the *bonding maps* for the solenoid

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Choose basepoints $x_\ell \in M_\ell$ with $p_\ell(x_\ell) = x_{\ell-1}$.

Set $G_\ell = \pi_1(M_\ell, x_\ell)$.

# McCord solenoids

There is a descending chain of groups and injective maps

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0$$

Set $q_\ell = p_\ell \circ \cdots \circ p_1 \colon M_\ell \longrightarrow M_0$.

**Definition:** $\mathcal{S}$ is a *McCord solenoid* for some fixed $\ell_0 \geq 0$, for all $\ell \geq \ell_0$ the image $G_\ell \to H_\ell \subset G_{\ell_0}$ is a normal subgroup of $G_{\ell_0}$.

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Our technique of proof of the main theorem is to "present" the matchbox manifold $\mathfrak{M}$ as an inverse limit of a normal tower of coverings.

# Effros Theorem

Let $X$ be a separable and metrizable topological space. Let $G$ be a topological group with identity $e$.

For $U \subseteq G$ and $x \in X$, let $Ux = \{gx \mid g \in U\}$.

**Definition:** An action of $G$ on $X$ is *transitive* if $Gx = X$ for all $x \in X$.

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Alternate proofs of have been given by [Ancel 1987] and [van Mill 2004]. Jan Van Mill showed that Effros Theorem is equivalent to the *Open Mapping Principle* of Functional Analysis – see [ American Mathematical Monthly, pages 801–806, 2004].

# Interpretation for compact metric spaces

The metric on the group $\mathrm{Homeo}(X)$ for $(X, d_X)$ a separable, locally compact, metric space is given by

$$\begin{aligned} d_H(f,g) \quad &:= \quad \sup\{d_X(f(x), g(x)) \mid x \in X\} \\ & \qquad + \sup\left\{d_X\left(f^{-1}(x), g^{-1}(x)\right) \mid x \in X\right\} \end{aligned}$$

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**Corollary:** Let $X$ be a homogeneous compact metric space. Then for any given $\epsilon > 0$ there is a corresponding $\delta > 0$ so that if $d_X(x,y) < \delta$, there is a homeomorphism $h \colon X \to X$ with $d_H(h, id_X) < \epsilon$ and $h(x) = y$.

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**Lemma:** Let $h\colon \mathfrak{M} \to \mathfrak{M}$ be a homeomorphism. Then $h$ maps the leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. That is, every $h \in \mathrm{Homeo}(\mathfrak{M})$ is foliation-preserving.

Proof: The leaves of $\mathcal{F}$ are the path components of $\mathfrak{M}$.

# Holonomy groupoids

Let $\varphi_i \colon U_i \to [-1,1]^n \times \mathfrak{T}_i$ for $1 \leq i \leq \nu$ be the covering of $\mathfrak{M}$ by foliation charts. For $U_i \cap U_j \neq \emptyset$ we obtain the holonomy transformation

$$h_{ji} \colon D(h_{ji}) \subset \mathfrak{T}_i \longrightarrow R(h_{ji}) \subset \mathfrak{T}_j$$

These transformations generate the holonomy pseudogroup $\mathcal{G}_{\mathcal{F}}$ of $\mathfrak{M}$, modeled on the transverse metric space $\mathfrak{T} = \mathfrak{T}_1 \cup \cdots \cup \mathfrak{T}_\nu$

Typical element of $\mathcal{G}_{\mathcal{F}}$ is a composition, for $\mathcal{I} = (i_0, i_1, \ldots, i_k)$ where $U_{i_\ell} \cap U_{i_{\ell-1}} \neq \emptyset$ for $1 \leq \ell \leq k$,

$$h_{\mathcal{I}} = h_{i_k i_{k-1}} \circ \cdots \circ h_{i_1 i_0} \colon D(h_{\mathcal{I}}) \subset \mathfrak{T}_{i_0} \longrightarrow R(h_{\mathcal{I}}) \subset \mathfrak{T}_{i_k}$$

$x \in \mathfrak{T}$ is a *point of holonomy* for $\mathcal{G}_{\mathcal{F}}$ if there exists some $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ with $x \in D(h_{\mathcal{I}})$ such that $h_{\mathcal{I}}(x) = x$ and the germ of $h_{\mathcal{I}}$ at $x$ is non-trivial.

We say $\mathcal{F}$ is *without holonomy* if there are no points of holonomy.

# Equicontinuous matchbox manifolds

**Definition:** $\mathfrak{M}$ is an *equicontinuous matchbox manifold* if it admits some covering by foliation charts as above, such that for all $\epsilon > 0$, there exists $\delta > 0$ so that for all $h_{\mathcal{I}} \in \mathcal{G}_{\mathcal{F}}$ we have

$$x, y \in D(h_{\mathcal{I}}) \text{ with } d_{\mathfrak{X}}(x, y) < \delta \quad \Longrightarrow \quad d_{\mathfrak{X}}(h_{\mathcal{I}}(x), h_{\mathcal{I}}(y)) < \epsilon$$

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**Theorem:** A homogeneous matchbox manifold $\mathfrak{M}$ is equicontinuous without holonomy.

The intuitive proof is *nice*; the precise proof is rather technical.

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The intuitive proof is *nice*; the precise proof is rather technical.

**Theorem:** An equicontinuous matchbox manifold $\mathfrak{M}$ is minimal.

The proof is a *pleasant* exercise.

# Three Structure Theorems

We can now state the three main structure theorems.

**Theorem 1:** Let $\mathfrak{M}$ be an equicontinuous matchbox manifold without holonomy. Then $\mathfrak{M}$ is homeomorphic to a solenoid

$$\mathcal{S} = \lim_{\leftarrow} \; \{p_\ell \colon M_\ell \to M_{\ell-1}\}$$

IoP: equicontinuity + transversally discrete $\Longrightarrow$ effective coding.

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**Theorem 2:** Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then the bonding maps above can be chosen so that $q_\ell \colon M_\ell \longrightarrow M_0$ is a normal covering for all $\ell \geq 0$. That is, $\mathcal{S}$ is McCord.

IoP: homogeneous solenoids have transitive Galois groups for subfactors, which are normal coverings.

# Three Structure Theorems

The third structure theorem is more technical:

**Theorem 3:** Let $\mathfrak{M}$ be a homogeneous matchbox manifold. Then there exists a clopen subset $V \subset \mathfrak{T}$ such that the restricted groupoid $\mathcal{H}(\mathcal{F}, V) \equiv \mathcal{G}_{\mathcal{F}}|V$ is a group, and $\mathfrak{M}$ is homeomorphic to the suspension of the action of $\mathcal{H}(\mathcal{F}, V)$ on $V$. Thus, the fibers of the map $q_\infty \colon \mathfrak{M} \to M_0$ are homeomorphic to a profinite completion of $\mathcal{H}(\mathcal{F}, V)$.

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This suggests a discrete analogue of Molino Structure Theory.

**Problem:** What types of towers of finitely-generated groups

$$\xrightarrow{p_{\ell+1}} G_\ell \xrightarrow{p_\ell} G_{\ell-1} \xrightarrow{p_{\ell-1}} \cdots \xrightarrow{p_2} G_1 \xrightarrow{p_1} G_0$$

arise from equicontinuous matchbox minimal sets of $C^r$-foliations?

The role of $r$ appears to be fundamental – say $r \geq 1$.

# Coding & Quasi-Tiling

Let $\mathfrak{M}$ be an equicontinuous matchbox manifold without holonomy.

Fix basepoint $w_0 \in int(\mathfrak{T}_1)$ with corresponding leaf $L_0 \subset \mathfrak{M}$.

The equivalence relation on $\mathfrak{T}$ induced by $\mathcal{F}$ is denoted $\Gamma$, and we have the following subsets:

$$\begin{aligned} \Gamma_W &= \left\{(w, w') \mid w \in W \ , \ w' \in \mathcal{O}(w)\right\} \\ \Gamma_W^W &= \left\{(w, w') \mid w \in W \ , \ w' \in \mathcal{O}(w) \cap W\right\} \\ \Gamma_0 &= \left\{w' \in W \mid (w_0, w') \in \Gamma_W^W\right\} = \mathcal{O}(w_0) \cap W \end{aligned}$$

Note that $\Gamma_W^W$ is a groupoid, with object space $W$. The assumption that $\mathcal{F}$ is without holonomy implies $\Gamma_W^W$ is equivalent to the groupoid of germs of local holonomy maps induced from the restriction of $\mathcal{G}_{\mathcal{F}}$ to $W$.

# Equicontinuity & uniform domains

**Proposition:** Let $\mathfrak{M}$ be an equicontinuous matchbox manifold without holonomy. Given $\epsilon_* > 0$, then there exists $\delta_* > 0$ such that:

- for all $(w, w') \in \Gamma_W^W$ the corresponding holonomy map $h_{w,w'}$ satisfies $D_{\mathfrak{T}}(w, \delta_*) \subset D(h_{w,w'})$
- $d_{\mathfrak{T}}(h_{w,w'}(z), h_{w,w'}(z')) < \epsilon_*$ for all $z, z' \in D_{\mathfrak{T}}(w, \delta_*)$.

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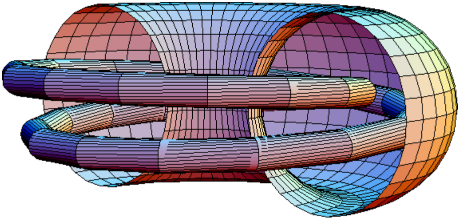
Let $W \subset \mathfrak{X}_1$ be a clopen subset with $w_0 \in W$. Decompose $W$ into clopen subsets of diameter $\epsilon_\ell > 0$,

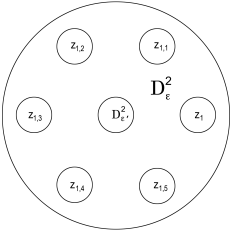
$$W = W_1^\ell \cup \cdots \cup W_{\beta_\ell}^\ell$$

Set $\eta_\ell = \min \left\{ d_{\mathfrak{X}}(W_i^\ell, W_j^\ell) \mid 1 \leq i \neq j \leq \beta_\ell \right\} > 0$ and let $\delta_\ell > 0$ be the constant of equicontinuity as above.

# An illustration

The idea is to translate data from "small tubes" to coding functions, as suggested in the pictures:





# The orbit coding function

- The code space $\mathcal{C}_\ell = \{1, \ldots, \beta_\ell\}$
- For $w \in W$, the $\mathcal{C}^\ell_w$-code of $u \in W$ is the function $C^\ell_{w,u} \colon \Gamma_w \to \mathcal{C}_\ell$ defined as: for $w' \in \Gamma_w$ set $C^\ell_{w,u}(w') = i$ if $h_{w,w'}(u) \in W^\ell_i$.
- Define $V^\ell = \left\{ u \in W^\ell_1 \mid C^\ell_{w_0,u}(w') = C^\ell_{w_0,w_0}(w') \text{ for all } w' \in \Gamma_0 \right\}$

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**Lemma:** If $u, v \in W$ with $d_{\mathfrak{X}}(u, v) < \delta_\ell$ then $C^\ell_{w,u}(w') = C^\ell_{w,v}(w')$ for all $w' \in \Gamma_w$. Hence, the function $C^\ell_w(u) = C^\ell_{w,u}$ is locally constant in $u$.

Thus, $V^\ell$ is open, and the translates of this set define a $\Gamma_0$-invariant clopen decomposition of $W$.

# The coding decomposition

The Thomas tube $\widetilde{\mathfrak{N}}_\ell$ for $\mathfrak{M}$ is the "saturation" of $V^\ell$ by $\mathcal{F}$.

The saturation is necessarily all of $\mathfrak{M}$. But the tube structure comes with a vertical fibration, which allows for collapsing the tube in foliation charts. This is the basis of the main technical result:

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**Theorem:** For $\mathrm{diam}(V^\ell)$ sufficiently small, there is a quotient map $\Pi_\ell \colon \widetilde{\mathfrak{N}}_\ell \to M_\ell$ whose fibers are the transversal sections isotopic to $V^\ell$, and whose base if a compact manifold. This yields compatible maps $\Pi_\ell \colon \mathfrak{M} \to M_\ell$ which induce the solenoid structure on $\mathfrak{M}$.

Furthermore, if $\mathfrak{M}$ is homogeneous, then $\mathrm{Homeo}(\mathfrak{M})$ acts transitively on the fibers of the tower induced by the maps $\Pi_\ell \colon \mathfrak{M} \to M_\ell$, hence the tower is normal.

# Three Conjectures

**Conjecture:** If $\mathfrak{M}$ is matchbox minimal set for $C^r$-foliation of codimension $q \geq 2$, then $\mathfrak{M}$ is modeled on a nilpotent group $G_0$. ($r = 1$, or $r \geq 3$?)

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**Conjecture:** Let $\mathfrak{M}$ be a matchbox manifold, then its transverse dynamics, and so its topological type, are determined by an "algebraic system" formed by an inverse system of graphs.

That is, the results of the paper "On the dynamics of $\mathbb{G}$-solenoids. Applications to Delone sets", by R. Benedetti and J.-M. Gambaudo, **Ergodic Theory Dyn. Syst.**, 2003, extend to general matchbox manifolds.

# The view from Sant Cugat

But we really are in the dark about all this...